Mathematics

ALGEBRA

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Mathematics for JEE/ISEET: Algebra

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Preface

While the paper-setting pattern and assessment methodology have been revised many times over and newer criteria devised to help develop more aspirant-friendly engineering entrance tests, the need to standardize the selection processes and their outcomes at the national level has always been felt. A combined national-level engineering entrance examination has finally been proposed by the Ministry of Human Resource Development, Government of India. The Joint Entrance Examination (JEE) to India's prestigious engineering institutions (IITs, IIITs, NITs, ISM, IISERs, and other engineering colleges) aims to serve as a common national-level engineering entrance test, thereby eliminating the need for aspiring engineers to sit through multiple entrance tests.

While the methodology and scope of an engineering entrance test are prone to change, there are two basic objectives that any test needs to serve:

1. The objective to test an aspirant's caliber, aptitude, and attitude for the engineering field and profession.
2. The need to test an aspirant's grasp and understanding of the concepts of the subjects of study and their applicability at the grassroots level.

Students appearing for various engineering entrance examinations cannot bank solely on conventional shortcut measures to crack the entrance examination. Conventional techniques alone are not enough as most of the questions asked in the examination are based on concepts rather than on just formulae. Hence, it is necessary for students appearing for joint entrance examination to not only gain a thorough knowledge and understanding of the concepts but also develop problem-solving skills to be able to relate their understanding of the subject to real-life applications based on these concepts.

This series of books is designed to help students to get an all-round grasp of the subject so as to be able to make its useful application in all its contexts. It uses a right mix of fundamental principles and concepts, illustrations which highlight the application of these concepts, and exercises for practice. The objective of each book in this series is to help students develop their problem-solving skills/accuracy, the ability to reach the crux of the matter, and the speed to get answers in limited time. These books feature all types of problems asked in the examination—be it MCQs (one or more than one correct), assertion-reason type, matching column type, comprehension type, or integer type questions. These problems have skillfully been set to help students develop a sound problem-solving methodology.

Not discounting the need for skilled and guided practice, the material in the books has been enriched with a number of fully solved concept application exercises so that every step in learning is ensured for the understanding and application of the subject. This whole series of books adopts a multi-faceted approach to mastering concepts by including a variety of exercises asked in the examination. A mix of questions helps stimulate and strengthen multi-dimensional problem-solving skills in an aspirant.

It is imperative to note that this book would be as profound and useful as you want it to be. Therefore, in order to get maximum benefit from this book, we recommend the following study plan for each chapter.

Step 1: Go through the entire opening discussion about the fundamentals and concepts.

Step 2: After learning the theory/concept, follow the illustrative examples to get an understanding of the theory/concept.

Overall the whole content of the book is an amalgamation of the theme of mathematics with ahead-of-time problems, which equips the students with the knowledge of the field and paves a confident path for them to accomplish success in the JEE.

With best wishes!

GHANSHYAM TEWANI

CHAPTER 1

Number System, Inequalities and Theory of Equations

- Constant and Variables
- What is Function
- Intervals
- Inequalities
- Generalized Method of Intervals for Solving Inequalities
- Absolute Value of x
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- Geometrical Meaning of Roots (Zeros) of an Equation
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- Finding the Range of a Function Involving Quadratic Expression
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CONSTANT AND VARIABLES

In *mathematics*, a **variable** is a *value* that may change within the scope of a given problem or set of operations.

In contrast, a **constant** is a value that remains unchanged, though often unknown or undetermined.

Dependent and Independent Variables

Variables are further distinguished as being either a **dependent variable** or an **independent variable**. Independent variables are regarded as inputs to a system and may take on different values freely.

Dependent variables are those values that change as a consequence to changes in other values in the system.

When one value is completely determined by another, or of several others, then it is called a function of the other value or values. In this case the value of the function is a dependent variable and the other values are independent variables. The notation $f(x)$ is used for the value of the function f with x representing the independent variable.

For example, $y = f(x) = 3x^2$, here we can take x as any real value, hence x is independent variable. But value of y depends on value of x, hence y is dependent variable.

WHAT IS FUNCTION

To provide the classical understanding of functions, think of a *function* as a kind of machine. You feed the machine raw materials, and the machine changes the raw materials into a finished product based on a specific set of instructions. The kinds of functions we consider here, for the most part, take in a real number, change it in a formulaic way, and give out a real number (possibly the same as the one it took in). Think of this as an *input–output machine*; you give the function an input, and it gives you an output.

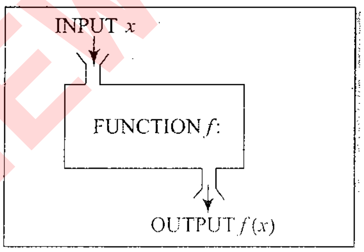


Fig. 1.1

For example, the squaring function takes the input 4 and gives the output value 16. The same squaring function takes the input 1 and gives the output value 1.

A function is always defined as "of a variable" which tells us what to replace in the formula for the function.

For example, $f(x) = 3x + 2$ tells us:

- The function f is a function of x.
- To evaluate the function at a certain number, replace the x with that number.
- Replacing x with that number in the right side of the function will produce the function's output for that certain input.
- In English, the above definition of f is interpreted, "Given *a* number, f will return *two more than the triple of that number*."

Thus, if we want to know the value (or output) of the function at 3:

$$f(x) = 3x + 2$$
$$f(3) = 3(3) + 2 = 11$$

Thus, the value of f at 3 is 11.

Note that $f(3)$ means the value of the dependent variable when "x" takes on the value of 3. So we see that the number 11 is the output of the function when we give the number 3 as the input. We refer to the input as the **argument** of the function (or the **independent variable**), and to the output as the **value** of the function at the given argument (or the **dependent variable**). A good way to think of it is the dependent variable $f(x)$ depends on the value of the independent variable x.

The formal definition of a function states that a function is actually a *rule* that associates elements of one set called the *domain* of the function with the elements of another set called the *range* of the function. For each value, we select from the domain of the function, there exists exactly one corresponding element in the range of the function. The definition of the function tells us which element in the range corresponds to the element we picked from the domain. Classically, the element picked from the domain is pictured as something that is fed into the function and the corresponding element in the range is pictured as the output. Since we "pick" the element in the domain whose corresponding element in the range we want to find, we have control over what element we pick and hence this element is also known as the "independent variable". The element mapped in the range is beyond our control and is "mapped to" by the function. This element is hence also known as the "dependent variable", for it depends on which independent variable we pick. Since the elementary idea of functions is better understood from the classical viewpoint, we shall use it hereafter. However, it is still important to remember the correct definition of functions at all times.

To make it simple, for the function $f(x)$, all of the possible x values constitute the domain, and all of the values $f(x)$ (y on the x-y plane) constitute the range.

Example 1.1 **A function is defined as $f(x) = x^2 - 3x$.**

(i) Find the value of $f(2)$.

(ii) Find the value of x for which $f(x) = 4$.

Sol.

(i) $f(2) = (2)^2 - 3(2) = -2$

(ii) $f(x) = 4$

$\Rightarrow x^2 - 3x = 4 \quad \Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow (x - 4)(x + 1) = 0 \quad \Rightarrow x = 4$ or -1

This means $f(4) = 4$ and $f(-1) = 4$.

Example 1.2 **If f is linear function and $f(2) = 4, f(-1) = 3$, then find $f(x)$.**

Sol. Let linear function is $f(x) = ax + b$

Given $f(2) = 4 \Rightarrow 2a + b = 4$ (1)

Also $f(-1) = 3 \Rightarrow -a + b = 3$ (2)

Solving (1) and (2) we get $a = \frac{1}{3}$ and $b = \frac{10}{3}$

Hence, $f(x) = \frac{x+10}{3}$

Example 1.3 **A function is defined as $f(x) = \frac{x^2+1}{3x-2}$. Can $f(x)$ take a value 1 for any real x?**

Also find the value/values of x for which $f(x)$ takes the value 2.

Sol. Here $f(x) = \frac{x^2+1}{3x-2} = 1$

$\Rightarrow \quad x^2 + 1 = 3x - 2$

$\Rightarrow \quad x^2 - 3x + 3 = 0.$

Now this equation has no real roots as $D < 0$.

Hence, value of $f(x)$ cannot be 1 for any real x.

For $f(x) = 2$ we have $\frac{x^2+1}{3x-2} = 2$

or $\quad x^2 + 1 = 6x - 4$ or $x^2 - 6x + 5 = 0$

or $\quad (x-1)(x-5) = 0$

or $\quad x = 1, 5$

Example 1.4 **Find the values of x for which the following functions are defined. Also find all possible values which functions take.**

(i) $f(x) = \frac{1}{x+1}$ **(ii)** $f(x) = \frac{x-2}{x-3}$ **(iii)** $f(x) = \frac{2x}{x-1}$

Sol.

(i) $f(x) = \frac{1}{x+1}$ is defined for all real values of x except when $x + 1 = 0$

Hence, $f(x)$ is defined for $x \in R - \{-1\}$.

Let $y = \frac{1}{x+1}$

Here we cannot find any real x for which $y = \frac{1}{x+1} = 0$

For y other than '0', there exists a real number x.

Hence, $\frac{1}{x+1} \in R - \{0\}$.

(ii) $f(x) = \frac{x-2}{x-3}$ is defined for all real values of x except when $x - 3 = 0$.

Hence, $f(x)$ is defined for $x \in R - \{3\}$

Let $y = \frac{x-2}{x-3}$

Here we cannot find any real x for which $y = \frac{x-2}{x-3} = 1$

Note: *When $\frac{x-2}{x-3} = 1$, we have $x - 2 = x - 3$ or $-2 = -3$ which is absurd.*

For y other than '1' there exists a real number x.

Hence, $\frac{1}{x+1} \in R - \{1\}$.

(iii) $f(x) = \frac{2x}{x-1}$ is defined for all real values of x except when $x - 1 = 0$

Hence, $f(x)$ is defined for $x \in R - \{1\}$

Let $y = \frac{2x}{x-1}$

Here we cannot find any real x for which $y = \frac{2x}{x-1} = 2$

Note: *When $\frac{2x}{x-1} = 2$, we have $2x = 2x - 2$ or $0 = -2$ which is absurd.*

For y other than '2' there exists a real number x.

Hence, $\frac{2x}{x-1} \in R - \{2\}$.

Example 1.5 **If $f(x) = \begin{cases} x^3, & x < 0 \\ 3x-2, & 0 \le x \le 2 \\ x^2+1, & x > 2 \end{cases}$, then find the value of $f(-1) + f(1) + f(3)$.**

Also find the value/values of x for which $f(x) = 2$.

Sol. Here function is differently defined for three different intervals mentioned.

For $x = -1$, consider $f(x) = x^3$

$\Rightarrow \quad f(-1) = -1$

For $x = 1$, consider $f(x) = 3x - 2$

$\Rightarrow \quad f(1) = 1$

For $x = 3$, consider $f(x) = x^2 + 1$

$\Rightarrow \quad f(3) = 10$

$\Rightarrow \quad f(-1) + f(1) + f(3) = -1 + 1 + 10 = 10$

Also when $f(x) = 2$,

for $x^3 = 2$, $x = 2^{1/3}$, which is not possible as $x < 0$.

for $3x - 2 = 2$, $x = 4/3$, which is possible as $0 \le x \le 2$.

For $x^2 + 1 = 2$, $x = \pm 1$, which is not possible as $x > 2$.

Hence, for $f(x) = 2$, we have $x = 4/3$.

INTERVALS

The set of numbers between any two real numbers is called interval. The following are the types of interval.

Closed Interval

$x \in [a, b] \equiv \{x : a \le x \le b\}$

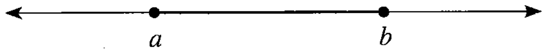


Fig. 1.2

Open Interval

$x \in (a, b)$ or $]a, b[\equiv \{x : a < x < b\}$

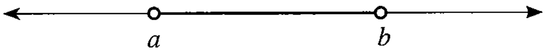


Fig. 1.3

Semi-Open or Semi Closed Interval

$x \in [a, b)$ or $[a, b[= \{x: a \le x < b\}$

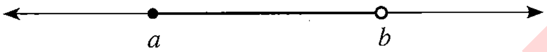


Fig. 1.4

$x \in]a, b]$ or $(a, b] = \{x: a < x \le b\}$

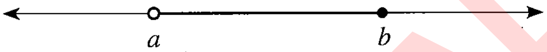


Fig. 1.5

Note:

- *A set of all real numbers can be expressed as* $(-\infty, \infty)$
- $x \in (-\infty, a) \cup (b, \infty) \Rightarrow x \in R - [a, b]$
- $x \in (-\infty, a] \cup [b, \infty) \Rightarrow x \in R - (a, b)$

INEQUALITIES

Some Important Facts about Inequalities

The following are some very useful points to remember:

(i) $a \le b$ either $a < b$ or $a = b$

(ii) $a < b$ and $b < c \Rightarrow a < c$ (transition property)

(iii) $a < b \Rightarrow -a > -b$, i.e., inequality sign reverses if both sides are multiplied by a negative number

(iv) $a < b$ and $c < d \Rightarrow a + c < b + d$ and $a - d < b - c$.

(v) If both sides of inequality are multiplied (or divided) by a positive number, inequality does not change. When both of its sides are multiplied (or divided) by a negative number, inequality gets reversed.

i.e., $a < b \Rightarrow ka < kb$ if $k > 0$ and $ka > kb$ if $k < 0$

(vi) $0 < a < b \Rightarrow a^r < b^r$ if $r > 0$ and $a^r > b^r$ if $r < 0$

(vii) $a + \dfrac{1}{a} \ge 2$ for $a > 0$ and equality holds for $a = 1$

(viii) $a + \dfrac{1}{a} \le -2$ for $a < 0$ and equality holds for $a = -1$

(ix) **Squaring an inequality:**

If $a < b$, then $a^2 < b^2$ does not follows always:

Consider the following illustrations:

$2 < 3 \Rightarrow 4 < 9$, but $-4 < 3 \Rightarrow 16 > 9$

Also if $x > 2 \Rightarrow x^2 > 4$, but for $x < 2 \Rightarrow x^2 \ge 0$

If $2 < x < 4 \Rightarrow 4 < x^2 < 16$

If $-2 < x < 4 \Rightarrow 0 \le x^2 < 16$

If $-5 < x < 4 \Rightarrow 0 \le x^2 < 25$

In fact $a < b \Rightarrow a^2 < b^2$ follows only when absolute value of a is less than the absolute value of b or distance of a from zero is less than the distance of b from zero on real number line.

(x) **Law of reciprocal:**

If both sides of inequality have same sign, while taking its reciprocal the sign of inequality gets reversed. i.e., $a > b > 0 \Rightarrow \dfrac{1}{a} < \dfrac{1}{b}$ and $a < b < 0 \Rightarrow \dfrac{1}{a} > \dfrac{1}{b}$

But if both sides of inequality have opposite sign, then while taking reciprocal sign of inequality does not change, i.e.

$a < 0 < b \Rightarrow \dfrac{1}{a} < \dfrac{1}{b}$

$$\text{If } x \in [a, b] \Rightarrow \begin{cases} \dfrac{1}{x} \in \left[\dfrac{1}{b}, \dfrac{1}{a}\right], \text{ if } a \text{ and } b \text{ have same sign} \\ \dfrac{1}{x} \in \left(-\infty, \dfrac{1}{a}\right] \cup \left[\dfrac{1}{b}, \infty\right), \text{ if } a \text{ and } b \text{ have opposite signs} \end{cases}$$

Example 1.6 **Find the values of x^2 for the given values of x.**

(i) $x < 2$ (ii) $x > -1$ (iii) $x \ge 2$ (iv) $x < -1$

Sol.

(i) When $x < 2$ we have $x \in (-\infty, 0) \cup [0, 2)$

for $x \in [0, 2)$, $x^2 \in [0, 4)$

for $x \in (-\infty, 0)$, $x^2 \in (0, \infty)$

$\Rightarrow$ for $x < 2$, $x^2 \in [0, 4) \cup (0, \infty)$

$\Rightarrow$ $x \in [0, \infty)$

(ii) When $x > -1$ we have $x \in (-1, 0) \cup [0, \infty)$

for $x \in (-1, 0)$, $x^2 \in (0, 1)$

for $x \in [0, \infty)$, $x^2 \in [0, \infty)$

$\Rightarrow$ for $x > -1$, $x^2 \in (0, 1) \cup [0, \infty)$

$\Rightarrow$ $x \in [0, \infty)$

(iii) Here $x \in [2, \infty)$

$\Rightarrow$ $x^2 \in [4, \infty)$

(iv) Here $x \in (-\infty, -1)$

$\Rightarrow$ $x^2 \in (1, \infty)$

Example 1.7 **Find the values of $1/x$ for the given values of x.**

(i) $x > 3$ (ii) $x < -2$ (iii) $x \in (-1, 3) - \{0\}$

Sol.

(i) We have $3 < x < \infty$

$\Rightarrow \dfrac{1}{3} > \dfrac{1}{x} > \dfrac{1}{\to \infty}$ ($\to \infty$ means tends to infinity)

$\Rightarrow 0 < \dfrac{1}{x} < \dfrac{1}{3}$

(ii) We have $-\infty < x < -2$

$\Rightarrow \quad \frac{1}{\to -\infty} > \frac{1}{x} > \frac{1}{-2}$

$\Rightarrow \quad \frac{1}{\to -\infty} > \frac{1}{x} > \frac{1}{-2}$

$\Rightarrow \quad 0 > \frac{1}{x} > -\frac{1}{2}$

(iii) $x \in (-1, 3) - \{0\}$

$\Rightarrow \quad x \in (-1, 0) \cup (0, 3)$

For $x \in (-1, 0)$

$\frac{1}{-1} > \frac{1}{x} > \frac{1}{\to 0^-}$

(here $\to 0^-$ means value of x approaches to 0 from its left hand side or negative side)

$\Rightarrow \quad -1 > \frac{1}{x} > -\infty$

$\Rightarrow \quad -\infty < \frac{1}{x} < -1 \qquad (1)$

For $x \in (0, 3)$

$\frac{1}{\to 0^+} > \frac{1}{x} > \frac{1}{3}$

(here $\to 0^+$ means value of x approaches to 0 from its right hand side or positive side)

$\Rightarrow \quad \infty > \frac{1}{x} > \frac{1}{3}$

$\Rightarrow \quad \frac{1}{3} < \frac{1}{x} < \infty \qquad (2)$

From (1) and (2), $\frac{1}{x} \in (-\infty, -1) \cup \left(\frac{1}{3}, \infty\right)$

Note: For $x \in R - \{0\}$, $\frac{1}{x} \in R - \{0\}$

Example 1.8 **Find all possible values of the following expressions:**

(i) $\frac{1}{x^2+2}$ **(ii)** $\frac{1}{x^2-2x+3}$ **(iii)** $\frac{1}{x^2-x-1}$

Sol.

(i) We know that $x^2 \geq 0\ x \in R$.

$\Rightarrow \quad x^2 + 2 \geq 2, \forall x \in R.$

or $\quad 2 \leq (x^2 + 2) < \infty$

$\Rightarrow \quad \frac{1}{2} \geq \frac{1}{x^2+2} > 0$

$\Rightarrow \quad 0 < \frac{1}{x^2+2} \leq \frac{1}{2}$

(ii) $\quad \frac{1}{x^2-2x+3} = \frac{1}{(x-1)^2+2}$

Now we know that $(x-1)^2 \geq 0\ \forall x \in R$.

$\Rightarrow \quad (x-1)^2 + 2 \geq 2\ \forall x \in R.$

or $\quad 2 \leq (x-1)^2 + 2 < \infty$

$\Rightarrow \quad \frac{1}{2} \geq \frac{1}{(x-1)^2+2} > 0$

$\Rightarrow \quad \frac{1}{x^2-2x+3} \in \left(0, \frac{1}{2}\right]$

(iii) $\quad \frac{1}{x^2-x-1} = \frac{1}{\left(x-\frac{1}{2}\right)^2 - \frac{5}{4}}$

$\left(x-\frac{1}{2}\right)^2 \geq 0, \forall x \in R$

$\Rightarrow \quad \left(x-\frac{1}{2}\right)^2 - \frac{5}{4} \geq -\frac{5}{4}, \forall x \in R$

For $\frac{1}{\left(x-\frac{1}{2}\right)^2 - \frac{5}{4}}$, we must have

$\left(x-\frac{1}{2}\right)^2 - \frac{5}{4} \in \left[-\frac{5}{4}, 0\right) \cup (0, \infty)$

$\Rightarrow \quad \frac{1}{\left(x-\frac{1}{2}\right)^2 - \frac{5}{4}} \in \left(-\infty, -\frac{4}{5}\right] \cup (0, \infty)$

Example 1.9 **Find all possible values of the following expressions:**

(i) $\sqrt{x^2-4}$ **(ii)** $\sqrt{9-x^2}$ **(iii)** $\sqrt{x^2-2x+10}$

Sol.

(i) $\sqrt{x^2-4}$

Least value of square root is 0 when $x^2 = 4$ or $x = \pm 2$. Also $x^2 - 4 \geq 0$

Hence, $\sqrt{x^2-4} \in [0, \infty)$.

(ii) $\sqrt{9-x^2}$

Least value of square root is 0 when $9 - x^2 = 0$ or $x = \pm 3$. Also, the greatest value of $9 - x^2$ is 9 when $x = 0$.

Hence, we have $0 \leq 9 - x^2 \leq 9 \Rightarrow \sqrt{9-x^2} \in [0, 3]$.

(iii) $\sqrt{x^2-2x+10} = \sqrt{(x-1)^2+9}$

Here, the least value of $\sqrt{(x-1)^2+9}$ is 3 when $x - 1 = 0$.

Also $(x-1)^2 + 9 \geq 9 \Rightarrow \sqrt{(x-1)^2+9} \geq 3$

Hence, $\sqrt{x^2-2x+10} \in [3, \infty)$.

GENERALIZED METHOD OF INTERVALS FOR SOLVING INEQUALITIES

Let $F(x) = (x-a_1)^{k_1}(x-a_2)^{k_2}\dots(x-a_{n-1})^{k_{n-1}}(x-a_n)^{k_n}$

where $k_1, k_2, \dots, k_n \in Z$ and $a_1, a_2, \dots, a_n$ are fixed real numbers satisfying the condition

$$a_1 < a_2 < a_3 < \dots < a_{n-1} < a_n$$

For solving $F(x) > 0$ or $F(x) < 0$, consider the following algorithm:

- We mark the numbers $a_1, a_2, \dots, a_n$ on the number axis and put the plus sign in the interval on the right of the largest of these numbers, i.e., on the right of a_n.
- Then we put the plus sign in the interval on the left of a_n if k_n is an even number and the minus sign if k_n is an odd number. In the next interval, we put a sign according to the following rule:
  - When passing through the point a_{n-1} the polynomial $F(x)$ changes sign if k_{n-1} is an odd number. Then we consider the next interval and put a sign in it using the same rule.
- Thus we consider all the intervals. The solution of the inequality $F(x) > 0$ is the union of all intervals in which we have put the plus sign and the solution of the inequality $F(x) < 0$ is the union of all intervals in which we have put the minus sign.

Frequently used Inequalities

(i) $(x-a)(x-b) < 0 \Rightarrow x \in (a, b)$, where $a < b$

(ii) $(x-a)(x-b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)$, where $a < b$

(iii) $x^2 \le a^2 \Rightarrow x \in [-a, a]$

(iv) $x^2 \ge a^2 \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

(v) If $ax^2 + bx + c < 0$, $(a > 0) \Rightarrow x \in (\alpha, \beta)$, where $\alpha, \beta\ (\alpha < \beta)$ are roots of the equation $ax^2 + bx + c = 0$

(vi) If $ax^2 + bx + c > 0$, $(a > 0) \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$, where $\alpha, \beta\ (\alpha < \beta)$ are roots of the equation $ax^2 + bx + c = 0$

Example 1.10 **Solve $x^2 - x - 2 > 0$.**

Sol. $x^2 - x - 2 > 0$

$\Rightarrow\ (x-2)(x+1) > 0$

Now $x^2 - x - 2 = 0 \Rightarrow x = -1, 2$.

Now on number line (x-axis) mark $x = -1$ and $x = 2$.

Now when $x > 2$, $x + 1 > 0$ and $x - 2 > 0$

$\Rightarrow\ (x+1)(x-2) > 0$

when $-1 < x < 2$, $x + 1 > 0$ but $x - 2 < 0$

$\Rightarrow\ (x+1)(x-2) < 0$

when $x < -1$, $x + 1 < 0$ and $x - 2 < 0$

$\Rightarrow\ (x+1)(x-2) > 0$

Hence, sign scheme of $x^2 - x - 2$ is

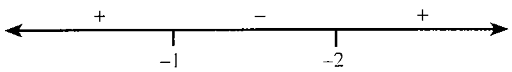


Fig. 1.6

From the figure, $x^2 - x - 2 > 0$, $x \in (-\infty, -1) \cup (2, \infty)$.

Example 1.11 **Solve $x^2 - x - 1 < 0$.**

Sol. Let's first factorize $x^2 - x - 1$.

For that let $x^2 - x - 1 = 0$

$$\Rightarrow\quad x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Now on number line (x-axis) mark $x = \dfrac{1 \pm \sqrt{5}}{2}$

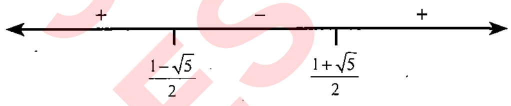


Fig. 1.7

From the sign scheme of $x^2 - x - 1$ which shown in the given figure,

$$x^2 - x - 1 < 0 \Rightarrow x \in \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$$

Example 1.12 **Solve $(x-1)(x-2)(1-2x) > 0$.**

Sol. We have $(x-1)(x-2)(1-2x) > 0$

or $\quad -(x-1)(x-2)(2x-1) > 0$

or $\quad (x-1)(x-2)(2x-1) < 0$

On number line mark $x = 1/2, 1, 2$

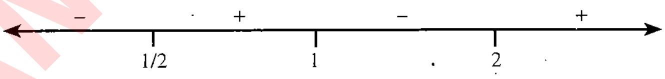


Fig. 1.8

When $x > 2$, all factors $(x-1)$, $(2x-1)$ and $(x-2)$ are positive.

Hence, $(x-1)(x-2)(2x-1) > 0$ for $x > 2$.

Now put positive and negative sign alternatively as shown in figure.

Hence, solution set of $(x-1)(x-2)(1-2x) > 0$ or $(x-1)(x-2)(2x-1) < 0$ is $(-\infty, 1/2) \cup (1, 2)$.

Example 1.13 **Solve $(2x+1)(x-3)(x+7) < 0$.**

Sol. $(2x+1)(x-3)(x+7) < 0$

Sign scheme of $(2x+1)(x-3)(x+7)$ is as follows:

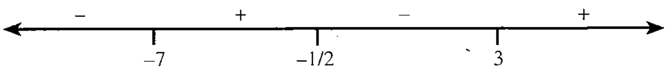


Fig. 1.9

Hence, solution is $(-\infty, -7) \cup (-1/2, 3)$.

Example 1.14 **Solve $\dfrac{2}{x} < 3$.**

Sol. $\dfrac{2}{x} < 3$

$\Rightarrow\quad \dfrac{2}{x} - 3 < 0$ (We cannot crossmultiply with x as x can be negative or positive)

$\Rightarrow \quad \dfrac{2-3x}{x} < 0$

$\Rightarrow \quad \dfrac{3x-2}{x} > 0$

$\Rightarrow \quad \dfrac{(x-2/3)}{x} > 0$

Sign scheme of $\dfrac{(x-2/3)}{x}$ is as follows:

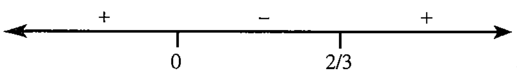


Fig. 1.10

$\Rightarrow \quad x \in (-\infty, 0) \cup (2/3, \infty)$

Example 1.15 Solve $\dfrac{2x-3}{3x-5} \geq 3$.

Sol. $\dfrac{2x-3}{3x-5} \geq 3$

$\Rightarrow \quad \dfrac{2x-3}{3x-5} - 3 \geq 0$

$\Rightarrow \quad \dfrac{2x-3-9x+15}{3x-5} \geq 0$

$\Rightarrow \quad \dfrac{-7x+12}{3x-5} \geq 0$

$\Rightarrow \quad \dfrac{7x-12}{3x-5} \leq 0$

Sign scheme of $\dfrac{7x-12}{3x-5}$ is as follows:

Fig. 1.11

$\Rightarrow \quad x \in (5/3, 12/7]$

$x = 5/3$ is not included in the solutions as at $x = 5/3$ denominator becomes zero.

Example 1.16 Solve $x > \sqrt{(1-x)}$.

Sol. Given inequality can be solved by squaring both sides.

But sometimes squaring gives extraneous solutions which do not satisfy the original inequality. Before squaring we must restrict x for which terms in the given inequality are well defined.

$x > \sqrt{(1-x)}$. Here x must be positive.

Here $\sqrt{1-x}$ is defined only when $1 - x \geq 0$ or $x \leq 1$ (1)

Squaring given inequality but sides $x^2 > 1 - x$

$\Rightarrow x^2 + x - 1 > 0 \Rightarrow \left(x - \dfrac{-1-\sqrt{5}}{2}\right)\left(x - \dfrac{-1+\sqrt{5}}{2}\right) > 0$

$\Rightarrow x < \dfrac{-1-\sqrt{5}}{2}$ or $x > \dfrac{-1+\sqrt{5}}{2}$ (2)

From (1) and (2) $x \in \left(\dfrac{\sqrt{5}-1}{2}, 1\right]$ (as x is +ve)

Example 1.17 Solve $\dfrac{2}{x^2-x+1} - \dfrac{1}{x+1} - \dfrac{2x-1}{x^3+1} \leq 0$.

Sol. $\dfrac{2}{x^2-x+1} - \dfrac{1}{x+1} - \dfrac{2x-1}{x^3+1} \geq 0$

$\Rightarrow \quad \dfrac{2(x+1)-(x^2-x+1)-(2x-1)}{(x+1)(x^2-x+1)} \geq 0$

$\Rightarrow \quad \dfrac{-(x^2-x-2)}{(x+1)(x^2-x+1)} \geq 0$

$\Rightarrow \quad \dfrac{-(x-2)(x+1)}{(x+1)(x^2-x+1)} \geq 0$

$\Rightarrow \quad \dfrac{2-x}{x^2-x+1} \geq 0$, where $x \neq 1$

$\Rightarrow \quad 2 - x \geq 0, x \neq -1$, (as $x^2 - x + 1 > 0$ for $\forall x \in R$)

$\Rightarrow \quad x \leq 2, x \neq -1$

Example 1.18 Solve $x(x+2)^2(x-1)^5(2x-3)(x-3)^4 \geq 0$.

Sol. Let $E = x(x+2)^2(x-1)^5(2x-3)(x-3)^4$.

Here for x, $(x-1)$, $(2x-3)$ exponents are odd, hence sign of E changes while crossing $x = 0, 1, 3/2$. Also for $(x+2)$, $(x-3)$ exponents are even, hence sign of E does not change while crossing $x = -2$ and $x = 3$.

Further for $x > 3$, all factors are positive, hence sign of the expression starts with positive sign from the right hand side.

The sign scheme of the expression is as shown in the following figure.

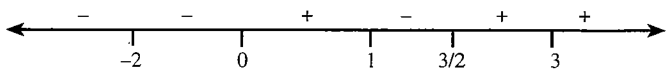


Fig. 1.12

Hence, for $E \geq 0$, we have $x \in [0, 1] \cup [3/2, \infty)$

Example 1.19 Solve $x(2^x - 1)(3^x - 9)(x - 3) < 0$.

Sol. Let $E = x(2^x - 1)(3^x - 9)(x - 3)$

Here $2^x - 1 = 0 \Rightarrow x = 0$ and when $3^x - 9 = 0 \Rightarrow x = 2$

Now mark $x = 0, 2$ and 3 on real number line.

Sign of E starts with positive sign from right hand side.

Also at $x = 0$, two factors are 0, x and $2^x - 1$, hence sign of E does not change while crossing $x = 0$.

Sign scheme of E is as shown in the following figure.

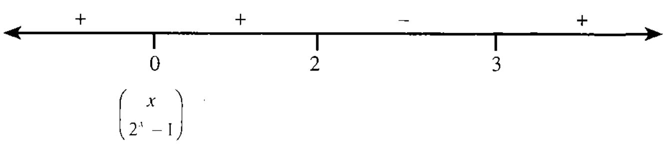


Fig. 1.13

From the figure, we have $E < 0$ for $x \in (2, 3)$.

Example 1.20 **Find all possible values of $\dfrac{x^2+1}{x^2-2}$.**

Sol. Let $y = \dfrac{x^2+1}{x^2-2}$

$\Rightarrow \quad yx^2 - 2y = x^2 + 1$

$\Rightarrow \quad x^2 = \dfrac{2y+1}{y-1}$

Now $x^2 \geq 0 \Rightarrow \dfrac{2y+1}{y-1} \geq 0$

Now $x^2 \geq 0 \Rightarrow \dfrac{2y+1}{y-1} \geq 0$

$\Rightarrow \quad y \leq -1/2$ or $y > 1$

Solving Irrational Inequalities

Example 1.21 **Solve $\sqrt{x-2} \geq -1$.**

Sol. We must have $x - 2 \geq 0$ for $\sqrt{x-2}$ to get defined, thus $x \geq 2$.

Now $\sqrt{x-2} \geq -1$, as square roots are always non-negative.

Hence, $x \geq 2$.

Note: *Some students solve it by squaring it both sides for which $x - 2 \geq 1$ or $x \geq 3$ which cause loss of interval [2, 3).*

Example 1.22 **Solve $\sqrt{x-1} > \sqrt{3-x}$.**

Sol. $\sqrt{x-1} > \sqrt{3-x}$ is meaningful if $x - 1 \geq 0$ and $3 - x \geq 0$

or $\quad 1 \leq x \leq 3$ (1)

Also $\quad \sqrt{x-1} > \sqrt{3-x}$

Squaring, we have $x - 1 > 3 - x$

$\Rightarrow \quad x > 2$ (2)

From (1) and (2), we have $2 < x \leq 3$.

Example 1.23 **Solve $x + \sqrt{x} \geq \sqrt{x} - 3$.**

Sol. $x + \sqrt{x} \geq \sqrt{x} - 3$ is meaningful only when $x \geq 0$ (1)

Now $x + \sqrt{x} \geq \sqrt{x} - 3$

$\Rightarrow \quad x \geq -3$ (2)

From (1) and (2), we have $x \geq 0$.

Example 1.24 **Solve $(x^2 - 4)\sqrt{x^2-1} < 0$.**

Sol. $\quad (x^2 - 4)\sqrt{x^2-1} < 0$

We must have $x^2 - 1 \geq 0$

or $\quad (x - 1)(x + 1) \geq 0$

or $\quad x \leq -1$ or $x \geq 1$ (1)

Also $(x^2 - 4)\sqrt{x^2-1} < 0$

$\Rightarrow \quad x^2 - 4 < 0$

$\Rightarrow \quad -2 < x < 2$ (2)

From (1) and (2), we have $x \in (-2, -1] \cup [1, 2)$

ABSOLUTE VALUE OF *x*

Absolute value of any real number x is denoted by $|x|$ (read as modulus of x).

The absolute value is closely related to the notions of *magnitude*, *distance*, and *norm* in various mathematical and physical contexts.

From an *analytic geometry* point of view, the absolute value of a real number is that number's *distance* from zero along the *real number line*, and more generally the absolute value of the difference of two real numbers is the distance between them.

Let's look at the number line:

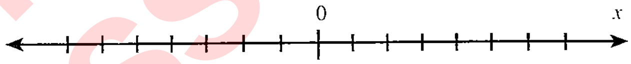


Fig. 1.14

The absolute value of x, denoted "$|x|$" (and which is read as "the absolute value of x"), is the distance of x from zero. This is why absolute value is never negative; absolute value only asks "how far?", not "in which direction?". This means not only that $|3| = 3$, because 3 is three units to the right of zero, but also that $|-3| = 3$, because -3 is three units to the left of zero.

When the number inside the absolute value (the "argument" of the absolute value) was positive anyway, we did not change the sign when we took the absolute value. But when the argument was negative, we did change the sign.

If $x > 0$ (that is, if x is positive), then the value would not change when you take the absolute value. For instance, if $x = 2$, then you have $|x| = |2| = 2 = x$. In fact, for *any* positive value of x (or if x equals zero), the sign would be unchanged, so:

For $x \geq 0$, $|x| = x$

On the other hand, if $x < 0$ (that is, if x is negative), then it will change its sign when you take the absolute value. For instance, if $x = -4$, then $|x| = |-4| = +4 = -(-4) = -x$. In fact, for *any* negative value of x, the sign would have to be changed, so:

For $x < 0$, $|x| = -x$

Thus $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Also $\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

i.e., $2 = \sqrt{2^2} = \sqrt{(-2)^2} = \left[(-2)^2\right]^{1/2} = -2$ is absurd as $\sqrt{x^2} = |x|$

$\Rightarrow \sqrt{(-2)^2} = |-2| = 2$

Thus square root exists only for non-negative numbers and its value is also non-negative.

Some students consider $\sqrt{4} = \pm 2$, which is wrong.

In fact $\sqrt{(-4)^2} = |-4| = 4$

$$\sqrt{(1-\sqrt{2})^2} = |1-\sqrt{2}| = \sqrt{2}-1 \text{ etc.}$$

Also some students write $\sqrt{x^2} = \pm x$ which is wrong, infact,

$$\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Also $a^2 < b^2 \Rightarrow \sqrt{a^2} < \sqrt{b^2} \Rightarrow |a| < |b|$

Graph of function $f(x) = y = |x|$

x	0	±1	±2	±3
y	0	1	4	9

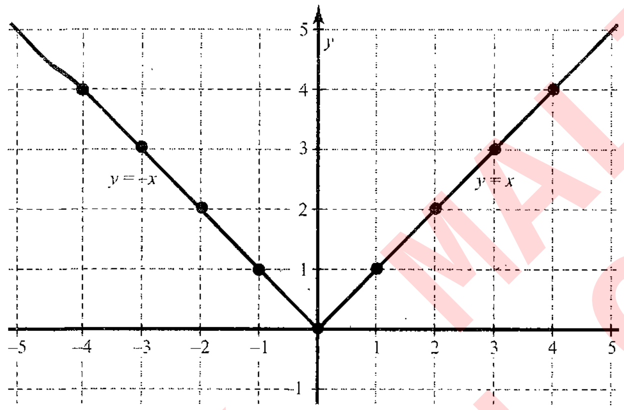


Fig. 1.15

We can see that graph of $y = |x|$ is in 1st and 2nd quadrant only where $y \geq 0$, hence $|x| \geq 0$.

Example 1.25 **Solve the following:**

(i) $|x| = 5$ **(ii)** $x^2 - |x| - 2 = 0$

Sol.

(i) $|x| = 5$, i.e., those points on real number line which are at distance 5 units from "0", which are –5 and 5.

Thus, $|x| = 5 \Rightarrow x = \pm 5$

(ii)

$$x^2 - |x| - 2 = 0$$
$$\Rightarrow |x|^2 - |x| - 2 = 0$$
$$\Rightarrow (|x| - 2)(|x| + 1) = 0$$
$$\Rightarrow |x| = 2 \; (\because |x| + 1 \neq 0)$$
$$\Rightarrow x = \pm 2$$

Example 1.26 **Find the value of x for which following expressions are defined:**

(i) $\dfrac{1}{\sqrt{x - |x|}}$ (ii) $\dfrac{1}{\sqrt{x + |x|}}$

Sol.

(i) $x - |x| = \begin{cases} x - x = 0, \text{ if } x \geq 0 \\ x + x = 2x, \text{ if } x < 0 \end{cases}$

$\Rightarrow x - |x| \leq 0$ for all x

$\Rightarrow \dfrac{1}{\sqrt{x - |x|}}$ does not take real values for any $x \in R$

$\Rightarrow \dfrac{1}{\sqrt{x - |x|}}$ is not defined for any $x \in R$.

(ii)

$$x + |x| = \begin{cases} x + x = 2x, \text{ if } x \geq 0 \\ x - x = 0, \text{ if } x < 0 \end{cases}$$

$\Rightarrow \dfrac{1}{\sqrt{x + |x|}}$ is defined only when $x > 0$

What is the geometric meaning of $|x - y|$?

$|x - y|$ is the distance between x and y on the real number line.

Example 1.27 **Solve the following:**

(i) $|x - 2| = 1$ **(ii)** $2|x + 1|^2 - |x + 1| = 3$

Sol.

(i) $|x - 2| = 1$, i.e., those points on real number line which are distance 1 units from 2.

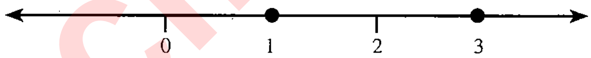


Fig. 1.16

As shown in the figure $x = 1$ and $x = 3$ are at distance 1 units from 2,

Hence, $x = 1$ or $x = 3$.

Thus $|x - 2| = 1$

$\Rightarrow x - 2 = \pm 1$

$\Rightarrow x = 1$ or $x = 3$

(ii)

$$2|x + 1|^2 - |x + 1| = 3$$
$$\Rightarrow 2|x + 1|^2 - |x + 1| - 3 = 0$$
$$\Rightarrow 2|x + 1|^2 - 3|x + 1| + 2|x + 1| - 3 = 0$$
$$\Rightarrow (2|x + 1| - 3)(|x + 1| + 1) = 0$$
$$\Rightarrow 2|x + 1| - 3 = 0$$
$$\Rightarrow |x + 1| = 3/2$$
$$\Rightarrow x + 1 = \pm 3/2$$
$$\Rightarrow x = 1/2 \text{ or } x = -5/2$$

$$|x - a| = \begin{cases} x - a, & x \geq a \\ a - x, & x < a \end{cases}, \quad \text{where } a > 0$$

In general, $|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$, where $y = f(x)$ is any real-valued function.

Example 1.28 **Solve the following:**

(i) $|x - 2| = (x - 2)$

(ii) $|x + 3| = -x - 3$

(iii) $|x^2 - x| = x^2 - x$

(iv) $|x^2 - x - 2| = 2 + x - x^2$

Sol.

(i) $|x - 2| = (x - 2)$, if $x - 2 \geq 0$ or $x \geq 2$

(ii) $|x + 3| = -x - 3$, if $x + 3 \leq 0$ or $x \leq -3$

(iii) $|x^2 - x| = x^2 - x$, if $x^2 - x \geq 0$

$\Rightarrow x(x - 1) \geq 0$

$\Rightarrow x \in (-\infty, 0] \cup [1, \infty)$

(iv) $|x^2 - x - 2| = 2 + x - x^2$

$\Rightarrow x^2 - x - 2 \leq 0$

$\Rightarrow (x - 2)(x + 1) \leq 0$

$\Rightarrow -1 \leq x \leq 2$

Example 1.29 **Solve** $1 - x = \sqrt{x^2 - 2x + 1}$.

Sol. $1 - x = \sqrt{x^2 - 2x + 1}$

$\Rightarrow 1 - x = \sqrt{(x-1)^2}$

$\Rightarrow 1 - x = |x - 1|$

$\Rightarrow 1 - x \geq 0$

$\Rightarrow x \leq 1$

Example 1.30 **Solve** $|3x - 2| = x$.

Sol. $|3x - 2| = x$

Case (i)

When $3x - 2 \geq 0$ or $x \geq 2/3$

For which we have $3x - 2 = x$ or $x = 1$.

Case (ii)

When $3x - 2 < 0$ or $x < 2/3$

For which we have $2 - 3x = x$ or $x = 1/2$.

Hence, solution set is $\{1/2, 1\}$.

Example 1.31 **Solve** $|x| = x^2 - 1$.

Sol. $x^2 - 1 = |x|$

$\Rightarrow x^2 - 1 = x$ when $x \geq 0$

or $x^2 - 1 = -x$ when $x < 0$

$x^2 - x - 1 = 0 \Rightarrow x = \dfrac{1+\sqrt{5}}{2}$ (as $x \geq 0$)

$x^2 + x - 1 = 0 \Rightarrow x = \dfrac{-1-\sqrt{5}}{2}$ (as $x < 0$)

Example 1.32 **Solve**

$$\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$$

Sol. $\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$

$\Rightarrow \sqrt{x-1-4\sqrt{x-1}+4} + \sqrt{x-1-6\sqrt{x-1}+9} = 1$

$\Rightarrow \sqrt{|\sqrt{x-1}-2|^2} + \sqrt{|\sqrt{x-1}-3|^2} = 1$

$\Rightarrow |\sqrt{x-1}-2| + |\sqrt{x-1}-3| = 1$

$\Rightarrow |\sqrt{x-1}-2| + |\sqrt{x-1}-3| = (\sqrt{x-1}-2) - (\sqrt{x-1}-3)$

$\Rightarrow \sqrt{x-1}-2 \geq 0$ and $\sqrt{x-1}-3 \leq 0$

$\Rightarrow 2 \leq \sqrt{x-1} \leq 3$

$\Rightarrow 4 \leq x - 1 \leq 9$

$\Rightarrow 5 \leq x \leq 10$

Example 1.33 **Prove that**

$$\sqrt{x^2+2x+1} - \sqrt{x^2-2x+1} = \begin{cases} -2, & x < -1 \\ 2x, & -1 \leq x \leq 1 \\ 2, & x > 1 \end{cases}$$

Sol. $\sqrt{x^2+2x+1} - \sqrt{x^2-2x+1}$

$= \sqrt{(x+1)^2} - \sqrt{(x-1)^2}$

$= |x+1| - |x-1|$

$$= \begin{cases} -x-1-(1-x), & x < -1 \\ x+1-(1-x), & -1 \leq x \leq 1 \\ x+1-(x-1), & x > 1 \end{cases}$$

$$= \begin{cases} -2, & x < -1 \\ 2x, & -1 \leq x \leq 1 \\ 2, & x > 1 \end{cases}$$

Example 1.34

(i) For $2 < x < 4$, find the values of $|x|$.

(ii) For $-3 \leq x \leq -1$, find the values of $|x|$.

(iii) For $-3 \leq x < 1$, find the values of $|x|$.

(iv) For $-5 < x < 7$, find the values of $|x - 2|$.

(v) For $1 \leq x \leq 5$, find the values of $|2x - 7|$.

Sol.

(i) $2 < x < 4$

Here values on real number line whose distance lies between 2 and 4.

Here values of x are positive $\Rightarrow |x| \in (2, 4)$

(ii) $-3 \leq x \leq -1$

Here values on real number line whose distance lies between 1 and 3 or at distance 1 or 3.

$\Rightarrow 1 \leq |x| \leq 3$

(iii) $-3 \leq x < 1$

For $-3 \leq x < 0$, $|x| \in (0, 3]$

For $0 \leq x < 1$, $|x| \in [0, 1)$

So for $-3 \leq x < 1$, $|x| \in [0, 1) \cup (0, 3]$ or $x \in [0, 3]$

(iv) $-5 < x < 7$

$\Rightarrow -7 < x - 2 < 5$

$\Rightarrow 0 \leq |x - 2| < 7$

(v) $1 \le x \le 5$

$\Rightarrow 2 \le 2x \le 10$

$\Rightarrow -5 \le 2x - 7 \le 3$

$\Rightarrow |2x - 7| \in [0, 5]$

Example 1.35 **For $x \in R$, find all possible values of**

(i) $|x - 3| - 2$ **(ii) $4 - |2x + 3|$**

Sol.

(i) We know that $|x - 3| \ge 0\ \forall\ x \in R$

$\Rightarrow \quad |x - 3| - 2 \ge -2$

$\Rightarrow \quad |x - 3| - 2 \in [-2, \infty)$

(ii) We know that $|2x + 3| \ge 0\ \forall\ x \in R$

$\Rightarrow \quad -|2x + 3| \le 0$

$\Rightarrow \quad 4 - |2x + 3| \le 4$

or $\quad 4 - |2x + 3| \in (-\infty, 4]$

Example 1.36 **Find all possible values of**

(i) $\sqrt{|x|-2}$ **(ii) $\sqrt{3-|x-1|}$** **(iii) $\sqrt{4-\sqrt{x^2}}$**

Sol.

(i) $\sqrt{|x|-2}$

We know that square roots are defined for non-negative values only.

It implies that we must have $|x| - 2 \ge 0$.

$\Rightarrow \sqrt{|x|-2} \ge 0$

(ii) $\sqrt{3-|x-1|}$ is defined when $3 - |x - 1| \ge 0$

But the maximum value of $3 - |x - 1|$ is 3 when $|x - 1|$ is 0.

Hence, for $\sqrt{3-|x-1|}$ to get defined, $0 \le 3 - |x - 1| \le 3$.

$\Rightarrow \quad \sqrt{3-|x-1|} \in \left[0, \sqrt{3}\right]$

Alternatively, $|x - 1| \ge 0$

$\Rightarrow \quad -|x - 1| \le 0$

$\Rightarrow \quad 3 - |x - 1| \le 3$

But for $\sqrt{3-|x-1|}$ to get defined, we must have

$0 \le 3 - |x - 1| \le 3 \Rightarrow 0 \le \sqrt{3-|x-1|} \le \sqrt{3}$

(iii) $\sqrt{4-\sqrt{x^2}} = \sqrt{4-|x|}$

$|x| \ge 0$

$\Rightarrow \quad -|x| \le 0$

$\Rightarrow \quad 4 - |x| \le 4$

But for $\sqrt{4-|x|}$ to get defined $0 \le 4 - |x| \le 4$

$\Rightarrow \quad 0 \le \sqrt{4-|x|} \le 2$

Example 1.37 **Solve $|x - 3| + |x - 2| = 1$.**

Sol. $|x - 3| + |x - 2| = 1$

$\Rightarrow \quad |x - 3| + |x - 2| = (3 - x) + (x - 2)$

$\Rightarrow \quad x - 3 \le 0$ and $x - 2 \ge 0$

$\Rightarrow \quad x \le 3$ and $x \ge 2$

$\Rightarrow \quad 2 \le x \le 3$

Inequalities Involving Absolute Value

(i) $|x| \le a$ (where $a > 0$)

It implies those values of x on real number line which are at distance a or less than a.

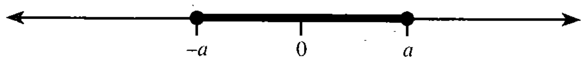


Fig. 1.17

$\Rightarrow -a \le x \le a$

e.g. $|x| \le 2 \Rightarrow -2 \le x \le 2$

$|x| < 3 \Rightarrow -3 < x < 3$

In general, $|f(x)| \le a$ (where $a > 0$) $\Rightarrow -a \le f(x) \le a$.

(ii) $|x| \ge a$ (where $a > 0$)

It implies those values of x on real number line which are at distance a or more than a

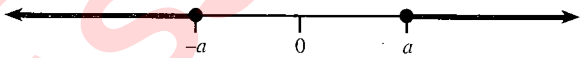


Fig. 1.18

$\Rightarrow x \le -a$ or $x \ge a$

e.g. $|x| \ge 3 \Rightarrow x \le -3$ or $x \ge 3$.

$|x| > 2 \Rightarrow x < -2$ or $x > 2$

In general, $|f(x)| \ge a \Rightarrow f(x) \le -a$ or $f(x) \ge a$.

(iii) $a \le |x| \le b$ (where $a, b > 0$)

It implies those value of x on real number line which are at distance equal a or b or between a and b.

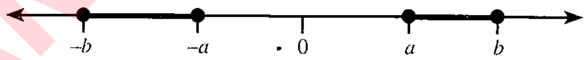


Fig. 1.19

$\Rightarrow [-b, -a] \cup [a, b]$

e.g. $2 \le |x| \le 4 \Rightarrow x \in [-4, -2]\ [2, 4]$

(iv) $|x + y| < |x| + |y|$ if x and y have opposite signs.

$|x - y| < |x| + |y|$ if x and y have same sign.

$|x + y| = |x| + |y|$ if x and y have same sign or at least one of x and y is zero.

$|x - y| = |x| + |y|$ if x and y have opposite signs or at least one of x and y is zero.

Example 1.38 **Solve $x^2 - 4|x| + 3 < 0$.**

Sol. $x^2 - 4|x| + 3 < 0$

$\Rightarrow \quad (|x| - 1)(|x| - 3) < 0$

$\Rightarrow \quad 1 < |x| < 3$

$\Rightarrow \quad -3 < x < -1$ or $1 < x < 3$

$\Rightarrow \quad x \in (-3, -1) \cup (1, 3)$

Example 1.39 **Solve $0 < |x| < 2$.**

Sol. We know that $|x| \ge 0,\ \forall\ x \in R$

But given $|x| > 0 \Rightarrow x \ne 0$

Now $0 < |x| < 2$

$\Rightarrow \quad x \in (-2, 2), x \ne 0$

$\Rightarrow \quad x \in (-2, 2) - \{0\}$

Example 1.40 **Solve $|3x - 2| < 4$.**

Sol. $|3x - 2| < 4$

$\Rightarrow \quad -4 < 3x - 2 < 4$

$\Rightarrow \quad -2 < 3x < 6$

$\Rightarrow \quad -2/3 < x < 2$

Example 1.41 **Solve $1 \le |x - 2| \le 3$.**

Sol. $1 \le |x - 2| \le 3$

$\Rightarrow \quad -3 \le x - 2 \le -1$ or $1 \le x - 2 \le 3$

$\Rightarrow \quad -1 \le x \le 1$ or $3 \le x \le 5$

$\Rightarrow \quad x \in [-1, 1] \cup [3, 5]$

Example 1.42 **Solve $0 < |x - 3| \le 5$.**

Sol. $0 < |x - 3| \le 5$

$\Rightarrow \quad -5 \le x - 3 < 0$ or $0 < x - 3 \le 5$

$\Rightarrow \quad -2 \le x < 3$ or $3 < x \le 8$

$\Rightarrow \quad x \in [-2, 3) \cup (3, 8]$

Example 1.43 **Solve $||x - 1| - 2| < 5$.**

Sol. $||x - 1| - 2| < 5$

$\Rightarrow \quad -5 < |x - 1| - 2 < 5$

$\Rightarrow \quad -3 < |x - 1| < 7$

$\Rightarrow \quad |x - 1| < 7$

$\Rightarrow \quad -7 < x - 1 < 7$

$\Rightarrow \quad -6 < x < 8$

Example 1.44 **Solve $|x - 3| \ge 2$.**

Sol. $|x - 3| \ge 2$

$\Rightarrow \quad x - 3 \le -2$ or $x - 3 \ge 2$

$\Rightarrow \quad x \le 1$ or $x \ge 5$

Example 1.45 **Solve $||x| - 3| > 1$.**

Sol. $||x| - 3| > 1$

$\Rightarrow \quad |x| - 3 < -1$ or $|x| - 3 > 1$

$\Rightarrow \quad |x| < 2$ or $|x| > 4$

$\Rightarrow \quad -2 < x < 2$ or $x < -4$ or $x > 4$

Example 1.46 **Solve $|x - 1| + |x - 2| \ge 4$.**

Sol. Let $f(x) = |x - 1| + |x - 2|$

A.	**B. $f(x)$**	**C. $f(x) \ge 4$**	**D. $A \cap C$**
$x < 1$	$1 - x + 2 - x = 3 - 2x$	$3 - 2x \ge 4 \Rightarrow x \le -2/3$	$x \le -1/2$
$1 \le x \le 2$	$x - 1 + 2 - x = 1$	$1 \ge 4$, not possible	
$x > 2$	$x - 1 + x - 2 = 2x - 3$	$2x - 3 \ge 4 \Rightarrow x \ge 7/2$	$x \ge 7/2$

Hence, solutions is $x \in (-\infty, -1/2] \cup [7/2, \infty)$.

Example 1.47 **Solve $|x + 1| + |2x - 3| = 4$.**

Sol. Let $f(x) = |x + 1| + |2x - 3|$

A.	**B. $f(x)$**	**C. $f(x) \ge 4$**	**D. $A \cap C$**
$x < -1$	$-1 - x + 3 - 2x$	$2 - 3x = 4 \Rightarrow x = -2/3$	No such x exists
$-1 \le x \le 3/2$	$x + 1 + 3 - 2x$	$4 - x = 4 \Rightarrow x = 0$	$x = 0$
$x > 3/2$	$x + 1 + 2x - 3$	$3x - 2 = 4 \Rightarrow x = 2$	$x = 2$

Hence, solutions set is $\{0, 2\}$

Example 1.48 **Solve $|x| + |x - 2| = 2$.**

Sol. We have $|x| + |x - 2| = 2$

$\Rightarrow \quad |x| + |x - 2| = x - (x - 2)$

$\Rightarrow \quad x(x - 2) \le 0$

$\Rightarrow \quad 0 \le x \le 2$

Example 1.49 **Solve $|2x - 3| + |x - 1| = |x - 2|$.**

Sol. $|2x - 3| + |x - 1| = |(2x - 3) - (x - 1)|$

$\Rightarrow \quad (2x - 3)(x - 1) \le 0$

$\Rightarrow \quad 1 \le x \le 3/2$

Example 1.50 **Solve $|x^2 + x - 4| = |x^2 - 4| + |x|$.**

Sol. $|x^2 + x - 4| = |x^2 - 4| + |x|$

$\Rightarrow \quad x(x^2 - 4) \ge 0$

$\Rightarrow \quad x(x - 2)(x + 2) \ge 0$

$\Rightarrow \quad x \in [-2, 0] \cup [2, \infty)$

Example 1.51 **If $|\sin x + \cos x| = |\sin x| + |\cos x|$ ($\sin x, \cos x \ne 0$), then in which quadrant does x lie?**

Sol. Here we have $|\sin x + \cos x| = |\sin x| + |\cos x|$.
It implies that $\sin x$ and $\cos x$ must have the same sign.
Therefore, x lies in the first or third quadrant.

Example 1.52 **Is $|\tan x + \cot x| < |\tan x| + |\cot x|$ true for any x ? If it is true, then find the values of x.**

Sol. Since $\tan x$ and $\cot x$ have always the same sign, $|\tan x + \cot x| < |\tan x| + |\cot x|$ does not hold true for any value of x.

Example 1.53 **Solve $\left|\dfrac{x - 3}{x + 1}\right| \le 1$.**

Sol. $\left|\dfrac{x - 3}{x + 1}\right| \le 1$

$$\Rightarrow \quad -1 \le \frac{x - 3}{x + 1} \le 1$$

$$\Rightarrow \quad \frac{x - 3}{x + 1} - 1 \le 0 \quad \text{and} \quad 0 \le \frac{x - 3}{x + 1} + 1$$

$$\Rightarrow \quad \frac{-4}{x + 1} \le 0 \quad \text{and} \quad 0 \le \frac{2x - 2}{x + 1}$$

$\Rightarrow$ $x > -1$ and $\{x < -1$ or $x \geq 1\}$

$\Rightarrow$ $x \geq 1$

Example 1.54 Solve $|x^2 - 2x| + |x - 4| > |x^2 - 3x + 4|$.

Sol. We have $|x^2 - 2x| + |4 - x| > |x^2 - 2x + 4 - x|$

$\Rightarrow$ $(x^2 - 2x)(4 - x) < 0$

$\Rightarrow$ $x\ (x - 2)(x - 4) > 0$

$\Rightarrow$ $x \in (0, 2) \cup (4, \infty)$

Concept Application Exercise 1.1

1. If $f(x) = \begin{cases} x+3, & x<1 \\ x^2, & 1 \leq x \leq 3 \\ 2-3x, & x>3 \end{cases}$, then which of the following is greatest ?
$f(0), f(3), f(4), f(2)$

2. If $f(x)$ is quadratic function such that $f(0) = -4$, $f(1) = -5$ and $f(-1) = -1$, then find the value of $f(3)$.

3. Find the value of x^2 for the following values of x:
(i) $[-5, -1]$ **(ii)** $(3, 6)$
(iii) $(-2, 3]$ **(iv)** $(-3, \infty)$ **(v)** $(-\infty, 4)$

4. Find the values of $1/x$ for the following values of x:
(i) $(2, 5)$ **(ii)** $[-5, -1)$
(iii) $(3, \infty)$ **(iv)** $(-\infty, -2]$
(v) $[-3, 4]$

5. Which of the following is always true?
(a) If $a < b$, then $a^2 < b^2$
(b) If $a < b$, then $\frac{1}{a} > \frac{1}{b}$
(c) If $a < b$, then $|a| < |b|$

6. Find the values of x which satisfy the inequalities simultaneously:
(i) $-3 < 2x - 1 < 19$ **(ii)** $-1 \leq \frac{2x+3}{5} \leq 3$

7. Find all the possible values which the following expressions take.
(i) $\frac{2-5x}{3x-4}$
(ii) $\sqrt{x^2 - 7x + 6}$
(iii) $\frac{x^2 - x - 6}{x - 3}$

8. Solve $\frac{x(3-4x)(x+1)}{(2x-5)} < 0$.

9. Solve $\frac{(2x+3)(4-3x)^3(x-4)}{(x-2)^2 x^5} \leq 0$.

10. Solve $\frac{(x-3)(x+5)(x-7)}{|x-4|(x+6)} \leq 0$.

11. Find all possible values of $f(x) = \frac{1-x^2}{x^2+3}$.

12. Solve $\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$.

13. Solve (i) $\frac{\sqrt{x-1}}{x-2} < 0$ (ii) $\sqrt{x-2} \leq 3$

14. Which of the following equations has maximum number of real roots?
(i) $x^2 - |x| - 2 = 0$
(ii) $x^2 - 2|x| + 3 = 0$
(iii) $x^2 - 3|x| + 2 = 0$
(iv) $x^2 + 3|x| + 2 = 0$

15. Find the number of solutions of the system of equation $x + 2y = 6$ and $|x - 3| = y$.

16. Find the values of x for which $f(x) = \sqrt{\frac{1}{|x-2|-(x-2)}}$ is defined.

17. Find all values of x for which $f(x) = x + \sqrt{x^2}$.

18. Solve $\left|\frac{x+2}{x-1}\right| = 2$.

19. If $|x^2 - 7| \leq 9$, then find the values of x.

20. Find the values of x for which $\sqrt{5 - |2x - 3|}$ is defined.

21. Solve $||x - 2| - 3| < 5$.

22. Which of the following is/are true?
(a) If $|x + y| = |x| + |y|$, then points (x, y) lie in 1st or 3rd quadrant or any of the x-axis or y-axis.
(b) If $|x + y| < |x| + |y|$, then points (x, y) lie in 2nd or 4th quadrant.
(c) If $|x - y| = |x| + |y|$, then points (x, y) lie in 2nd or 4th quadrant.

23. Solve $|x^2 - x - 2| + |x + 6| = |x^2 - 2x - 8|$.

24. Solve $|x| = 2x - 1$.

25. Solve $|2^x - 1| + |2^x + 1| = 2$.

26. Solve $|x^2 - 4x + 3| = x + 1$.

27. Solve $|x^2 - 1| + |x^2 - 4| > 3$.

28. Solve $|x - 1| - |2x - 5| = 2x$.

SOME DEFINITIONS

Real Polynomial

Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x is a real variable. Then, $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is called a real polynomial of real variable x with real coefficients.

Complex Polynomial

If $a_0, a_1, a_2, \ldots, a_n$ are complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is called a complex polynomial or a polynomial of complex coefficients.

Rational Expression or Rational Function

An expression of the form

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials in x is called a rational expression.

In the particular case when $Q(x)$ is a non-zero constant,

$$\frac{P(x)}{Q(x)}$$

reduces to a polynomial. Thus every polynomial is a rational expression but the converse is not true. Some of the examples are as follows:

(1) $\frac{x^2-5x+4}{x-2}$ (2) $x^2 - 5x + 4$

(3) $\frac{1}{x-2}$ (4) $x+\frac{1}{x}$, i.e., $\frac{x^2+1}{x}$.

Degree of a Polynomial

A polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, real or complex, is a polynomial of degree n, if $a_n \neq 0$.

The polynomials $2x^3 - 7x^2 + x + 5$ and $(3 - 2i)x^2 - ix + 5$ are polynomials of degree 3 and 2, respectively

A polynomial of second degree is generally called a quadratic polynomial, and polynomials of degree 3 and 4 are known as cubic and bi-quadratic polynomials, respectively.

Polynomial Equation

If $f(x)$ is a polynomial, then $f(x) = 0$ is called a polynomial equation.

If $f(x)$ is a quadratic polynomial, then $f(x) = 0$ is called a quadratic equation. The general form of a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$. Here, x is the variable and a, b and c are called coefficients, real or imaginary.

Roots of an Equation

The values of the variable satisfying a given equation are called its roots.

Thus, $x = \alpha$ is a root of the equation $f(x) = 0$, if $f(\alpha) = 0$.

For example, $x = 1$ is a root of the equation $x^3 - 6x^2 + 11x - 6 = 0$, because $1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 0$.

Similarly, $x = \omega$ and $x = \omega^2$ are roots of the equation $x^2 + x + 1 = 0$ as they satisfy it (where ω is the complex cube root of unity).

Solution Set

The set of all roots of an equation, in a given domain, is called the solution set of the equation.

For example, the set $\{1, 2, 3\}$ is the solution set of the equation $x^3 - 6x^2 + 11x - 6 = 0$.

Solving an equation means finding its solution set. In other words, solving an equation is the process of obtaining all its roots.

Example 1.55 **If $x = 1$ and $x = 2$ are solutions of the equation $x^3 + ax^2 + bx + c = 0$ and $a + b = 1$, then find the value of b.**

Sol. Since $x = 1$ is a root of the given equation it satisfies the equation.

Hence, putting $x = 1$ in the given equation, we get

$$a + b + c = -1 \quad (1)$$

but given that

$$a + b = 1 \quad (2)$$

$\Rightarrow \quad c = -2$

Now put $x = 2$ in the given equation, we have

$8 + 4a + 2b - 2 = 0$

$\Rightarrow \quad 6 + 2a + 2(a + b) = 0$

$\Rightarrow \quad 6 + 2a + 2 = 0$

$\Rightarrow \quad a = -4$

$\Rightarrow \quad b = 5$

Example 1.56 **Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$. It is known that $f(5) = -3f(2)$ and that 3 is a root of $f(x) = 0$, then find the other root of $f(x) = 0$.**

Sol. $f(x) = ax^2 + bx + c$

Given that $f(5) = -3f(2)$

$25a + 5b + c = -3(4a + 2b + c)$

or $\quad 37a + 11b + 4c = 0 \quad (1)$

Also $x = 3$ satisfies $f(x) = 0$

$\therefore \quad 9a + 3b + c = 0 \quad (2)$

or $\quad 36a + 12b + 4c = 0 \quad (3)$

[Multiplying Eq. (2) by 4]

Subtracting (3) from (1), we have

$a - b = 0$

$\Rightarrow \quad a = b \quad \Rightarrow$ In (2) put $b = a$,

$\Rightarrow \quad 12a + c = 0$ or $c = -12a$

Hence, equation $f(x) = 0$ becomes

$ax^2 + ax - 12a = 0$

or $\quad x^2 + x - 12 = 0$

or $\quad (x - 3)(x + 4) = 0 \quad$ or $\quad x = -4, 3$

Example 1.57 **A polynomial in x of degree three vanishes when $x = 1$ and $x = -2$, and has the values 4 and 28 when $x = -1$ and $x = 2$, respectively. Then find the value of polynomial when $x = 0$.**

Sol. From the given data $f(x) = (x - 1)(x + 2)(ax + b)$

Now $f(-1) = 4$ and $f(2) = 28$

$\Rightarrow \quad (-1 - 1)(-1 + 2)(-a + b) = 4$

and $\quad (2 - 1)(2 + 2)(2a + b) = 28$

$\Rightarrow \quad a - b = 2$ and $2a + b = 7$

Solving, $a = 3$ and $b = 1$

$\Rightarrow \quad f(x) = (x - 1)(x + 2)(3x + 1)$

$\Rightarrow \quad f(0) = -2$

Example 1.58 **If $(1 - p)$ is a root of quadratic equation $x^2 + px + (1 - p) = 0$, then find its roots.**

Sol. Since $(1 - p)$ is the root of quadratic equation

$$x^2 + px + (1 - p) = 0 \quad (1)$$

So $(1 - p)$ satisfies the above equation

$\therefore \quad (1 - p)^2 + p(1 - p) + (1 - p) = 0$

$\therefore \quad (1-p)[1 - p + p + 1] = 0$

$\therefore \quad (1 - p)(2) = 0$

$\Rightarrow \quad p = 1$

On putting this value of p in Eq. (1), we get

$x^2 + x = 0$

$\Rightarrow \quad x(x + 1) = 0$

$\Rightarrow \quad x = 0, -1$

Example 1.59 **The quadratic polynomial $p(x)$ has the following properties:**

$p(x)$ can be positive or zero for all real numbers

$p(1) = 0$ and $p(2) = 2$.

Then find the quadratic polynomial.

Sol. $p(x)$ is positive or zero for all real numbers

also $\quad p(1) = 0$

then we have $p(x) = k(x - 1)^2$, where $k > 0$

Now $\quad p(2) = 2$

$\Rightarrow \quad k = 2$

$\therefore \quad p(x) = 2(x - 1)^2$

GEOMETRICAL MEANING OF ROOTS (ZEROS) OF AN EQUATION

We know that that a real number k is a zero of the polynomial $f(x)$ if $f(k) = 0$. But why are the zeroes of a polynomial so important? To answer this, first we will see the *geometrical* representations of polynomials and the geometrical meaning of their zeroes.

We know that graph of the linear function $y = f(x) = ax + b$ is a straight line.

Consider the function $f(x) = x + 3$.

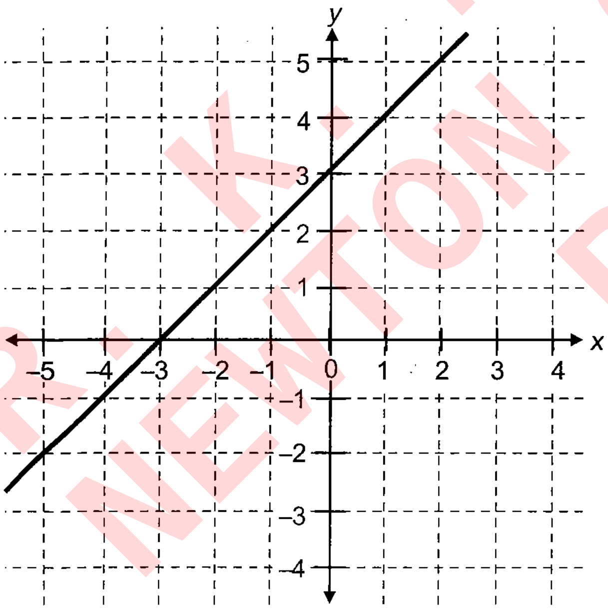


Fig. 1.20

Now we can see that this graph cuts the x-axis at $x = -3$, where value of $y = 0$ or we can say $x + 3 = 0$ (or $y = 0$) when value of $x = -3$. Thus, $x = -3$ which is a root (zero) of equation $x + 3 = 0$ is actually the value of x where graph of $y = f(x) = x + 3$ intersects the x-axis.

Consider the function $f(x) = x^2 - x - 2$, now for $f(x) = 0$ or $x^2 - x - 2 = 0$, we have $(x - 2)(x + 1) = 0$ or $x = -1$ or $x = 2$. Then graph of $f(x) = x^2 - x - 2$ cuts the x-axis at two values of x, $x = -1$ and $x = 2$.

Following is the graph of $y = f(x)$.

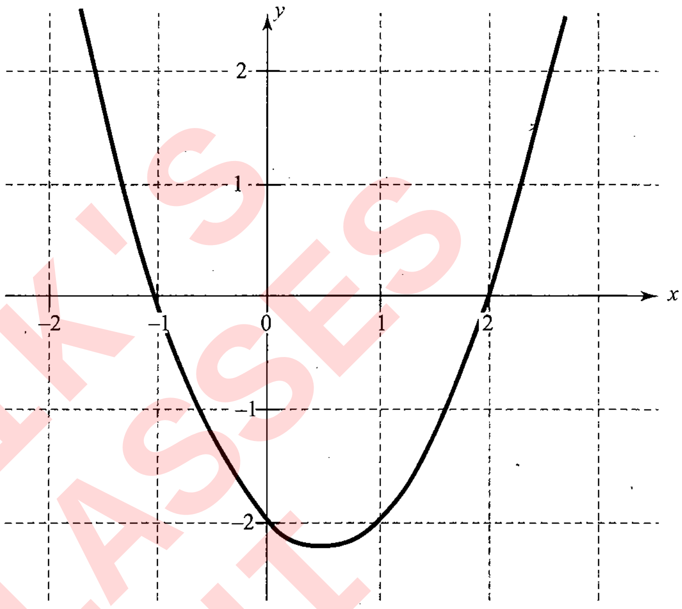


Fig. 1.21

Consider the function $f(x) = x^3 - 6x^2 + 11x - 6$, now for $f(x) = 0$ we have $(x - 1)(x - 2)(x - 3) = 0$ or $x = 1, 2, 3$. Then graph of $y = f(x)$ cuts x-axis at three values of x, $x = 1, 2, 3$.

Following is the graph of $y = f(x)$.

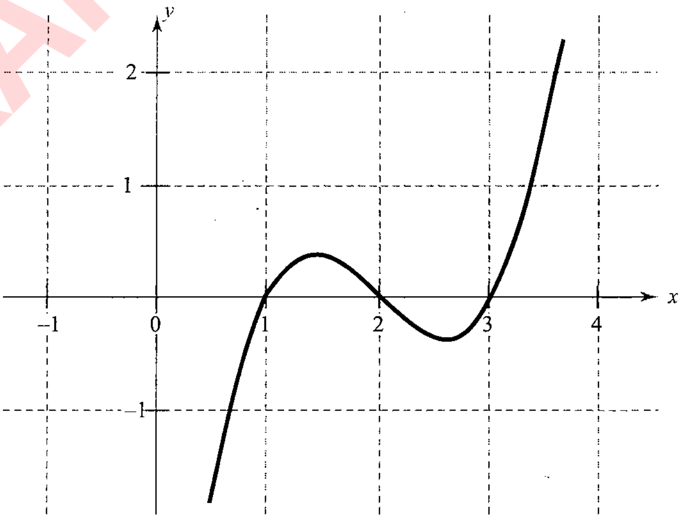


Fig. 1.22

Consider the function $f(x) = (x^2 - 3x + 2)(x^2 - x + 1)$, now for $f(x) = 0$ we have $x = 1$ or $x = 2$, as $x^2 - x + 1 = 0$ is not possible for any real value of x. Hence, $f(x) = 0$ has only two real roots and cuts x-axis for only two values of x, $x = 1$ and $x = 2$.

Following is the graph of $y = f(x)$.

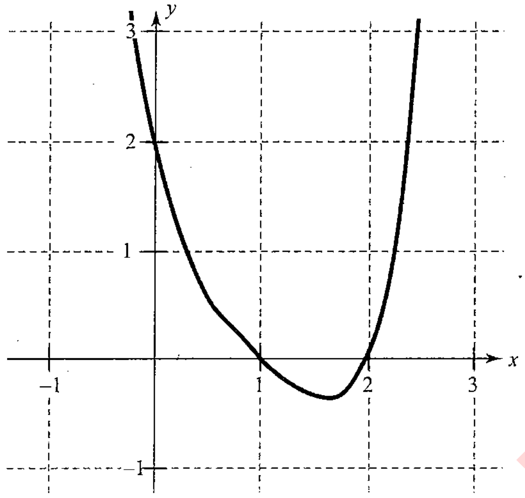


Fig. 1.23

Thus, roots of equation $f(x) = 0$ are actually those values of x where graph $y = f(x)$ meets x-axis.

Roots (Zeros) of the Equation $f(x) = g(x)$

Now we know that zeros of the equation $f(x) = 0$ are the x-coordinates of the points where graph of $y = f(x)$ intersect the x-axis, where $y = 0$ or zeros are x-coordinate of the point of intersection of $y = f(x)$ and $y = 0$ (x-axis)

Consider the equation $x + 5 = 2$.

Let's draw the graph of $y = x + 5$ and $y = 2$, which are as shown in the following figure.

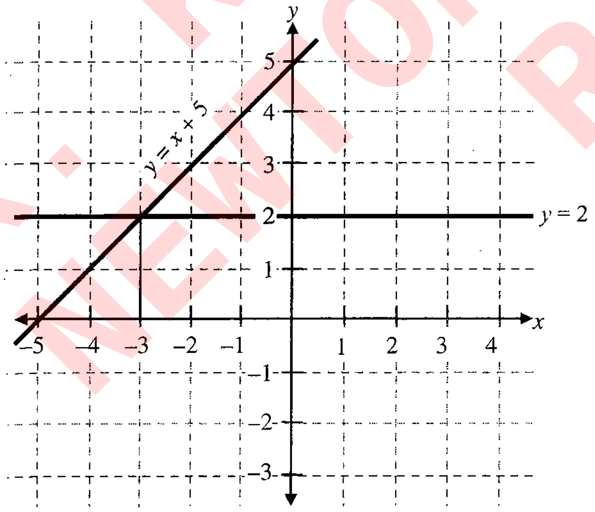


Fig. 1.24

Graph of $y = 2$ is a line parallel to x-axis at height 2 unit above x-axis. Now in the figure, we can see that graphs of $y = x + 5$ and $y = 2$ intersect at point $(-3, 2)$ where value of $x = -3$.

Also from $x + 5 = 2$, we have $x = 2 - 5$ or $x = -3$, which is a root of the equation $x + 2 = 5$. Thus root of the equation $x + 5 = 2$ occurs at point of intersection of graphs $y = x + 5$ and $y = 2$.

Consider the another example $x^2 - 2x = 2 - x$. Let's draw the graph of $y = x^2 - 2x$ and $y = 2 - x$ as shown in the following figure.

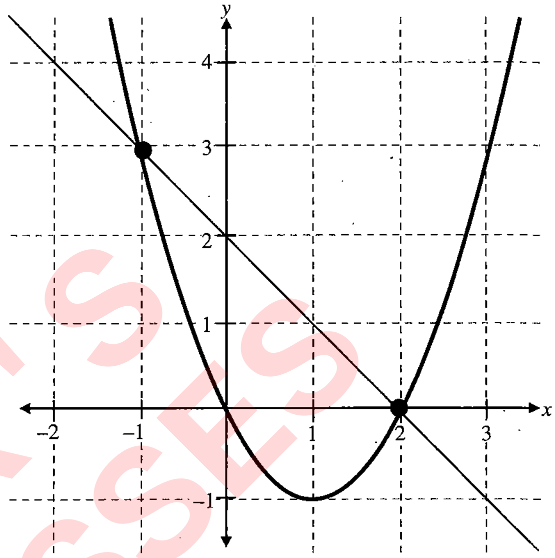


Fig. 1.25

Now in the figure, we can see that graphs of $y = x^2 - 2x$ and $y = 2 - x$ intersect at points $(-1, 3)$ and $(2, 0)$ or where values of x are $x = -1$ and $x = 2$, which are in fact zeros or roots of the equation $x^2 - 2x = 2 - x$ or $x^2 - x - 2 = 0$.

The given equation simplifies to $x^2 - x - 2 = 0$. So one can also locate the roots of the same equation by plotting the graph of $y = x^2 - x - 2$, then the roots of equation are x-coordinates of points where graph of $y = x^2 - x - 2$ intersects with the x-axis (where $y = 0$), as shown in the following figure.

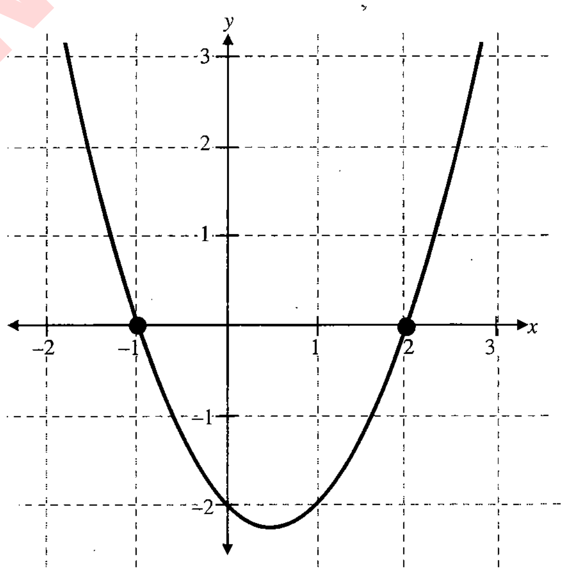


Fig. 1.26

From the above discussion we understand that roots of the equation $f(x) = g(x)$ are the x-coordinate of the points of intersection of graphs $y = f(x)$ and $y = g(x)$.

Example 1.60 **In how many points graph of $y = x^3 - 3x^2 + 5x - 3$ intersect x-axis?**

Sol. Number of point in which $y = x^3 - 3x^2 + 5x - 3$ intersect the x-axis is same as number of real roots of the equation $x^3 - 3x^2 + 5x - 3 = 0$.

Now we can see that $x = 1$ satisfies the equation, hence one root of the equation is $x = 1$.

Now dividing $x^3 - 3x^2 + 5x - 3$ by $x - 1$, we have quotient $x^2 - 2x + 3$.

Hence equation reduces to $(x - 1)(x^2 - 2x + 3) = 0$.

Now $x^2 - 2x + 3 = 0$ or $(x - 1)^2 + 2 = 0$ is not true for any real value of x.

Hence, the only root of the equation is $x = 1$.

Therefore, the graph of $y = x^3 - 3x^2 + 5x - 3$ cuts the x-axis in one point only.

Example 1.61 **In the following diagram, the graph of $y = f(x)$ is given.**

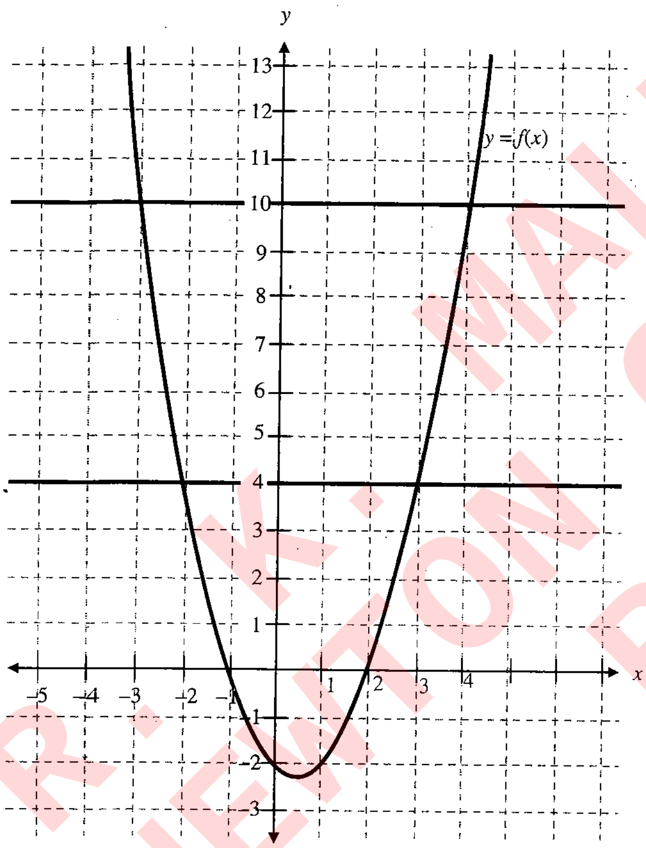


Fig. 1.27

Answer the following questions:

(a) what are the roots of the $f(x) = 0$?

(b) what are the roots of the $f(x) = 4$?

(c) what are the roots of the $f(x) = 10$?

Sol.

(a) The root of the equation $f(x) = 0$ occurs for the values of x where the graphs of $y = f(x)$ and $y = 0$ intersect.

From the diagram, for these point of intersection $x = -1$ and $x = 2$. Hence, roots of the equation $f(x) = 0$ are $x = -1$ and $x = 2$.

(b) The root of the equation $f(x) = 4$ occurs for the values of x where the graphs of $y = f(x)$ and $y = 4$ intersect.

From the diagram, for these point of intersection $x = -2$ and $x = 3$. Hence, roots of the equation $f(x) = 0$ are $x = -2$ and $x = 3$.

(c) Also roots of the equation $f(x) = 10$ are -3 and 4.

Example 1.62 **Which of the following pair of graphs intersect?**

(i) $y = x^2 - x$ and $y = 1$

(ii) $y = x^2 - 2x + 3$ and $y = \sin x$

(iii) $y = x^2 - x + 1$ and $y = x - 4$

Sol. $y = x^2 - x$ and $y = 1$ intersect if $x^2 - x = 1 \Rightarrow x^2 - x - 1 = 0$, which has real roots.

$y = x^2 - 2x + 3$ and $y = \sin x$ intersect if $x^2 - 2x + 3 = \sin x$ or $(x - 1)^2 + 2 = \sin x$, which is not possible as L.H.S. has the least value 2, while R.H.S. has the maximum value 1.

$y = x^2 - x + 1$ and $y = x - 4$ intersect if $x^2 - x + 1 = x - 4$ or $x^2 - 2x + 5 = 0$, which has non-real roots. Hence, graphs do not intersect.

Example 1.63 **Prove that graphs $y = 2x - 3$ and $y = x^2 - x$ never intersect.**

Sol. $y = 2x - 3$ and $y = x^2 - x$ intersect only when $x^2 - x = 2x - 3$ or $x^2 - 3x + 3 = 0$

Now discriminant $D = (-3)^2 - 4(3) = -3 < 0$

Hence, roots of the equation are not real, or we can say that there is no real number for which $2x - 3$ and $x^2 - x$ are equal (or $y = 2x - 3$ and $y = x^2 - x$ intersect).

Hence, proved.

KEY POINTS IN SOLVING AN EQUATION

Domain of Equation

It is a set of the values of independent variables x for which each function used in the equation is defined, i.e., it takes up finite real values. In other words, the final solution obtained while solving any equation must satisfy the domain of the expression of the parent equation.

Example 1.64 **Solve $\dfrac{x^2 - 2x - 3}{x + 1} = 0$.**

Sol. Equation $\dfrac{x^2 - 2x - 3}{x + 1} = 0$ is solvable over $R - \{-1\}$

Now $\dfrac{x^2 - 2x - 3}{x + 1} = 0$

$\Rightarrow \quad x^2 - 2x - 3 = 0$ or $(x - 3)(x + 1) = 0$

$\Rightarrow \quad x = 3$ (as $x \in R - \{-1\}$)

Example 1.65 **Solve $(x^3 - 4x)\sqrt{x^2 - 1} = 0$.**

Sol. Given equation is solvable for $x^2 - 1 \geq 0$

or $\quad x \in (-\infty, -1]\ [1, \infty)$

$(x^3 - 4x)\sqrt{x^2 - 1} = 0$

$\Rightarrow \quad x(x - 2)(x + 2)\sqrt{x^2 - 1} = 0$

$\Rightarrow \quad x = 0, -2, 2, -1, 1$

But $x \in (-\infty, -1] \cup [1, \infty)$

$\Rightarrow$ $x = \pm 1, \pm 2$

Example 1.66 Solve $\dfrac{2x-3}{x-1} + 1 = \dfrac{6x-x^2-6}{x-1}$.

Sol. $\dfrac{2x-3}{x-1} + 1 = \dfrac{6x-x^2-6}{x-1}, x \neq 1$

$\Rightarrow$ $\dfrac{3x-4}{x-1} = \dfrac{6x-x^2-6}{x-1}, x \neq 1$

$\Rightarrow$ $3x - 4 = 6x - x^2 - 6, x \neq 1$

$\Rightarrow$ $x^2 - 3x + 2 = 0, x \neq 1$

$\Rightarrow$ $x = 2$

Extraneous Roots

While simplifying the equation, the domain of the equation may expand and give the extraneous roots.

For example, consider the equation $\sqrt{x} = x - 2$.

For solving, we first square it

so $\sqrt{x} = x - 2$

$\Rightarrow$ $x = (x-2)^2$ [on squaring both sides]

$\Rightarrow$ $x^2 - 5x + 4 = 0$

$\Rightarrow$ $(x-1)(x-4) = 0$

$\Rightarrow$ $x = 1, 4$

We observe that $x = 4$ satisfies the given equation but $x = 1$ does not satisfy it.

Hence, $x = 4$ is the only solution of the given equation.

The domain of actual equation is $[2, \infty)$.

While squaring the equation, domain expands to R, which gives extra root $x = 1$.

Loss of Root

Cancellation of common factors from both sides of equation leads to loss of root.

For example, consider an equation $x^2 - 2x = x - 2$

$\Rightarrow$ $x(x-2) = x - 2$

$\Rightarrow$ $x = 1$

Here we have cancelled factor $x - 2$ which causes the loss of root, $x = 2$

The correct way of solving is

$x^2 - 2x = x - 2$

$\Rightarrow$ $x^2 - 3x + 2 = 0$

$\Rightarrow$ $(x-1)(x-2) = 0$

$\Rightarrow$ $x = 1$ and $x = 2$.

GRAPHS OF POLYNOMIAL FUNCTIONS

When the polynomial function is written in standard form, $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $(a_n \neq 0)$, the leading term is $a_n x^n$. In other words, the leading term is the term that the variable has its highest exponent. The degree of a term of a polynomial function is the exponent on the variable. The degree of the polynomial is the largest degree of all of its terms.

For drawing the graph of the polynomial function, we consider the following tests.

Test 1: Leading Co-efficient

If n is odd and the leading coefficient a_n is positive, then the graph falls to the left and rises to the right:

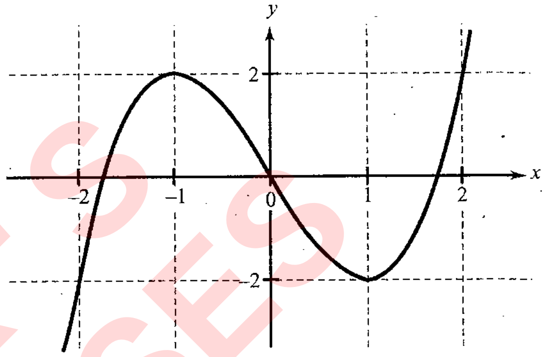


Fig. 1.28

If n is odd and the leading coefficient a_n is negative, the graph rises to the left and falls to the right.

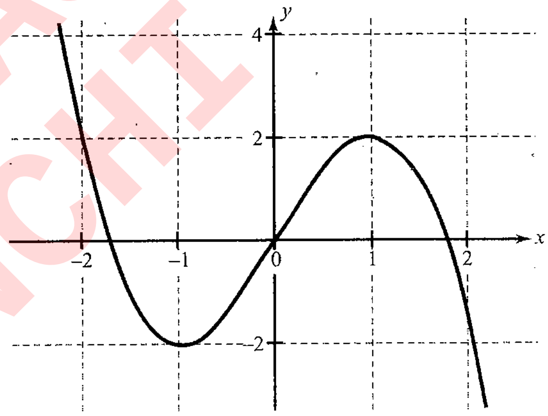


Fig. 1.29

If n is even and the leading coefficient a_n is positive, the graph rises to the left and to the right.

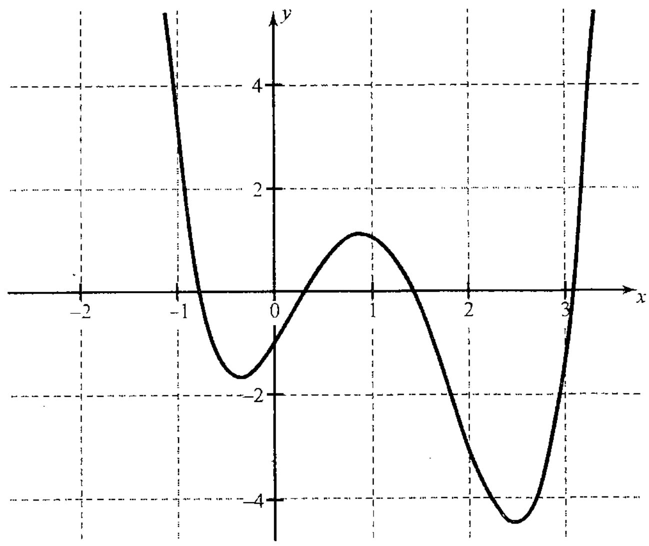


Fig. 1.30

If n is even and the leading coefficient a_n is negative, the graph falls to the left and to the right

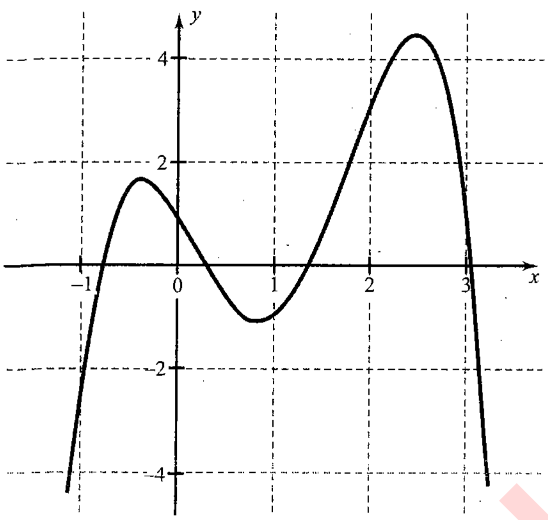


Fig. 1.31

Test 2: Roots (Zeros) of Polynomial

In other words, when a polynomial function is set equal to zero and has been completely factored and each different factor is written with the highest appropriate exponent, depending on the number of times that factor occurs in the product, the exponent on the factor that the zero is a solution for it gives the multiplicity of that zero.

The exponent indicates how many times that factor would be written out in the product, this gives us a multiplicity.

Multiplicity of Zeros and the x-Intercept

If r is a zero of even multiplicity:

This means the graph touches the x-axis at r and turns around.

This happens because the sign of $f(x)$ does not change from one side to the other side of r.

See the graph of $f(x) = (x-2)^2(x-1)(x+1)$.

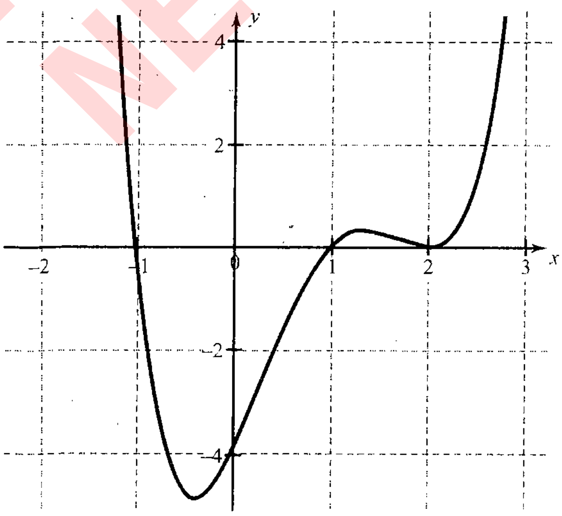


Fig. 1.32

If r is a zero of odd multiplicity:

This means the graph crosses (also touches if exponent is more than 1) the x-axis at r. This happens because the sign of $f(x)$ changes from one side to the other side or r.

See the graph of $f(x) = (x-1)(3x-2)(x-3)^3$

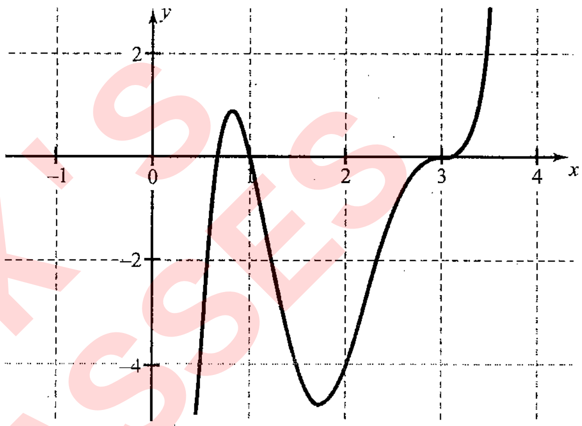


Fig. 1.33

Thus, in general, polynomial function graphs consist of a smooth line with a series of hills and valleys. The hills and valleys are called **turning points**. The maximum possible number of turning points is one less than the degree of the polynomial. The point where graph has turning point, derivative of function $f(x)$ becomes zero, which provides point of local minima or local maxima. Knowledge of derivative provides great help in drawing the graph of the function, hence finding its point of intersection with x-axis or roots of the equation $f(x) = 0$. Also we know that geometrically the derivative of function at any point of the graph of the function is equal to the slope of tangent at that point to the curve.

Consider the following graph of the function $y = f(x)$ as shown in the following figure.

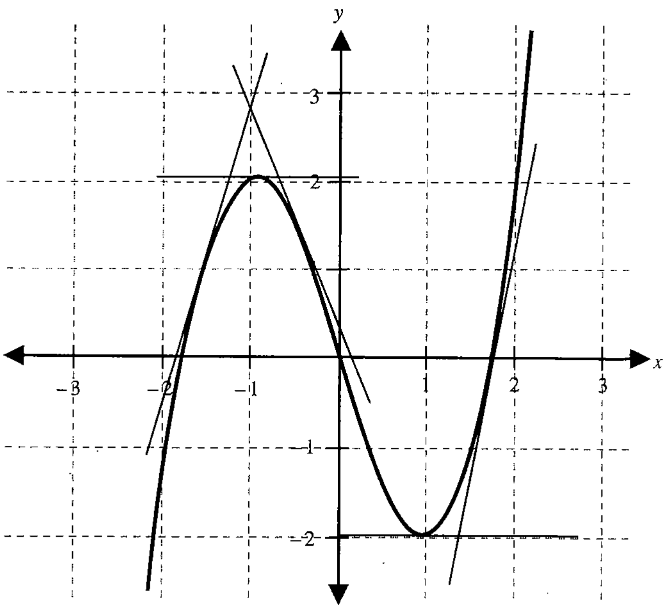


Fig. 1.34

In the figure, we can see that tangent to the curve at point for which $x < -1$ and $x > 1$ makes acute angle with the positive direction of x-axis, hence derivative is positive for these points. For $-1 < x < 1$, tangent to the curve makes obtuse angle with the positive direction of x-axis, hence derivative is negative at these points. At $x = -1$ and $x = 1$, tangent is parallel to x-axis, where derivative is zero.

Here $x = -1$ is called point of maxima, where derivative changes sign from positive to negative (from left to right), and $x = 1$ is called point of minima, where derivative changes sign from negative to positive (from left to right).

At point of maxima and minima, derivative of the function is zero.

Example 1.67 **Using differentiation method check how many roots of the equation $x^3 - x^2 + x - 2 = 0$ are real?**

Sol. Let $y = f(x) = x^3 - x^2 + x - 2$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 2x + 1$$

Let $3x^2 - 2x + 1 = 0$, now this equation has non-real roots, i.e., derivative never becomes zero or graph of $y = f(x)$ has no turning point.

Also when $x \to \infty, f(x) \to \infty$ and when $x \to -\infty, f(x) \to -\infty$

Further $3x^2 - 2x + 1 > 0 \;\forall\; x \in R$

Thus graph of the function is as shown in the following figure.

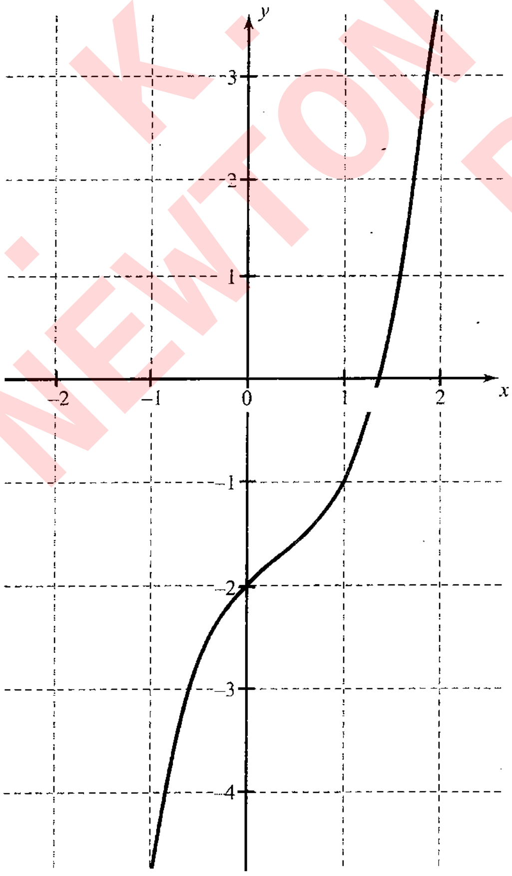


Fig. 1.35

Also $f(0) = -2$, hence graph cuts the x-axis for some positive value of x.

Hence, the only root of the equation is positive.

Thus we can see that differentiation and then graph of the function is much important in analyzing the equation.

Example 1.68 **Analyze the roots of the following equations:**

(i) $2x^3 - 9x^2 + 12x - (9/2) = 0$

(ii) $2x^3 - 9x^2 + 12x - 3 = 0$

Sol.

(i) Let $f(x) = 2x^3 - 9x^2 + 12x - (9/2)$

Then $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$

Now $f'(x) = 0 \Rightarrow x = 1$ and $x = 2$.

Hence, graph has turn at $x = 1$ and at $x = 2$.

Also $f(1) = 2 - 9 + 12 - (9/2) > 0$

and $f(2) = 16 - 36 + 24 - (9/2) < 0$

Hence, graph of the function $y = f(x)$ is as shown in the following figure.

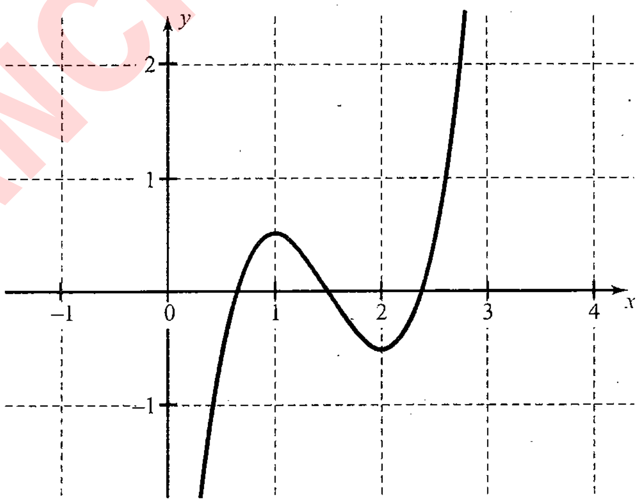


Fig. 1.36(a)

As shown in the figure, graph cuts x-axis at three distinct point.

Hence, equation $f(x) = 0$ has three distinct roots.

(ii) For $2x^3 - 9x^2 + 12x - 3 = 0$, $f(x) = 2x^3 - 9x^2 + 12x - 3$

$f'(x) = 0 \Rightarrow x = 1$ and $x = 2$

Also $f(1) = 2 - 9 + 12 - 3 = 2$ and $f(2) = 16 - 36 + 24 - 3 = 1$

Hence, graph of $y = f(x)$ is as shown in the following figure.

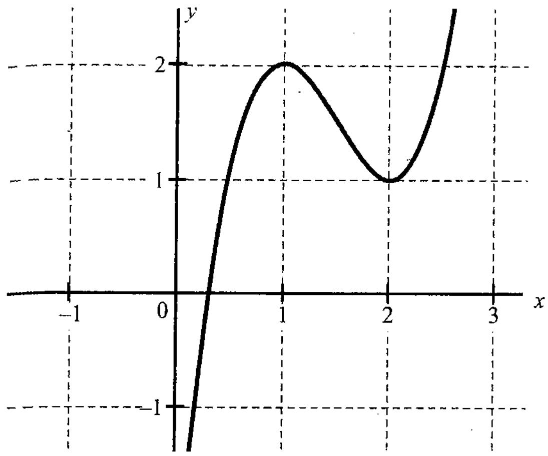


Fig. 1.36(b)

Thus from the graph, we can see that $f(x) = 0$ has only one real root, though $y = f(x)$ has two turning points.

Example 1.69 **Find how many roots of the equation $x^4 + 2x^2 - 8x + 3 = 0$ are real.**

Sol. Let $f(x) = x^4 + 2x^2 - 8x + 3$

$\Rightarrow \quad f'(x) = 4x^3 + 4x - 8 = 4(x-1)(x^2+x+2)$

Now $f'(x) = 0 \Rightarrow x = 1$

Hence graph of $y = f(x)$ has only one turn (maxima/minima).

Now $f(1) = 1 + 2 - 8 + 3 < 0$

Also when $x \to \pm\infty, f(x) \to \infty$

Then graph of the function is as shown in the following figure.

Fig. 1.37

Hence, equation $f(x) = 0$ has only two real roots.

EQUATIONS REDUCIBLE TO QUADRATIC

Example 1.70 **Solve $\sqrt{5x^2-6x+8} - \sqrt{5x^2-6x-7} = 1$.**

Sol. Let $5x^2 - 6x = y$. Then,

$$\sqrt{5x^2-6x+8} - \sqrt{5x^2-6x-7} = 1$$

$\Rightarrow \quad \sqrt{y+8} - \sqrt{y-7} = 1$

$\Rightarrow \quad (\sqrt{y+8} - \sqrt{y-7})^2 = 1$

$\Rightarrow \quad y = \sqrt{y^2+y-56}$

$\Rightarrow \quad y^2 = y^2 + y - 56$

$\Rightarrow \quad y = 56$

$\Rightarrow \quad 5x^2 - 6x = 56 \qquad [\because y = 5x^2 - 6x]$

$\Rightarrow \quad 5x^2 - 6x - 56 = 0$

$\Rightarrow \quad (5x + 14)(x - 4) = 0$

$\Rightarrow \quad x = 4, \dfrac{-14}{5}$

Clearly, both the values satisfy the given equation. Hence, the roots of the given equation are 4 and −14/5.

Example 1.71 **Solve $(x^2 - 5x + 7)^2 - (x - 2)(x - 3) = 1$.**

Sol. We have,

$(x^2 - 5x + 7)^2 - (x - 2)(x - 3) = 1$

$\Rightarrow \quad (x^2 - 5x + 7)^2 - (x^2 - 5x + 7) = 0$

$\Rightarrow \quad y^2 - y = 0$, where $-y = x^2 - 5x + 7$

$\Rightarrow \quad y(y - 1) = 0$

$\Rightarrow \quad y = 0, 1$

Now,

$y = 0$

$\Rightarrow \quad x^2 - 5x + 7 = 0$

$\Rightarrow x = \dfrac{5 \pm \sqrt{25-28}}{2} = \dfrac{5 \pm \sqrt{-3}}{2} = \dfrac{5 \pm i\sqrt{3}}{2}$

where $i = \sqrt{-1}$

and

$y = 1$

$\Rightarrow \quad x^2 - 5x + 6 = 0$

$\Rightarrow \quad (x - 3)(x - 2) = 0$

$\Rightarrow \quad 3, 2$

Hence, the roots of the equation are 2, 3, $(5+i\sqrt{3})/2$ and $(5-i\sqrt{3})/2$.

Example 1.72 **Solve the equation $4^x - 5 \times 2^x + 4 = 0$.**

Sol. We have,

$4^x - 5 \times 2^x + 4 = 0$

$\Rightarrow \quad (2^x)^2 - 5(2^x) + 4 = 0$

$\Rightarrow \quad y^2 - 5y + 4 = 0$, where $y = 2^x$

$\Rightarrow \quad (y - 4)(y - 1) = 0$

$\Rightarrow \quad y = 1, 4$

$\Rightarrow \quad 2^x = 1, 2^x = 4$

$\Rightarrow \quad 2^x = 2^0, 2^x = 2^2$

$\Rightarrow \quad x = 0, 2$

Hence, the roots of the given equation are 0 and 2.

Example 1.73 **Solve the equation $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$.**

Sol. The given equation is

$12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$

Dividing by x^2, we get

$$12x^2 - 56x + 89 - \frac{56}{x} + \frac{12}{x^2} = 0$$

$$\Rightarrow \quad 12\left(x^2 + \frac{1}{x^2}\right) - 56\left(x + \frac{1}{x}\right) + 89 = 0$$

$$\Rightarrow \quad 12\left[\left(x + \frac{1}{x}\right)^2 - 2\right] - 56\left(x + \frac{1}{x}\right) + 89 = 0$$

$$\Rightarrow \quad 12\left(x + \frac{1}{x}\right)^2 - 56\left(x + \frac{1}{x}\right) + 65 = 0$$

$$\Rightarrow \quad 12y^2 - 56y + 65 = 0, \text{ where } y = x + \frac{1}{x}$$

$$\Rightarrow \quad 12y^2 - 26y - 30y + 65 = 0$$

$$\Rightarrow \quad (6y - 13)(2y - 5) = 0$$

$$\Rightarrow \quad y = \frac{13}{6} \text{ or } y = \frac{5}{2}$$

If $y = 13/6$, then

$$x + \frac{1}{x} = \frac{13}{6}$$

$$\Rightarrow \quad 6x^2 - 13x + 6 = 0$$

$$\Rightarrow \quad (3x - 2)(2x - 3) = 0$$

$$\Rightarrow \quad x = \frac{2}{3}, \frac{3}{2}$$

If $y = 5/2$, then

$$x + \frac{1}{x} = \frac{5}{2}$$

$$\Rightarrow \quad 2x^2 - 5x + 2 = 0$$

$$\Rightarrow \quad (x - 2)(2x - 1) = 0$$

$$\Rightarrow \quad x = 2, \frac{1}{2}$$

Hence, the roots of the given equation are 2, 1/2, 2/3, 3/2.

Example 1.74 **Solve the equation $3^{x^2-x} + 4^{x^2-x} = 25$.**

Sol. We have,

$$3^{x^2-x} + 4^{x^2-x} = 25$$

$$\Rightarrow \quad 3^{x^2-x} + 4^{x^2-x} = 3^2 + 4^2$$

$$\Rightarrow \quad x^2 - x = 2$$

$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad (x - 2)(x + 1) = 0$$

$$\Rightarrow \quad x = -1, 2$$

Hence, the roots of the given equation are -1 and 2.

Example 1.75 **Solve the equation $(x - 1)^4 + (x - 5)^4 = 82$.**

Sol. Let

$$y = \frac{(x-1)+(x-5)}{2} = x - 3$$

$$\Rightarrow \quad x = y + 3$$

Putting $x = y + 3$ in the given equation, we obtain

$$(y + 2)^4 + (y - 2)^4 = 82$$

$$\Rightarrow \quad (y^2 + 4y + 4)^2 + (y^2 - 4y + 4)^2 = 82$$

$$\Rightarrow \quad \{(y^2 + 4)^2 + 4y\}^2 + \{(y^2 + 4) - 4y\}^2 = 82$$

$$\Rightarrow \quad 2\{(y^2 + 4)^2 + 16y^2\} = 82$$

$$[\because (a + b)^2 + (a - b)^2 = 2(a^2 + b^2)]$$

$$\Rightarrow \quad y^4 + 8y^2 + 16 + 16y^2 = 41$$

$$\Rightarrow \quad y^4 + 24y^2 - 25 = 0$$

$$\Rightarrow \quad (y^2 + 25)(y^2 - 1) = 0$$

$$\Rightarrow \quad y^2 + 25 = 0, y^2 - 1 = 0$$

$$\Rightarrow \quad y = \pm 5i, y = \pm 1 \qquad (\text{where } i = \sqrt{-1})$$

$$\Rightarrow \quad x - 3 = \pm 5i, x - 3 = \pm 1$$

$$\Rightarrow \quad x = 3 \pm 5i, x = 4, 2 \qquad [\because y = x - 3]$$

Hence, the roots of the given equation are $3\pm5i$, 2 and 4.

Example 1.76 **Solve the equation $(x + 2)(x + 3)(x + 8) \times (x + 12) = 4x^2$.**

Sol. $(x + 2)(x + 3)(x + 8)(x + 12) = 4x^2$

$$\Rightarrow \quad \{(x + 2)(x + 12)\}\{(x + 3)(x + 8)\} = 4x^2$$

$$\Rightarrow \quad (x^2 + 14x + 24)(x^2 + 11x + 24) = 4x^2$$

Dividing throughout by x^2, we get

$$\left(x + 14 + \frac{24}{x}\right)\left(x + 11 + \frac{24}{x}\right) = 4$$

$$\Rightarrow \quad (y + 14)(y + 11) = 4, \text{ where } x + \frac{24}{x} = y$$

$$\Rightarrow \quad y^2 + 25y + 154 = 4$$

$$\Rightarrow \quad y^2 + 25y + 150 = 0$$

$$\Rightarrow \quad (y + 15)(y + 10) = 0$$

$$\Rightarrow \quad y = -15, -10$$

If $y = -15$, then

$$x + \frac{24}{x} = -15$$

$$\Rightarrow \quad x^2 + 15x + 24 = 0$$

$$\Rightarrow \quad x = \frac{-15 - \sqrt{129}}{2}$$

If $y = -10$, then

$$x + \frac{24}{x} = -10$$

$$\Rightarrow \quad x^2 + 10x + 24 = 0$$

$$\Rightarrow \quad (x + 4)(x + 6) = 0$$

$$\Rightarrow \quad x = -4, -6$$

Hence, the roots of the given equation are -4, -6, $(-15 - \sqrt{129})/2$.

Example 1.77 **Evaluate $\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots \infty}}}$.**

Sol. Let $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots \infty}}}$. Then,

$$x = \sqrt{6 + x}$$

$$\Rightarrow \quad x^2 = 6 + x$$

$$\Rightarrow \quad x^2 - x - 6 = 0$$

$$\Rightarrow \quad (x - 3)(x + 2) = 0$$

$$\Rightarrow \quad x = 3 \text{ or } x = -2$$

But, the given expression is positive. So, $x = 3$. Hence, the value of the given expression is 3.

Example 1.78 **Solve $\sqrt{x+5}+\sqrt{x+21}=\sqrt{6x+40}$.**

Sol. $\sqrt{x+5}+\sqrt{x+21}=\sqrt{6x+40}$

$\Rightarrow$ $(\sqrt{x+5}+\sqrt{x+21})^2=6x+40$

$\Rightarrow$ $(x+5)+(x+21)+2\sqrt{(x+5)(x+21)}=6x+40$

$\Rightarrow$ $\sqrt{(x+5)(x+21)}=2x+7$

$\Rightarrow$ $(x+5)(x+21)=(2x+7)^2$

$\Rightarrow$ $3x^2+2x-56=0$

$\Rightarrow$ $(3x+14)(x-4)=0$

$\Rightarrow$ $x=4$ or $x=-14/3$

Clearly, $x=-14/3$ does not satisfy the given equation. Hence, $x=4$ is the only root of the given equation.

Concept Application Exercise 1.2

1. Prove that graph of $y=x^2+2$ and $y=3x-4$ never intersect.
2. In how many points the line $y+14=0$ cuts the curve whose equation is $x(x^2+x+1)+y=0$?
3. Consider the following graphs:

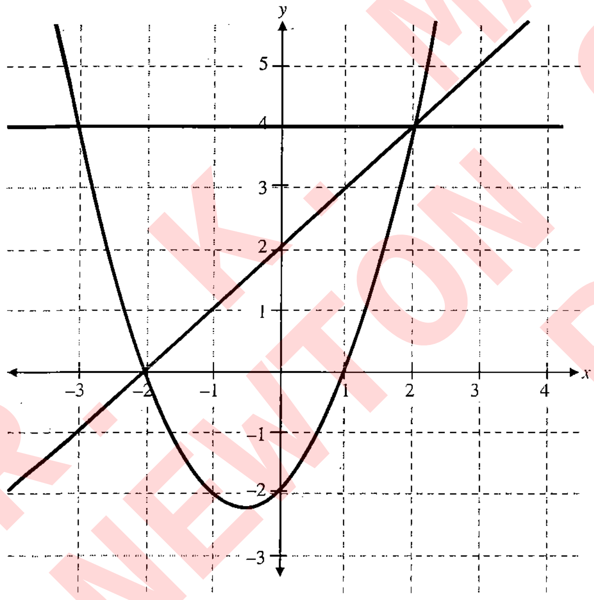


Fig. 1.38

Answer the following questions:

(i) sum of roots of the equation $f(x)=0$

(ii) product of roots of the equation $f(x)=4$

(iii) the absolute value of the difference of the roots of equation $f(x)=x+2$

4. Solve $\dfrac{x^2+3x+2}{x^2-6x-7}=0$.
5. Solve $\sqrt{x-2}+\sqrt{4-x}=2$.
6. Solve $\sqrt{x-2}(x^2-4x-5)=0$.
7. Solve the equation $x(x+2)(x^2-1)=-1$.
8. Find the value of $2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots\infty}}}$.
9. Solve $4^x+6^x=9^x$.
10. Solve $3^{2x^2-7x+7}=9$.
11. Find the number of real roots of the equation $(x-1)^2+(x-2)^2+(x-3)^2=0$.
12. Solve $\sqrt{3x^2-7x-30}+\sqrt{2x^2-7x-5}=x+5$.
13. If $x=\sqrt{7+4\sqrt{3}}$, prove that $x+1/x=4$
14. Solve $\sqrt{5x^2-6x+8}-\sqrt{5x^2-6x-7}=1$.
15. Solve $\sqrt{x^2+4x-21}+\sqrt{x^2-x-6}=\sqrt{6x^2-5x-39}$.
16. How many roots of the equation $3x^4+6x^3+x^2+6x+3=0$ are real ?
17. Find the value of k if $x^3-3x+a=0$ has three real distinct roots.
18. Analyze the roots of the equation $(x-1)^3+(x-2)^3+(x-3)^3+(x-4)^3+(x-5)^3=0$ by differentiation method.
19. In how many points the graph of $f(x)=x^3+2x^2+3x+4$ meets x-axis.

REMAINDER AND FACTOR THEOREMS

Remainder Theorem

The remainder theorem states that if a polynomial $f(x)$ is divided by a linear function $x-k$, then the remainder is $f(k)$.

Proof:

In any division,

Dividend = Divisor × Quotient + Remainder

Let $Q(x)$ be the quotient and R be the remainder. Then,

$f(x)=(x-k)\,Q(x)+R$

$\Rightarrow$ $f(k)=(k-k)\,Q(x)+R=0+R=R$

Note: *If a n-degree polynomial is divided by a m-degree polynomial, then the maximum degree of the remainder polynomial is m – 1.*

Example 1.79 **Find the remainder when x^3+4x^2-7x+6 is divided by $x-1$.**

Sol. Let $f(x)=x^3+4x^2-7x+6$. The remainder when $f(x)$ is divided by $x-1$ is

$f(1)=1^3+4\times(1)^2-7+6=4$

Example 1.80 **If the expression $ax^4+bx^3-x^2+2x+3$ has remainder $4x+3$ when divided by x^2+x-2, find the value of a and b.**

Sol. Let $f(x)=ax^4+bx^3-x^2+2x+3$.

Now, $x^2+x-2=(x+2)\times(x-1)$.

Given, $f(-2)=a(-2)^4+b(-2)^3-(-2)^2+2(-2)+3$

$= 4(-2) + 3$

$\Rightarrow \quad 16a - 8b - 4 - 4 + 3 = -5$

$\Rightarrow \quad 2a - b = 0 \qquad (1)$

Also,

$f(1) = a + b - 1 + 2 + 3 = 4(1) + 3$

$\Rightarrow \quad a + b = 3 \qquad (2)$

From (1) and (2), $a = 1$, $b = 2$.

Factor Theorem

Factor Theorem Is a Special Case of Remainder Theorem

Let,

$$f(x) = (x - k)\, Q(x) + R$$

$$\Rightarrow \quad f(x) = (x - k)\, Q(x) + f(k)$$

When $f(k) = 0$, $f(x) = (x - k)\, Q(x)$. Therefore, $f(x)$ is exactly divisible by $x - k$.

Example 1.81 **Given that $x^2 + x - 6$ is a factor of $2x^4 + x^3 - ax^2 + bx + a + b - 1$, find the values of a and b.**

Sol. We have,

$$x^2 + x - 6 = (x + 3)(x - 2)$$

Let,

$$f(x) = 2x^4 + x^3 - ax^2 + bx + a + b - 1$$

Now,

$$f(-3) = 2(-3)^4 + (-3)^3 - a(-3)^2 - 3b + a + b - 1 = 0$$

$\Rightarrow \quad 134 - 8a - 2b = 0$

$\Rightarrow \quad 4a + b = 67 \qquad (1)$

$\Rightarrow \quad f(2) = 2(2)^4 + 2^3 - a(2)^2 + 2b + a + b - 1 = 0$

$\Rightarrow \quad 39 - 3a + 3b = 0$

$\Rightarrow \quad a - b = 13 \qquad (2)$

From (1) and (2), $a = 16$, $b = 3$.

Example 1.82 **Use the factor theorem to find the value of k for which $(a + 2b)$, where $a, b \neq 0$ is a factor of $a^4 + 32b^4 + a^3b(k + 3)$.**

Sol. Let $f(a) = a^4 + 32b^4 + a^3b(k + 3)$. Now,

$$f(-2b) = (-2b)^4 + 32b^4 + (-2b)^3b(k + 3) = 0$$

$\Rightarrow \quad 48b^4 - 8b^4(k + 3) = 0$

$\Rightarrow \quad 8b^4[6 - (k + 3)] = 0$

$\Rightarrow \quad 8b^4(3 - k) = 0$

Since $b \neq 0$, so, $3 - k = 0$ or $k = 3$.

Example 1.83 **If c, d are the roots of the equation $(x - a)(x - b) - k = 0$, prove that a, b are the roots of the equation $(x - c)(x - d) + k = 0$.**

Sol. Since c and d are the roots of the equation $(x - a)(x - b) - k = 0$, therefore,

$$(x - a)(x - b) - k = (x - c)(x - d)$$

$\Rightarrow \quad (x - a)(x - b) = (x - c)(x - d) + k$

$\Rightarrow \quad (x - c)(x - d) + k = (x - a)(x - b)$

Clearly, a and b are roots of the equation $(x - a)(x - b) = 0$. Hence, a, b are roots of $(x - c)(x - d) + k = 0$.

Concept Application Exercise 1.3

1. Given that the expression $2x^3 + 3px^2 - 4x + p$ has a remainder of 5 when divided by $x + 2$, find the value of p.
2. Determine the value of k for which $x + 2$ is a factor of $(x + 1)^7 + (2x + k)^3$.
3. Find the value of p for which $x + 1$ is a factor of $x^4 + (p - 3)x^3 - (3p - 5)x^2 + (2p - 9)x + 6$. Find the remaining factors for this value of p.
4. If $x^2 + ax + 1$ is a factor of $ax^3 + bx + c$, then find the conditions.
5. If $f(x) = x^3 - 3x^2 + 2x + a$ is divisible by $x - 1$, then find the remainder when $f(x)$ is divided by $x - 2$.
6. If $f(x) = x^3 - x^2 + ax + b$ is divisible by $x^2 - x$, then find the value of $f(2)$.

Identity

A relation which is true for every value of the variable is called an identity.

Example 1.84 **If $(a^2 - 1)x^2 + (a - 1)x + a^2 - 4a + 3 = 0$ be an identity in x, then find the value of a.**

Sol. The given relation is satisfied for all real values of x, so all the coefficients must be zero. Then,

$$\left.\begin{array}{r} a^2 - 1 = 0 \Rightarrow a = \pm 1 \\ a - 1 = 0 \Rightarrow a = 1 \\ a^2 - 4a + 3 = 0 \Rightarrow a = 1, 3 \end{array}\right\} \text{common value of } a \text{ is } 1$$

Example 1.85 **Show that $\dfrac{(x+b)(x+c)}{(b-a)(c-a)} + \dfrac{(x+c)(x+a)}{(c-b)(a-b)} + \dfrac{(x+a)(x+b)}{(a-c)(b-c)} = 1$ is an identity.**

Sol. Given relation is

$$\frac{(x+b)(x+c)}{(b-a)(c-a)} + \frac{(x+c)(x+a)}{(c-b)(a-b)} + \frac{(x+a)(x+b)}{(a-c)(b-c)} = 1 \qquad (1)$$

When $x = -a$,

$$\text{L.H.S.} = \frac{(b-a)(c-a)}{(b-a)(c-a)} = 1 = \text{R.H.S.}$$

Similarly, when $x = -b$,

$$\text{L.H.S.} = \frac{(c-b)(a-b)}{(c-b)(a-b)} = 1 = \text{R.H.S.}$$

When $x = -c$,

$$\text{L.H.S.} = \frac{(a-c)(b-c)}{(a-c)(b-c)} = 1 = \text{R.H.S.}$$

Thus, the highest power of x occurring in relation (1) is 2 and this relation is satisfied by three distinct values a, b and c of x; therefore, it is an equation but an identity.

Example 1.86 **A certain polynomial $P(x)$, $x \in R$ when divided by $x - a$, $x - b$ and $x - c$ leaves remainders a, b and c, respectively. Then find the remainder when $P(x)$ is divided by $(x - a)$ $(x - b)$ $(x - c)$ where a, b, c are distinct.**

Sol. By remainder theorem, $P(a) = a$, $P(b) = b$ and $P(c) = c$.

Let the required remainder be $R(x)$. Then,

$$P(x) = (x - a)(x - b)(x - c)Q(x) + R(x)$$

where $R(x)$ is a polynomial of degree at most 2. We get $R(a) = a$, $R(b) = b$ and $R(c) = c$. So, the equation $R(x) - x = 0$ has three roots a, b and c. But its degree is at most 2. So, $R(x) - x$ must be zero polynomial (or identity). Hence $R(x) = x$.

QUADRATIC EQUATION

Quadratic Equation with Real Coefficients

Consider the quadratic equation

$$ax^2 + bx + c = 0 \qquad (1)$$

where $a, b, c \in R$ and $a \neq 0$.

Roots of the equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now, we observe that the nature of the roots depend upon the value of the quantity $b^2 - 4ac$. This quantity is generally denoted by D and is known as the discriminant of the quadratic equation [Eq.(1)].

We also observe the following results:

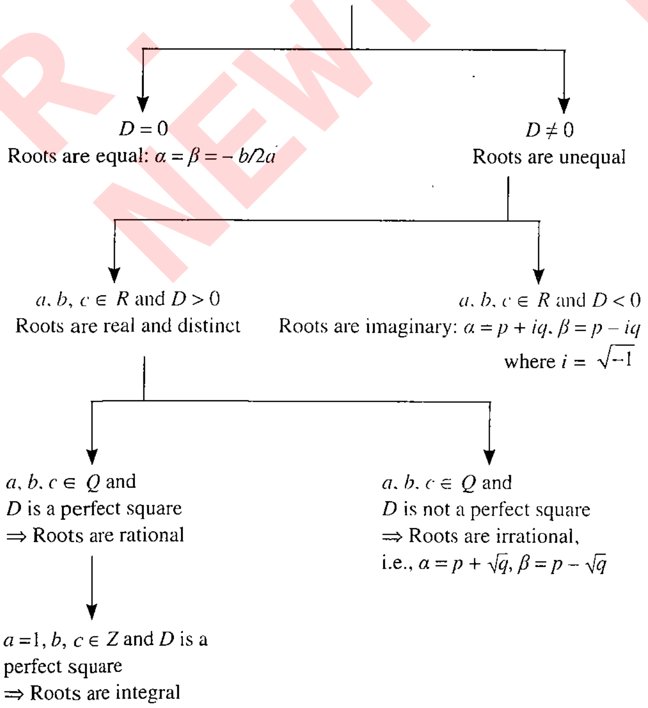


Fig. 1.39

Note:

- *If $a, b, c \in Q$ and $b^2 - 4ac$ is positive but not a perfect square, then roots are irrational and they always occur in conjugate pair like $2 + \sqrt{3}$ and $2 - \sqrt{3}$. However, if a, b, c are irrational numbers and $b^2 - 4ac$ is positive but not a perfect square, then the roots may not occur in conjugate pairs. For example, the roots of the equation $x^2 - (5 + \sqrt{2})x + 5\sqrt{2} = 0$ are 5 and $\sqrt{2}$, which do not form a conjugate pair.*
- *If $b^2 - 4ac < 0$, then roots of equations are complex. If a, b and c are real then complex roots occur in conjugate pair such as of the form $p + iq$ and $p - iq$. If all the coefficients are not real then complex roots may not conjugate.*

Example 1.87 **If $a, b, c \in R^+$ and $2b = a + c$, then check the nature of roots of equation $ax^2 + 2bx + c = 0$.**

Sol. Given equation is $ax^2 + 2bx + c = 0$. Hence,

$$\begin{aligned} D &= 4b^2 - 4ac \\ &= (a + c)^2 - 4ac \\ &= (a - c)^2 > 0 \end{aligned}$$

Thus, the roots are real and distinct.

Example 1.88 **If the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, show that $2/b = 1/a + 1/c$.**

Sol. Since the roots of the given equations are equal, therefore its discriminant is zero, i.e.,

$$b^2(c - a)^2 - 4a(b - c)c(a - b) = 0$$

$$\Rightarrow b^2(c^2 + a^2 - 2ac) - 4ac(ba - ca - b^2 + bc) = 0$$

$$\Rightarrow a^2b^2 + b^2c^2 + 4a^2c^2 + 2b^2ac - 4a^2bc - 4abc^2 = 0$$

$$\Rightarrow (ab + bc - 2ac)^2 = 0$$

$$\Rightarrow ab + bc - 2ac = 0$$

$$\Rightarrow ab + bc = 2ac$$

$$\Rightarrow \frac{1}{c} + \frac{1}{a} = \frac{2}{b} \quad \text{[Dividing both sides by } abc\text{]}$$

$$\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

Example 1.89 **Prove that the roots of the equation $(a^4 + b^4)x^2 + 4abcdx + (c^4 + d^4) = 0$ cannot be different, if real.**

Sol. The discriminant of the given equation is

$$\begin{aligned} D &= 16a^2b^2c^2d^2 - 4(a^4 + b^4)(c^4 + d^4) \\ &= -4[(a^4 + b^4)(c^4 + d^4) - 4a^2b^2c^2d^2] \\ &= -4[a^4c^4 + a^4d^4 + b^4c^4 + b^4d^4 - 4a^2b^2c^2d^2] \\ &= -4[(a^4c^4 + b^4d^4 - 2a^2b^2c^2d^2) + (a^4d^4 + b^4c^4 - 2a^2b^2c^2d^2)] \\ &= -4[(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2] \qquad (1) \end{aligned}$$

Since roots of the given equation are real, therefore

$$D \geq 0$$

$$\Rightarrow -4[(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2] \geq 0$$

$\Rightarrow$ $(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2 \le 0$

$\Rightarrow$ $(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2 = 0$ (2)

(since sum of two positive quantities cannot be negative)

From (1) and (2), we get $D = 0$. Hence, the roots of the given quadratic equation are not different, if real.

Example 1.90 **If the roots of the equation $x^2 - 8x + a^2 - 6a = 0$ are real distinct, then find all possible values of a.**

Sol. Since the roots of the given equation are real and distinct, we must have

$D > 0$

$\Rightarrow$ $64 - 4(a^2 - 6a) > 0$

$\Rightarrow$ $4[16 - a^2 + 6a] > 0$

$\Rightarrow$ $-4(a^2 - 6a - 16) > 0$

$\Rightarrow$ $a^2 - 6a - 16 < 0$

$\Rightarrow$ $(a - 8)(a + 2) < 0$

$\Rightarrow$ $-2 < a < 8$

Hence, the roots of the given equation are real if a lies between -2 and 8.

Example 1.91 **Find the quadratic equation with rational coefficients whose one root is $1/(2 + \sqrt{5})$.**

Sol. If the coefficients are rational, then irrational roots occur in conjugate pair. Given that if one root is $\alpha = 1/(2 + \sqrt{5}) = \sqrt{5} - 2$, then the other root is $\beta = 1/(2 - \sqrt{5}) = -(2 + \sqrt{5})$.

Sum of roots $\alpha + \beta = -4$ and product of roots $\alpha\beta = -1$. Thus, required equation is $x^2 + 4x - 1 = 0$.

Example 1.92 **If $f(x) = ax^2 + bx + c$, $g(x) = -ax^2 + bx + c$, where $ac \ne 0$, then prove that $f(x)\,g(x) = 0$ has at least two real roots.**

Sol. Let D_1 and D_2 be discriminants of $ax^2 + bx + c = 0$ and $-ax^2 + bx + c = 0$, respectively. Then,

$D_1 = b^2 - 4ac$, $D_2 = b^2 + 4ac$

Now,

$ac \ne 0 \Rightarrow$ either $ac > 0$ or $ac < 0$

If $ac > 0$, then $D_2 > 0$. Therefore, roots of $-ax^2 + bx + c = 0$ are real.

If $ac < 0$, then $D_1 > 0$. Therefore, roots of $ax^2 + bx + c = 0$ are real.

Thus, $f(x)\,g(x)$ has at least two real roots.

Example 1.93 **If $a, b, c \in R$ such that $a + b + c = 0$ and $a \ne c$, then prove that the roots of $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ are real and distinct.**

Sol. Given equation is

$(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$

or

$(-2a)x^2 + (-2b)x + (-2c) = 0$

or

$ax^2 + bx + c = 0$

$\Rightarrow$ $D = b^2 - 4ac$

$= (-c - a)^2 - 4ac$

$= (c - a)^2$

> 0

Hence, roots are real and distinct.

Example 1.94 **If $\cos\theta$, $\sin\phi$, $\sin\theta$ are in G.P., then check the nature of roots of $x^2 + 2\cot\phi \cdot x + 1 = 0$.**

Sol. We have,

$\sin^2\phi = \cos\theta\sin\theta$

The discriminant of the given equation is

$$D = 4\cot^2\phi - 4$$

$$= 4\left[\frac{\cos^2\varphi - \sin^2\varphi}{\sin^2\varphi}\right]$$

$$= \frac{4(1 - 2\sin^2\varphi)}{\sin^2\varphi}$$

$$= \frac{4(1 - 2\sin\theta\cos\theta)}{\sin^2\varphi}$$

$$= \left[\frac{2(\sin\theta - \cos\theta)}{\sin\varphi}\right]^2 \ge 0$$

Example 1.95 **If a, b and c are odd integers, then prove that roots of $ax^2 + bx + c = 0$ cannot be rational.**

Sol. Discriminant $D = b^2 - 4ac$. Suppose the roots are rational. Then, D will be a perfect square.

Let $b^2 - 4ac = d^2$. Since a, b and c are odd integers, d will be odd. Now,

$b^2 - d^2 = 4ac$

Let $b = 2k + 1$ and $d = 2m + 1$. Then

$b^2 - d^2 = (b - d)(b + d)$

$= 2(k - m)\,2(k + m + 1)$

Now, either $(k - m)$ or $(k + m + 1)$ is always even. Hence $b^2 - d^2$ is always a multiple of 8. But, $4ac$ is only a multiple of 4 (not of 8), which is a contradiction. Hence, the roots of $ax^2 + bx + c = 0$ cannot be rational.

Quadratic Equations with Complex Coefficients

Consider the quadratic equation $ax^2 + bx + c = 0$, where a, b, c are complex numbers and $a \ne 0$. Roots of equation are given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Here nature of roots should not be analyzed by sign of $b^2 - 4ac$.

Note: *In case of quadratic equations with real coefficients, imaginary (complex) roots always occur in conjugate pairs. However, it is not true for quadratic equations with complex coefficients. For example, the equation $4x^2 - 4ix - 1 = 0$ has both roots equal to $1/(2i)$.*

Concept Application Exercise 1.4

1. Find the values of a for which the roots of the equation $x^2 + a^2 = 8x + 6a$ are real.
2. Find the condition if the roots of $ax^2 + 2bx + c = 0$ and $bx^2 - 2\sqrt{acx} + b = 0$ are simultaneously real.
3. If $a < c < b$, then check the nature of roots of the equation $(a - b)^2 x^2 + 2(a + b - 2c)\,x + 1 = 0$.
4. If $a + b + c = 0$ then check the nature of roots of the equation $4ax^2 + 3bx + 2c = 0$ where $a, b, c \in R$.
5. Find the greatest value of a non-negative real number λ for which both the equations $2x^2 + (\lambda - 1)x + 8 = 0$ and $x^2 - 8x + \lambda + 4 = 0$ have real roots.

Relations Between Roots and Coefficients

Let α and β be the roots of quadratic equation $ax^2 + bx + c = 0$. Then by factor theorem,

$$ax^2 + bx + c = a(x - \alpha)(x - \beta) = a(x^2 - (\alpha + \beta)x + \alpha\beta)$$

Comparing coefficients, we have $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$. Thus, we find that

$$\alpha + \beta = \frac{-b}{a} = -\frac{\text{coeff of } x}{\text{coeff of } x^2} \text{ and } \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coeff of } x^2}$$

Also, if sum of roots is S and product is P, then quadratic equation is given by $x^2 - Sx + P = 0$.

Example 1.96 **Form a quadratic equation whose roots are –4 and 6.**

Sol. We have sum of the roots, $S = -4 + 6 = 2$ and, product of the roots, $P = -4 \times 6 = -24$. Hence, the required equation is

$$x^2 - Sx + P = 0$$

$$\Rightarrow \quad x^2 - 2x - 24 = 0$$

Example 1.97 **Form a quadratic equation with real coefficients whose one root is $3 - 2i$.**

Sol. Since the complex roots always occur in pairs, so the other root is $3 + 2i$. The sum of the roots is $(3 + 2i) + (3 - 2i) = 6$. The product of the roots is $(3 + 2i)(3 - 2i) = 9 - 4i^2 = 9 + 4 = 13$. Hence, the equation is

$$x^2 - Sx + P = 0$$

$$\Rightarrow \quad x^2 - 6x + 13 = 0$$

Example 1.98 **If roots of the equation $ax^2 + bx + c = 0$ are α and β, find the equation whose roots are**

(i) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$

(ii) $-\alpha, -\beta$

(iii) $\dfrac{1-\alpha}{1+\alpha}, \dfrac{1-\beta}{1+\beta}$

Sol. Here in all cases functions of α and β are symmetric.

(i) Let $\dfrac{1}{\alpha} = y \Rightarrow \alpha = \dfrac{1}{y}$

Now α is a root of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0$$

$$\frac{a}{y^2} + \frac{b}{y} + c = 0$$

$$\Rightarrow \quad cy^2 + by + a = 0$$

Hence, the required equation is $cx^2 + bx + a = 0$.

We get same equation if we start with $1/\beta$.

(ii) Let $-\alpha = y \Rightarrow \alpha = -y$

Now α is root of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0$$

$$\Rightarrow \quad a(-y)^2 + b(-y) + c = 0$$

Hence, the required equation is $ax^2 - bx + c = 0$.

(iii) Let $\dfrac{1-\alpha}{1+\alpha} = y \Rightarrow \alpha = \dfrac{1-y}{1+y}$

Now α is root of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0$$

$$\Rightarrow \quad a\left(\frac{1-y}{1+y}\right)^2 + b\left(\frac{1-y}{1+y}\right) + c = 0$$

Hence required equation is $a(1 - x)^2 + b(1 - x^2) + c(1 + x)^2 = 0$.

Example 1.99 **If a, b and c are in A.P. and one root of the equation $ax^2 + bx + c = 0$ is 2, then find the other root.**

Sol. Let α be the other root. Then,

$$4a + 2b + c = 0 \text{ and } 2b = a + c$$

$$\Rightarrow \quad 5a + 2c = 0$$

$$\Rightarrow \quad \frac{c}{a} = \frac{-5}{2}$$

Now,

$$2 \times \alpha = \frac{c}{a} = -\frac{5}{2}$$

$$\therefore \quad \alpha = -\frac{5}{4}$$

Example 1.100 **If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30° and tan 15°, respectively, then find the value of $2 + q - p$.**

Sol. The equation $x^2 + px + q = 0$ has roots tan 30° and tan 15°. Therefore,

$$\tan 30° + \tan 15° = -p \qquad (1)$$

$$\tan 30° \tan 15° = q \qquad (2)$$

Now,

$$\tan 45° = \tan(30° + 15°)$$

$$\Rightarrow \quad 1 = \frac{\tan 30 + \tan 15}{1 - \tan 30 \tan 15}$$

$$\Rightarrow \quad 1 = \frac{-p}{1-q} \quad \text{[Using (1) and (2)]}$$

$$\Rightarrow \quad 1 - q = -p \Rightarrow q - p = 1$$

$\Rightarrow \quad 2 + q - p = 3$

Example 1.101 **If the sum of the roots of the equation $1/(x + a) + 1/(x + b) = 1/c$ is zero, then prove that the product of the roots is $(-1/2)(a^2 + b^2)$.**

Sol. We have,

$$\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$$

$\Rightarrow \quad x^2 + (a + b - 2c)x + (ab - bc - ca) = 0$

Let α, β be the roots of this equation. Then,

$\alpha + \beta = -(a + b - 2c)$ and $\alpha\beta = ab - bc - ca$

It is given that

$\alpha + \beta = 0$

$\Rightarrow \quad -(a + b - 2c) = 0$

$\Rightarrow \quad c = \dfrac{a+b}{2}$ (1)

$\therefore \quad \alpha\beta = ab - bc - ca$

$= ab - c(a + b)$

$= ab - \left(\dfrac{a+b}{2}\right)(a+b)$ [Using (1)]

$= \dfrac{2ab-(a+b)^2}{2} = -\dfrac{1}{2}(a^2+b^2)$

Example 1.102 **Solve the equation $x^2 + px + 45 = 0$. It is given that the squared difference of its roots is equal to 144.**

Sol. Let α, β be the roots of the equation $x^2 + px + 45 = 0$. Then,

$\alpha + \beta = -p$ (1)

$\alpha\beta = 45$ (2)

It is given that

$(\alpha - \beta)^2 = 144$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = 144$

$\Rightarrow \quad p^2 - 4 \times 45 = 144$

$\Rightarrow \quad p^2 = 324$

$\Rightarrow \quad p = \pm 18$

Substituting $p = 18$ in the given equation, we obtain

$x^2 + 18x + 45 = 0$

$\Rightarrow \quad (x + 3)(x + 15) = 0$

$\Rightarrow \quad x = -3, -15$

Substituting $p = -18$ in the given equation, we obtain

$x^2 + 18x + 45 = 0$

$\Rightarrow \quad (x - 3)(x - 15) = 0$

$\Rightarrow \quad x = 3, 15$

Hence, the roots of the given equation are $-3, -15$ or $3, 15$.

Example 1.103 **If the ratio of the roots of the equation $x^2 + px + q = 0$ are equal to the ratio of the roots of the equation $x^2 + bx + c = 0$, then prove that $p^2c = b^2q$.**

Sol. Let α, β be the roots of $x^2 + px + q = 0$ and γ, δ be the roots of the equation $x^2 + bx + c = 0$. Then,

$\alpha + \beta = -p, \alpha\beta = q$ (1)

$\gamma + \delta = -b, \gamma\delta = c$ (2)

We have,

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \Rightarrow \frac{\alpha+\beta}{\alpha-\beta} = \frac{\gamma+\delta}{\gamma-\delta}$$

[Using componendo and dividendo]

$\Rightarrow \quad \dfrac{(\alpha-\beta)^2}{(\alpha+\beta)^2} = \dfrac{(\gamma-\delta)^2}{(\gamma+\delta)^2}$

$\Rightarrow \quad \dfrac{(\alpha+\beta)^2-4\alpha\beta}{(\alpha+\beta)^2} = \dfrac{(\gamma+\delta)^2-4\gamma\delta}{(\gamma+\delta)^2}$

$\Rightarrow \quad 1-\dfrac{4\alpha\beta}{(\alpha+\beta)^2} = 1-\dfrac{4\gamma\delta}{(\gamma+\delta)^2}$

$\Rightarrow \quad \dfrac{\alpha\beta}{(\alpha+\beta)^2} = \dfrac{\gamma\delta}{(\gamma+\delta)^2}$

$\Rightarrow \quad \dfrac{q}{p^2} = \dfrac{c}{b^2}$

$\Rightarrow \quad p^2c = b^2q$

Example 1.104 **If $\sin\theta$, $\cos\theta$ be the roots of $ax^2 + bx + c = 0$, then prove that $b^2 = a^2 + 2ac$.**

Sol. We have,

$$\sin\theta + \cos\theta = -\frac{b}{a}, \sin\theta\cos\theta = \frac{c}{a}$$

Now, we know that

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \quad (\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta = 1$

$\Rightarrow \quad \dfrac{b^2}{a^2} = 1 + 2\dfrac{c}{a} \Rightarrow b^2 = a^2 + 2ac$

Example 1.105 **If a and b ($\neq 0$) are the roots of the equation $x^2 + ax + b = 0$, then find the least value of $x^2 + ax + b$ ($x \in R$).**

Sol. Since a and b are the roots of the equation $x^2 + ax + b = 0$, so

$a + b = -a, ab = b$

Now,

$ab = b \Rightarrow (a - 1)b = 0 \Rightarrow a = 1 \quad (\because b \neq 0)$

Putting $a = 1$ in $a + b = -a$, we get $b = -2$. Hence,

$x^2 + ax + b = x^2 + x - 2 = (x + 1/2)^2 - 1/4 - 2$

$= (x + 1/2)^2 - 9/4$

which has a minimum value $-9/4$.

Example 1.106 **If the sum of the roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ is -1, then find the product of the roots.**

Sol. Let α, β be roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$. Then,

$$\alpha + \beta = -1 \Rightarrow -\left(\frac{2a+3}{a+1}\right) = -1 \Rightarrow a = -2$$

Now, product of the roots is $(3a + 4)/(a + 1) = (-6 + 4)/(-2 + 1) = 2$.

Example 1.107 **Find the value of 'a' for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other.**

Sol. Let the roots be α and 2α. Then,

$$\alpha + 2\alpha = \frac{1-3a}{a^2-5a+3},\ \alpha \times 2\alpha = \frac{2}{a^2-5a+3}$$

$$\Rightarrow\ 2\left[\frac{1}{9}\frac{(1-3a)^2}{(a^2-5a+3)^2}\right] = \frac{2}{a^2-5a+3}$$

$$\Rightarrow\ \frac{(1-3a)^2}{a^2-5a+3} = 9 \Rightarrow 9a^2 - 6a + 1 = 9a^2 - 45a + 27$$

$$\Rightarrow\ 39a = 26 \Rightarrow a = \frac{2}{3}$$

Example 1.108 **If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then find the set of possible values of a.**

Sol. If α, β are roots of $x^2 + ax + 1 = 0$, then

$$|\alpha - \beta| < \sqrt{5}$$

$$\Rightarrow\ \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} < \sqrt{5}$$

$$\Rightarrow\ \sqrt{a^2-4} < \sqrt{5}$$

$$\Rightarrow\ \sqrt{a^2-4} < \sqrt{5}$$

$$\Rightarrow\ a^2 - 4 < 5$$

$$\Rightarrow\ a^2 < 9$$

$$\Rightarrow\ -3 < a < 3$$

$$\therefore\ a \in (-3, 3)$$

Example 1.109 **Find the values of the parameter a such that the roots α, β of the equation $2x^2 + 6x + a = 0$ satisfy the inequality $\alpha/\beta + \beta/\alpha < 2$.**

Sol. We have $\alpha + \beta = -3$ and $\alpha\beta = a/2$. Now,

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} < 2$$

$$\Rightarrow\ \frac{\alpha^2+\beta^2}{\alpha\beta} < 2$$

$$\Rightarrow\ \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} < 2$$

$$\Rightarrow\ \frac{9-a}{a/2} < 2$$

$$\Rightarrow\ \frac{9-a}{a} < 1$$

$$\Rightarrow\ \frac{9-a}{a} - 1 < 0$$

$$\Rightarrow\ \frac{9-2a}{a} < 0$$

$$\Rightarrow\ \frac{2a-9}{a} > 0$$

$$\Rightarrow\ a < 0 \text{ or } a > 9/2$$

Example 1.110 **If the harmonic mean between roots of $(5 + \sqrt{2})x^2 - bx + 8 + 2\sqrt{5} = 0$ is 4, then find the value of b.**

Sol. Let α, β be the roots of the given equation whose H.M. is 4. Then,

$$4 = \frac{2\alpha\beta}{\alpha+\beta}$$

$$\Rightarrow\ 4 = 2 \times \frac{\frac{8+2\sqrt{5}}{5+\sqrt{2}}}{\frac{b}{5+\sqrt{2}}}$$

$$\Rightarrow\ 2 = \frac{8+2\sqrt{5}}{b} \Rightarrow b = 4 + \sqrt{5}$$

Example 1.111 **If α, β are the roots of the equation $2x^2 - 3x - 6 = 0$, find the equation whose roots are $\alpha^2 + 2$ and $\beta^2 + 2$.**

Sol. Since α, β are roots of the equation $2x^2 - 3x - 6 = 0$, so

$$\alpha + \beta = 3/2 \text{ and } \alpha\beta = -3$$

$$\Rightarrow\ \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \frac{9}{4} + 6 = \frac{33}{4}$$

Now,

$$(\alpha^2 + 2) + (\beta^2 + 2) = (\alpha^2 + \beta^2) + 4 = \frac{33}{4} + 4 = \frac{49}{4}$$

and

$$(\alpha^2 + 2)(\beta^2 + 2) = \alpha^2\beta^2 + 2(\alpha^2 + \beta^2) + 4$$

$$= (3)^2 + 2\left(\frac{33}{4}\right) + 4$$

$$= \frac{59}{2}$$

So, the equation whose roots are $\alpha^2 + 2$ and $\beta^2 + 2$ is

$$x^2 - x[(\alpha^2 + 2) + (\beta^2 + 2)] + (\alpha^2 + 2)(\beta^2 + 2) = 0$$

$$\Rightarrow\ x^2 - \frac{49}{4}x + \frac{59}{2} = 0$$

$$\Rightarrow\ 4x^2 - 49x + 118 = 0$$

Example 1.112 **If $\alpha \neq \beta$ and $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$, find the equation whose roots are α/β and β/α.**

Sol. We have $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$. Hence, α, β are roots of $x^2 = 5x - 3$, i.e., $x^2 - 5x + 3 = 0$. Therefore,

$$\alpha + \beta = 5 \text{ and } \alpha\beta = 3$$

Now,

$$S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{25-6}{3} = \frac{19}{3}$$

and

$$P = \frac{\alpha}{\beta}\frac{\beta}{\alpha} = 1$$

So, the required equation is

$$x^2 - Sx + P = 0$$

$$\Rightarrow \quad x^2 - \frac{19}{3}x + 1 = 0$$

$$\Rightarrow \quad 3x^2 - 19x + 3 = 0$$

Example 1.113 **If α, β are the roots of the equation $ax^2 + bx + c = 0$, then find the roots of the equation $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$ in terms of α and β.**

Sol. $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$

$$\Rightarrow \quad \frac{ax^2}{(1-x)^2} + \frac{bx}{1-x} + c = 0 \qquad (1)$$

Now, α is a root of $ax^2 + bx + c = 0$. Then let

$$\alpha = \frac{x}{1-x}$$

$$\Rightarrow \quad x = \frac{\alpha}{\alpha + 1}$$

Hence, the roots of (1) are $\alpha/(1 + \alpha)$, $\beta/(1 + \beta)$.

Concept Application Exercise 1.5

1. If the product of the roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ is 2, then find the sum of roots.
2. Find the value of a for which the sum of the squares of the roots of the equation $x^2 - (a - 2)x - a - 1 = 0$ assumes the least value.
3. If x_1 and x_2 are the roots of $x^2 + (\sin\theta - 1)x - 1/2\cos^2\theta = 0$, then find the maximum value of $x_1^2 + x_2^2$.
4. If $\tan\theta$ and $\sec\theta$ are the roots of $ax^2 + bx + c = 0$, then prove that $a^4 = b^2(4ac - b^2)$.
5. If the roots of the equation $x^2 - bx + c = 0$ be two consecutive integers, then find the value of $b^2 - 4c$.
6. If the roots of the equation $12x^2 - mx + 5 = 0$ are in the ratio 2:3, then find the value of m.
7. If α, β are the roots of $x^2 + px + 1 = 0$ and γ, δ are the roots of $x^2 + qx + 1 = 0$, then prove that $q^2 - p^2 = (\alpha - \gamma)(\beta - \gamma)(\alpha + \delta) \times (\beta + \delta)$.
8. If the equation formed by decreasing each root of $ax^2 + bx + c = 0$ by 1 is $2x^2 + 8x + 2 = 0$, find the condition.
9. If α, β be the roots of $x^2 - a(x - 1) + b = 0$, then find the value of $1/(\alpha^2 - a\alpha) + 1/(\beta^2 - b\beta) + 2/a + b$.
10. If α, β are roots of $375x^2 - 25x - 2 = 0$ and $s_n = \alpha^n + \beta^n$, then find the value of $\lim_{n\to\infty} \sum_{r=1}^{n} s_r$.
11. If α and β are the roots of the equation $2x^2 + 2(a + b)x + a^2 + b^2 = 0$, then find the equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$.
12. If the sum of the roots of an equation is 2 and sum of their cubes is 98, then find the equation.
13. Let α, β be the roots of $x^2 + bx + 1 = 0$. Then find the equation whose roots are $-(\alpha + 1/\beta)$ and $-(\beta + 1/\alpha)$.

COMMON ROOT(S)

Condition for One Common Root

Let us find the condition that the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ may have a common root. Let α be the common root of the given equations. Then,

$$a_1\alpha^2 + b_1\alpha + c_1 = 0$$

and

$$a_2\alpha^2 + b_2\alpha + c_2 = 0$$

Solving these two equations by cross-multiplication, we have

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \quad \alpha^2 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad \text{(from first and third)}$$

and

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \quad \text{(from second and third)}$$

$$\Rightarrow \quad \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} = \left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)^2$$

$$\Rightarrow \quad (c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$$

This condition can easily be remembered by cross-multiplication method as shown in the following figure.

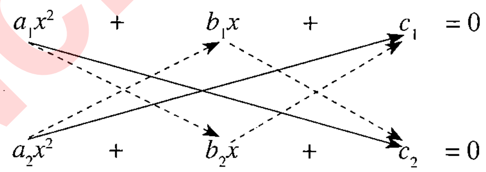


Fig. 1.40

(Bigger cross product)^2
= Product of the two smaller crosses

This is the condition required for a root to be common to two quadratic equations. The common root is given by

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

or

$$\alpha = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$$

Note: *The common root can also be obtained by making the coefficient of x^2 common to the two given equations and then subtracting the two equations. The other roots of the given equations can be determined by using the relations between their roots and coefficients.*

Condition for Both the Common Roots

Let α, β be the common roots of the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$. Then, both the equations are identical, hence,

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

Note:

- *If two quadratic equations with real coefficients have a non-real complex common root then both the roots will be common, i.e. both the equations will be the same. So the coefficients of the corresponding powers of x will have proportional values.*
- *If two quadratic equations with rational coefficients have a common irrational root $p+\sqrt{q}$ then both the roots will be common, i.e. no two different quadratic equations with rational coefficients can have a common irrational root $p+\sqrt{q}$.*

Example 1.114 **Determine the values of m for which the equations $3x^2+4mx+2=0$ and $2x^2+3x-2=0$ may have a common root.**

Sol. Let α be the common root of the equations $3x^2+4mx+2=0$ and $2x^2+3x-2=0$. Then, α must satisfy both the equations. Therefore,

$$3\alpha^2+4m\alpha+2=0$$
$$2\alpha^2+3\alpha-2=0$$

Using cross-multiplication method, we have

$$(-6-4)^2=(9-8m)(-8m-6)$$
$$\Rightarrow\quad 50=(8m-9)(4m+3)$$
$$\Rightarrow\quad 32m^2-12m-77=0$$
$$\Rightarrow\quad 32m^2-56m+44m-77=0$$
$$\Rightarrow\quad 8m(4m-7)+11(4m-7)=0$$
$$\Rightarrow\quad (8m+11)(4m-7)=0$$
$$\Rightarrow\quad m=-\frac{11}{8},\frac{7}{4}$$

Example 1.115 **If $x^2+3x+5=0$ and $ax^2+bx+c=0$ have common root/roots and $a, b, c\in N$, then find the minimum value of $a+b+c$.**

Sol. The roots of $x^2+3x+5=0$ are non-real. Thus given equations will have two common roots. We have,

$$\frac{a}{1}=\frac{b}{3}=\frac{c}{5}=\lambda$$
$$\Rightarrow\quad a+b+c=9\lambda$$

Thus minimum value of $a+b+c$ is 9.

Example 1.116 **If $ax^2+bx+c=0$ and $bx^2+cx+a=0$ have a common root and a, b and c are non-zero real numbers then find the value of $(a^3+b^3+c^3)/abc$.**

Sol. Given that $ax^2+bx+c=0$ and $bx^2+cx+a=0$ have a common root. Hence,

$$(bc-a^2)^2=(ab-c^2)(ac-b^2)$$
$$\Rightarrow\quad b^2c^2+a^4-2a^2bc=a^2bc-ab^3-ac^3+b^2c^2$$
$$\Rightarrow\quad a^4+ab^3+ac^3=3a^2bc$$
$$\Rightarrow\quad \frac{a^3+b^3+c^3}{abc}=3$$

Example 1.117 **a, b, c are positive real numbers forming a G.P. If $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root, then prove that $d/a, e/b, f/c$ are in A.P.**

Sol. For first equation $D=4b^2-4ac=0$ (as given a, b, c are in G.P.). The equation has equal roots which are equal to $-b/a$ each. Thus, it should also be the root of the second equation. Hence,

$$d\left(\frac{-b}{a}\right)^2+2e\left(\frac{-b}{a}\right)+f=0$$
$$\Rightarrow\quad d\frac{b^2}{a^2}-2\frac{be}{a}+f=0$$
$$\Rightarrow\quad d\frac{ac}{a^2}-2\frac{be}{a}+f=0 \qquad (\because b^2=ac)$$
$$\Rightarrow\quad \frac{d}{a}+\frac{f}{c}=2\frac{eb}{ac}=2\frac{e}{b}$$

Example 1.118 **If the equations $x^2+ax+12=0$, $x^2+bx+15=0$ and $x^2+(a+b)x+36=0$ have a common positive root, then find the values of a and b.**

Sol. We have,

$$x^2+ax+12=0 \qquad (1)$$
$$x^2+bx+15=0 \qquad (2)$$

Adding (1) and (2), we get

$$2x^2+(a+b)x+27=0$$

Now subtracting it from the third given equation, we get

$$x^2-9=0\Rightarrow x=3,-3$$

Thus, common positive root is 3. Hence,

$$9+12+3a=0$$
$$\Rightarrow\quad a=-7 \text{ and } 9+3b+15=0$$
$$\Rightarrow\quad b=-8$$

Example 1.119 **The equations $ax^2+bx+a=0$ and $x^3-2x^2+2x-1=0$ have two roots common. Then find the value of $a+b$.**

Sol. By observation, $x=1$ is a root of equation $x^3-2x^2+2x-1=0$. Thus we have

$$(x-1)(x^2-x+1)=0$$

Now roots of $x^2-x+1=0$ are non-real.

Then equation $ax^2+bx+a=0$ has both roots common with $x^2-x+1=0$. Hence, we have

$$\frac{a}{1}=\frac{b}{-1}=\frac{a}{1}$$

or $\quad a+b=0$

Concept Application Exercise 1.6

1. If $x^2+ax+bc=0$ and $x^2+bx+ca=0$ $(a\neq b)$ have a common root, then prove that their other roots satisfy the equation $x^2+cx+ab=0$.

2. Find the condition that the expressions $ax^2 + bxy + cy^2$ and $a_1x^2 + b_1xy + c_1y^2$ may have factors $y - mx$ and $my - x$, respectively.
3. If $a, b, c \in R$ and equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 9 = 0$ have a common root, then find $a:b:c$.
4. Find the condition on a, b, c, d such that equations $2ax^3 + bx^2 + cx + d = 0$ and $2ax^2 + 3bx + 4c = 0$ have a common root.
5. Let $f(x)$, $g(x)$ and $h(x)$ be the quadratic polynomials having positive leading coefficients and real and distinct roots. If each pair of them has a common root, then find the roots of $f(x) + g(x) + h(x) = 0$.

RELATION BETWEEN COEFFICIENT AND ROOTS OF *n*-DEGREE EQUATIONS

- Let α and β be roots of quadratic equation $ax^2 + bx + c = 0$. Then by factor theorem
$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$
$$= a(x^2 - (\alpha + \beta)x + \alpha\beta)$$
Comparing coefficients, we have
$$\alpha + \beta = -b/a \text{ and } \alpha\beta = c/a$$
- Let α, β, γ are roots of cubic equation $ax^3 + bx^2 + cx + d = 0$. Then,
$$ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)$$
$$= a(x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma)$$
Comparing coefficients, we have
$$\alpha + \beta + \gamma = -b/a$$
$$\alpha\beta + \beta\gamma + \alpha\gamma = c/a$$
$$\alpha\beta\gamma = -d/a$$
- If $\alpha, \beta, \gamma, \delta$ are roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$, then
$$\alpha + \beta + \gamma + \delta = -b/a$$
$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = c/a$$
(sum of product taking two at a time)
$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -d/a$$
(sum of product taking three at a time)
$$\alpha\beta\gamma\delta = e/a$$
In general, if $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$, then sum of the roots is
$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n = -\frac{a_1}{a_0}$$
Sum of the product taken two at a time is
$$\left.\begin{array}{r}\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_1\alpha_n \\ \cdots + \alpha_2\alpha_3 + \cdots + \alpha_2\alpha_n \\ \cdots + \alpha_{n-1}\alpha_n\end{array}\right\} = \frac{a_2}{a_0}$$
Sum of the product taken three at a time is $-a_3/a_0$ and so on.
Product of all the root is
$$\alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

Note:
- *A polynomial equation of degree n has n roots (real or imaginary).*
- *If all the coefficients are real then the imaginary roots occur in conjugate pairs, i.e., number of imaginary roots is always even.*
- *If the degree of a polynomial equation is odd, then the number of real roots will also be odd. It follows that at least one of the roots will be real.*

SOLVING CUBIC EQUATION

By using factor theorem together with some intelligent guessing, we can factorise polynomials of higher degree.

In summary, to solve a cubic equation of the form $ax^3 + bx^2 + cx + d = 0$,
1. obtain one factor $(x - \alpha)$ by trial and error
2. factorize $ax^3 + bx^2 + cx + d = 0$ as $(x - \alpha)(hx^2 + kx + s) = 0$
3. solve the quadratic expression for other roots

Example 1.120 **If α, β, γ are the roots of the equation $x^3 + 4x + 1 = 0$, then find the value of $(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1}$.**

Sol. For the given equation $\alpha + \beta + \gamma = 0$,
$$\alpha\beta + \beta\gamma + \alpha\gamma = 4, \quad \alpha\beta\gamma = -1$$
Now,
$$(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1} = (-\gamma)^{-1} + (-\alpha)^{-1} + (-\beta)^{-1}$$
$$= -\frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma}$$
$$= -\frac{4}{(-1)}$$
$$= 4$$

Example 1.121 **Let $\alpha + i\beta$ ($\alpha, \beta \in R$) be a root of the equation $x^3 + qx + r = 0$, $q, r \in R$. Find a real cubic equation, independent of α and β, whose one root is 2α.**

Sol. If $\alpha + i\beta$ is a root then, $\alpha - i\beta$ will also be a root. If the third root is γ, then
$$(\alpha + i\beta) + (\alpha - i\beta) + \gamma = 0$$
$$\Rightarrow \quad \gamma = -2\alpha$$
But γ is a root of the given equation $x^3 + qx + r = 0$. Hence,
$$(-2\alpha)^3 + q(-2\alpha) + r = 0$$
$$\Rightarrow \quad (2\alpha)^3 + q(2\alpha) - r = 0$$
Therefore, 2α is a root of $t^3 + qt - r = 0$, which is independent of α and β.

Example 1.122 **In equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ if two of its roots are equal in magnitude but opposite in sign, find the roots.**

Sol. Given that $\alpha + \beta = 0$ but $\alpha + \beta + \gamma + \delta = 2$. Hence,
$$\gamma + \delta = 2$$
Let $\alpha\beta = p$ and $\gamma\delta = q$. Therefore, given equation is equivalent to $(x^2 + p)(x^2 - 2x + q) = 0$. Comparing the coefficients, we get

$p+q=4$, $-2p=6$, $pq=-21$. Therefore, $p=-3$, $q=7$ and they satisfy $pq=-21$. Hence,

$$(x^2-3)(x^2-2x+7)=0$$

Therefore, the roots are $\pm\sqrt{3}$ and $1\pm i\sqrt{6}$. (where $i=\sqrt{-1}$)

Example 1.123 **Solve the equation $x^3-13x^2+15x+189=0$ if one root exceeds the other by 2.**

Sol. Let the roots be α, $\alpha+2$, β. Sum of roots is $2\alpha+\beta+2=13$.

$$\therefore\quad \beta=11-2\alpha \qquad (1)$$

Sum of the product of roots taken two at a time is

$$\alpha(\alpha+2)+(\alpha+2)\beta+\beta\alpha=15$$

or

$$\alpha^2+2\alpha+2(\alpha+1)\beta=15 \qquad (2)$$

Product of the roots is

$$\alpha\beta(\alpha+2)=-189 \qquad (3)$$

Eliminating β from (1) and (2), we get

$$\alpha^2+2\alpha+2(\alpha+1)(11-2\alpha)=15$$

or

$$3\alpha^2-20\alpha-7=0$$

$$\therefore\quad (\alpha-7)(3\alpha+1)=0$$

$$\therefore\quad \alpha=7 \text{ or } -\frac{1}{3}$$

$$\therefore\quad \beta=-3,\ \frac{35}{3}$$

Out of these values, $\alpha=7$, $\beta=-3$ satisfy the third relation $\alpha\beta(\alpha+2)=-189$, i.e., $(-21)(9)=-189$. Hence, the roots are 7, 7 + 2, –3 or 7, 9, –3.

REPEATED ROOTS

In equation $f(x)=0$, where $f(x)$ is a polynomial function, and if it has roots $\alpha,\alpha,\beta,\ldots$ or α is a repeated root, then $f(x)=0$ is equivalent to $(x-\alpha)^2(x-\beta)\cdots=0$, from which we can conclude that $f'(x)=0$ or $2(x-\alpha)[(x-\beta)\cdots]+(x-\alpha)^2[(x-\beta)\cdots]'=0$ or $(x-\alpha)[2\{(x-\beta)\cdots\}+(x-\alpha)\{(x-\beta)\cdots\}']=0$ has root α.

Thus if α root occurs twice in equation then it is common in equations $f(x)=0$ and $f'(x)=0$.

Similarly, if root α occurs thrice in equation, then it is common in the equations $f(x)=0$, $f'(x)=0$ and $f''(x)=0$.

Example 1.124 **If $x-c$ is a factor of order m of the polynomial $f(x)$ of degree n $(1<m<n)$, then find the polynomials for which $x=c$ is a root.**

Sol. From the given information we have $f(x)=(x-c)^m g(x)$, where $g(x)$ is polynomial of degree $n-m$. Then $x=c$ is common root for the equations $f(x)=0$, $f^1(x)=0$, $f^2(x)=0$, $\ldots$, $f^{m-1}(x)=0$, where $f^r(x)$ represents r^{th} derivative of $f(x)$ w.r.t. x.

Example 1.125 **If $a_1x^3+b_1x^2+c_1x+d_1=0$ and $a_2x^3+b_2x^2+c_2x+d_2=0$ have a pair of repeated roots common, then prove that**

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2b_1-a_1b_2 & c_1a_2-c_2a_1 & d_1a_2-d_2a_1 \end{vmatrix}=0$$

Sol. If $f(x)=a_1x^3+b_1x^2+c_1x+d_1=0$ has roots α,α,β, then $g(x)=a_2x^3+b_2x^2+c_2x+d_2=0$ must have roots α,α,γ. Hence,

$$a_1\alpha^3+b_1\alpha^2+c_1\alpha+d_1=0 \qquad (1)$$

$$a_2\alpha^3+b_2\alpha^2+c_2\alpha+d_2=0 \qquad (2)$$

Now, α is also a root of equations $f'(x)=3a_1x^2+2b_1x+c_1=0$ and $g'(x)=3a_2x^2+2b_2x+c_2=0$. Therefore,

$$3a_1\alpha^2+2b_1\alpha+c_1=0 \qquad (3)$$

$$3a_2\alpha^2+2b_2\alpha+c_2=0 \qquad (4)$$

Also, from $a_2\times(1)-a_1\times(2)$, we have

$$(a_2b_1-a_1b_2)\alpha^2+(c_1a_2-c_2a_1)\alpha+d_1a_2-d_2a_1=0 \qquad (5)$$

Eliminating α from (3), (4) and (5), we have

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2b_1-a_1b_2 & c_1a_2-c_2a_1 & d_1a_2-d_2a_1 \end{vmatrix}=0$$

Concept Application Exercise 1.7

1. If $b^2<2ac$, then prove that $ax^3+bx^2+cx+d=0$ has exactly one real root.
2. If two roots of $x^3-ax^2+bx-c=0$ are equal in magnitude but opposite in signs, then prove that $ab=c$.
3. If α, β and γ are the roots of $x^3+8=0$, then find the equation whose roots are α^2, β^2 and γ^2.
4. If α, β, γ are the roots of the equation $x^3-px+q=0$, then find the cubic equation whose roots are $\alpha/(1+\alpha)$, $\beta/(1+\beta)$, $\gamma/(1+\gamma)$.
5. If the roots of equation $x^3+ax^2+b=0$ are α_1, α_2 and α_3 $(a, b\neq 0)$, then find the equation whose roots are
$$\frac{\alpha_1\alpha_2+\alpha_2\alpha_3}{\alpha_1\alpha_2\alpha_3},\ \frac{\alpha_2\alpha_3+\alpha_3\alpha_1}{\alpha_1\alpha_2\alpha_3},\ \frac{\alpha_1\alpha_3+\alpha_1\alpha_2}{\alpha_1\alpha_2\alpha_3}$$

QUADRATIC EXPRESSION IN TWO VARIABLES

The general quadratic expression $ax^2+2hxy+by^2+2gx+2fy+c$ can be factorized into two linear factors. Given quadratic expression is

$$ax^2+2hxy+by^2+2gx+2fy+c \qquad (1)$$

Corresponding equation is

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

or

$$ax^2+2(hy+g)x+by^2+2fy+c=0 \qquad (2)$$

$$\therefore\quad x=\frac{-2(hy+g)\pm\sqrt{4(hy+g)^2-4a(by^2+2fy+c)}}{2a}$$

$$\Rightarrow\quad x=\frac{-(hy+g)\pm\sqrt{h^2y^2+g^2+2ghy-aby^2-2afy-ac}}{a}$$

$$\Rightarrow ax+hy+g=\pm\sqrt{h^2y^2+g^2+2ghy-aby^2-2afy-ac} \qquad (3)$$

Now, expression (1) can be resolved into two linear factors if $(h^2-ab)y^2+2(gh-af)y+g^2-ac$ is a perfect square and $h^2-ab>0$. But $(h^2-ab)y^2+2(gh-af)y+g^2-ac$ will be a perfect square if

$\Rightarrow$ $4(gh-af)^2-4(h^2-ab)(g^2-ac)=0$ and $h^2-ab>0$

$\Rightarrow$ $g^2h^2+a^2f^2-2afgh-h^2g^2+abg^2+ach^2-a^2bc=0$

and

$h^2-ab>0$

$\Rightarrow$ $abc+2fgh-af^2-bg^2-ch^2=0$

and

$h^2-ab>0$

This is the required condition.

Example 1.126 **Find the values of m for which the expression $2x^2+mxy+3y^2-5y-2$ can be resolved into two rational linear factors.**

Sol. We know that $ax^2+2hxy+by^2+2gx+2fy+c$ can be resolved into two linear factors if and only if

$$abc+2fgh-af^2-bg^2-ch^2=0$$

Given expression is

$$2x^2+mxy+3y^2-5y-2 \quad (1)$$

Here, $a=2$, $h=m/2$, $b=3$, $g=0$, $f=-5/2$, $c=-2$. Therefore, expression $2x^2+mxy+3y^2-5y-2$ will have two linear factors if and only if

$$abc+2fgh-af^2-bg^2-ch^2=0$$

$$\Rightarrow \quad 2\times3(-2)+2\left(\frac{-5}{2}\right)(0)\left(\frac{m}{2}\right)-2\left(\frac{-5}{2}\right)^2-3\times0^2-(-2)\left(\frac{m}{2}\right)^2=0$$

$$\Rightarrow \quad -12-\frac{25}{2}+\frac{m^2}{2}=0$$

$$\Rightarrow \quad m^2=49 \Rightarrow m=\pm7$$

Example 1.127 **Find the linear factors of $2x^2-y^2-x+xy+2y-1$.**

Sol. Given expression is

$$2x^2-y^2-x+xy+2y-1 \quad (1)$$

Its corresponding equation is

$$2x^2-y^2-x+xy+2y-1=0$$

or

$$2x^2-(1-y)x-(y^2-2y+1)=0$$

$$\therefore \quad x=\frac{1-y\pm\sqrt{(1-y)^2+4.2(y^2-2y+1)}}{4}$$

$$=\frac{1-y\pm\sqrt{(1-y)^2+8(y-1)^2}}{4}$$

$$=\frac{1-y\pm\sqrt{9(1-y)^2}}{4}$$

$$=\frac{1-y\pm3(1-y)}{4}$$

$$=1-y,-\frac{1-y}{2}$$

Hence, the required linear factors are $(x+y-1)$ and $(2x-y+1)$.

FINDING THE RANGE OF A FUNCTION INVOLVING QUADRATIC EXPRESSION

In this section, some examples are given to illustrate the range of a function involving quadratic expression.

Example 1.128 **Find the range of the function $f(x)=x^2-2x-4$.**

Sol. Let

$$x^2-2x-4=y$$

$$\Rightarrow \quad x^2-2x-4-y=0$$

Now if x is real, then

$$D\geq0$$

$$\Rightarrow \quad (-2)^2-4(1)(-4-y)\geq0$$

$$\Rightarrow \quad 4+16+4y\geq0$$

$$\Rightarrow \quad y\geq-5$$

Hence range of $f(x)$ is $[-5,\infty)$.

Alternative method:

$$f(x)=x^2-2x-4$$
$$=(x-1)^2-5$$
$$\geq-5$$

Hence, range is

$$[-5,\infty)$$

Example 1.129 **Find the least value of $\dfrac{(6x^2-22x+21)}{(5x^2-18x+17)}$ for real x.**

Sol. Let,

$$\frac{6x^2-22x+21}{5x^2-18x+17}=y$$

$$\Rightarrow \quad (6-5y)x^2-2x(11-9y)+21-17y=0$$

Since x is real

$$4(11-9y)^2-4(6-5y)(21-17y)\geq0$$

$$\Rightarrow \quad -4y^2+9y-5\geq0$$

$$\Rightarrow \quad 4y^2-9y+5\leq0$$

$$\Rightarrow \quad 4(y-1)(y-5/4)\leq0$$

$$\Rightarrow \quad 1\leq y\leq5/4$$

Hence, the least value of the given expression is 1.

Example 1.130 **Prove that if the equation $x^2+9y^2-4x+3=0$ is satisfied for real value of x and y, then x must lie between 1 and 3 and y must lie between $-1/3$ and $1/3$.**

Sol. Given equation is

$$x^2+9y^2-4x+3=0 \quad (1)$$

$$\Rightarrow \quad x^2-4x+9y^2+3=0$$

Since x is real,

$$(-4)^2-4(9y^2+3)\geq0$$

$$\Rightarrow \quad 16-4(9y^2+3)\geq0$$

$$\Rightarrow \quad 4-9y^2-3\geq0$$

$$\Rightarrow \quad 9y^2-1\leq0$$

$$\Rightarrow \quad 9y^2\leq1$$

$$\Rightarrow \quad y^2\leq\frac{1}{9}$$

$$\Rightarrow \quad -\frac{1}{3}\leq y\leq\frac{1}{3} \quad (2)$$

Equation (1) can also be written as

$$9y^2+0y+x^2-4x+3=0 \quad (3)$$

Since y is real, so

$$0^2 - 4.9(x^2 - 4x + 3) \geq 0$$

or

$$x^2 - 4x + 3 \leq 0 \qquad (4)$$

or

$$(x - 3)(x - 1) \leq 0$$

or

$$1 \leq x \leq 3$$

Example 1.131 **Find the domain and the range of**

$$f(x) = \sqrt{3 - 2x - x^2}.$$

Sol. $f(x) = \sqrt{3 - 2x - x^2}$ is defined if

$$3 - 2x - x^2 \geq 0$$

$$\Rightarrow x^2 + 2x - 3 \leq 0$$

$$\Rightarrow (x - 1)(x + 3) \leq 0$$

$$\Rightarrow x \in [-3, 1]$$

Also, $f(x) = \sqrt{4 - (x+1)^2}$ has maximum value when $x + 1 = 0$. Hence range is $[0, 2]$.

Example 1.132 **Find the domain and range of**

$$f(x) = \sqrt{x^2 - 3x + 2}.$$

Sol. $x^2 - 3x + 2 \geq 0$

$$\Rightarrow (x - 1)(x - 2) \geq 0$$

$$\Rightarrow x \in (-\infty, 1] \cup [2, \infty)$$

Now,

$$f(x) = \sqrt{x^2 - 3x + 2}$$

$$= \sqrt{\left(x - \frac{3}{2}\right)^2 + 2 - \frac{9}{4}}$$

$$= \sqrt{\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}}$$

Now, the least permissible value of $(x - 3/2)^2 - 1/4$ is 0 when $(x - 3/2) = \pm 1/2$. Hence, the range is $[0, \infty)$.

Concept Application Exercise 1.8

1. Find the range of $f(x) = x^2 - x - 3$.
2. Find the range of

   (i) $f(x) = \dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$

   (ii) $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1}$
3. Find the range of $f(x) = \sqrt{x - 1} + \sqrt{5 - x}$.
4. Find the range of the function $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$.
5. Find the domain and range of $f(x) = \sqrt{x^2 - 4x + 6}$.

QUADRATIC FUNCTION

Let $f(x) = ax^2 + bx + c$, where $a, b, c, \in R$ and $a \neq 0$. We have,

$$f(x) = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

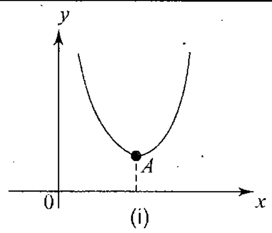
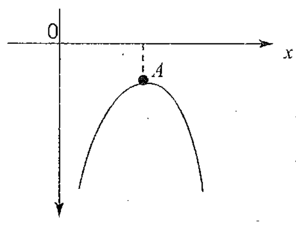
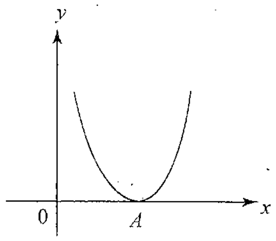
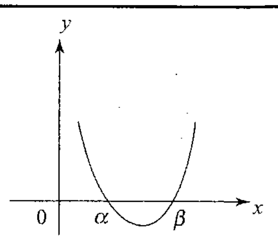
$$= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

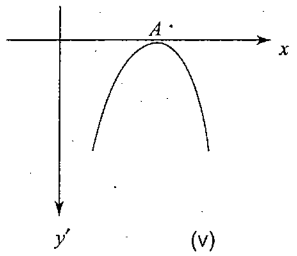
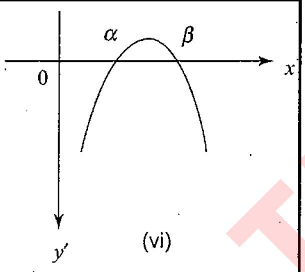
$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$$

$$\Rightarrow \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

Thus $y = f(x)$ represents a parabola whose axis is parallel to y-axis and vertex is $A(-b/2a, -D/4a)$. For some values of x, $f(x)$ may be positive, negative or zero and for $a > 0$, the parabola opens upwards and for $a < 0$, the parabola opens downwards. This gives the following cases:

1. $a > 0$ and $D < 0$, so $f(x) > 0$, $\forall x \in R$, i.e., $f(x)$ is positive for all values of x. Range of function is $[-D/(4a), \infty)$. $x = -b/(2a)$ is a point of minima.	 (i)
2. $a < 0$ and $D < 0$ so $f(x) < 0$, $\forall x \in R$, i.e., $f(x)$ is negative for all values of x. Range of function is $(-\infty, -D/(4a)]$. $x = -b/(2a)$ is a point of maxima.	 (ii)
3. $a > 0$ and $D = 0$, so $f(x) \geq 0$, $\forall x \in R$, i.e., $f(x)$ is positive for all values of x except at vertex where $f(x) = 0$.	 (iii)
4. $a > 0$ and $D > 0$. Let $f(x) = 0$ have two real roots α and β. If $\alpha < \beta$, then $f(x) > 0$, $\forall x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) < 0$, $\forall x \in (\alpha, \beta)$.	 (iv)

5. $a < 0$ and $D = 0$, so $f(x) \le 0$, $\forall\, x \in R$, i.e., $f(x)$ is negative for all values of x except at vertex where $f(x) = 0$.	 (v)
6. $a < 0$ and $D > 0$. Let $f(x) = 0$ have two roots α and β ($\alpha < \beta$). Then $f(x) < 0$, $\forall\, x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0$, $\forall\, x \in (\alpha, \beta)$.	 (vi)

Note: *If $f(x) \ge 0$, $\forall\, x \in R$, then $a > 0$ and $D \le 0$ and if $f(x) \le 0$, $\forall\, x \in R$, then $a < 0$ and $D \le 0$.*

Example 1.133 **What is the minimum height of any point on the curve $y = x^2 - 4x + 6$ above the x-axis?**

Sol. $y = x^2 - 4x + 6$

$= (x - 2)^2 + 2$

Now $(x - 2)^2 \ge 0$ for all real x.

Then $(x - 2)^2 + 2 \ge 2$ for all real x.

Hence, the minimum value of y (or $x^2 - 4x + 6$) is 2, which is the height of the graph above the x-axis.

Example 1.134 **What is the maximum height of any point on the curve $y = -x^2 + 6x - 5$ above the x-axis?**

Sol. $y = -x^2 + 6x - 5$

$= 4 - (x - 3)^2$

Now $(x - 3)^2 \ge 0$ for all real x,

Hence, the maximum value of y (or $-x^2 + 6x - 5$) is 4, which is the height of the graph above x-axis.

Example 1.135 **Find the largest natural number 'a' for which the maximum value of $f(x) = a - 1 + 2x - x^2$ is smaller than the minimum value of $g(x) = x^2 - 2ax + 10 - 2a$.**

Sol. $f(x) = a - 1 + 2x - x^2$

$= a - (x^2 - 2x + 1)$

$= a - (x - 1)^2$

Hence, the maximum value of $f(x)$ is "a" when $(x - 1)^2 = 0$ or $x = 1$

$g(x) = x^2 - 2ax + 10 - 2a$

$= (x - a)^2 + 10 - 2a - a^2$

Hence, the minimum value of $g(x)$ is $10 - 2a - a^2$ when $(x - a)^2 = 0$ or $x = a$.

Now given that maximum of $f(x)$ is smaller than the minimum of $g(x)$

$a < -a^2 + 10 - 2a$

$\Rightarrow$ $a^2 + 3a - 10 < 0$

$\therefore$ $(a + 5)(a - 2) < 0$

The largest natural number $a = 1$.

Example 1.136 **Find the least value of n such that $(n - 2)x^2 + 8x + n + 4 > 0$, $\forall\, x \in R$, where $n \in N$.**

Sol. $(n - 2)x^2 + 8x + n + 4 > 0$, $\forall\, x \in R$

$\Rightarrow$ $64 - 4(n - 2)(n + 4) < 0$ and $n - 2 > 0$

$\Rightarrow$ $16 - (n^2 + 2n - 8) < 0$ and $n > 2$

$\Rightarrow$ $n^2 + 2n - 24 > 0$ and $n > 2$

$\Rightarrow$ $(n + 6)(n - 4) > 0$ and $n > 2$

$\Rightarrow$ $n > 4$ as $n \in N$ and $n > 2$

$\Rightarrow$ $n \ge 5$

Hence, the least value of n is 5.

Example 1.137 **If the inequality $(mx^2 + 3x + 4)/(x^2 + 2x + 2) < 5$ is satisfied for all $x \in R$, then find the values of m.**

Sol. We have,

$x^2 + 2x + 2 = (x + 1)^2 + 1 > 0$, $\forall\, x \in R$

Therefore,

$$\frac{mx^2 + 3x + 4}{x^2 + 2x + 2} < 5$$

$\Rightarrow$ $(m - 5)x^2 - 7x - 6 < 0$, $\forall\, x \in R$

$\Rightarrow$ $m - 5 < 0$ and $D < 0$

$\Rightarrow$ $m < 5$ and $49 + 24(m - 5) < 0$

$\Rightarrow$ $m < \dfrac{71}{24}$

Example 1.138 **If $f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2$, then prove that $(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$.**

Sol. Given,

$$f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2 \qquad (1)$$

or

$$f(x) = (a_1^2 + a_2^2 + \cdots + a_n^2)x^2 + 2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)x + (b_1^2 + b_2^2 + \cdots + b_n^2) \qquad (2)$$

From (1), $f(x) \ge 0$, $\forall\, x \in R$. Hence, from (2), we have

$$(a_1^2 + a_2^2 + \cdots + a_n^2)x^2 + 2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)x + (b_1^2 + b_2^2 + \cdots + b_n^2) \ge 0 \ \forall\, x \in R$$

Discriminant of its corresponding equation is

$D \le 0$ ($\because$ coefficient of x^2 is positive)

$\Rightarrow$ $4(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le 4(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$

$\Rightarrow$ $(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$

Example 1.139 **If c is positive and $2ax^2 + 3bx + 5c = 0$ does not have any real roots, then prove that $2a - 3b + 5c > 0$.**

Sol. Given $c > 0$ and $2ax^2 + 3bx + 5c = 0$ does not have real roots. Let

$$f(x) = 2ax^2 + 3bx + 5c$$

$\Rightarrow$ $f(x) > 0, \forall x \in R$, if $a > 0$ or $f(x) < 0 \forall x \in R$, if $a < 0$

But

$5c = f(0) > 0$

$\Rightarrow$ $f(x) > 0, \forall x \in R$

$\Rightarrow$ $2ax^2 + 3bx + 5c > 0, \forall x \in R$

$\Rightarrow$ $2a - 3b + 5c > 0$ (for $x = -1$)

Example 1.140 **If $ax^2 + bx + 6 = 0$ does not have distinct real roots, then find the least value of $3a + b$.**

Sol. Given equation $ax^2 + bx + 6 = 0$ does not have distinct real roots. Hence,

$\Rightarrow$ $f(x) = ax^2 + bx + 6 \le 0, \forall x \in R$, if $a < 0$

or

$f(x) = ax^2 + bx + 6 \ge 0, \forall x \in R$, if $a > 0$

But

$f(0) = 6 > 0$

$\Rightarrow$ $f(x) = ax^2 + bx + 6 \ge 0, \forall x \in R$

$\Rightarrow$ $f(3) = 9a + 3b + 6 \ge 0$

$\Rightarrow$ $3a + b \ge -2$

Therefore, the least value of $3a + b$ is -2.

Example 1.141 **A quadratic trinomial $P(x) = ax^2 + bx + c$ is such that the equation $P(x) = x$ has no real roots. Prove that in this case the equation $P(P(x)) = x$ has no real roots either.**

Sol. Since the equation $ax^2 + bx + c = x$ has no real roots, the expression $P(x) - x = ax^2 + (b - 1)x + c$ assumes values of one sign $\forall x \in R$, say $P(x) - x > 0$. Then

$$P(P(x_0)) - P(x_0) > 0$$

for any $x = x_0$, i.e., $P(x_0) > x_0$ and hence $P(P(x_0)) > x_0$. Therefore, x_0 cannot be a root of the 4th degree equation $P(P(x)) = x$.

Example 1.142 **Prove that for real values of x the expression $(ax^2 + 3x - 4)/(3x - 4x^2 + a)$ may have any value provided a lies between 1 and 7.**

Sol. Let,

$$y = \frac{ax^2 + 3x - 4}{3x - 4x^2 + a}$$

$\Rightarrow$ $(a + 4y)x^2 + (3 - 3y)x - 4 - ay = 0$

Now, x is real. So,

$D \ge 0$

$\Rightarrow$ $9(1 - y)^2 + 4(a + 4y)(4 + ay) \ge 0$

$\Rightarrow$ $(9 + 16a)y^2 + (-18 + 4a^2 + 64)y + (9 + 16a) \ge 0$, $\forall y \in R$ ($\because$ y takes any real value)

$\Rightarrow$ $9 + 16a > 0$ and $(4a^2 + 46)^2 - 4(9 + 16a)^2 \le 0$

$\Rightarrow$ $a > -\frac{9}{16}$ and $(4a^2 + 46 - 18 - 32a)(4a^2 + 46 + 18 + 32a) \le 0$

$\Rightarrow$ $a > -\frac{9}{16}$ and $(a^2 - 8a + 7)(a^2 + 8a + 16) \le 0$

$\Rightarrow$ $a > -\frac{9}{16}$ and $1 \le a \le 7$ or $a = -4$

$\Rightarrow$ $1 \le a \le 7$

Example 1.143 **Let a, b and c be real numbers such that $a + 2b + c = 4$. Find the maximum value of $(ab + bc + ca)$.**

Sol. Given,

$$a + 2b + c = 4 \Rightarrow a = 4 - 2b - c$$

Let,

$ab + bc + ca = x \Rightarrow a(b + c) + bc = x$

$\Rightarrow$ $(4 - 2b - c)(b + c) + bc = x$

$\Rightarrow$ $4b + 4c - 2b^2 - 2bc - bc - c^2 + bc = x$

$\Rightarrow$ $2b^2 - 4b + 2bc - 4c + c^2 + x = 0$

$\Rightarrow$ $2b^2 + 2(c - 2)b - 4c + c^2 + x = 0$

Since $b \in R$, so

$4(c - 2)^2 - 4 \times 2(-4c + c^2 + x) \ge 0$

$\Rightarrow$ $c^2 - 4c + 4 + 8c - 2c^2 - 2x \ge 0$

$\Rightarrow$ $c^2 - 4c + 2x - 4 \le 0$

Since $c \in R$, so

$16 - 4(2x - 4) \ge 0 \Rightarrow x \le 4$

$\therefore$ $\max(ab + bc + ac) = 4$

Example 1.144 **Prove that for all real values of x and y, $x^2 + 2xy + 3y^2 - 6x - 2y \ge -11$.**

Sol. Let,

$x^2 + 2xy + 3y^2 - 6x - 2y + 11 \ge 0, \forall x, y \in R$

$\Rightarrow$ $x^2 + (2y - 6)x + 3y^2 - 2y + 11 \ge 0, \forall x \in R$

$\Rightarrow$ $(2y - 6)^2 - 4(3y^2 - 2y + 11) \le 0, \forall y \in R$

$\Rightarrow$ $(y - 3)^2 - (3y^2 - 2y + 11) \le 0, \forall y \in R$

$\Rightarrow$ $2y^2 + 4y + 2 \ge 0, \forall y \in R$

$\Rightarrow$ $(y + 1)^2 \ge 0, \forall y \in R$, which is always true

Concept Application Exercise 1.9

1. If $f(x) = \sqrt{x^2 + ax + 4}$ is defined for all x, then find the values of a.
2. If $ax^2 + bx + c = 0$, $a, b, c \in R$ has no real zeros, and if $c < 0$, then which of the following is true?
   (i) $a < 0$
   (ii) $a + b + c > 0$
   (iii) $a > 0$
3. If $ax^2 + bx + c = 0$ has imaginary roots and $a + c < b$, then prove that $4a + c < 2b$.

4. Let $x, y, z \in R$ such that $x + y + z = 6$ and $xy + yz + zx = 7$. Then find the range of values of x, y and z.
5. If x is real and $(x^2 + 2x + c)/(x^2 + 4x + 3c)$ can take all real values, then show that $0 \le c \le 1$.
6. If $x \in R$, and a, b, c are in ascending or descending order of magnitude, show that $(x - a)(x - c)/(x - b)$ (where $x \ne b$) can assume any real value.
7. Find the complete set of values of a such that $(x^2 - x)/(1 - ax)$ attains all real values.
8. If the quadratic equation $ax^2 + bx + 6 = 0$ does not have real roots and $b \in R^+$, then prove that
$$a > \max\left\{\frac{b^2}{24}, b - 6\right\}$$
9. If x be real and the roots of the equation $ax^2 + bx + c = 0$ are imaginary, then prove that $a^2x^2 + abx + ac$ is always positive.

LOCATION OF ROOTS

In some problems, we want the roots of the equation $ax^2 + bx + c = 0$ to lie in a given interval. For this we impose conditions on a, b and c.

1. $\alpha, \beta > 0$

Conditions:
(a) sum of roots, $\alpha + \beta > 0$
(b) product of roots, $\alpha\beta > 0$
(c) $D \ge 0$

Graphically:

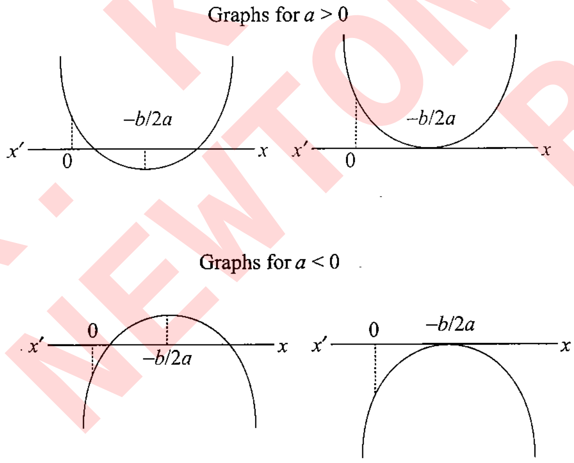


Fig. 1.41

Conditions:
(a) $af(0) > 0$ ($\because$ when $a > 0, f(0) > 0$ and when $a < 0, f(0) < 0$)
(b) $-b/2a > 0$
(c) $D \ge 0$

2. $\alpha, \beta < 0$

Conditions:
(a) sum of roots, $\alpha + \beta < 0$
(b) product of roots, $\alpha\beta > 0$
(c) $D \ge 0$

Graphically:

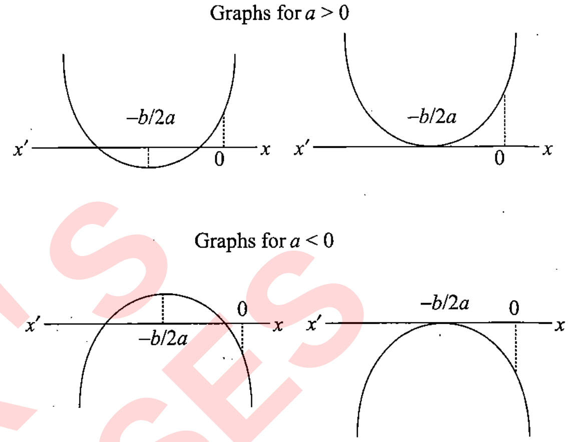


Fig. 1.42

Conditions:
(a) $af(0) > 0$
(b) $-b/2a < 0$
(c) $D \ge 0$

3. $\alpha < 0 < \beta$ (roots of opposite sign)
Product of roots, $\alpha\beta < 0$

Note That when $\alpha\beta = \frac{C}{a} 0,\ ac < 0$

$\Rightarrow\quad D = b^2 - 4ac > 0.$

Graphically:

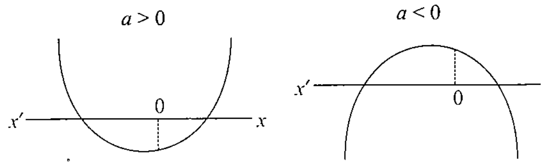


Fig. 1.43

When $a > 0, f(0) < 0$ and when $a < 0$, then $f(0) > 0$
$\Rightarrow\quad af(0) < 0.$

4. $\alpha, \beta > k$

Graphically:

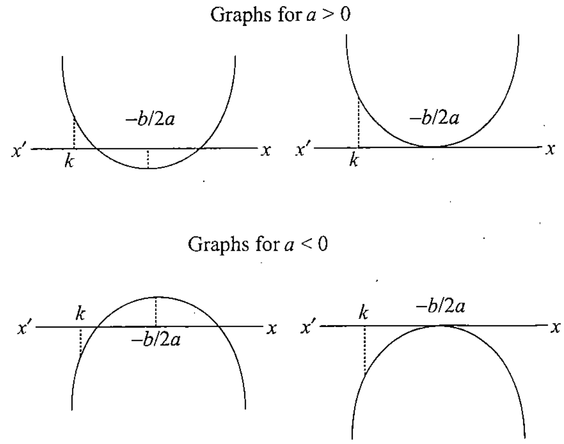


Fig. 1.44

Conditions:
(a) $af(k) > 0$ ($\because$ when $a > 0, f(k) > 0$ and when $a < 0, f(k) < 0$)
(b) $-b/2a > k$
(c) $D \geq 0$

5. $\alpha, \beta < k$
Graphically:

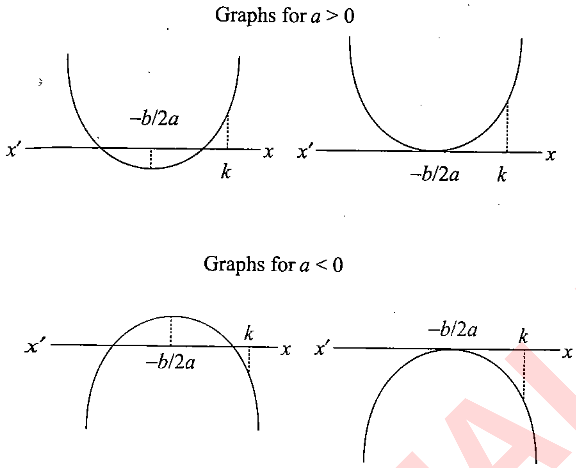


Fig. 1.45

Conditions:
(a) $af(k) > 0$
($\because$ when $a > 0, f(k) > 0$ and when $a < 0, f(k) < 0$)
(b) $-b/(2a) < k$
(c) $D \geq 0$

6. $\alpha < k < \beta$ (one root is smaller than k and other root is greater than k)

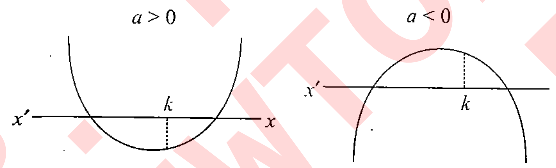


Fig. 1.46

When $a > 0, f(k) < 0$ and when $a < 0$, then $f(k) > 0$
$\Rightarrow \quad af(k) < 0$.

7. Exactly one root lying in (k_1, k_2)

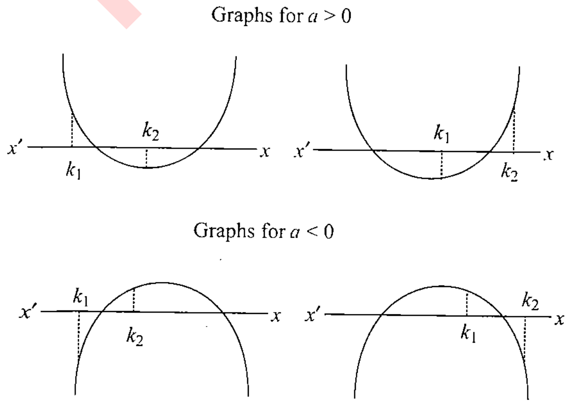


Fig. 1.47

From the graphs, we can see that $f(k_1)$ and $f(k_2)$ have opposite sign. Hence, $f(k_1)f(k_2) < 0$.

8. Both the roots lying in the interval (k_1, k_2).

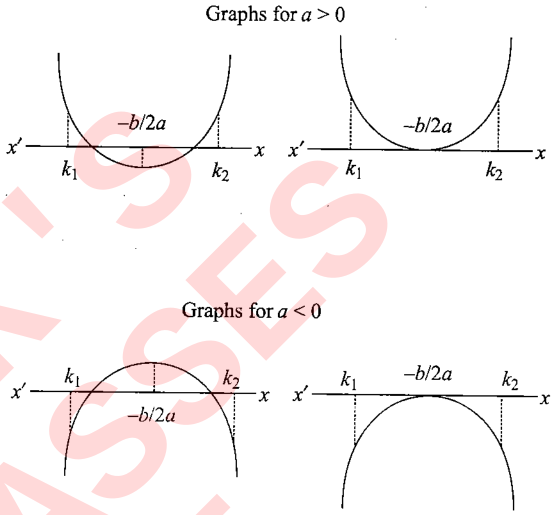


Fig. 1.48

From the graphs,
(a) $af(k_1) > 0$ and $af(k_2) > 0$
(b) $k_1 < -b/(2a) < k_2$
(c) $D \geq 0$

9. One root is smaller than k_1 and other root is greater than k_2
In this case k_1 and k_2 lie between the roots.

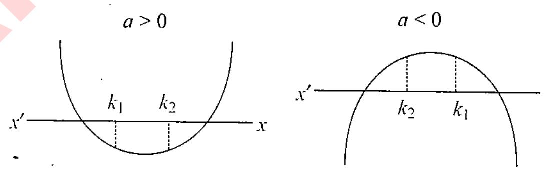


Fig. 1.49

From the graphs, $af(k_1) < 0$ and $af(k_2) < 0$.

Example 1.145 **Let $x^2 - (m - 3)x + m = 0$ ($m \in R$) be a quadratic equation. Find the values of m for which the roots are**

(i) **real and distinct**
(ii) **equal**
(iii) **not real**
(iv) **opposite in sign**
(v) **equal in magnitude but opposite in sign**
(vi) **positive**
(vii) **negative**
(viii) **such that at least one is positive**
(ix) **one root is smaller than 2 and the other root is greater than 2**
(x) **both the roots are greater than 2**

(xi) both the roots are smaller than 2

(xii) exactly one root lies in the interval (1, 2)

(xiii) both the roots lie in the interval (1, 2)

(xiv) at least one root lies in the interval (1, 2)

(xv) one root is greater than 2 and the other root is smaller than 1

Sol. Let $f(x) = x^2 - (m - 3)x + m = 0$

(i)

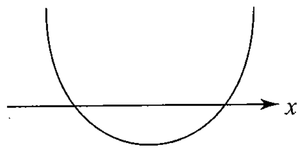


Fig. 1.50

Both the roots are real and distinct. So,

$D > 0$

$\Rightarrow \quad (m-3)^2 - 4m > 0$

$\Rightarrow \quad m^2 - 10m + 9 > 0$

$\Rightarrow \quad (m-1)(m-9) > 0$

$\Rightarrow \quad m \in (-\infty, 1) \cup (9, \infty)$

(ii)

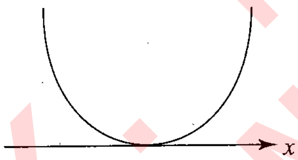


Fig. 1.51

Both the roots are equal. So,

$D = 0 \Rightarrow m = 9$ or $m = 1$

(iii)

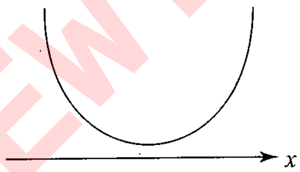


Fig. 1.52

Both the roots are imaginary. So,

$D < 0$

$\Rightarrow \quad (m-1)(m-9) < 0$

$\Rightarrow \quad m \in (1, 9)$

(iv)

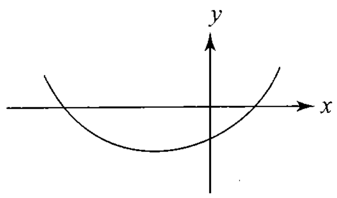


Fig. 1.53

The roots are opposite in sign. Hence, the product of roots is negative. So,

$m < 0 \Rightarrow m \in (-\infty, 0)$

(v)

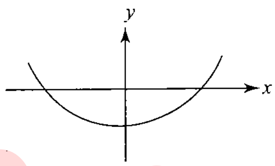


Fig. 1.54

Roots are equal in magnitude but opposite in sign. Hence, sum of roots is zero as well as $D \geq 0$. So,

$m \in (-\infty, 1) \cup (9, \infty)$ and $m - 3 = 0$, i.e., $m = 3$

$\Rightarrow$ no such m exists, so $m \in \phi$.

(vi)

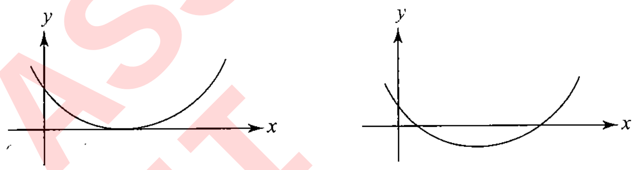


Fig. 1.55

Both the roots are positive. Hence, $D \geq 0$ and both the sum and the product of roots are positive. So,

$m - 3 > 0$, $m > 0$ and $m\ (-\infty, 1] \cup [9, \infty)$

$m \in [9, \infty)$

(vii)

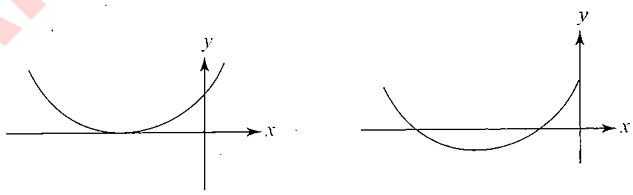


Fig. 1.56

Both the roots are negative. Hence, $D \geq 0$, and sum is negative but product is positive. So,

$m - 3 < 0$, $m > 0$, $m \in (-\infty, 1] \cup [9, \infty)$

$\Rightarrow \quad m \in (0, 1]$

(viii) At least one root is positive. Hence, either one root is positive or both roots are positive. So,

$m \in (-\infty, 0) \cup [9, \infty)$

(ix)

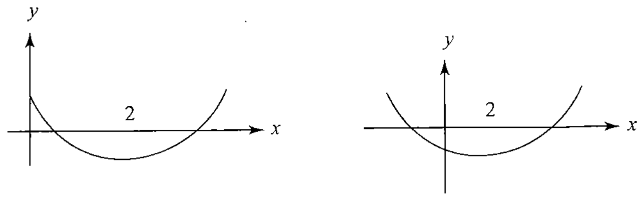


Fig. 1.57

One root is smaller than 2 and the other root is greater than 2, i.e., 2 lies between the roots. So,

$$f(2) < 0$$

$\Rightarrow \quad 4 - 2(m - 3) + m < 0$

$\Rightarrow \quad m > 10$

(x)

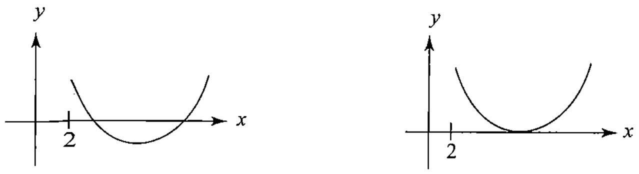


Fig. 1.58

Both the roots are greater than 2. So,

$$f(2) > 0, D \ge 0, -\frac{b}{2a} > 2$$

$\Rightarrow \quad m < 10$ and $m \in (-\infty, 1] \cup [9, \infty)$ and $m - 3 > 4$

$\Rightarrow \quad m \in [9, 10)$

(xi)

Fig. 1.59

Both the roots are smaller than 2. So,

$$f(2) > 0, D \ge 0, -\frac{b}{2a} < 2$$

$\Rightarrow \quad m \in (-\infty, 1]$

(xii)

Fig. 1.60

Exactly one root lies in (1, 2). So,

$$f(1)f(2) < 0$$

$\Rightarrow \quad 4(10 - m) < 0$

$\Rightarrow \quad m \in (10, \infty)$

(xiii) Both the roots lie in the interval (1, 2). Then,

$$D \ge 0 \Rightarrow (m - 1)(m - 9) \ge 0 \Rightarrow m \le 1 \text{ or } m \ge 9 \quad (1)$$

Also

$$f(1) > 0 \text{ and } f(2) > 0 \Rightarrow 10 > m \quad (2)$$

and

$$1 < -\frac{b}{2a} < 2 \Rightarrow 5 < m < 7 \quad (3)$$

Thus, no such m exists.

(xiv) **Case I:** Exactly one root lies in (1, 2). So,

$$f(1)f(2) < 0 \Rightarrow m > 10$$

Case II: Both the roots lie in (1, 2). So, from (xiii), $m \in \phi$. Hence, $m \in (10, \infty)$.

(xv) For one root greater than 2 and the other root smaller than 1,

$$f(1) < 0 \quad (1)$$

$$f(2) < 0 \quad (2)$$

From (1), $f(1) < 0$, but $f(1) = 4$, which is not possible. Thus, no such m exists.

Example 1.146 **Find the values of a for which the equation $\sin^4 x + a\sin^2 x + 1 = 0$ will have a solution.**

Sol. Let

$$t = \sin^2 x \Rightarrow t \in [0, 1]$$

Hence, $t^2 + at + 1 = 0$ should have at least one solution in [0, 1]. Since product of roots is positive and equal to one, $t^2 + at + 1 = 0$ must have exactly one root in [0, 1]. Hence,

$$f(1) < 0$$

$\Rightarrow \quad 2 + a < 0$

$\Rightarrow \quad a \in (-\infty, -2)$

Example 1.147 **If $(x^2 + x + 2)^2 - (a - 3)(x^2 + x + 1)(x^2 + x + 2) + (a - 4)(x^2 + x + 1)^2 = 0$ has at least one root, then find the complete set of values of a.**

Sol. Let,

$$t = x^2 + x + 1 \Rightarrow t \in \left[\frac{3}{4}, \infty\right)$$

Hence,

$$(t + 1)^2 - (a - 3)\, t(t + 1) + (a - 4)\, t^2 = 0$$

$\Rightarrow \quad t^2 + 2t + 1 - (a - 3)(t^2 + t) + (a - 4)t^2 = 0$

$\Rightarrow \quad t(2 - a + 3) + 1 = 0$

$\Rightarrow \quad t = \dfrac{1}{(a - 5)}$

$\Rightarrow \quad \dfrac{1}{a - 5} \ge \dfrac{3}{4}$

$\Rightarrow \quad \dfrac{19 - 3a}{(a - 5)} \ge 0$

$\Rightarrow \quad a \in \left(5, \dfrac{19}{3}\right]$

Example 1.148 **If α is a real root of the quadratic equation $ax^2 + bx + c = 0$ and β is a real root of $-ax^2 + bx + c = 0$,**

then show that there is a root γ of the equation $(a/2)x^2 + bx + c = 0$ which lies between α and β.

Sol. Let,

$$f(x) = \frac{a}{2}x^2 + bx + c$$

$$\Rightarrow \quad f(\alpha) = \frac{a}{2}\alpha^2 + b\alpha + c$$

$$= a\alpha^2 + b\alpha + c - \frac{a}{2}\alpha^2$$

$$= -\frac{a}{2}\alpha^2 \qquad (\because \alpha \text{ is a root of } ax^2 + bx + c = 0)$$

$$f(\beta) = \frac{a}{2}\beta^2 + b\beta + c$$

$$= -a\beta^2 + b\beta + c + \frac{3}{2}a\beta^2$$

$$= \frac{3}{2}a\beta^2 \qquad (\because \beta \text{ is a root of } -ax^2 + bx + c = 0)$$

Now,

$$f(\alpha)f(\beta) = \frac{-3}{4}a^2\alpha^2\beta^2 < 0$$

Hence, $f(x) = 0$ has one real root between α and β.

Example 1.149 **For what real values of a do the roots of the equation $x^2 - 2x - (a^2 - 1) = 0$ lie between the roots of the equation $x^2 - 2(a + 1)x + a(a - 1) = 0$.**

Sol. $x^2 - 2x - (a^2 - 1) = 0$ (1)

$x^2 - 2(a + 1)x + a(a - 1) = 0$ (2)

From Eq. (1),

$$x = \frac{2 \pm \sqrt{4 + 4(a^2 - 1)}}{2} = 1 \pm a$$

Now, roots of Eq. (1) lie between roots of Eq. (2). Hence, graphs of expressions for Eqs. (1) and (2) are as follows:

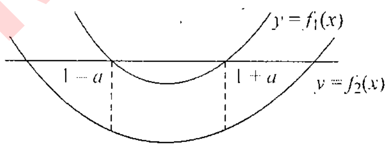


Fig. 1.61

$$f_1(x) = x^2 - 2x - (a^2 - 1)$$
$$f_2(x) = x^2 - 2(a + 1)x + a(a - 1)$$

From the graph, we have

$$f_2(1 - a) < 0 \text{ and } f_2(1 + a) < 0$$

$$\Rightarrow \quad (1 - a)^2 - 2(a + 1)(1 - a) + a(a - 1) < 0$$

$$\Rightarrow \quad (1 - a)[(1 - a) - 2a - 2 - a] < 0$$

$$\Rightarrow \quad (1 - a)(-4a - 1) < 0$$

$$\Rightarrow \quad (a - 1)(4a + 1) < 0$$

$$\Rightarrow \quad -\frac{1}{4} < a < 1 \qquad (3)$$

and

$$\Rightarrow \quad (1 + a)^2 - 2(a + 1)(a + 1) + a(a - 1) < 0$$

$$\Rightarrow -(a + 1)^2 + a(a - 1) < 0$$

$$\Rightarrow \quad -a^2 - 2a - 1 + a^2 - a < 0$$

$$\Rightarrow \quad 3a + 1 > 0$$

$$\Rightarrow \quad a > -\frac{1}{3} \qquad (4)$$

From (3) and (4), the required values of a lies in the range $-1/4 < a < 1$.

SOLVING INEQUALITIES USING LOCATION OF ROOTS

Example 1.150 **Find the value of a for which $ax^2 + (a - 3)x + 1 < 0$ for at least one positive real x.**

Sol. Let $f(x) = ax^2 + (a - 3)x + 1$

Case I:

If $a > 0$, then $f(x)$ will be negative only for those values of x which lie between the roots. From the graphs, we can see that $f(x)$ will be less than zero for at least one positive real x, when $f(x) = 0$ has distinct roots and at least one of these roots is a positive real root.

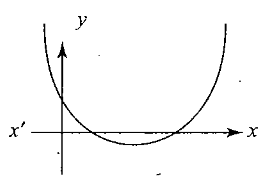


Fig. 1.62

Since $f(0) = 1 > 0$, the favourable graph according to the question is shown in the figure given above. From the graph, we can see that both the roots are non-negative. For this,

(i) $D > 0 \Rightarrow (a - 3)^2 - 4a > 0$

$\Rightarrow \quad a < 1$ or $a > 9$ (1)

(ii) sum > 0 and product ≥ 0

$\Rightarrow \quad -(a - 3) > 0$ and $1/a > 0$

$\Rightarrow \quad 0 < a < 3$ (2)

From (1) and (2), we have

$a \in (0, 1)$

Case II: $a < 0$

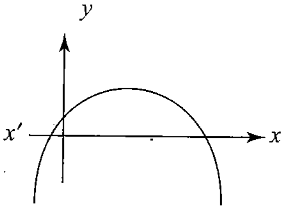


Fig. 1.63

Since $f(0) = 1 > 0$, then graph is as shown in the figure, which shows that $ax^2 + (a-3)x + 1 < 0$, for at least one positive x.

Case III: $a = 0$

If $a = 0$,

$$f(x) = -3x + 1$$

$\Rightarrow$ $f(x) < 0, \forall x > 1/3$

Thus, from all the cases, the required set of values of a is $(-\infty, 1)$.

Example 1.151 **If $x^2 + 2ax + a < 0 \ \forall x \in [1, 2]$, then find the values of a.**

Sol. Given,

$$x^2 + 2ax + a < 0, \forall x \in [1, 2]$$

Hence, 1 and 2 lie between the roots of the equation $x^2 + 2ax + a = 0$,

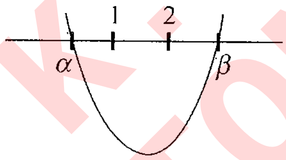


Fig. 1.64

$\Rightarrow$ $f(1) < 0$ and $f(2) < 0$

$\Rightarrow$ $1 + 2a + a < 0, 4 + 4a + a < 0$

$\Rightarrow$ $a < -\frac{1}{3}, a < -\frac{4}{5}$

$\Rightarrow$ $a \in \left(-\infty, -\frac{4}{5}\right)$

Example 1.152 **If $(y^2 - 5y + 3)(x^2 + x + 1) < 2x$ for all $x \in R$, then find the interval in which y lies.**

Sol. $(y^2 - 5y + 3)(x^2 + x + 1) < 2x, \forall x \in R$

$$\Rightarrow \quad y^2 - 5y + 3 < \frac{2x}{x^2 + x + 1} \quad (\because \quad x^2 + x + 1 > 0 \ \forall x \in R)$$

L.H.S. must be less than the least value of R.H.S. Now lets find the range of R.H.S.

Let

$$\frac{2x}{x^2 + x + 1} = p$$

$\Rightarrow$ $px^2 + (p-2)x + p = 0$

Since x is real,

$$(p-2)^2 - 4p^2 \geq 0$$

$\Rightarrow$ $-2 \leq p \leq \frac{2}{3}$

The minimum value of $2x/(x^2 + x + 1)$ is -2. So,

$$y^2 - 5y + 3 < -2$$

$\Rightarrow$ $y^2 - 5y + 5 < 0$

$\Rightarrow$ $y \in \left(\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right)$

Example 1.153 **Find the values of a for which $4^t - (a-4)2^t + (9/4)a < 0, \forall t \in (1, 2)$.**

Sol. Let $2^t = x$ and $f(x) = x^2 - (a-4)x + (9/4)a$. We want $f(x) < 0, \forall x \in (2^1, 2^2)$, i.e., $\forall x \in (2, 4)$.

(i) Since coefficient of x^2 in $f(x)$ is positive, $f(x) < 0$ for some x only when roots of $f(x) = 0$ are real and distinct. So,

$$D > 0$$

$\Rightarrow$ $a^2 - 17a + 16 > 0$ (1)

(ii) Since we want $f(x) < 0 \ \forall x \in (2, 4)$, one of the roots of $f(x) = 0$ should be smaller than 2 and the other must be greater than 4, i.e.,

$$f(2) < 0 \text{ and } f(4) < 0$$

$\Rightarrow$ $a < -48$ and $a > 128/7$

which is not possible. Hence, no such a exists.

Concept Application Exercise 1.10

1. Find the values of a if $x^2 - 2(a-1)x + (2a+1) = 0$ has positive roots.
2. If the equation $(a-5)x^2 + 2(a-10)x + a + 10 = 0$ has roots of opposite sign, then find the values of a.
3. If both the roots of $x^2 - ax + a = 0$ are greater than 2, then find the values of a.
4. If both the roots of $ax^2 + ax + 1 = 0$ are less than 1, then find exhaustive range of values of a.
5. If both the roots of $x^2 + ax + 2 = 0$ lies in the interval $(0, 3)$, then find exhaustive range of values of a.
6. If α, β are the roots of $x^2 - 3x + a = 0$, $a \in R$ and $\alpha < 1 < \beta$, then find the values of a.
7. If α is the root (having the least absolute value) of the equation $x^2 - bx - 1 = 0$ ($b \in R^+$), then prove that $-1 < \alpha < 0$.
8. If $a < b < c < d$, then show that the quadratic equation $\mu(x-a)(x-c) + \lambda(x-b)(x-d) = 0$ has real roots for all real μ and λ.

EXERCISES

Subjective Type

Solutions on page 1.60

1. Solve the following:

$$\left(\sqrt{x^2-5x+6}+\sqrt{x^2-5x+4}\right)^{x/2}+\left(\sqrt{x^2-5x+6}-\sqrt{x^2-5x+4}\right)^{x/2}=2^{\frac{x+4}{4}}$$

2. Show that the equation $A^2/(x-a)+B^2/(x-b)+C^2/(x-c)+\cdots+H^2/(x-h)=k$ has no imaginary root, where $A, B, C, \ldots, H$ and $a, b, c, \ldots, h$ and $k \in R$.

3. Given that a, b, c are distinct real numbers such that the expression ax^2+bx+c, bx^2+cx+a and cx^2+ax+b are always non-negative. Prove that the quantity $(a^2+b^2+c^2)/(ab+bc+ca)$ can never lie in $(-\infty, 1] \cup [4, \infty)$.

4. Find the number of quadratic equations, which are unchanged by squaring their roots.

5. If α and β are the roots of $x^2-p(x+1)-c=0$, then show that $(\alpha+1)(\beta+1)=1-c$. Hence, prove that

$$\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}=1$$

6. If α, β are the roots of the equation $ax^2+bx+c=0$ and $S_n=\alpha^n+\beta^n$, then show that $aS_{n+1}+bS_n+cS_{n-1}=0$ and hence find S_5.

7. If α be a root of the equation $4x^2+2x-1=0$, then prove that $4\alpha^3-3\alpha$ is the other root.

8. If $(ax^2+bx+c)y+(a'x^2+b'x+c')=0$ and x is a rational function of y, then prove that $(ac'-a'c)^2=(ab'-a'b)\times(bc'-b'c)$.

9. If the roots of the equation $x^2-ax+b=0$ are real and differ by a quantity which is less than c $(c>0)$, then show that b lies between $(a^2-c^2)\,4$ and $a^2/4$.

10. The equation $ax^2+bx+c=0$ has real and positive roots. Prove that the roots of the equation $a^2x^2+a(3b-2c)x+(2b-c)(b-c)+ac=0$ are real and positive.

11. If $x^2+px-444p=0$ has integral roots where p is a prime number, then find the value(s) of p.

12. If a and c are odd prime numbers and $ax^2+bx+c=0$ has rational roots where $b \in I$. Prove that one root of the equation will be independent of a, b, c.

13. If $2x^2-3xy-2y^2=7$, then prove that there will be only two integral pairs (x, y) satisfying the above relation.

14. Let $a, b \in N$ and $a>1$. Also p is a prime number. If $ax^2+bx+c=p$ for two distinct integral values of x, then prove that $ax^2+bx+c \neq 2p$ for any integral value of x.

15. Show that minimum value of $(x+a)(x+b)/(x+c)$, where $a>c$, $b>c$ is $(\sqrt{a-c}+\sqrt{b-c})^2$ for real values of $x>-c$.

16. If $x \in R$, then prove that maximum value of $2(a-x)(x+\sqrt{x^2+b^2})$ is a^2+b^2.

17. If $f(x)=x^3+bx^2+cx+d$ and $f(0), f(-1)$ are odd integers, prove that $f(x)=0$ cannot have all integral roots.

18. Find the values of k for which

$$\left|\frac{x^2+kx+1}{x^2+x+1}\right|<2, \forall x \in R$$

19. Solve the equation $\sqrt{a(2^x-2)+1}=1-2^x, x \in R$.

20. For $a<0$, determine all real roots of the equation $x^2-2a|x-a|-3a^2=0$.

21. Find the integral part of the greatest root of equation $x^3-10x^2-11x-100=0$.

22. Find the values of a for which all the roots of the equation $x^4-4x^3-8x^2+a=0$ are real.

Objective Type

Solutions on page 1.64

Each question has four choices a, b, c and d, out of which *only one* is correct. Find the correct answer.

1. If $x=2+2^{2/3}+2^{1/3}$, then the value of x^3-6x^2+6x is
   a. 3 **b.** 2 **c.** 1 **d.** −2

2. The least value of the expression $x^2+4y^2+3z^2-2x-12y-6z+14$ is
   a. 1 **b.** no least value **c.** 0 **d.** none of these

3. Number of positive integers n for which n^2+96 is a perfect square is
   a. 8 **b.** 12 **c.** 4 **d.** infinite

4. If $x, y \in R$ satisfy the equation $x^2+y^2-4x-2y+5=0$, then the value of the expression $[(\sqrt{x}-\sqrt{y})^2+4\sqrt{xy}]/(x+\sqrt{xy})$ is
   a. $\sqrt{2}+1$ **b.** $\frac{\sqrt{2}+1}{2}$
   c. $\frac{\sqrt{2}-1}{2}$ **d.** $\frac{\sqrt{2}+1}{\sqrt{2}}$

5. The number of real roots of the equation $x^2-3|x|+2=0$ is
   a. 2 **b.** 1 **c.** 4 **d.** 3

6. If $x=1+i$ is a root of the equation $x^3-ix+1-i=0$, then the other real root is
   a. 0 **b.** 1 **c.** −1 **d.** none of these

7. The number of roots of the equation $\sqrt{x-2}(x^2-4x+3)=0$ is
   a. three **b.** four
   c. one **d.** two

8. The curve $y=(\lambda+1)x^2+2$ intersects the curve $y=\lambda x+3$ in exactly one point, if λ equals
   a. $\{-2, 2\}$ **b.** $\{1\}$

c. $\{-2\}$ d. $\{2\}$

9. If the expression $x^2 + 2(a + b + c)x + 3(bc + ca + ab)$ is a perfect square, then

a. $a = b = c$ b. $a = \pm b = \pm c$
c. $a = b \neq c$ d. none of these

10. If one root of the equation $ax^2 + bx + c = 0$ is square of the other, then $a(c - b)^3 = cX$, where X is

a. $a^3 - b^3$ b. $a^3 + b^3$
c. $(a - b)^3$ d. none of these

11. If $x^2 + px + 1$ is a factor of the expression $ax^3 + bx + c$, then

a. $a^2 - c^2 = ab$ b. $a^2 + c^2 = -ab$
c. $a^2 - c^2 = -ab$ d. none of these

12. Sum of the non-real roots of $(x^2 + x - 2)(x^2 + x - 3) = 12$ is

a. -1 b. 1
c. -6 d. 6

13. If $(ax^2 + c)y + (a'x^2 + c') = 0$ and x is a rational function of y and ac is negative, then

a. $ac' + a'c = 0$ b. $a/a' = c/c'$
c. $a^2 + c^2 = a'^2 + c'^2$ d. $aa' + cc' = 1$

14. Let p and q be roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmetic progression, then the values of A and B are

a. $3, -77$ b. $3, 77$
c. $-3, -77$ d. $-3, 77$

15. The number of irrational roots of the equation $4x/(x^2 + x + 3) + 5x/(x^2 - 5x + 3) = -3/2$ is

a. 4 b. 0
c. 1 d. 2

16. Let a, b and c be real numbers such that $4a + 2b + c = 0$ and $ab > 0$. Then the equation $ax^2 + bx + c = 0$ has

a. complex roots b. exactly one root
c. real roots d. none of these

17. If α, β are the roots of the equation $x^2 - 2x + 3 = 0$. Then the equation whose roots are $P = \alpha^3 - 3\alpha^2 + 5\alpha - 2$ and $Q = \beta^3 - \beta^2 + \beta + 5$ is

a. $x^2 + 3x + 2 = 0$ b. $x^2 - 3x - 2 = 0$
c. $x^2 - 3x + 2 = 0$ d. none of these

18. If α, β be the roots of the equation $2x^2 - 35x + 2 = 0$, then the value of $(2\alpha - 35)^3 (2\beta - 35)^3$ is equal to

a. 8 b. 1
c. 64 d. none of these

19. If a, b, c are three distinct positive real numbers, then the number of real roots of $ax^2 + 2b|x| - c = 0$ is

a. 0 b. 4
c. 2 d. none of these

20. If $p(q - r)x^2 + q(r - p)x + r(p - q) = 0$ has equal roots, then $2/q =$

a. $\frac{1}{p} + \frac{1}{r}$ b. $p + r$
c. $p^2 + r^2$ d. $\frac{1}{p^2} + \frac{1}{r^2}$

21. Let $a \neq 0$ and $p(x)$ be a polynomial of degree greater than 2. If $p(x)$ leaves remainders a and $-a$ when divided respectively by $x + a$ and $x - a$, the remainder when $p(x)$ is divided by $x^2 - a^2$ is

a. $2x$ b. $-2x$
c. x d. $-x$

22. The quadratic $x^2 + ax + b + 1 = 0$ has roots which are positive integers, then $(a^2 + b^2)$ can be equal to

a. 50 b. 37
c. 61 d. 19

23. The sum of values of x satisfying the equation $(31+8\sqrt{15})^{x^2-3}+1=(32+8\sqrt{15})^{x^2-3}$ is

a. 3 b. 0
c. 2 d. none of these

24. If α, β be the roots of the equation $ax^2 + bx + c = 0$, then value of $(a\alpha^2+c)/(a\alpha+b)+(a\beta^2+c)/(a\beta+b)$ is

a. $\dfrac{b(b^2-2ac)}{4a}$ b. $\dfrac{b^2-4ac}{2a}$
c. $\dfrac{b(b^2-2ac)}{a^2c}$ d. none of these

25. A quadratic equation whose product of roots x_1 and x_2 is equal to 4 and satisfying the relation $x_1/(x_1-1)+x_2/(x_2-1)=2$ is

a. $x^2 - 2x + 4 = 0$ b. $x^2 + 2x + 4 = 0$
c. $x^2 + 4x + 4 = 0$ d. $x^2 - 4x + 4 = 0$

26. If $a, b, c, d \in R$, then the equation $(x^2 + ax - 3b)(x^2 - cx + b)(x^2 - dx + 2b) = 0$ has

a. 6 real roots b. at least 2 real roots
c. 4 real roots d. 3 real roots

27. If α and β are the roots of the equation $x^2 + px + q = 0$, and α^4 and β^4 are the roots of $x^2 - rx + q = 0$, then the roots of $x^2 - 4qx + 2q^2 - r = 0$ are always

a. both non-real b. both positive
c. both negative d. opposite in sign

28. If the roots of the equation $(a - 1)(x^2 + x + 1)^2 = (a + 1)(x^4 + x^2 + 1)$ are real and distinct then the value of $a \in$

a. $(-\infty, 3]$ b. $(-\infty, -2) \cup (2, \infty)$
c. $[-2, 2]$ d. $[-3, \infty)$

29. If $b_1b_2 = 2(c_1 + c_2)$, then at least one of the equations $x^2 + b_1x + c_1 = 0$ and $x^2 + b_2x + c_2 = 0$ has

a. imaginary roots b. real roots
c. purely imaginary roots d. none of these

30. The integral values of m for which the roots of the equation $mx^2 + (2m-1)x + (m-2) = 0$ are rational are given by the expression [where n is integer]
a. n^2
b. $n(n+2)$
c. $n(n+1)$
d. none of these

31. Suppose A, B, C are defined as $A = a^2b + ab^2 - a^2c - ac^2$, $B = b^2c + bc^2 - a^2b - ab^2$ and $C = a^2c + ac^2 - b^2c - bc^2$, where $a > b > c > 0$ and the equation $Ax^2 + Bx + C = 0$ has equal roots, then a, b, c are in
a. A.P.
b. G.P.
c. H.P.
d. A.G.P.

32. The coefficient of x in the equation $x^2 + px + q = 0$ was wrongly written as 17 in place of 13 and the roots thus found was -2 and -15. Then the roots of the correct equation are
a. $-3, 10$
b. $-3, -10$
c. $3, -10$
d. none of these

33. If $a(p+q)^2 + 2bpq + c = 0$ and $a(p+r)^2 + 2bpr + c = 0$ $(a \neq 0)$, then
a. $qr = p^2$
b. $qr = p^2 + \dfrac{c}{a}$
c. $qr = -p^2$
d. none of these

34. If the roots of the equation $ax^2 - bx + c = 0$ are α, β then the roots of the equation $b^2cx^2 - ab^2x + a^3 = 0$ are
a. $\dfrac{1}{\alpha^3+\alpha\beta}, \dfrac{1}{\beta^3+\alpha\beta}$
b. $\dfrac{1}{\alpha^2+\alpha\beta}, \dfrac{1}{\beta^2+\alpha\beta}$
c. $\dfrac{1}{\alpha^4+\alpha\beta}, \dfrac{1}{\beta^4+\alpha\beta}$
d. none of these

35. If α and β, α and γ, α and δ are the roots of the equations $ax^2 + 2bx + c = 0$, $2bx^2 + cx + a = 0$ and $cx^2 + ax + 2b = 0$, respectively, where a, b and c are positive real numbers, then $\alpha + \alpha^2 =$
a. abc
b. $a + 2b + c$
c. -1
d. 0

36. $x^2 - xy + y^2 - 4x - 4y + 16 = 0$ represents
a. a point
b. a circle
c. a pair of straight lines
d. none of these

37. If α, β be the non-zero roots of $ax^2 + bx + c = 0$ and α^2, β^2 be the roots of $a^2x^2 + b^2x + c^2 = 0$, then a, b, c are in
a. G.P.
b. H.P.
c. A.P.
d. none of these

38. If the roots of the equation $ax^2 + bx + c = 0$ are of the form $(k+1)/k$ and $(k+2)/(k+1)$, then $(a+b+c)^2$ is equal to
a. $2b^2 - ac$
b. Σa^2
c. $b^2 - 4ac$
d. $b^2 - 2ac$

39. If α, β are the roots of $ax^2 + bx + c = 0$ and $\alpha + h, \beta + h$ are the roots of $px^2 + qx + r = 0$, then $h =$
a. $-\dfrac{1}{2}\left(\dfrac{a}{b} - \dfrac{p}{q}\right)$
b. $\left(\dfrac{b}{a} - \dfrac{q}{p}\right)$
c. $\dfrac{1}{2}\left(\dfrac{b}{a} - \dfrac{q}{p}\right)$
d. none of these

40. If α, β be the roots of the equation $(x-a)(x-b) + c = 0$ $(c \neq 0)$, then the roots of the equation $(x-c-\alpha)(x-c-\beta) = c$ are
a. $a + c$ and $b + c$
b. $a - c$ and $b - c$
c. a and $b + c$
d. $a + c$ and b

41. If α, β are the roots of $ax^2 + c = bx$, then the equation $(a + cy)^2 = b^2y$ in y has the roots
a. $\alpha\beta^{-1}, \alpha^{-1}\beta$
b. α^{-2}, β^{-2}
c. α^{-1}, β^{-1}
d. α^2, β^2

42. If the roots of the equation, $x^2 + 2ax + b = 0$, are real and distinct and they differ by at most $2m$, then b lies in the interval
a. $(a^2, a^2 + m^2)$
b. $(a^2 - m^2, a^2)$
c. $[a^2 - m^2, a^2)$
d. none of these

43. If the ratio of the roots of $ax^2 + 2bx + c = 0$ is same as the ratio of the $px^2 + 2qx + r = 0$, then
a. $\dfrac{2b}{ac} = \dfrac{q^2}{pr}$
b. $\dfrac{b}{ac} = \dfrac{q}{pr}$
c. $\dfrac{b^2}{ac} = \dfrac{q^2}{pr}$
d. none of these

44. If one root of $x^2 - x - k = 0$ is square of the other, then $k =$
a. $2 \pm \sqrt{5}$
b. $2 \pm \sqrt{3}$
c. $3 \pm \sqrt{2}$
d. $5 \pm \sqrt{2}$

45. If α and β be the roots of the equation $x^2 + px - 1/(2p^2) = 0$ where $p \in R$. Then the minimum value of $\alpha^4 + \beta^4$ is
a. $2\sqrt{2}$
b. $2 - \sqrt{2}$
c. 2
d. $2 + \sqrt{2}$

46. If α, β are real and α^2, β^2 are the roots of the equation $a^2x^2 + x + 1 - a^2 = 0$ $(a > 1)$, then $\beta^2 =$
a. a^2
b. $1 - \dfrac{1}{a^2}$
c. $1 - a^2$
d. $1 + a^2$

47. If α and β are the roots of the equation $x^2 - ax + b = 0$ and $A_n = \alpha^n + \beta^n$, then which of the following is true?
a. $A_{n+1} = aA_n + bA_{n-1}$
b. $A_{n+1} = bA_n + aA_{n-1}$
c. $A_{n+1} = aA_n - bA_{n-1}$
d. $A_{n+1} = bA_n - aA_{n-1}$

48. The value of m for which one of the roots of $x^2 - 3x + 2m = 0$ is double of one of the roots of $x^2 - x + m = 0$ is
a. -2
b. 1
c. 2
d. none of these

49. If the equations $ax^2 + bx + c = 0$ and $x^3 + 3x^2 + 3x + 2 = 0$ have two common roots, then
a. $a = b = c$
b. $a = b \neq c$
c. $a = -b = c$
d. none of these

50. Number of values of a for which equations $x^3 + ax + 1 = 0$ and $x^4 + ax^2 + 1 = 0$ have a common root
a. 0
b. 1
c. 2
d. infinite

51. Let $p(x) = 0$ be a polynomial equation of the least possible degree, with rational coefficients, having $\sqrt[3]{7}+\sqrt[3]{49}$ as one of its roots. Then the product of all the roots of $p(x) = 0$ is

a. 56 **b.** 63
c. 7 **d.** 49

52. If $\alpha, \beta, \gamma, \sigma$ are the roots of the equation $x^4 + 4x^3 - 6x^2 + 7x - 9 = 0$, then the value of $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$ is

a. 9 **b.** 11
c. 13 **d.** 5

53. If $(m_r, 1/m_r)$, $r = 1, 2, 3, 4$ be four pairs of values of x and y that satisfy the equation $x^2 + y^2 + 2gx + 2fy + c = 0$, then value of $m_1 m_2 m_3 m_4$ is

a. 0 **b.** 1
c. -1 **d.** none of these

54. If roots of an equation $x^n - 1 = 0$ are $1, a_1, a_2, \ldots, a_{n-1}$, then the value of $(1 - a_1)(1 - a_2)(1 - a_3) \cdots (1 - a_{n-1})$ will be

a. n **b.** n^2
c. n^n **d.** 0

55. If $\tan\theta_1, \tan\theta_2, \tan\theta_3$ are the real roots of the $x^3 - (a + 1)x^2 + (b - a)x - b = 0$, where $\theta_1 + \theta_2 + \theta_3 \in (0, \pi)$, then $\theta_1 + \theta_2 + \theta_3$ is equal to

a. $\pi/2$ **b.** $\pi/4$
c. $3\pi/4$ **d.** π

56. If α, β, γ are the roots of $x^3 - x^2 - 1 = 0$ then the value of $(1+\alpha)/(1-\alpha) + (1+\beta)/(1-\beta) + (1+\gamma)/(1-\gamma)$ is equal to

a. -5 **b.** -6
c. -7 **d.** -2

57. Let r, s and t be the roots of the equation, $8x^3 + 1001x + 2008 = 0$. The value of $(r + s)^3 + (s + t)^3 + (t + r)^3$ is

a. 251 **b.** 751
c. 735 **d.** 753

58. If x be real, then $x/(x^2-5x+9)$ lies between

a. -1 and $-1/11$ **b.** 1 and $-1/11$
c. 1 and $1/11$ **d.** none of these

59. If x is real, then the maximum value of $(3x^2+9x+17)/(3x^2+9x+7)$ is

a. 1/4 **b.** 41
c. 1 **d.** 17/7

60. If $a, b \in R, a \neq 0$ and the quadratic equation $ax^2 - bx + 1 = 0$ has imaginary roots then $(a + b + 1)$ is

a. positive **b.** negative
c. zero **d.** dependent on the sign of b

61. If the expression $[mx - 1 + (1/x)]$ is non-negative for all positive real x, then the minimum value of m must be

a. $-1/2$ **b.** 0
c. 1/4 **d.** 1/2

62. Suppose that $f(x)$ is a quadratic expression positive for all real x. If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x (where $f'(x)$ and $f''(x)$ represent 1st and 2nd derivative respectively)

a. $g(x) < 0$ **b.** $g(x) > 0$
c. $g(x) = 0$ **d.** $g(x) \geq 0$

63. Let $f(x) = ax^2 - bx + c^2$, $b \neq 0$ and $f(x) \neq 0$ for all $x \in R$. Then

a. $a + c^2 < b$ **b.** $4a + c^2 > 2b$
c. $9a - 3b + c^2 < 0$ **d.** none of these

64. x_1 and x_2 are the roots of $ax^2 + bx + c = 0$ and $x_1x_2 < 0$. Roots of $x_1(x - x_2)^2 + x_2(x - x_1)^2 = 0$ are

a. real and of opposite sign **b.** negative
c. positive **d.** non-real

65. If a, b, c, d are four consecutive terms of an increasing A.P. then the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are

a. non-real complex **b.** real and equal
c. integers **d.** real and distinct

66. If roots of $x^2 - (a - 3)x + a = 0$ are such that at least one of them is greater than 2, then

a. $a \in [7, 9]$ **b.** $a \in [7, \infty)$
c. $a \in [9, \infty)$ **d.** $a \in [7, 9)$

67. Let $f(x) = ax^2 + bx + c$, $a, b, c \in R$. If $f(x)$ takes real values for real values of x and non-real values for non-real values of x, then

a. $a = 0$
b. $b = 0$
c. $c = 0$
d. nothing can be said about a, b, c.

68. All the values of m for which both the roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4, lie in the interval

a. $-2 < m < 0$ **b.** $m > 3$
c. $-1 < m < 3$ **d.** $1 < m < 4$

69. If the roots of the quadratic equation $(4p - p^2 - 5)x^2 - (2p - 1)x + 3p = 0$ lie on either side of unity, then the number of integral values of p is

a. 1 **b.** 2
c. 3 **d.** 4

70. The interval of a for which the equation $\tan^2 x - (a - 4)\tan x + 4 - 2a = 0$ has at least one solution $\forall\, x \in [0, \pi/4]$

a. $a \in (2, 3)$ **b.** $a \in [2, 3]$
c. $a \in (1, 4)$ **d.** $a \in [1, 4]$

71. The range of a for which the equation $x^2 + ax - 4 = 0$ has its smaller root in the interval $(-1, 2)$ is

a. $(-\infty, -3)$ **b.** $(0, 3)$

c. $(0, \infty)$ d. $(-\infty, -3) \cup (0, \infty)$

72. If both roots of the equation $ax^2 + x + c - a = 0$ are imaginary and $c > -1$, then
a. $3a > 2 + 4c$ b. $3a < 2 + 4c$
c. $c < a$ d. none of these

73. The set of all possible real values of a such that the inequality $(x - (a - 1))\ (x - (a^2 + 2)) < 0$ holds for all $x \in (-1, 3)$ is
a. $(0, 1)$ b. $(\infty, -1]$
c. $(-\infty, -1)$ d. $(1, \infty)$

74. Consider the equation $x^2 + 2x - n = 0$, where $n \in N$ and $n \in [5, 100]$. Total number of different values of 'n' so that the given equation has integral roots is
a. 8 b. 3
c. 6 d. 4

75. Total number of values of a so that $x^2 - x - a = 0$ has integral roots, where $a \in N$ and $6 \le a \le 100$, is equal to
a. 2 b. 4
c. 6 d. 8

76. Total number of integral values of 'a' so that $x^2 - (a + 1)x + a - 1 = 0$ has integral roots is equal to
a. 1 b. 2
c. 4 d. none of these

77. The number of values of k for which $[x^2 - (k - 2)x + k^2] \times [x^2 + kx + (2k - 1)]$ is a perfect square is
a. 2 b. 1
c. 0 d. none of these

78. If α, β are the roots of $x^2 + px + q = 0$ and $x^{2n} + p^n x^n + q^n = 0$ and if $(\alpha/\beta), (\beta/\alpha)$ are the roots of $x^n + 1 + (x + 1)^n = 0$, then $n (\in N)$
a. must be an odd integer b. may be any integer
c. must be an even integer d. cannot say anything

79. The number of positive integral solutions of $x^4 - y^4 = 3789108$ is
a. 0 b. 1
c. 2 d. 4

80. If $xy = 2(x + y)$, $x \le y$ and $x, y \in N$, then the number of solutions of the equation are
a. two b. three
c. no solution d. infinitely many solutions

81. If α, β, γ are such that $\alpha + \beta + \gamma = 2$, $\alpha^2 + \beta^2 + \gamma^2 = 6$, $\alpha^3 + \beta^3 + \gamma^3 = 8$, then $\alpha^4 + \beta^4 + \gamma^4$ is
a. 18 b. 10
c. 15 d. 36

82. The number of integral values of a for which the quadratic equation $(x + a)\ (x + 1991) + 1 = 0$ has integral roots are
a. 3 b. 0
c. 1 d. 2

83. The number of real solutions of the equation $(9/10)^x = -3 + x - x^2$ is
a. 2 b. 0
c. 1 d. none of these

84. If a, b and c are real numbers such that $a^2 + b^2 + c^2 = 1$, then $ab + bc + ca$ lies in the interval
a. $[1/2, 2]$ b. $[-1, 2]$
c. $[-1/2, 1]$ d. $[-1, 1/2]$

85. If the equation $\cot^4 x - 2\operatorname{cosec}^2 x + a^2 = 0$ has at least one solution then, sum of all possible integral values of a is equal to
a. 4 b. 3
c. 2 d. 0

86. Let x, y, z, t be real numbers $x^2 + y^2 = 9$, $z^2 + t^2 = 4$ and $xt - yz = 6$. Then the greatest value of $P = xz$ is
a. 2 b. 3
c. 4 d. 6

87. If a, b, c be distinct positive numbers, then the nature of roots of the equation $1/(x-a) + 1/(x-b) + 1/(x-c) = 1/x$ is
a. all real and distinct
b. all real and at least two are distinct
c. at least two real
d. all non-real

88. If $(b^2 - 4ac)^2 (1 + 4a^2) < 64a^2$, $a < 0$, then maximum value of quadratic expression $ax^2 + bx + c$ is always less than
a. 0 b. 2
c. −1 d. −2

89. For $x^2 - (a + 3)\,|x| + 4 = 0$ to have real solutions, the range of a is
a. $(-\infty, -7] \cup [1, \infty)$ b. $(-3, \infty)$
c. $(-\infty, -7]$ d. $[1, \infty)$

90. If the quadratic equation $4x^2 - 2(a + c - 1)x + ac - b = 0$ $(a > b > c)$
a. both roots are greater than a
b. both roots are less than c
c. both roots lie between $c/2$ and $a/2$
d. exactly one of the roots lies between $c/2$ and $a/2$

91. If the equation $x^2 + ax + b = 0$ has distinct real roots and $x^2 + a|x| + b = 0$ has only one real root, then which of the following is true
a. $b = 0, a > 0$ b. $b = 0, a < 0$
c. $b > 0, a < 0$ d. $b < 0, a > 0$

92. The equation $2^{2x} + (a-1)2^{x+1} + a = 0$ has roots of opposite signs then exhaustive set of values of a is

a. $a \in (-1, 0)$ **b.** $a < 0$
c. $a \in (-\infty, 1/3)$ **d.** $a \in (0, 1/3)$

93. If the equation $|x^2 + bx + c| = k$ has four real roots, then

a. $b^2 - 4c > 0$ and $0 < k < \dfrac{4c - b^2}{4}$

b. $b^2 - 4c < 0$ and $0 < k < \dfrac{4c - b^2}{4}$

c. $b^2 - 4c > 0$ and $k > \dfrac{4c - b^2}{4}$

d. none of these

94. $P(x)$ is a polynomial with integral coefficients such that for four distinct integers a, b, c, d; $P(a) = P(b) = P(c) = P(d) = 3$. If $P(e) = 5$ (e is an integer), then

a. $e = 1$ **b.** $e = 3$
c. $e = 4$ **d.** no real value of e

95. The number of integral values of x satisfying $\sqrt{-x^2 + 10x - 16} < x - 2$ is

a. 0 **b.** 1
c. 2 **d.** 3

96. If $x^2 + ax - 3x - (a+2) = 0$ has real and distinct roots, then minimum value of $(a^2+1)/(a^2+2)$ is

a. 1 **b.** 0
c. $\dfrac{1}{2}$ **d.** $\dfrac{1}{4}$

97. The set of values of a for which $(a-1)x^2 - (a+1)x + a - 1 \geq 0$ is true for all $x \geq 2$

a. $(-\infty, 1)$ **b.** $\left(1, \dfrac{7}{3}\right)$
c. $\left(\dfrac{7}{3}, \infty\right)$ **d.** none of these

98. The value of the expression $x^4 - 8x^3 + 18x^2 - 8x + 2$ when $x = 2 + \sqrt{3}$

a. 2 **b.** 1
c. 0 **d.** 3

Multiple Correct Answers Type

Solutions on page 1.72

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. If $x, y \in R$ and $2x^2 + 6xy + 5y^2 = 1$, then

a. $|x| \leq \sqrt{5}$ **b.** $|x| \geq \sqrt{5}$
c. $y^2 \leq 2$ **d.** $y^2 \leq 4$

2. If the equation whose roots are the squares of the roots of the cubic $x^3 - ax^2 + bx - 1 = 0$ is identical with the given cubic equation, then

a. $a = 0, b = 3$
b. $a = b = 0$
c. $a = b = 3$
d. a, b are roots of $x^2 + x + 2 = 0$

3. If the equation $ax^2 + bx + c = 0$, $a, b, c \in R$ have non-real roots, then

a. $c(a - b + c) > 0$ **b.** $c(a + b + c) > 0$
c. $c(4a - 2b + c) > 0$ **d.** none of these

4. If $c \neq 0$ and the equation $p/(2x) = a/(x+c) + b/(x-c)$ has two equal roots, then p can be

a. $(\sqrt{a} - \sqrt{b})^2$ **b.** $(\sqrt{a} + \sqrt{b})^2$
c. $a + b$ **d.** $a - b$

5. If the equations $4x^2 - x - 1 = 0$ and $3x^2 + (\lambda + \mu)x + \lambda - \mu = 0$ have a root common then the rational values of λ and μ are

a. $\lambda = \dfrac{-3}{4}$ **b.** $\lambda = 0$
c. $\mu = \dfrac{3}{4}$ **d.** $\mu = 0$

6. If the equation $ax^2 + bx + c = 0$ $(a > 0)$ has two real roots α and β such that $\alpha < -2$ and $\beta > 2$, then which of the following statements is/are true?

a. $a - |b| + c < 0$ **b.** $c < 0$, $b^2 - 4ac > 0$
c. $4a - 2|b| + c < 0$ **d.** $9a - 3|b| + c < 0$

7. If the following figure shows the graph of $f(x) = ax^2 + bx + c$, then

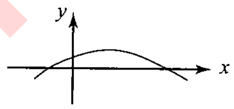


Fig. 1.65

a. $ac < 0$ **b.** $bc > 0$
c. $ab > 0$ **d.** $abc < 0$

8. If $\cos x - y^2 - \sqrt{y - x^2 - 1} \geq 0$, then

a. $y \geq 1$ **b.** $x \in R$
c. $y = 1$ **d.** $x = 0$

9. The value of x satisfying the equation $2^{2x} - 8 \times 2^x = -12$ is

a. $1 + \dfrac{\log 3}{\log 2}$ **b.** $\dfrac{1}{2}\log 6$
c. $1 + \log\dfrac{3}{2}$ **d.** 1

10. If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$ then which of the following expression will be the symmetric function of roots

a. $\left|\log\dfrac{\alpha}{\beta}\right|$ **b.** $\alpha^2\beta^5 + \beta^2\alpha^5$
c. $\tan(\alpha - \beta)$ **d.** $\left(\log\dfrac{1}{\alpha}\right)^2 + (\log\beta)^2$

11. If the quadratic equation $ax^2 + bx + c = 0$ $(a > 0)$ has $\sec^2\theta$ and $\operatorname{cosec}^2\theta$ as its roots, then which of the following must hold good?

a. $b + c = 0$ **b.** $b^2 - 4ac \geq 0$
c. $c \geq 4a$ **d.** $4a + b \geq 0$

12. Let $a, b, c \in Q^+$ satisfying $a > b > c$. Which of the following statement(s) hold true for the quadratic polynomial $f(x) = (a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b)$?

a. The mouth of the parabola $y = f(x)$ opens upwards
b. Both roots of the equation $f(x) = 0$ are rational
c. x-coordinate of vertex of the graph is positive
d. Product of the roots is always negative

13. The graph of the quadratic trinomial $y = ax^2 + bx + c$ has its vertex at $(4, -5)$ and two x-intercepts one positive and one negative. Which of the following holds good?

a. $a > 0$ **b.** $b < 0$
c. $c < 0$ **d.** $8a = b$

14. If the roots of the equation, $x^3 + px^2 + qx - 1 = 0$ form an increasing G.P., where p and q are real, then

a. $p + q = 0$
b. $p \in (-3, \infty)$
c. one of the root is unity
d. one root is smaller than 1 and one root is greater than 1

15. If $(\sin \alpha)\, x^2 - 2x + b \geq 2$, for all real values of $x \leq 1$ and $\alpha \in (0, \pi/2) \cup (\pi/2, \pi)$, then possible real values of b is/are

a. 2 **b.** 3 **c.** 4 **d.** 5

16. If every pair from among the equations $x^2 + ax + bc = 0$, $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ has a common root, then

a. the sum of the three common roots is $-1/2\,(a+b+c)$
b. the sum of the three common roots is $2(a + b + c)$
c. the product of the three common roots is abc
d. the product of the three common roots is $a^2b^2c^2$

17. If a, b, c are in G.P. then the roots of the equation $ax^2 + bx + c = 0$ are in the ratio

a. $\frac{1}{2}(-1+i\sqrt{3})$ **b.** $\frac{1}{2}(1-i\sqrt{3})$
c. $\frac{1}{2}(-1-i\sqrt{3})$ **d.** $\frac{1}{2}(1+i\sqrt{3})$

18. If $ax^2 + (b - c)x + a - b - c = 0$ has unequal real roots for all $c \in R$, then

a. $b < 0 < a$ **b.** $a < 0 < b$
c. $b < a < 0$ **d.** $b > a > 0$

19. Given that α, γ are roots of the equation $Ax^2 - 4x + 1 = 0$, and β, δ the roots of the equation of $Bx^2 - 6x + 1 = 0$, such that α, β, γ and δ are in H.P., then

a. $A = 3$ **b.** $A = 4$
c. $B = 2$ **d.** $B = 8$

20. If the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root, then it must be equal to

a. $\frac{pq' - p'q}{q - q'}$ **b.** $\frac{q - q'}{p' - p}$
c. $\frac{p' - p}{q - q'}$ **d.** $\frac{pq' - p'q}{p - p'}$

21. If $x^3 + 3x^2 - 9x + c$ is of the form $(x - \alpha)^2 (x - \beta)$, then c is equal to

a. 27 **b.** -27 **c.** 5 **d.** -5

22. If the equations $x^2 + bx - a = 0$ and $x^2 - ax + b = 0$ have a common root, then

a. $a + b = 0$ **b.** $a = b$
c. $a - b = 1$ **d.** $a + b = 1$

23. If $(x^2+ax+3)/(x^2+x+a)$ takes all real values for possible real values of x, then

a. $4a^3 + 39 < 0$ **b.** $4a^3 + 39 \geq 0$
c. $a \geq \frac{1}{4}$ **d.** $a < \frac{1}{4}$

24. If $\cos^4\theta + \alpha$, $\sin^4\theta + \alpha$ are the roots of the equation $x^2 + 2bx + b = 0$ and $\cos^2\theta + \beta$, $\sin^2\theta + \beta$ are the roots of the equation $x^2 + 4x + 2 = 0$, then values of b are

a. 2 **b.** -1 **c.** -2 **d.** 1

25. If the roots of the equation $x^2 + ax + b = 0$ are c and d, then roots of the equation $x^2 + (2c + a)x + c^2 + ac + b = 0$ are

a. c **b.** $d - c$ **c.** $2c$ **d.** 0

26. If $a, b, c \in R$ and $abc < 0$, then the equation $bcx^2 + 2(b + c - a)x + a = 0$, has

a. both positive roots
b. both negative roots
c. real roots
d. one positive and one negative root

27. For the quadratic equation $x^2 + 2(a + 1)x + 9a - 5 = 0$, which of the following is/are true?

a. If $2 < a < 5$, then roots are of opposite sign.
b. If $a < 0$, then roots are of opposite sign.
c. If $a > 7$, then both roots are negative.
d. If $2 \leq a \leq 5$, then roots are unreal.

28. Let $P(x) = x^2 + bx + c$, where b and c are integer. If $P(x)$ is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, then

a. $P(x) = 0$ has imaginary roots
b. $P(x) = 0$ has roots of opposite sign
c. $P(1) = 4$
d. $P(1) = 6$

30. If $|ax^2 + bx + c| \leq 1$ for all x in $[0, 1]$, then

a. $|a| \leq 8$ **b.** $|b| > 8$
c. $|c| \leq 1$ **d.** $|a| + |b| + |c| \leq 17$

31. Let $f(x) = ax^2 + bx + c$. Consider the following diagram. Then

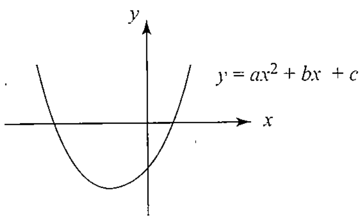


Fig. 1.66

a. $c < 0$ **b.** $b > 0$
c. $a + b - c > 0$ **d.** $abc < 0$

Reasoning Type

Solutions on page 1.76

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of Statement 1.

b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.

c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. Consider the function $f(x) = \log_e (ax^3 + (a+b)x^2 + (b+c)x + c)$.

Statement 1: Domain of the functions is $(-1, \infty) \sim \{-(b/2a)\}$, where $a > 0$, $b^2 - 4ac = 0$.

Statement 2: $ax^2 + bx + c = 0$ has equal roots when $b^2 - 4ac = 0$.

2. Statement 1: If $a > 0$ and $b^2 - ac < 0$, then domain of the function $f(x) = \sqrt{ax^2+2bx+c}$ is R.

Statement 2: If $b^2 - ac < 0$, then $ax^2 + 2bx + c = 0$ has imaginary roots.

3. Statement 1: If equations $ax^2 + bx + c = 0$ and $x^2 - 3x + 4 = 0$ have exactly one root common, then at least one of a, b, c is imaginary.

Statement 2: If a, b, c are not all real, then equation $ax^2 + bx + c = 0$ can have one root real and one root imaginary.

4. Statement 1: If $\cos^2 \pi/8$ is a root of the equation $x^2 + ax + b = 0$ where $a, b \in Q$, then ordered pair (a, b) is $[-1, (1/8)]$.

Statement 2: If $a + mb = 0$ and m is irrational, then $a, b = 0$.

5. Statement 1: If $a^2 + b^2 + c^2 < 0$, then if roots of the equation $ax^2 + bx + c = 0$ are imaginary, then they are not complex conjugates.

Statement 2: Equation $ax^2 + bx + c = 0$ has complex conjugate roots when a, b, c are real.

6. Statement 1: Equation $ix^2+(i-1)x-(1/2)-i=0$ has imaginary roots.

Statement 2: If $a = i$, $b = i - 1$ and $c=-(1/2)-i$, then $b^2 - 4ac < 0$.

7. Statement 1: If $f(x)$ is a quadratic polynomial satisfying $f(2) + f(4) = 0$. If unity is a root of $f(x) = 0$, then the other root is 3.5.

Statement 2: If $g(x) = px^2 + qx + r = 0$ has roots α, β, then $\alpha + \beta = -q/p$ and $\alpha\beta = (r/p)$.

8. Let $f(x) = -x^2 + (a+1)x + 5$.

Statement 1: $f(x)$ is positive for some $\alpha < x < \beta$ and for all $a \in R$.

Statement 2: $f(x)$ is positive for all $x \in R$ and for some real a.

9. Let a, b, c be real such that $ax^2 + bx + c = 0$ and $x^2 + x + 1 = 0$ have a common root.

Statement 1: $a = b = c$

Statement 2: Two quadratic equations with real coefficients cannot have only one imaginary root common.

10. Statement 1: The equation $(x-p)(x-r) + \lambda(x-q)(x-s) = 0$, where $p < q < r < s$, has non-real roots.

Statement 2: The equation $px^2 + qx + r = 0$ $(p, q, r \in R)$ has non-real roots if $q^2 - 4pr < 0$.

11. Statement 1: If $px^2 + qx + r = 0$ is a quadratic equation $(p, q, r \in R)$ such that its roots are α, β and $p + q + r < 0$, $p - q + r < 0$ and $r > 0$, then $[\alpha] + [\beta] = -1$, where $[\cdot]$ denotes greatest integer function.

Statement 2: If for any two real numbers a and b, function $f(x)$ is such that $f(a) f(b) < 0 \Rightarrow f(x)$ has at least one real root lying in (a, b).

12. Statement 1: If $0<\alpha<(\pi/4)$, then the equation $(x - \sin\alpha) \times (x - \cos\alpha) - 2 = 0$ has both roots in $(\sin\alpha, \cos\alpha)$.

Statement 2: If $f(a)$ and $f(b)$ possess opposite signs, then there exist at least one solution of the equation $f(x) = 0$ in open interval (a, b).

13. Statement 1: If all real values of x obtained from the equation $4^x - (a-3)\,2^x + (a-4) = 0$ are non-positive, then $a \in (4, 5]$.

Statement 2: If $ax^2 + bx + c$ is non-positive for all real values of x, then $b^2 - 4ac$ must be negative or zero and 'a' must be negative.

14. Statement 1: If $(a^2 - 4)x^2 + (a^2 - 3a + 2)x + (a^2 - 7a + 10) = 0$ is an identity, then the value of a is 2.

Statement 2: If $a - b = 0$, then $ax^2 + bx + c = 0$ is an identity.

15. Statement 1: If the roots of $x^5 - 40x^4 + Px^3 + Qx^2 + Rx + S = 0$ are in G.P. and sum of their reciprocal is 10, then $|S| = 64$.

Statement 2: $x_1 x_2 x_3 x_4 x_5 = -S$, where x_1, x_2, x_3, x_4, x_5 are the roots of given equation.

16. Statement 1: If a, b, c, a_1, b_1, c_1 are rational and equations $ax^2 + 2bx + c = 0$ and $a_1x^2 + 2b_1x + c_1 = 0$ have one and only one root in common, then both $b^2 - ac$ and $b_1^2 - a_1c_1$ must be perfect squares.

Statement 2: If two quadratic equations with rational coefficients have a common irrational root $p+\sqrt{q}$, then both roots will be common.

17. **Statement 1:** If $a, b, c \in Z$ and $ax^2 + bx + c = 0$ has an irrational root, then $|f(\lambda)| \geq 1/q^2$, where $\lambda \in (\lambda = \frac{p}{q}; p, q \in Z)$ and $f(x) = ax^2 + bx + c$.

Statement 2: If $a, b, c \in Q$ and $b^2 - 4ac$ is positive but not a perfect square, then roots of equation $ax^2 + bx + c = 0$ are irrational and always occur in conjugate pair like $2+\sqrt{3}$ and $2-\sqrt{3}$.

18. **Statement 1:** The number of values of a for which $(a^2 - 3a + 2)x^2 + (a^2 - 5a + 6)x + a^2 - 4 = 0$ is an identity in x is 2.

Statement 2: If $a = b = c = 0$, then equation $ax^2 + bx + c = 0$ is an identity in x.

19. **Statement 1:** If roots of the equation $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c = 1$.

Statement 2: If a, b, c are odd integer, then the roots of the equation $4\,abc\,x^2 + (b^2 - 4ac)\,x - b = 0$ are real and distinct.

20. Let $ax^2 + bx + c = 0$, $a \neq 0$ $(a, b, c \in R)$ has no real roots and $a + b + 2c = 2$.

Statement 1: $ax^2 + bx + c > 0$, $\forall\, x \in R$.

Statement 2: $a + b$ is positive.

21. Consider a general expression of degree 2 in two variables as $f(x, y) = 5x^2 + 2y^2 - 2xy - 6x - 6y + 9$.

Statement 1: $f(x, y)$ can be resolved into two linear factors over real coefficients.

Statement 2: If we compare $f(x, y)$ with $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, we have $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

22. **Statement 1:** The equation $x^2 + (2m + 1)x + (2n + 1) = 0$, where m and n are integers cannot have any rational roots.

Statement 2: The quantity $(2m + 1)^2 - 4(2n + 1)$, where m, $n \in I$ can never be a perfect square.

Linked Comprehension Type

Solutions on page 1.78

Based upon each paragraph, some multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

Consider an unknown polynomial which when divided by $(x - 3)$ and by $(x - 4)$ leaves remainders as 2 and 1, respectively. Let $R(x)$ be the remainder when this polynomial is divided by $(x - 3)(x - 4)$.

1. If equation $R(x) = x^2 + ax + 1$ has two distinct real roots, then exhaustive values of a are

a. $(-2, 2)$ **b.** $(-\infty, -2) \cup (2, \infty)$
c. $(-2, \infty)$ **d.** all real numbers

2. If $R(x) = px^2 + (q - 1)x + 6$ has no distinct real roots and $p > 0$, then least value of $3p + q$ is

a. -2 **b.** $2/3$
c. $-1/3$ **d.** none of these

3. Range of $f(x) = [R(x)]/(x^2 - 3x + 2)$ is

a. $[-2, 2]$
b. $(-\infty, -2-\sqrt{3}] \cup [-2+\sqrt{3}, \infty)$
c. $(-\infty, -7-4\sqrt{3}] \cup [-7+4\sqrt{3}, \infty)$
d. none of these

For Problems 4–6

Consider the quadratic equation $ax^2 - bx + c = 0$, $a, b, c \in N$, which has two distinct real roots belonging to the interval $(1, 2)$.

4. The least value of a is

a. 4 **b.** 6 **c.** 7 **d.** 5

5. The least value of b is

a. 10 **b.** 11 **c.** 13 **d.** 15

6. The least value of c is

a. 4 **b.** 6 **c.** 7 **d.** 5

For Problems 7–9

Consider the equation $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$, where $a \in R$. Also range of function $f(x) = x + 1/x$ is $(-\infty, -2] \cup [2, \infty)$.

7. If equation has at least two distinct positive real roots then all possible values of a are

a. $(-\infty, -1/4)$ **b.** $(5/4, \infty)$
c. $(-\infty, -3/4)$ **d.** none of these

8. If equation has at least two distinct negative real roots, then all possible values of a are

a. $(3/4, \infty)$ **b.** $(-5/4, \infty)$
c. $(-\infty, 1/4)$ **d.** none of these

9. If exactly two roots are positive and two roots are negative, then number of integral values of a is

a. 2 **b.** 1 **c.** 0 **d.** 3

For Problems 10–12

Let $f(x) = x^2 + b_1x + c_1$, $g(x) = x^2 + b_2x + c_2$. Let the real roots of $f(x) = 0$ be α, β and real roots of $g(x) = 0$ be $\alpha + h, \beta + h$. The least value of $f(x)$ is $-1/4$. The least value of $g(x)$ occurs at $x = 7/2$

10. The least value of $g(x)$ is

a. $-\frac{1}{4}$ **b.** -1 **c.** $-\frac{1}{3}$ **d.** $-\frac{1}{2}$

11. The value of b_2 is

a. -5 **b.** 9 **c.** -8 **d.** -7

12. The roots of $f(x) = 0$ are

a. 3, -4 **b.** -3, 4 **c.** 3, 4 **d.** -3, -4

For Problems 13–15

In the given figure, vertices of ΔABC lie on $y = f(x) = ax^2 + bx + c$. The ΔABC is right angled isosceles triangle whose hypotenuse $AC = 4\sqrt{2}$ units.

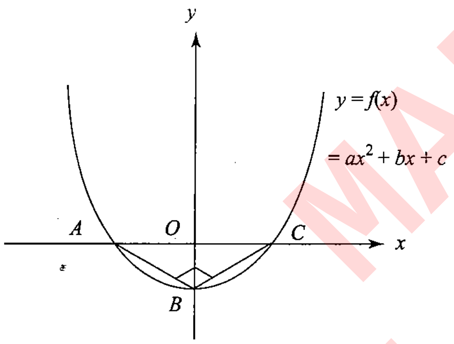


Fig. 1.67

13. $y = f(x)$ is given by

a. $y = x^2 - 2\sqrt{2}$ **b.** $y = x^2 - 12$

c. $y = \frac{x^2}{2} - 2$ **d.** $y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$

14. Minimum value of $y = f(x)$ is

a. -4 **b.** -2

c. $-2\sqrt{2}$ **d.** none of these

15. Number of integral values of k for which one root of $f(x) = 0$ is more than k and other less than k

a. 6 **b.** 4 **c.** 5 **d.** 7

For Problems 16–18

Consider the inequality $9^x - a3^x - a + 3 \le 0$, where 'a' is a real parameter.

16. The given inequality has at least one negative solution for $a \in$

a. $(-\infty, 2)$ **b.** $(3, \infty)$

c. $(-2, \infty)$ **d.** $(2, 3)$

17. The given inequality has at least one positive solution for $a \in$

a. $(-\infty, -2)$ **b.** $[3, \infty)$

c. $(2, \infty)$ **d.** $[-2, \infty)$

18. The given inequality has at least one real solution for $a \in$

a. $(-\infty, 3)$ **b.** $[2, \infty)$

c. $(3, \infty)$ **d.** $[-2, \infty)$

For Problems 19–21

Consider the inequation $x^2 + x + a - 9 < 0$.

19. The values of the real parameter 'a' so that the given inequation has at least one positive solution:

a. $(-\infty, 37/4)$ **b.** $(-\infty, \infty)$

c. $(3, \infty)$ **d.** $(-\infty, 9)$

20. The values of the real parameter 'a' so that the given inequation has at least one negative solution:

a. $(-\infty, 9)$ **b.** $(37/4, \infty)$

c. $\left(-\infty, \frac{37}{4}\right)$ **d.** none of these

21. The values of the real parameter 'a' so that the given inequation is true $\forall\, x \in (-1, 3)$:

a. $(-\infty, -3)$ **b.** $(-3, \infty)$

c. $[9, \infty)$ **d.** $(-\infty, 37/4)$

For Problems 22–24

'$af(\mu) < 0$' is the necessary and sufficient condition for a particular real number μ to lie between the roots of a quadratic equation $f(x) = 0$, where $f(x) = ax^2 + bx + c$. Again if $f(\mu_1)f(\mu_2) < 0$, then exactly one of the roots will lie between μ_1 and μ_2.

22. If $|b| > |a + c|$, then

a. one root of $f(x) = 0$ is positive, the other is negative

b. exactly one of the roots of $f(x) = 0$ lies in $(-1, 1)$

c. 1 lies between the roots of $f(x) = 0$

d. both the roots of $f(x) = 0$ are less than 1

23. If $a(a + b + c) < 0 < (a + b + c)c$, then

a. one root is less than 0, the other is greater than 1

b. exactly one of the roots lies in $(0, 1)$

c. both the roots lie in $(0, 1)$

d. at least one of the roots lies in $(0, 1)$

24. If $(a + b + c)c < 0 < a(a + b + c)$, then

a. one root is less than 0, the other is greater than 1

b. one root lies in $(-\infty, 0)$ and other in $(0, 1)$

c. both the roots lie in $(0, 1)$

d. one root lies in $(0, 1)$ and other in $(1, \infty)$

For Problems 25–26

The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P.

25. All possible values of β are

a. $\left(-\infty, \frac{1}{3}\right)$ **b.** $\left(-\infty, -\frac{1}{3}\right)$

c. $\left(\frac{1}{3}, \infty\right)$ **d.** $\left(-\frac{1}{3}, \infty\right)$

26. All possible values of γ are

a. $\left(-\frac{1}{9}, \infty\right)$ **b.** $\left(-\frac{1}{27}, +\infty\right)$

c. $\left(\frac{2}{9}, +\infty\right)$ **d.** none of these

Matrix-Match Type

Solutions on page 1.81

Each question contains statements given in two columns, which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a → p, a → s, b → q, b → r, c → p, c → q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1. Match the following for the equation $x^2 + a|x| + 1 = 0$, where a is a parameter.

Column I	Column II
a. No real roots	**p.** $a < -2$
b. Two real roots	**q.** ϕ
c. Three real roots	**r.** $a = -2$
d. Four distinct real roots	**s.** $a \geq 0$

2.

Column I	Column II
a. $y = \frac{x^2-2x+4}{x^2+2x+4}, x \in R$, then y can be	**p.** 1
b. $y = \frac{x^2-3x-2}{2x-3}, x \in R$, then y can be	**q.** 4
c. $y = \frac{2x^2-2x+4}{x^2-4x+3}, x \in R$, then y can be	**r.** -3
d. $x^2-(a-3)x+2<0, \forall x \in (-2, 3)$, then a can be	**s.** -10

3.

Column I	Column II
a. If a, b, c and d are four zero real number such that $(d+a-b)^2+(d+b-c)^2=0$ and the roots of the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ are real and equal then	**p.** $a+b+c=0$
b. If the roots of the equation $(a^2+b^2)x^2-2b(a+c)x+(b^2+c^2)=0$ are real and equal, then	**q.** a, b, c are in A.P.
c. If the equation $ax^2+bx+c=0$ and $x^3-3x^2+3x-1=0$ have a common real root, then	**r.** a, b, c are in G.P.
d. Let a, b, c be positive real numbers such that the expression $bx^2+\left(\sqrt{(a+c)^2+4b^2}\right)x+(a+c)$ is non-negative $\forall\ x \in R$, then	**s.** a, b, c are in H.P.

4.

Column I (Number of positive integers for which)	Column II
a. one root is positive and the other is negative for the equation $(m-2)x^2-(8-2m)x-(8-3m)=0$	**p.** 0
b. exactly one root of equation $x^2-m(2x-8)-15=0$ lies in interval (0, 1)	**q.** infinite
c. the equation $x^2+2(m+1)x+9m-5=0$ has both roots negative	**r.** 1
d. the equation $x^2+2(m-1)x+m+5=0$ has both roots lying on either sides of 1	**s.** 2

5.

Column I	Column II
a. If $x^2+ax+b=0$ has roots α, β and $x^2+px+q=0$ has roots $-\alpha, \gamma$, then	**p.** $(1-bq)^2=(a-pb)(p-aq)$
b. If $x^2+ax+b=0$ has roots α, β and $x^2+px+q=0$ has roots $1/\alpha$, γ, then	**q.** $(4-bq)^2=(4a+2pb)(-2p-aq)$
c. If $x^2+ax+b=0$ has roots α, β and $x^2+px+q=0$ has roots $-2/\alpha, \gamma$, then	**r.** $(1-4bq)^2=(a+2bp)(-2p-4aq)$
d. If $x^2+ax+b=0$ has roots α, β and $x^2+px+q=0$ has roots $-1/(2\alpha), \gamma$, then	**s.** $(q-b)^2=(aq+bp)(p-a)$

Integer Type

Solutions on page 1.83

1. Let 'a' is a real number satisfying $a^3+\frac{1}{a^3}=18$. Then the value of $a^4+\frac{1}{a^4}-39$ is.

2. Let $P(x) = \frac{5}{3} - 6x - 9x^2$ and $Q(y) = -4y^2 + 4y + \frac{13}{2}$. If there exist unique pair of real numbers (x, y) such that $P(x)\,Q(y) = 20$, then the value of $(6x + 10y)$ is.

3. Let $P(x) = x^3 - 8x^2 + cx - d$ be a polynomial with real coefficients and with all its roots being distinct positive integers. Then number of possible value of 'c' is.

4. Let α_1, β_1 are the roots of $x^2 - 6x + p = 0$ and α_2, β_2 are the roots of $x^2 - 54x + q = 0$. If $\alpha_1, \beta_1, \alpha_2, \beta_2$ form an increasing G.P., then sum of the digits of the value of $(q - p)$ is.

5. If the equation $2x^2 + 4xy + 7y^2 - 12x - 2y + t = 0$ where 't' is a parameter has exactly one real solution of the form (x, y). Then the sum of $(x + y)$ is equal to.

6. Polynomial $P(x)$ contains only terms of odd degree. When $P(x)$ is divided by $(x - 3)$, the remainder is 6. If $P(x)$ is divided by $(x^2 - 9)$, then the remainder is $g(x)$. Then the value of $g(2)$ is.

7. If set of values of 'a' for which $f(x) = ax^2 - (3 + 2a)x + 6$, $a \neq 0$ is positive for exactly three distinct negative integral values of x is $(c, d]$, then the value of $(c^2 + 4|d|)$ is equal to.

8. Given α and β are the roots of the quadratic equation $x^2 - 4x + k = 0$ $(k \neq 0)$. If $\alpha\beta$, $\alpha\beta^2 + \alpha^2\beta$, $\alpha^3 + \beta^3$ are in geometric progression, then the value of $7k/2$ equals.

9. If the equation $x^2 + 2(\lambda + 1)x + \lambda^2 + \lambda + 7 = 0$ has only negative roots, then the least value of λ equals.

10. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial such that $P(1) = 1$, $P(2) = 8$, $P(3) = 27$, $P(4) = 64$, then the value of $P(5)$ is divisible by prime number.

11. If $\sqrt{\sqrt{\sqrt{x}}} = \sqrt[4]{\sqrt[4]{\sqrt[4]{3x^4} + 4}}$, then the value of x^4 is.

12. Number of positive integers x for which $f(x) = x^3 - 8x^2 + 20x - 13$ is a prime number is.

13. If equation $x^4 - (3m + 2)x^2 + m^2 = 0$ $(m > 0)$ has four real solutions which are in A.P., then the value of 'm' is.

14. The quadratic polynomial $p(x)$ has the following properties: $p(x) \geq 0$ for all real numbers, $p(1) = 0$ and $p(2) = 2$. Find the value of $p(3)$ is.

15. $f: R \to R, f(x) = \dfrac{3x^2 + mx + n}{x^2 + 1}$. If the range of this function is $[-4, 3)$, then find the value of $|m + n|$ is.

16. If a and b are positive numbers and each of the equations $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ has real roots, then the smallest possible value of $(a + b)$ is.

17. Suppose a, b, c are the roots of the cubic $x^3 - x^2 - 2 = 0$. Then the value of $a^3 + b^3 + c^3$ is.

18. Given that $x^2 - 3x + 1 = 0$, then the value of the expression $y = x^9 + x^7 + x^{-9} + x^{-7}$ is divisible by prime number.

19. Suppose $a, b, c \in I$ such that greatest common divisor of $x^2 + ax + b$ and $x^2 + bx + c$ is $(x + 1)$ and the least common multiple of $x^2 + ax + b$ and $x^2 + bx + c$ is $(x^3 - 4x^2 + x + 6)$. Then the value of $|a + b + c|$ is equal to.

20. If the roots of the cubic, $x^3 + ax^2 + bx + c = 0$ are three consecutive positive integers. Then the value of $\dfrac{a^2}{b+1}$ is equal to.

21. If $x + y + z = 12$ and $x^2 + y^2 + z^2 = 96$ and $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 36$. Then the value $x^3 + y^3 + z^3$ is divisible by prime number.

22. Let α and β be the solutions of the quadratic equation $x^2 - 1154x + 1 = 0$, then the value of $\sqrt[4]{\alpha} + \sqrt[4]{\beta}$ is equal to.

23. If $a^2 - 4a + 1 = 4$, then the value of $\dfrac{a^3 - a^2 + a - 1}{a^2 - 1}$ $(a^2 \neq 1)$ is equal to.

24. The function $f(x) = ax^3 + bx^2 + cx + d$ has three positive roots. If the sum of the roots of $f(x)$ is 4, the largest possible integral values of c/a is.

25. Let $x^2 + y^2 + xy + 1 \geq a(x + y)\ \forall\, x, y \in R$, then the number of possible integer(s) in the range of a is.

26. a, b and c are all different and non-zero real numbers in arithmetic progression. If the roots of quadratic equation $ax^2 + bx + c = 0$ are α and β such that $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$, $\alpha + \beta$ and $\alpha^2 + \beta^2$ are in geometric progression, then the value of a/c will be.

27. All the values of k for which the quadratic polynomial $f(x) = -2x^2 + kx + k^2 + 5$ has two distinct zeroes and only one of them satisfying $0 < x < 2$, lie in the interval (a, b). The value of $(a + 10b)$ is.

28. The quadratic equation $x^2 + mx + n = 0$ has roots which are twice those of $x^2 + px + m = 0$ and m, n and $p \neq 0$. Then the value of n/p is.

29. a, b, c are reals such that $a + b + c = 3$ and $\dfrac{1}{a+b} + \dfrac{1}{b+c} + \dfrac{1}{c+a} = \dfrac{10}{3}$. The value of $\dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b}$ is.

30. Let a, b and c be real numbers which satisfy the equations $a + \dfrac{1}{bc} = \dfrac{1}{5}$, $b + \dfrac{1}{ac} = \dfrac{-1}{15}$ and $c + \dfrac{1}{ab} = \dfrac{1}{3}$. The value of $\dfrac{c-b}{c-a}$ is equal to.

31. If a, b, c are non-zero real numbers, then the minimum value of the expression $\left(\dfrac{(a^4 + 3a^2 + 1)(b^4 + 5b^2 + 1)(c^4 + 7c^2 + 1)}{a^2 b^2 c^2}\right)$ is not divisible by prime number.

32. If $a, b \in R$ such that $a + b = 1$ and $(1 - 2ab)(a^3 + b^3) = 12$. The value of $(a^2 + b^2)$ is equal to.

33. If the cubic $2x^3 - 9x^2 + 12x + k = 0$ has two equal roots then maximum value of $|k|$ is

34. Let a, b and c be distinct non zero real numbers such that $\frac{1-a^3}{a} = \frac{1-b^3}{b} = \frac{1-c^3}{c}$. The value of $(a^3 + b^3 + c^3)$, is

Archives

Solutions on page 1.86

Subjective Type

1. Solve for x: $4^x - 3^{x-1/2} = 3^{x+1/2} - 2^{2x-1}$. **(IIT-JEE, 1978)**

2. Solve for x: $\sqrt{x+1} - \sqrt{x-1} = 1$. **(IIT-JEE, 1978)**

3. Solve the following equation for x: $2\log_x a + \log_{ax} a + 3\log_{a^2x} a = 0$, $a > 0$. **(IIT-JEE, 1978)**

4. Show that the square of $\left(\sqrt{26-15\sqrt{3}}\right)/\left(5\sqrt{2}-\sqrt{38+5\sqrt{3}}\right)$ is a rational number. **(IIT-JEE, 1978)**

5. Find all integers x for which $(5x - 1) < (x + 1)^2 < (7x - 3)$. **(IIT-JEE, 1978)**

6. If α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s. Deduce the condition that the equation has a common root. **(IIT-JEE, 1979)**

7. Show that for any triangle with sides a, b and c, $3(ab + bc + ca) < (a + b + c)^2 < 4(bc + ca + ab)$. When are the first two expressions equal? **(IIT-JEE, 1979)**

8. Let $y = \sqrt{((x+1)(x-3))/(x-2)}$. Find all the real values of x for which y takes real values. **(IIT-JEE, 1980)**

9. For what values of m, does the system of equations $3x + my = m$, $2x - 5y = 20$ has solution satisfying the conditions $x > 0$, $y > 0$. **(IIT-JEE, 1980)**

10. Find the solution set of the system

$x + 2y + z = 1$

$2x - 3y - w = 2$

$x \geq 0, y \geq 0, z \geq 0, w \geq 0$ **(IIT-JEE, 1980)**

11. Show that the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has no real solution. **(IIT-JEE, 1982)**

12. mn squares of equal size are arranged to form a rectangle of dimension m by n, where m and n are natural numbers. Two square will be called 'neighbours' if they have exactly one common side. A natural is written in each square such that the number written in any square is the arithmetic mean of the numbers written in its neighbouring squares. Show that this is possible only if all the numbers used are equal. **(IIT-JEE, 1982)**

13. If one root of the quadratic equation $ax^2 + bx + c = 0$ is equal to the n^{th} power of the other, then show that

$(ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} + b = 0$ **(IIT-JEE, 1983)**

14. Find all real values of x which satisfy $x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \leq 0$. **(IIT-JEE, 1983)**

15. Solve for $x: (5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10$. **(IIT-JEE, 1985)**

16. For $a \leq 0$, determine all real roots of the equation $x^2 - 2a|x - a| - 3a^2 = 0$ **(IIT-JEE, 1985)**

17. Find the set of all x for which $2x/(2x^2+5x+2) > 1/(x+1)$. **(IIT-JEE, 1987)**

18. Solve $|x^2 + 4x + 3| + 2x + 5 = 0$. **(IIT-JEE, 1988)**

19. Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α and β, where $\alpha < -1$ and $\beta > 1$, then show that

$1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$ **(IIT-JEE, 1995)**

20. The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P. Find the intervals in which β and γ lie. **(IIT-JEE, 1996)**

21. Let S be a square of unit area. Consider any quadrilateral, which has one vertex on each side of S. If a, b, c and d denote the lengths of the sides of the quadrilateral, prove that $2 \leq a^2 + b^2 + c^2 + d^2 \leq 4$. **(IIT-JEE, 1997)**

22. Let $f(x) = Ax^2 + Bx + C$, where A, B, C are real numbers. Prove that if $f(x)$ is an integer whenever x is an integer, then the numbers $2A$, $A + B$ and C are all integers. Conversely, prove that if the number $2A$, $A + B$ and C are all integers, then $f(x)$ is an integer whenever x is an integer. **(IIT-JEE, 1998)**

23. If α, β are the roots of $ax^2 + bx + c = 0$ $(a \neq 0)$ and $\alpha + \delta, \beta + \delta$ are the roots of $Ax^2 + Bx + C = 0$ $(A \neq 0)$ for some constant δ, then prove that $(b^2 - 4ac)/a^2 = (B^2 - 4AC)/A^2$. **(IIT-JEE, 2000)**

24. Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equations $ax^2 + bx + c = 0$. Express the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α, β. **(IIT-JEE, 2001)**

25. If $x^2 + (a - b)x + (1 - a - b) = 0$ where $a, b \in R$, then find the values of a for which equation has unequal real roots for all values of b. **(IIT-JEE, 2003)**

26. Let a and b be the roots of the equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ are c, d. Then find the value of $a + b + c + d$, when $a \neq b \neq c \neq d$. **(IIT-JEE, 2006)**

Objective Type

Fill in the blanks

1. The coefficient of x^{99} in the polynomial $(x - 1)(x - 2) \cdots (x - 100)$ is ________. **(IIT-JEE, 1982)**

2. If $2 + i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, where p and q are real, then $(p, q) = ($________, ________$)$.

3. If the product of the roots of the equation $x^2 - 3kx + 2e^{2\ln k} - 1 = 0$ is 7, then the roots are real for = ________.

4. If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $(a \neq b)$ have a common root, then the numerical value of $a + b$ is ________. **(IIT-JEE, 1986)**

5. If $x < 0, y < 0, x + y + (x/y) = (1/2)$ and $(x + y)(x/y) = -(1/2)$, then x = ________ and y = ________. **(IIT-JEE, 1982)**

6. The sum of all real roots of the equation $|x - 2|^2 + |x - 2| - 2 = 0$ is ________. **(IIT-JEE, 1997)**

7. The solution of the equation $\log_7 \log_5 (\sqrt{x+5} + \sqrt{x}) = 0$ is ________. **(IIT-JEE, 1986)**

True or false

1. The equation $2x^2 + 3x + 1 = 0$ has an irrational root. **(IIT-JEE, 1983)**

2. If $a < b < c < d$, then the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are real and distinct. **(IIT-JEE, 1984)**

3. If $n_1, n_2, \ldots, n_p$ are p positive integers, whose sum is an even number, then the number of odd integers among them is odd. **(IIT-JEE, 1985)**

4. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$, then $P(x)Q(x) = 0$ has at least two real roots. **(IIT-JEE, 1985)**

Multiple choice questions with one correct answer

1. If l, m, n are real $l \neq m$, then the roots of the equation $(l - m)x^2 - 5(l + m)x - 2(l - m) = 0$ are
   a. real and equal **b.** complex
   c. real and unequal **d.** none of these
   (IIT-JEE, 1979)

2. If x, y and z are real and different and $u = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$, then u is always
   a. non-negative **b.** zero
   c. non-positive **d.** none of these
   (IIT-JEE, 1979)

3. If $a > 0$, $b > 0$ and $c > 0$ then the roots of the equation $ax^2 + bx + c = 0$
   a. are real and negative **b.** have positive real parts
   c. have negative real parts **d.** none of these
   (IIT-JEE, 1979)

4. Both the roots of the equation $(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b) = 0$ are always
   a. positive **b.** real
   c. negative **d.** none of these
   (IIT-JEE, 1980)

5. If $(x^2 + px + 1)$ is a factor of $(ax^3 + bx + c)$, then
   a. $a^2 + c^2 = -ab$ **b.** $a^2 - c^2 = -ab$
   c. $a^2 - c^2 = ab$ **d.** none of these
   (IIT-JEE, 1980)

6. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is
   a. 4 **b.** 1 **c.** 2 **d.** 0 **(IIT-JEE, 1982)**

7. Two towns A and B are 60 km apart. A school is to be built to serve 150 students in town A and 50 students in town B. If the total distance to be travelled by all 200 students is to be as small as possible, then the school be built at
   a. town B **b.** 45 km from town A
   c. town A **d.** 45 km from town B
   (IIT-JEE, 1982)

8. The largest interval for which $x^{12} - x^9 + x^4 - x + 1 > 0$ is
   a. $-4 < x \leq 0$ **b.** $0 < x < 1$
   c. $-100 < x < 100$ **d.** $-\infty < x < \infty$
   (IIT-JEE, 1982)

9. The equation $x - 2/(x-1) = 1 - 2/(x-1)$ has
   a. no root **b.** one root
   c. two equals roots **d.** infinitely many roots
   (IIT-JEE, 1984)

10. If $a^2 + b^2 + c^2 = 1$, then $ab + bc + ca$ lies in the interval
    a. $\left[\frac{1}{2}, 2\right]$ **b.** $[-1, 2]$

c. $\left[-\frac{1}{2}, 1\right]$ d. $\left[-1, \frac{1}{2}\right]$

(IIT-JEE, 1984)

11. If α and β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always

a. one positive and one negative root

b. two positive roots

c. two negative roots

d. cannot say anything **(IIT-JEE, 1989)**

12. Let a, b, c be real numbers, $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$. β is the root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies

a. $\gamma = \frac{\alpha+\beta}{2}$ **b.** $\gamma = \alpha + \frac{\beta}{2}$

c. $\gamma = \alpha$ **d.** $\alpha < \gamma < \beta$

(IIT-JEE, 1989)

13. The number of solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is

a. 0 **b.** 1

c. 2 **d.** infinitely many

14. Let α, β be the roots of the equation $(x - a)(x - b) = c, c \neq 0$. Then the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are

a. a, c **b.** b, c

c. a, b **d.** $a + c, b + c$

(IIT-JEE, 1992)

15. The number of points of intersection of two curves $y = 2 \sin x$ and $y = 5x^2 + 2x + 3$ is

a. 0 **b.** 1

c. 2 **d.** ∞ **(IIT-JEE, 1994)**

16. If p, q, r are +ve and are in A.P., in the roots of quadratic equation $px^2 + qx + r = 0$ are all real for

a. $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$ **b.** $\left|\frac{p}{r} - 7\right| \geq 4\sqrt{3}$

c. all p and r **d.** no p and r

(IIT-JEE, 1994)

17. The equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ has

a. no solution **b.** one solution

c. two solutions **d.** more than two solutions

(IIT-JEE, 1997)

18. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then

a. $a < 2$ **b.** $2 \leq a \leq 3$

c. $3 < a \leq 4$ **d.** $a > 4$ **(IIT-JEE, 1999)**

19. If α and β $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then

a. $0 < \alpha < \beta$ **b.** $\alpha < 0 < \beta < |\alpha|$

c. $\alpha < \beta < 0$ **d.** $\alpha < 0 < |\alpha| < \beta$

(IIT-JEE, 2000)

20. If $b > a$, then the equation $(x - a)(x - b) - 1 = 0$ has

a. both roots in (a, b)

b. both roots in $(-\infty, a)$

c. both roots in $(b, +\infty)$

d. one root in $(-\infty, a)$ and the other in $(b, +\infty)$

(IIT-JEE, 2000)

21 For the equation $3x^2 + px + 3 = 0$, $p > 0$, if one of the root is square of the other, then p is equal to

a. 1/3 **b.** 1 **c.** 3 **d.** 2/3 **(IIT-JEE, 2000)**

22. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is

a. [0, 1] **b.** $\left(0, \frac{1}{2}\right]$

c. $\left[\frac{1}{2}, 1\right]$ **d.** (0, 1]

(IIT-JEE, 2001)

23. Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in G.P., then the integral values of p and q, respectively are

a. $-2, -32$ **b.** $-2, 3$ **c.** $-6, 3$ **d.** $-6, -32$

(IIT-JEE, 2001)

24. The set of all real numbers x for which $x^2 - |x + 2| + x > 0$ is

a. $(-\infty, -2)$ **b.** $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

c. $(-\infty, -1) \cup (1, \infty)$ **d.** $(\sqrt{2}, \infty)$

(IIT-JEE, 2002)

25. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ are such that $\min f(x) > \max g(x)$, then the relation between b and c is

a. no relation **b.** $0 < c < b/2$

c. $|c| < |b|\sqrt{2}$ **d.** $|c| > |b|\sqrt{2}$ **(IIT-JEE, 2003)**

26. For all x, $x^2 + 2ax + 10 - 3a > 0$, then the interval in which a lies is

a. $a < -5$ **b.** $-5 < a < 2$

c. $a > 5$ **d.** $2 < a < 5$

(IIT-JEE, 2004)

27. If one root is square of the other root of the equation $x^2 + px + q = 0$, then the relation between p and q is

a. $p^3 - q(3p - 1) + q^2 = 0$

b. $p^3 - q(3p + 1) + q^2 = 0$

c. $p^3 + q(3p - 1) + q^2 = 0$

d. $p^3 + q(3p + 1) + q^2 = 0$ **(IIT-JEE, 2004)**

28. Let α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$ and $\Delta = b^2 - 4ac$. If $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$ are in G.P. Then

a. $\Delta = 0$ **b.** $\Delta \neq 0$ **c.** $b\Delta = 0$ **d.** $c\Delta = 0$ **(IIT-JEE, 2005)**

29. Let a, b, c be the sides of a triangle, where $a \neq b \neq c$ and $\lambda \in R$. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real. Then

a. $\lambda < \frac{4}{3}$ **b.** $\lambda > \frac{5}{3}$

c. $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ **d.** $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$ **(IIT-JEE, 2006)**

30. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\alpha/2, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is

a. $\frac{2}{9}(p-q)(2q-p)$ **b.** $\frac{2}{9}(q-p)(2p-q)$

c. $\frac{2}{9}(q-2p)(2q-p)$ **d.** $\frac{2}{9}(2p-q)(2q-p)$

(IIT-JEE, 2007)

31. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are non-zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having α/β and β/α as its roots is

a. $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$

b. $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

c. $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$

d. $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

(IIT-JEE, 2010)

32. A value of b for which the equations $x^2 + bx - 1 = 0$, $x^2 + x + b = 0$ have one root in common is

a. $-\sqrt{2}$ **b.** $-i\sqrt{3}$ **c.** $\sqrt{2}$ **d.** $\sqrt{3}$

(IIT-JEE, 2011)

33. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is

a. 1 **b.** 2 **c.** 3 **d.** 4

(IIT-JEE, 2011)

Multiple choice questions with one or more than one correct answer

1. For real x, then function $(x-a)(x-b)/(x-c)$ will assume all real values provided

a. $a > b > c$ **b.** $a < b < c$

c. $a > c > b$ **d.** $a < c < b$ **(IIT-JEE, 1984)**

2. If S is the set of all real x such that $(2x-1)/(2x^3+3x^2+x)$ is positive, then S contains

a. $\left(-\infty, -\frac{3}{2}\right)$ **b.** $\left(-\frac{3}{2}, -\frac{1}{4}\right)$

c. $\left(-\frac{1}{4}, \frac{1}{2}\right)$ **d.** $\left(\frac{1}{2}, 3\right)$

e. none of these **(IIT-JEE, 1986)**

3. The equation $x^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$ has

a. at least one real solution

b. exactly three solutions

c. exactly one irrational solution

d. complex roots **(IIT-JEE, 1989)**

Assertion and reasoning

1. Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and $1/\beta$, are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$.

Statement 1: $(p^2 - q)(b^2 - ac) \geq 0$

Statement 2: $b \neq pa$ or $c \neq qa$

a. Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.

b. Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1.

c. Statement 1 is true, statement 2 is false.

d. Statement 1 is false, statement 2 is true.

(IIT-JEE, 2008)

Integer type

1. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is. **(IIT-JEE, 2011)**

ANSWERS AND SOLUTIONS

Subjective Type

1. Let,

$$\left(\sqrt{x^2-5x+6}+\sqrt{x^2-5x+4}\right)^{x/2}=A$$

and

$$\left(\sqrt{x^2-5x+6}-\sqrt{x^2-5x+4}\right)^{x/2}=B$$

We have,

$$A+B=2^{\frac{x+4}{4}}=2\times 2^{\frac{x}{4}} \text{ and } AB=2^{\frac{x}{2}}$$

Therefore,

$$(A-B)^2 = (A+B)^2-4AB$$

$$=4\times 2^{\frac{x}{2}}-4\times 2^{\frac{x}{2}}=0$$

$\Rightarrow \quad A=B$

Since powers in A and B are same, either the power is equal to zero or the bases are the same. Hence,

$$x=0 \text{ or } \sqrt{x^2-5x+6}+\sqrt{x^2-5x+4}$$

$$=\sqrt{x^2-5x+6}-\sqrt{x^2-5x+4}$$

$\Rightarrow \quad x=0$ or $\sqrt{x^2-5x+4}=0$

$\Rightarrow \quad x=0$ or $x=4$ or 1

2. Suppose one root of the equation is $u+iv$, then the other root would be $u-iv$. Hence,

$$\Rightarrow \frac{A^2}{(u-a)+iv}+\frac{B^2}{(u-b)+iv}+\cdots+\frac{H^2}{(u-h)+iv}=k \quad (1)$$

and

$$\frac{A^2}{(u-a)-iv}+\frac{B^2}{(u-b)-iv}+\cdots+\frac{H^2}{(u-h)-iv}=k \quad (2)$$

From (1) − (2), we get

$$iv\left[\frac{A^2}{(u-a)^2+v^2}+\frac{B^2}{(u-b)^2+v^2}+\cdots+\frac{H^2}{(u-h)^2+v^2}\right]=0$$

This is possible only when $v=0$, and for this case there is no imaginary root.

3. Given, $a\neq b\neq c$, $a, b, c \in R$. Now,

$ax^2+bx+c\geq 0$

$\Rightarrow \quad b^2-4ac\leq 0$ and $a>0$ (1)

$bx^2+cx+a\geq 0$

$\Rightarrow \quad c^2-4ab\leq 0$ and $b>0$ (2)

$cx^2+ax+b\geq 0$

$\Rightarrow \quad a^2-4bc\leq 0$ and $c>0$ (3)

Equality cannot hold simultaneously in (1), (2) and (3).

$$\therefore \quad \frac{a^2+b^2+c^2}{ab+bc+ca}<4 \quad (4)$$

Now,

$(a-b)^2+(b-c)^2+(c-a)^2>0$ ($\because$ a, b, c are distinct)

$$\Rightarrow \quad \frac{a^2+b^2+c^2}{ab+bc+ca}>1 \quad (5)$$

From (4) and (5), $(a^2+b^2+c^2)/(ab+bc+ca)$ can never lie in $(-\infty, 1]\cup[4,\infty)$.

4. Let α, β be the root of a quadratic and α^2, β^2 be the roots of another quadratic. Since the quadratics remain same, we have

$\alpha+\beta=\alpha^2+\beta^2$ (1)

$\alpha\beta=\alpha^2\beta^2$ (2)

Now,

$\alpha\beta=\alpha^2\beta^2$

$\Rightarrow \quad \alpha\beta(1-\alpha\beta)=0$

$\Rightarrow \quad \alpha=0$ or $\beta=0$ or $\alpha\beta=1$

Case I:

When $\alpha=0$, from (1),

$\alpha+\beta=\alpha^2+\beta^2$

$\Rightarrow \quad \beta=\beta^2$

$\Rightarrow \quad \beta(1-\beta)=0$

$\Rightarrow \quad \beta=0$ or $\beta=1$

Thus, we get two sets of values of α and β, viz., $\alpha=0, \beta=0$ and $\alpha=0, \beta=1$

Case II:

When $\beta=0$, from (1),

$\alpha+\beta=\alpha^2+\beta^2$

$\Rightarrow \quad \alpha=\alpha^2$

$\Rightarrow \quad \alpha(1-\alpha)=0$

$\Rightarrow \quad \alpha=0$ or $\alpha=1$

Thus, we get two sets of values of α and β, viz., $\alpha=0, \beta=0$ and $\alpha=1, \beta=0$.

Case III:

When $\alpha\beta=1$, from (1),

$\alpha+\beta=\alpha^2+\beta^2$

$$\Rightarrow \quad \alpha+\frac{1}{\alpha}=\alpha^2+\frac{1}{\alpha^2} \qquad \left[\because \alpha\beta=1\Rightarrow\beta=\frac{1}{\alpha}\right]$$

$$\Rightarrow \quad \alpha+\frac{1}{\alpha}=\left(\alpha+\frac{1}{\alpha}\right)^2-2$$

$$\Rightarrow \quad \left(\alpha+\frac{1}{\alpha}\right)^2-\left(\alpha+\frac{1}{\alpha}\right)-2=0$$

$\Rightarrow \quad y^2-y-2=0$, where $y=\alpha+\dfrac{1}{\alpha}$

$\Rightarrow \quad (y-2)(y+1)=0$

$\Rightarrow \quad y=2$ or $y=-1$

Now,

$y=2$

$\Rightarrow \quad \alpha+\dfrac{1}{\alpha}=2$

$\Rightarrow \quad \alpha=1$

Similarly,

$y=-1$

$\Rightarrow \quad \alpha+\dfrac{1}{\alpha}=-1$

$\Rightarrow \quad \alpha^2+\alpha+1=0$

$\Rightarrow \quad \alpha = \dfrac{-1 \pm i\sqrt{3}}{2} = \omega, \omega^2$ (where ω, ω^2 are cube roots of unity. See in Chapter 3 for more details)

When

$\alpha = 1, \alpha\beta = 1 \Rightarrow \beta = 1$

When

$\alpha = \omega, \alpha\beta = 1 \Rightarrow \beta = \dfrac{1}{\omega} = \omega^2$

When

$\alpha = \omega^2, \alpha\beta = 1 \Rightarrow \beta = \dfrac{1}{\omega^2} = \omega$

Thus, when $\alpha\beta = 1$, we get two sets of values of α and β, viz.,

$\alpha = 1, \beta = 1; \alpha = \omega, \beta = \omega^2$.

Hence, there are four sets of values of α and β, viz. $\alpha = 0$, $\beta = 0$; $\alpha = 1, \beta = 0$; $\alpha = 1, \beta = 1$ and $\alpha = \omega, \beta = \omega^2$. Consequently, there are four quadratic equations, which do not change by squaring their roots.

5. The given equation is $x^2 - px - (p + c) = 0$.

$\therefore \quad \alpha + \beta = p, \alpha\beta = -(p + c)$

So,

$$(\alpha + 1)(\beta + 1)$$
$$= \alpha\beta + (\alpha + \beta) + 1$$
$$= -(p + c) + p + 1$$
$$= 1 - c \qquad (1)$$

Now,

$$\frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + c} + \frac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + c}$$

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2 - (1-c)} + \frac{(\beta+1)^2}{(\beta+1)^2 - (1-c)}$$

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2 - (\alpha+1)(\beta+1)} + \frac{(\beta+1)^2}{(\beta+1)^2 - (\alpha+1)(\beta+1)} \quad \text{[Using (1)]}$$

$$= \frac{\alpha+1}{\alpha-\beta} + \frac{\beta+1}{\beta-\alpha} = \frac{(\alpha+1)-(\beta+1)}{\alpha-\beta} = 1$$

6. Since α, β are the roots of equation $ax^2 + bx + c = 0$, therefore,

$$a\alpha^2 + b\alpha + c = 0$$
$$a\beta^2 + b\beta + c = 0$$
$$\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$$

Given, $S_n = \alpha^n + \beta^n$. Now,

$$aS_{n+1} + bS_n + cS_{n-1}$$
$$= a(\alpha^{n+1} + \beta^{n+1}) + b(\alpha^n + \beta^n) + c(\alpha^{n-1} + \beta^{n-1})$$
$$= \alpha^{n-1}(a\alpha^2 + b\alpha + c) + \beta^{n-1}(a\beta^2 + b\beta + c)$$
$$= \alpha^{n-1} \times 0 + \beta^{n-1} \times 0$$
$$= 0$$

$$\therefore \quad S_{n+1} = -\frac{b}{a}S_n - \frac{c}{a}S_{n-1} \qquad (1)$$

Putting $n = 4$ in (1), we get

$$S_5 = -\frac{b}{a}S_4 - \frac{c}{a}S_3$$

$$= -\frac{b}{a}\left(-\frac{b}{a}S_3 - \frac{c}{a}S_2\right) - \frac{c}{a}S_3 \qquad \text{[From (1), when } n = 3\text{]}$$

$$= \left(\frac{b^2}{a^2} - \frac{c}{a}\right)S_3 + \frac{bc}{a^2}S_2$$

$$= \left(\frac{b^2}{a^2} - \frac{c}{a}\right)\left(-\frac{b}{a}S_2 - \frac{c}{a}S_1\right) + \frac{bc}{a^2}S_2$$

$$= \left(-\frac{b^3}{a^3} + \frac{2abc}{a^3}\right)S_2 - \left(\frac{b^2}{a^2} - \frac{c}{a}\right)\frac{c}{a}S_1$$

$$= -\frac{b(b^2 - 2ac)}{a^3}\left[\left(-\frac{b}{a}\right)^2 - \frac{2c}{a}\right] - \left(\frac{b^2 - ac}{a^2}\right)\frac{c}{a}\left(-\frac{b}{a}\right)$$

$$[\because S_2 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta \text{ and } S_1 = \alpha + \beta = -\frac{b}{a}]$$

$$= -\frac{b}{a^5}(b^2 - 2ac)^2 + \frac{(b^2 - ac)bc}{a^4}$$

7. $4x^2 + 2x - 1 = 0$

$$\therefore \quad \alpha + \beta = -\frac{1}{2}, \alpha\beta = -\frac{1}{4} \qquad (1)$$

Also, $4\alpha^2 + 2\alpha - 1 = 0$ as α is a root, and we have to prove that $\beta = 4\alpha^3 - 3\alpha$. Now,

$$4\alpha^3 - 3\alpha = 4\alpha^2\,\alpha - 3\alpha$$
$$= \alpha(1 - 2\alpha) - 3\alpha$$
$$= -2\alpha^2 - 2\alpha$$
$$= -\frac{1}{2}[4\alpha^2 + 4\alpha]$$
$$= -\frac{1}{2}[1 - 2\alpha + 4\alpha]$$
$$= -\frac{1}{2}(1 + 2\alpha) = -\frac{1}{2} - \alpha = \beta$$

Now,

$$a + \beta = -\frac{1}{2} \qquad \text{[From (1)]}$$

Hence, the other root β is $4\alpha^3 - 3\alpha$.

8. The given equation can be written as $(ay + a')x^2 + (by + b')x + (cy + c') = 0$. Since roots are rational, therefore, D is a perfect square. Hence, $(by + b')^2 - 4(ay + a')(cy + c')$ is a perfect square. That is, $y^2(b^2 - 4ac) + [2bb' - 4(ac' + a'c)]y + (b'^2 - 4a'c')$ is a perfect square. In other words, the roots of the above equation are equal so that $D = 0$.

$\therefore \quad [2bb' - 4(ac' + a'c)]^2 - 4(b^2 - 4ac)(b'^2 - 4a'c') = 0$

On simplifying, we get the required result.

9. Given roots are real and distinct. So,

$$a^2 - 4b > 0 \Rightarrow b < a^2/4 \qquad (1)$$

Again α and β differ by a quantity less than c ($c > 0$). Hence,

$$|\alpha - \beta| < c \text{ or } (\alpha - \beta)^2 < c^2$$
$$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta < c^2$$
$$\Rightarrow \quad a^2 - 4b < c^2$$
$$\Rightarrow \quad \frac{a^2 - c^2}{4} < b \qquad (2)$$
$$\Rightarrow \quad \frac{a^2 - c^2}{4} < b < \frac{a^2}{4} \qquad \text{[From (1) and (2)]}$$

10. Given $ax^2 + bx + c = 0$ has real and positive roots. Then,

$$b^2 - 4ac \ge 0 \quad (1)$$

Sum of roots is

$$-b/a > 0 \text{ or } b/a < 0 \quad (2)$$

i.e., a and b have opposite signs.

Product of root is

$$c/a > 0 \quad (3)$$

i.e., a and c have same sign.

Now, for equation $a^2x^2 + a(3b - 2c)x + (2b - c)(b - c) + ac = 0$, we have

$$D = a^2(3b - 2c)^2 - 4a^2[(2b - c)(b - c) + ac]$$
$$= a^2[9b^2 - 12bc + 4c^2 - 4(2b^2 - 3bc + c^2 + ac)]$$
$$= a^2[9b^2 - 12bc + 4c^2 - 8b^2 + 12bc - 4c^2 - 4ac]$$
$$= a^2(b^2 - 4ac) \ge 0 \quad \text{[Using (1)]}$$

Hence, the roots are real. Also, sum of roots,

$$\frac{-a(3b-2c)}{a^2} = -\left(\frac{3b}{a} - \frac{2c}{a}\right) > 0 \quad \text{[Using (2) and (3)]}$$

Product of roots,

$$\frac{(2b-c)(b-c)+ac}{a^2} = \left(2\frac{b}{a} - \frac{c}{a}\right)\left(\frac{b}{a} - \frac{c}{a}\right) + \frac{c}{a} > 0 \quad \text{[Using (2) and (3)]}$$

Hence, the roots are positive.

11. $$x = \frac{-p - \sqrt{p^2 + 4 \times 444p}}{2}$$

Since $p = 2$ does not give the integral roots, so D must be a perfect square of an odd integer, i.e.,

$$D^2 = p^2 + 1776p = p(p + 1776)$$

Since D is perfect square, hence $p + 1776$ must be a multiple of p, i.e., 1776 must be a multiple of p. Now, $1776 = 2^4 \times 3 \times 37$ hence $p = 2$ or 3 or 37.

(i) If $p = 2$, then $p(p + 1776) = 2(3 + 1776) = 3556 = 4 \times 7 \times 127$, which is not a perfect square.

(ii) If $p = 3$, then $p(p + 1776) = 3(3 + 1776) = 5337$, which is not a perfect square as its last digit is 7.

(iii) If $p = 37$, then $p(p + 1776) = 37(37 + 1776) = 37^2 \times 7^2$, which is odd. Hence, $p = 37$.

12. Consider the equation

$$ax^2 + bx + c = 0$$

Since the roots are rational, discriminant of the given quadratic equation will be a perfect square. Hence,

$$b^2 - 4ac = \lambda^2, \lambda \in I$$

$\Rightarrow \quad b^2 - \lambda^2 = 4ac \Rightarrow (b + \lambda)(b - \lambda) = 4ac$

Then, we have the following possibilities:

$b + \lambda = 2a, b - \lambda = 2c$

$\Rightarrow \quad b + \lambda = 2c, b - \lambda = 2a$

$\Rightarrow \quad b + \lambda = -2a, b - \lambda = -2c$

$\Rightarrow \quad b + \lambda = -2c, b - \lambda = -2a$

(as $b + \lambda$ and $b - \lambda$ both should be even)

Solving the above cases, we get

$$b = \pm(a + c), \lambda = \pm(a - c)$$

Hence, the roots are $-(b-\lambda)/2a$. Clearly, one of the roots is $-1/2$ (put the value of λ and simplify to get $-1/2$).

13. We are given that $(2x + y)(x - 2y) = 7$. Since x and y are to be integers, hence, L.H.S. is the product of two integers and R.H.S. is also the product of two integers, viz, 7 and 1, or 1 and 7, or -7 and -1, or -1 and -7. Hence, we can choose

$2x + y = 7$ and $x - 2y = 1$ (1)

$2x + y = 1$ and $x - 2y = 7$ (2)

$2x + y = -7$ and $x - 2y = -1$ (3)

$2x + y = -1$ and $x - 2y = -7$ (4)

Solving them as usual we find only (1) and (3) give integral solutions as 3, 1 for (1) and $-3, -1$ for (3). Both (2) and (4) when solved do not give integral values of x and y.

14. Given $ax^2 + bx + c - p = 0$ has integral roots. Let α, β be the roots. Then,

$$ax^2 + bx + c - p = a(x - \alpha)(x - \beta) \quad (1)$$

Now from $ax^2 + bx + c = 2p$, we have

$$ax^2 + bx + c - p = p$$

$\Rightarrow \quad a(x - \alpha)(x - \beta) = p$ [Using (1)]

In above equation, L.H.S. has three factors but R.H.S. is prime number, which is contradiction. Hence, $ax^2 + bx^2 + c = 2p$ cannot have integral roots.

15. Given expression is $(x+a)(x+b)/(x+c)$. Let $x + c = y$. Then,

$$\frac{(x+a)(x+b)}{(x+c)} = \frac{(y+(a-c))(y+(b-c))}{y}$$
$$= \frac{y^2 + [(a-c)+(b-c)]y + (a-c)(b-c)}{y}$$
$$= y + \frac{(a-c)(b-c)}{y} + (a-c) + (b-c)$$
$$= \left[\sqrt{y} - \sqrt{\frac{(a-c)(b-c)}{y}}\right]^2 + \left[\sqrt{a-c} + \sqrt{b-c}\right]^2$$
$$\ge \left[\sqrt{a-c} + \sqrt{b-c}\right]^2$$

Hence, the least value is $\left[\sqrt{a-c} + \sqrt{b-c}\right]^2$.

16. Let

$$t = x + \sqrt{x^2 + b^2} \quad (1)$$

$\Rightarrow \quad \sqrt{x^2 + b^2} - x = \dfrac{b^2}{t} \quad (2)$

$\therefore \quad 2x = t - \dfrac{b^2}{t}$ [Subtracting (2) from (1)]

$\Rightarrow \quad x = \dfrac{1}{2}\left(\dfrac{t^2 - b^2}{t}\right)$

Now,

$$y = 2(a - x)t$$
$$= 2\left(a - \left(\frac{t^2 - b^2}{2t}\right)\right)t$$
$$= (2at - t^2 + b^2)$$
$$= b^2 - (t^2 - 2at + a^2 - a^2)$$
$$= a^2 + b^2 - (t - a)^2$$

$\Rightarrow \quad y \le a^2 + b^2$

17. $f(0) = d, f(-1) = -1 + b - c + d$

$\Rightarrow$ $d =$ odd and $-1 + b - c + d =$ odd

$\Rightarrow$ $b - c = 1 +$ odd $- d$

$= (1 + \text{odd}) - (\text{odd}) = \text{even} - \text{odd} = \text{odd}$ (1)

Thus, both d and $b - c$ are odd.

If possible let the three roots α, β, γ be all integers. Now,

$$\alpha\beta\gamma = -\frac{d}{1} = -d = \text{negative odd integer} \quad (2)$$

$\Rightarrow$ α, β, γ are three integers whose product is odd

$\Rightarrow$ α, β, γ all are odd

Again

$\alpha + \beta + \gamma = -b$ and $\alpha\beta + \beta\gamma + \alpha\gamma = c$ (3)

$\Rightarrow$ b and c both will be odd

$\Rightarrow$ $(b - c)$ will be even which contradicts with (1)

Hence, the three roots cannot be all integers.

18. We have,

$|x| < a \Rightarrow -a < x < +a$

Therefore, the given inequality implies

$$-2 < \frac{x^2+kx+1}{x^2+x+1} < 2 \quad (1)$$

Now, $x^2 + x + 1 = (x+1/x)^2 + (3/4)$ is positive for all values of x.

Multiplying (1) by $x^2 + x + 1$, we get

$-2(x^2 + x + 1) < x^2 + kx + 1 < 2(x^2 + x + 1)$

This yields two inequalities, viz.,

$3x^2 + (2 + k)x + 3 > 0$

and

$x^2 + (2 - k)x + 1 > 0$

For these quadratic expressions to be positive for all values of x, their discriminants must be negative. Hence,

$(2 + k)^2 - 36 < 0$ and $(2 - k)^2 - 4 < 0$ (2)

$\Rightarrow$ $(k + 8)(k - 4) < 0$ and $k(k - 4) < 0$ (3)

$\Rightarrow$ $-8 < k < 4$ and $0 < k < 4$

For both these conditions to be satisfied, $0 < k < 4$.

19. Given equation is

$$\sqrt{a(2^x - 2) + 1} = 1 - 2^x \quad (1)$$

$\Rightarrow$ $\sqrt{a(y-2)+1} = 1 - y$, where $y = 2^x > 0$ (2)

$\Rightarrow$ $a(y - 2) + 1 = (1 - y)^2 = 1 - 2y + y^2$

$\Rightarrow$ $y^2 - (2 + a)y + 2a = 0$

$$\Rightarrow \quad y = \frac{(2+a) - \sqrt{(2+a)^2 - 8a}}{2}$$

$$= \frac{2 + a - (2 - a)}{2} = 2, a$$

$y = 2$ does not satisfy Eq. (2) because in that case R.H.S. of Eq. (2) is negative and L.H.S. is positive. When $y = a$, from (2),

$\sqrt{a(a-2)+1} = 1 - a$

$\Rightarrow$ $\sqrt{(a-1)^2} = 1 - a$ or $|a - 1| = 1 - a$

$\Rightarrow$ $a - 1 \le 0 \Rightarrow a \le 1$

$\Rightarrow$ $0 < a \le 1$ $[\because y > 0]$

Now,

$y = a \Rightarrow 2^x = a \Rightarrow x = \log_2 a,$

where $0 < a \le 1$. When $a > 1$, given equation has no solution.

20. Case I:

Suppose $x \ge a$. Then the given equation becomes

$x^2 - 2a(x - a) - 3a^2 = 0$

$\Rightarrow$ $x^2 - 2ax - a^2 = 0$

$$\Rightarrow x = \frac{2a \pm 2\sqrt{2}a}{2} = (1 \pm \sqrt{2})a$$

As $a < 0$ and $1 + \sqrt{2} > 1$, so $(1 + \sqrt{2})a < a$, therefore $x \ne (1 + \sqrt{2})a$.

Next, as $1 - \sqrt{2} < 1$, so $(1 - \sqrt{2})a > a$, therefore $x = (1 - \sqrt{2})a$.

Case II:

Suppose $x < a$. Then the given equation becomes

$x^2 - 2a(a - x) - 3a^2 = 0$

$\Rightarrow$ $x^2 + 2ax - 5a^2 = 0$

$$\Rightarrow \quad x = \frac{-2a - \sqrt{24a}}{2} = (-1 \pm \sqrt{6})a$$

As $a < 0$, $-1 - \sqrt{6} < 0 < 1$, so $(-1 - \sqrt{6})a > a$, therefore $x \ne (-1 - \sqrt{6})a$ ($\because x < a$). Next, as $a < 0$, $-1 + \sqrt{6} > 1$ and $(-1 + \sqrt{6})a < a$, therefore, $x = (-1 + \sqrt{6})a = (\sqrt{6} - 1)a$.

21. Given equation is

$x^3 - 10x^2 - 11x - 100 = 0$

Let

$f(x) = x^3 - 10x^2 - 11x - 100$

$\Rightarrow f'(x) = 3x^2 - 20x - 11$

For $3x^2 - 20x - 11 = 0$, we have

$$x = \frac{20 - \sqrt{400 + 132}}{6} = \frac{10 - \sqrt{133}}{3}$$

Hence, graph of $y = f(x)$ is

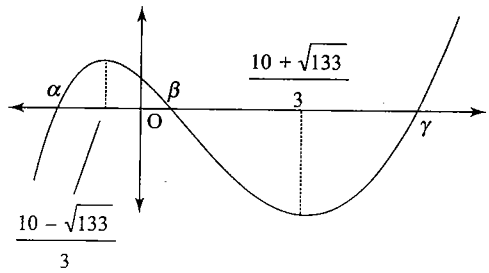


Fig. 1.68

Now, $(10 + \sqrt{133})/3 \cong 7.16$.

$f(8) = 8^3 - 10(8)^2 - 11(8) - 100 < 0$

$f(9) = 9^3 - 10(9)^2 - 11(9) - 100 < 0$

$f(10) = 10^3 - 10(10)^2 - 11(10) - 100 < 0$

$f(11) = 11^3 - 10(11)^2 - 11(11) - 100 < 0$

$f(12) = 12^3 - 10(12)^2 - 11(12) - 100 > 0$

$\Rightarrow$ $\gamma \in (11, 12)$

$\Rightarrow$ $[\gamma] = 11$

22. Given,

$x^4 - 4x^3 - 8x^2 + a = 0$

Let,

$f(x) = x^4 - 4x^3 - 8x^2$

$\Longrightarrow \quad f'(x) = 4x^3 - 12x^2 - 16x = 4x(x^2 - 3x - 4) = 4x(x-4)(x+1)$

Hence, graph of $y = f(x)$ is

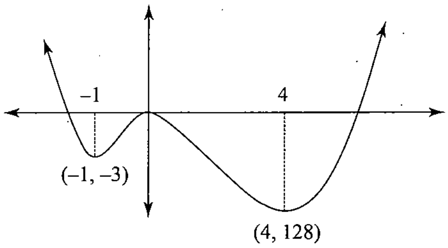


Fig. 1.69

Now, graph of $y = f(x) + a$ moves up or down depending on values of a. It is clear that if equation $f(x) + a = 0$ has four real roots, then

$0 \le a < 3$.

Objective Type

1. b. Given,

$x - 2 = 2^{2/3} + 2^{1/3}$

Cubing both sides, we get

$(x-2)^3 = 2^2 + 2 + 3 \times 2^{2/3} \times 2^{1/3}(x-2) = 6 + 6(x-2)$

or

$x^3 - 6x^2 + 12x - 8 = -6 + 6x$

$\therefore \quad x^3 - 6x^2 + 6x = 2$

2. a. Let,

$f(x, y, z) = x^2 + 4y^2 + 3z^2 - 2x - 12y - 6z + 14$

$= (x-1)^2 + (2y-3)^2 + 3(z-1)^2 + 1$

For the least value of $f(x, y, z)$,

$x - 1 = 0, 2y - 3 = 0$ and $z - 1 = 0$

$\therefore \quad x = 1, y = 3/2, z = 1$

Hence the least value of $f(x, y, z)$ is $f(1, 3/2, 1) = 1$.

3. c. Let m be a positive integer for which

$n^2 + 96 = m^2$

$\Rightarrow \quad m^2 - n^2 = 96 \Rightarrow (m+n)(m-n) = 96$

$\Rightarrow \quad (m+n)\{(m+n) - 2n\} = 96$

$\Rightarrow \quad m + n$ and $m - n$ must be both even

As $96 = 2 \times 48$ or 4×24 or 6×16 or 8×12, hence, number of solutions is 4.

4. d. $f(x, y) = (x-2)^2 + (y-1)^2 = 0$

$\Rightarrow \quad x = 2$ and $y = 1$

$$\therefore \quad E = \frac{(\sqrt{2}-1)^2 + 4\sqrt{2}}{2+\sqrt{2}} = \frac{(\sqrt{2}+1)^2}{\sqrt{2}(\sqrt{2}+1)} = \frac{\sqrt{2}+1}{\sqrt{2}}$$

5. c. We have,

$|x|^2 - 3|x| + 2 = 0$

$\Rightarrow \quad (|x| - 1)(|x| - 2) = 0$

$\Rightarrow \quad |x| = 1, 2 \Rightarrow x = \pm 1, \pm 2$

6. c. Clearly, $x = -1$ satisfies the equation.

7. d. Given equation is satisfied by $x = 1, 2, 3$. But for $x = 1$, $\sqrt{x-2}$ is not defined. Hence, number of roots is 2 and the roots are $x = 2$ and 3.

8. c. As $(\lambda + 1)x^2 + 2 = \lambda x + 3$ has only one solution, so

$D = 0$

$\Rightarrow \quad \lambda^2 - 4(\lambda + 1)(-1) = 0$

$\Rightarrow \quad \lambda^2 + 4\lambda + 4 = 0$

$\Rightarrow \quad (\lambda + 2)^2 = 0$

$\therefore \quad \lambda = -2$

9. a. Given quadratic expression is $x^2 + 2(a+b+c)x + 3(bc + ca + ab)$. this quadratic expression will be a perfect square if the discriminant of its corresponding equation is zero. Hence,

$4(a+b+c)^2 - 4 \times 3(bc + ca + ab) = 0$

$\Rightarrow \quad (a+b+c)^2 - 3(bc + ca + ab) = 0$

$\Rightarrow \quad a^2 + b + c^2 + 2ab + 2bc + 2ca - 3(bc + ca + ab) = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 - ab \;\; bc - ca = 0$

$$\Rightarrow \quad \frac{1}{2}[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca] = 0$$

$$\Rightarrow \quad \frac{1}{2}[(a^2+b^2-2ab)+(b^2+c^2-2bc)+(c^2+a^2-2ca)]=0$$

$$\Rightarrow \quad \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$

which is possible only when $(a-b)^2 = 0$, $(b-c)^2 = 0$ and $(c-a)^2 = 0$, i.e., $a = b = c$.

10. c. If one root is square of the other root of the equation $ax^2 + bx + c = 0$, then

$\beta = \alpha^2 \Rightarrow \alpha^2 + \alpha = -b/a$ and $\alpha^2 a = c/a$

By eliminating α, we get

$b^3 + ac^2 + a^2c = 3abc$

which can be written in the form $a(c-b)^3 = c(a-b)^3$.

Alternative solution:

Let the roots be 2 and 4. Then the equation is $x^2 - 6x + 8 = 0$.

Here obviously,

$$X = \frac{a(c-b)^3}{c} = \frac{1(14)^3}{8} = \frac{14}{2} \times \frac{14}{2} \times \frac{14}{2} = 7^3$$

which is given by $(a-b)^3 = 7^3$.

11. a. Given that $x^2 + px + 1$ is a factor of $ax^3 + bx + c$. Then let $ax^3 + bx + c = (x^2 + px + 1)(ax + \lambda)$, where λ is a constant. Then equating the coefficients of like powers of x on both sides, we get

$0 = ap + \lambda, b = p\lambda + a, c = \lambda$

$$\Rightarrow \quad p = -\frac{\lambda}{a} = -\frac{c}{a}$$

Hence,

$$b = \left(-\frac{c}{a}\right)c + a$$

or

$ab = a^2 - c^2$

12. a. Put $x^2 + x = y$, so that Eq. (1) becomes

$(y-2)(y-3) = 12$

$\Rightarrow \quad y^2 - 5y - 6 = 0$

$\Rightarrow \quad (y-6)(y+1) = 0 \Rightarrow y = 6, -1$

When $y = 6$, we get

$x^2 + x - 6 = 0$

$\Rightarrow \quad (x+3)(x-2) = 0$ or $x = -3, 2$

When $y = -1$, we get

$x^2 + x + 1 = 0$

which has non-real roots and sum of roots is -1.

13. b. Given,

$$(ax^2 + c)\,y + (a'x^2 + c') = 0$$

or

$$x^2 (ay + a') + (cy + c') = 0.$$

If x is rational, then the discriminant of the above equation must be a perfect square. Hence,

$0 - 4\,(ay + a')\,(cy + c')$ must be a perfect square

$\Rightarrow$ $-acy^2 - (ac' + a'c)\,y - a'c'$ must be a perfect square

$\Rightarrow$ $(ac' + a'c)^2 - 4ac\,a'c' = 0$ $\quad [\because D = 0]$

$\Rightarrow$ $(ac' - a'c)^2 = 0$

$\Rightarrow$ $ac' = a'c$

$\Rightarrow$ $\dfrac{a}{a'} = \dfrac{c}{c'}$

14. d. Let the four numbers in A.P. be $p = a - 3d$, $q = a - d$, $r = a + d$, $s = a + 3d$. Therefore,

$p + q = 2$, $r + s = 18$

Given that $pq = A$, $rs = B$.

$\therefore$ $p + q + r + s = 4a = 20$

$\Rightarrow$ $a = 5$

Now,

$p + q = 2 \Rightarrow 10 - 4d = 2$

$r + s = 18 \Rightarrow 10 + 4d = 18$

$\therefore$ $d = 2$

Hence, the numbers are $-1, 3, 7, 11$.

$pq = A = -3$, $rs = B = 77$

15. d. Here, $x = 0$ is not a root. Divide both the numerator and denominator by x and put $x + 3/x = y$ to obtain

$$\frac{4}{y+1} + \frac{5}{y-5} = -\frac{3}{2} \Rightarrow y = -5, 3$$

$x + 3/x = -5$ has two irrational roots and $x + 3/x = 3$ has imaginary roots.

16. c. Clearly, $x = 2$ is a root of the equation and imaginary roots always occur in pairs. Therefore, the other root is also real.

17. c. Given, α, β are roots of equation

$x^2 - 2x + 3 = 0$

$\Rightarrow \alpha^2 - 2\alpha + 3 = 0$ $\quad$ (1)

and

$\beta^2 - 2\beta + 3 = 0$ $\quad$ (2)

$\Rightarrow \alpha^2 = 2\alpha - 3 \Rightarrow \alpha^3 = 2\alpha^2 - 3\alpha$

$\Rightarrow P = (2\alpha^2 - 3\alpha) - 3\alpha^2 + 5\alpha - 2$

$= -\alpha^2 + 2\alpha - 2 = 3 - 2 = 1,$ $\quad$ [Using (1)]

Similarly, we have $Q = 2$.

Now, sum of roots is 3 and product of roots is 2. Hence, the required equation is $x^2 - 3x + 2 = 0$.

18. c. Since α, β are the roots of the equation $2x^2 - 35x + 2 = 0$, therefore,

$$2\alpha^2 - 35\alpha = -2 \text{ or } 2\alpha - 35 = \frac{-2}{\alpha}$$

and

$$2\beta^2 - 35\beta = -2 \text{ or } 2\beta - 35 = \frac{-2}{\beta}$$

Now,

$$(2\alpha - 35)^3 (2\beta - 35)^3 = \left(\frac{-2}{\alpha}\right)^3 \left(\frac{-2}{\beta}\right)^3$$

$$= \frac{8 \times 8}{\alpha^3 \beta^3} = \frac{64}{1} = 64 \qquad (\because \alpha\beta = 1)$$

19. c. Here a, b, c are positive. So,

$$|x| = -b + \sqrt{b^2 + ac}$$

Hence, x has two real values, neglecting $|x| = -b - \sqrt{b^2 + ac}$, as $|x| \ge 0$.

20. a. Since $p(q - r) + q(r - p) + r(p - q) = 0$, so one root is 1 and the other root is $r(p - q)/[p(q - r)]$. Since both the roots are equal, we have

$\dfrac{rp - rq}{pq - pr} = 1$

$\Rightarrow$ $rp - rq = pq - pr$

$\Rightarrow$ $2\,rp = q\,(p + r)$

$\Rightarrow$ $\dfrac{2}{q} = \dfrac{p + r}{pr} = \dfrac{1}{p} + \dfrac{1}{r}$

21. d. We are given that $p(-a) = a$ and $p(a) = -a$

[since when a polynomial $f(x)$ is divided by $x - a$, remainder is $f(a)$]. Let the remainder, when $p(x)$ is divided by $x^2 - a^2$, be $Ax + B$. Then,

$p(x) = Q(x)\,(x^2 - a^2) + Ax + B$ $\quad$ (1)

where $Q(x)$ is the quotient. Putting $x = a$ and $-a$ in (1), we get

$p(a) = 0 + Aa + B \Rightarrow -a = Aa + B$ $\quad$ (2)

and

$p(-a) = 0 - aA + B \Rightarrow a = -aA + B$ $\quad$ (3)

Solving (2) and (3), we get

$B = 0$ and $A = -1$

Hence, the required remainder is $-x$.

22. a. $x^2 + ax + b + 1 = 0$ has positive integral roots α and β. Hence,

$(\alpha + \beta) = -a$ and $\alpha\beta = b + 1$

$\Rightarrow$ $(\alpha + \beta)^2 + (\alpha\beta - 1)^2 = a^2 + b^2$

$\Rightarrow$ $a^2 + b^2 = (\alpha^2 + 1)\,(\beta^2 + 1)$.

$\Rightarrow$ $a^2 + b^2$ can be equal to 50 (since other options have prime numbers)

23. b. $(31 + 8\sqrt{15})^{x^2-3} + 1 = (32 + 8\sqrt{15})^{x^2-3}$

$\Rightarrow$ $(31 + 8\sqrt{15})^{x^2-3} + 1^{x^2-3} = (32 + 8\sqrt{15})^{x^2-3}$

$\Rightarrow$ $x^2 - 3 = 1$ or $x = \pm 2$ $[\because a^n + b^n = (a + b)^n]$

24. c. $a\alpha^2 + c = -b\alpha$, $a\alpha + b = -\dfrac{c}{\alpha}$

Hence, the given expression is

$$\frac{b}{c}(\alpha^2 + \beta^2) = \frac{b(b^2 - 2ac)}{a^2 c}$$

25. a. We have,

$x_1 x_2 = 4$

$\Rightarrow$ $x_2 = \dfrac{4}{x_1}$

$\therefore$ $\dfrac{x_1}{x_1 - 1} + \dfrac{x_2}{x_2 - 1} = 2$

$\Rightarrow$ $\dfrac{x_1}{x_1 - 1} + \dfrac{\frac{4}{x_1}}{\frac{4}{x_1} - 1} = 2$

$\Rightarrow$ $\dfrac{x_1}{x_1 - 1} + \dfrac{4}{4 - x_1} = 2$

$\Rightarrow \quad 4x_1 - x_1^2 + 4x_1 - 4 = 2(x_1 - 1)(4 - x_1)$

$\Rightarrow \quad x_1^2 - 2x_1 + 4 = 0$

$\Rightarrow \quad x^2 - 2x + 4 = 0$

26. b. The discriminants of the given equations are $D_1 = a^2 + 12b$, $D_2 = c^2 - 4b$ and $D_3 = d^2 - 8b$.

$\therefore \quad D_1 + D_2 + D_3 = a^2 + c^2 + d^2 \geq 0$

Hence, at least one of D_1, D_2, D_3 is non-negative. Therefore, the equation has at least two real roots.

27. d. α, β are roots of $x^2 + px + q = 0$. Hence,

$\alpha + \beta = -p$

$\alpha\beta = q$

Now,

α^4, β^4 are roots $x^2 - px + q = 0$. Hence,

$\alpha^4 + \beta^4 = r, \alpha^4\beta^4 = q$

Now, for equation $x^2 - 4qx + 2q^2 - r = 0$, product of roots is

$2q^2 - r = 2(\alpha\beta)^2 - (\alpha^4 + \beta^4)$

$= -(\alpha^2 - \beta^2)^2$

< 0

As product of roots is negative, so the roots must be real.

28. b. $x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2$

$= (x^2 + x + 1)(x^2 - x + 1)$

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \neq 0 \ \forall\, x$$

Therefore, we can cancel this factor and we get

$(a - 1)(x^2 - x + 1) = (a + 1)(x^2 - x + 1)$

or

$x^2 - ax + 1 = 0$

It has real and distinct roots if $D = a^2 - 4 > 0$.

29. b. Let D_1 and D_2 be discriminants of $x^2 + b_1x + c_1 = 0$ and $x^2 + b_2x + c_2 = 0$, respectively. Then,

$D_1 + D_2 = b_1^2 - 4c_1 + b_2^2 - 4c_2$

$= (b_1^2 + b_2^2) - 4(c_1 + c_2)$

$= b_1^2 + b_2^2 - 2b_1b_2 \quad [\because b_1 b_2 = 2(c_1 + c_2)]$

$= (b_1 - b_2)^2 \geq 0$

$\Rightarrow \quad D_1 \geq 0$ or $D_2 \geq 0$ or D_1 and D_2 both are positive

Hence, at least one of the equations has real roots.

30. c. Discriminant $D = (2m - 1)^2 - 4(m - 2)m = 4m + 1$ must be perfect square. Hence,

$4m + 1 = k^2$, say for some $k \in I$

$$\Rightarrow \quad m = \frac{(k-1)(k+1)}{4}$$

Clearly, k must be odd. Let $k = 2n + 1$.

$$\therefore \quad m = \frac{2n(2n+2)}{4} = = n(n + 1),\ n \in I$$

31. c. $A = a(b - c)(a + b + c)$

$B = b(c - a)(a + b + c)$

$C = c(a - b)(a + b + c)$

Now,

$Ax^2 + Bx + C = 0$

$\Rightarrow \quad (a + b + c)\{a(b - c)x^2 + b(c - a)x + c(a - b)\} = 0$

Given that roots are equal. Hence,

$D = 0$

$\Rightarrow \quad b^2(c - a)^2 - 4ac(b - c)(a - b) = 0$

$\Rightarrow \quad b^2c^2 - 2ab^2c + b^2a^2 - 4a^2bc + 4acb^2 + 4a^2c^2 - 4abc^2 = 0$

$\Rightarrow \quad b^2c^2 + b^2a^2 + 4a^2c^2 + 2ab^2c - 4a^2bc - 4abc^2 = 0$

$\Rightarrow \quad (bc + ab - 2ac)^2 = 0$

$\Rightarrow \quad bc + ab = 2ac$

$$\Rightarrow \quad \frac{1}{a} + \frac{1}{c} = \frac{2}{b}$$

$\Rightarrow \quad a, b, c$ are in H.P.

32. b. Correct equation is

$x^2 + 13x + q = 0 \quad (1)$

Incorrect equation is

$x^2 + 17x + q = 0 \quad (2)$

Given that roots of Eq. (1) are -2 and -15. Therefore, product of the roots of incorrect equation is $q = (-2)(-15) = 30$. From (1), the correct equation is

$x^2 + 13x + 30 = 0$

$\therefore \quad x = -3, -10$

33. b. Given,

$a(p + q)^2 + 2bpq + c = 0$ and $a(p + r)^2 + 2bpr + c = 0$

$\Rightarrow q$ and r satisfy the equation $a(p + x)^2 + 2bpx + c = 0$

$\Rightarrow q$ and r are the roots of

$ax^2 + 2(ap + bp)x + c + ap^2 = 0$

$$\Rightarrow \quad qr = \text{product of roots} = \frac{c + ap^2}{a} = p^2 + \frac{c}{a}$$

34. b. Multiplying the given equation by c/a^3, we get

$$\frac{b^2c^2}{a^3}x^2 - \frac{b^2c}{a^2}x + c = 0$$

$$\Rightarrow \quad a\left(\frac{bc}{a^2}x\right)^2 - b\left(\frac{bc}{a^2}\right)x + c = 0$$

$$\Rightarrow \quad \frac{bc}{a^2}x = \alpha, \beta$$

$\Rightarrow \quad (\alpha + \beta)\alpha\beta\, x = \alpha, \beta$

$$\Rightarrow \quad x = \frac{1}{(\alpha+\beta)\alpha}, \frac{1}{(\alpha+\beta)\beta}$$

35. c. Since α is root of all equations

$a\alpha^2 + 2b\alpha + c = 0$

$2b\alpha^2 + c\alpha + \alpha = 0$

$c\alpha^2 + a\alpha + 2b = 0$

Adding we get $(a + 2b + c)(\alpha^2 + \alpha + 1) = 0$

$a + 2b + c \neq 0 \quad$ as $\quad a, b, c > 0$

$\Rightarrow \quad \alpha^2 + \alpha + 1 = 0 \quad$ or $\quad \alpha^2 + \alpha = -1$

36. a. Given equation is

$x^2 - (y + 4)x + y^2 - 4y + 16 = 0$

Since x is real, so,

$D \geq 0$

$\Rightarrow \quad (y + 4)^2 - 4(y^2 - 4y + 16) \geq 0$

$\Rightarrow \quad -3y^2 + 24y - 48 \geq 0$

$\Rightarrow \quad y^2 - 8y + 16 \leq 0$

$\Rightarrow \quad (y - 4)^2 \leq 0$

$\Rightarrow \quad y - 4 = 0$

$\Rightarrow \quad y = 4$

Since the equation is symmetric in x and y, therefore $x = 4$ only.

37. a. α, β are roots of $ax^2 + bx + c = 0$. Hence,

$$\alpha + \beta = -\frac{b}{a} \quad (1)$$

$$\alpha\beta = \frac{c}{a} \quad (2)$$

α^2, β^2 are roots of $a^2x^2 + b^2x + c^2 = 0$. Hence,

$$\alpha^2 + \beta^2 = -\frac{b^2}{a^2} \quad (3)$$

$$\alpha^2\beta^2 = \frac{c^2}{a^2} \quad (4)$$

Now, from (3),

$$(\alpha + \beta)^2 - 2\alpha\beta = -\frac{b^2}{a^2}$$

$$\Rightarrow \left(\frac{-b}{a}\right)^2 - 2\frac{c}{a} = \frac{-b^2}{a^2}$$

$$\Rightarrow 2\frac{b^2}{a^2} = \frac{2c}{a}$$

$\Rightarrow$ $b^2 = ac \Rightarrow a, b, c$ are in G.P.

38. c. We have,

$$\frac{k+1}{k} + \frac{k+2}{k+1} = -\frac{b}{a} \quad (1)$$

and

$$\frac{k+1}{k} \cdot \frac{k+2}{k+1} = \frac{c}{a}$$

$$\Rightarrow \frac{k+2}{k} = \frac{c}{a} \text{ or } \frac{2}{k} = \frac{c}{a} - 1 = \frac{c-a}{a} \text{ or } k = \frac{2a}{c-a} \quad (2)$$

Now, eliminate k. Putting the value of k in Eq. (1), we get

$$\frac{c+a}{2a} + \frac{2c}{c+a} = -\frac{b}{a}$$

$\Rightarrow$ $(c + a)^2 + 4ac = -2b(a + c)$

$\Rightarrow$ $(a + c)^2 + 2b(a + c) = -4ac$

Adding b^2 to both sides, we have

$(a + b + c)^2 = b^2 - 4ac$

39. c. We have,

$$\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$$

$$\alpha + h + \beta + h = -\frac{q}{p}, (\alpha + h)(\beta + h) = \frac{r}{p}$$

$$\Rightarrow \alpha + \beta + 2h = -\frac{q}{p}$$

$$\Rightarrow -\frac{b}{a} + 2h = \frac{-q}{p} \qquad [\because a + b = -\frac{b}{a}]$$

$$\Rightarrow h = \frac{1}{2}\left(\frac{b}{a} - \frac{q}{p}\right)$$

40. a. Given, α, β are roots of the equation $(x - a)(x - b) + c = 0$.

Then, by factor theorem,

$(x - a)(x - b) + c = (x - \alpha)(x - \beta)$

Replacing x by $x - c$,

$(x - c - a)(x - c - b) + c = (x - c - \alpha)(x - c - \beta)$

$\Rightarrow (x - c - \alpha)(x - c - \beta) - c = [x - (c + a)][x - (c + b)]$

Then, again by factor theorem roots of the equation $(x - c - \alpha)(x - c - \beta) - c = 0$ are $a + c$ and $b + c$.

41. b. $ax^2 - bx + c = 0$

$$\alpha + \beta = \frac{b}{a}, \alpha\beta = \frac{c}{a}$$

Also,

$(a + cy)^2 = b^2y$

$\Rightarrow$ $c^2y^2 - (b^2 - 2ac)y + a^2 = 0$

$$\Rightarrow \left(\frac{c}{a}\right)^2 y^2 - \left(\left(\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)\right)y + 1 = 0$$

$\Rightarrow$ $(\alpha\beta)^2 y^2 - (\alpha^2 + \beta^2)y + 1 = 0$

$\Rightarrow$ $y^2 - (\alpha^{-2} + \beta^{-2})y + \alpha^{-2}\beta^{-2} = 0$

$\Rightarrow$ $(y - \alpha^{-2})(y - \beta^{-2}) = 0$

Hence the roots are α^{-2}, β^{-2}.

42. c. Let the roots be α, β.

$\therefore$ $\alpha + \beta = -2a$ and $\alpha\beta = b$

Given,

$|\alpha - \beta| \le 2m$

$\Rightarrow$ $|\alpha - \beta|^2 \le (2m)^2$

$\Rightarrow$ $(\alpha + \beta)^2 - 4ab \le 4m^2$

$\Rightarrow$ $4a^2 - 4b \le 4m^2$

$\Rightarrow$ $a^2 - m^2 \le b$ and discriminant $D > 0$ or $4a^2 - 4b > 0$

$\Rightarrow$ $a^2 - m^2 \le b$ and $b < a^2$

Hence, $b \in [a^2 - m^2, a^2)$.

43. c. Let roots of the equation $ax^2 + 2bx + c = 0$ be α and β and roots of the equation $px^2 + 2qx + r = 0$ be γ and δ. Given,

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \Rightarrow \frac{\alpha}{\gamma} = \frac{\beta}{\delta}$$

$$\Rightarrow \frac{\alpha + \beta}{\gamma + \delta} = \sqrt{\frac{\alpha\beta}{\gamma\delta}}$$

$$\Rightarrow \frac{-\frac{2b}{a}}{-\frac{2q}{p}} = \sqrt{\frac{\frac{c}{a}}{\frac{r}{p}}}$$

$$\Rightarrow \frac{b^2}{ac} = \frac{q^2}{pr}$$

44. a. Let α and α^2 be the roots of $x^2 - x - k = 0$. Then,

$\alpha + \alpha^2 = 1$ and $\alpha^3 = -k$

$\Rightarrow$ $(-k)^{1/3} + (-k)^{2/3} = 1$

$\Rightarrow$ $-k^{1/3} + k^{2/3} = 1$

$\Rightarrow$ $(k^{2/3} - k^{1/3})^3 = 1$

$\Rightarrow$ $k^2 - k - 3k(k^{2/3} - k^{1/3}) = 1$

$\Rightarrow$ $k^2 - k - 3k(1) = 1$

$\Rightarrow$ $k^2 - 4k - 1 = 0$

$\Rightarrow$ $k = 2 \pm \sqrt{5}$

45. d. Here,

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$$

$$= \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2(\alpha\beta)^2$$

$$= \left(p^2 + \frac{1}{p^2}\right)^2 - \frac{1}{2p^4}$$

$$= p^4 + \frac{1}{2p^4} + 2$$

$$= \left(p^2 - \frac{1}{\sqrt{2}p^2}\right)^2 + 2 + \sqrt{2} \ge 2 + \sqrt{2}$$

Thus, the minimum value of $\alpha^4 + \beta^4$ is $2 + \sqrt{2}$.

46. b. We have,

$$x = \frac{-1 \pm \sqrt{1 - 4a^2(1 - a^2)}}{2a^2}$$

$$= \frac{-1 \pm (2a^2 - 1)}{2a^2}$$

$$= 1 - \frac{1}{a^2} \text{ or } - a^2$$

$$\Rightarrow \quad \beta^2 = 1 - \frac{1}{a^2}$$

47. c. $A_{n+1} = \alpha^{n+1} + \beta^{n+1}$

$= \alpha^{n+1} + \alpha^n\beta + \beta^{n+1} + \alpha\beta^n - \alpha^n\beta - \alpha\beta^n$

$= \alpha^n(\alpha + \beta) + \beta^n(\beta + \alpha) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$

$= \alpha^n(\alpha + \beta) + \beta^n(\beta + \alpha) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$

$= (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$

$= aA_n - bA_{n-1}$

48. a. Let α be the root of $x^2 - x + m = 0$ and 2α be the root of $x^2 - 3x + 2m = 0$. Then,

$\alpha^2 - \alpha + m = 0$ and $4\alpha^2 - 6\alpha + 2m = 0$

Eliminating α, $m^2 = -2m \Rightarrow m = 0, m = -2$

49. a. By observation $x = -2$ satisfies equation $x^3 + 3x^2 + 3x + 2 = 0$

then we have $(x + 2)(x^2 + x + 1) = 0$

$x^2 + x + 1 = 0$ has non-real roots.

Since non-real roots occur in conjugate pair, $x^2 + x + 1 = 0$ and $ax^2 + bx + c = 0$ are identical

$\Rightarrow \quad a = b = c$

50. b. Given equations are

$x^3 + ax + 1 = 0$

or

$x^4 + ax^2 + x = 0 \qquad (1)$

and

$x^4 + ax^2 + 1 = 0 \qquad (2)$

From (1) – (2), we get $x = 1$. Thus, $x = 1$ is the common roots. Hence,

$1 + a + 1 = 0 \Rightarrow a = -2$

51. a. $x = \sqrt[3]{7} + \sqrt[3]{49}$

$\Rightarrow \quad x^3 = 7 + 49 + 3\sqrt[3]{7}\ \sqrt[3]{49}(\sqrt[3]{7} + \sqrt[3]{49}) = 56 + 21x$

$\Rightarrow \quad x^3 - 21x - 56 = 0$

Therefore, the product of roots is 56.

52. c. Since $\alpha, \beta, \gamma, \sigma$ are the roots of the given equation, therefore

$x^4 + 4x^3 - 6x^2 + 7x - 9 = (x - \alpha)(x - \beta)(x - \gamma)(x - \sigma)$

Putting $x = i$ and then $x = -i$, we get

$1 - 4i + 6 + 7i - 9 = (i - \alpha)(i - \beta)(i - \gamma)(i - \sigma)$

and

$1 + 4i + 6 - 7i - 9 = (-i - \alpha)(-i - \beta)(-i - \gamma)(-i - \sigma)$

Multiplying these two equations, we get

$(-2 + 3i)(-2 - 3i) = (1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$

$\Rightarrow \quad 13 = (1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$

53. b. If $[m_r, (1/m_r)]$ satisfy the given equation $x^2 + y^2 + 2gx + 2fy + c = 0$, then

$$m_r^2 + \frac{1}{m_r^2} + 2gm_r + \frac{2f}{m_r} + c = 0$$

$\Rightarrow \quad m_r^4 + 2gm_r^3 + cm_r^2 + 2fm_r + 1 = 0$

Now, roots of given equation are m_1, m_2, m_3, m_4. The product of roots

$$m_1 m_2 m_3 m_4 = \frac{\text{constant term}}{\text{coefficient of } m_r^4} = \frac{1}{1} = 1$$

54. a. Clearly,

$x^n - 1 = (x - 1)(x - a_1)(x - a_2) \cdots (x - a_{n-1})$

$\Rightarrow \quad \dfrac{x^n - 1}{x - 1} = (x - a_1)(x - a_2) \cdots (x - a_{n-1})$

$\Rightarrow \quad 1 + x + x^2 + \cdots + x^{n-1} = (x - a_1)(x - a_2) \cdots (x - a_{n-1})$

$\Rightarrow \quad n = (1 - a_1)(1 - a_2) \cdots (1 - a_{n-1}) \qquad$ [putting $x = 1$]

55. b. $\tan\theta_1 + \tan\theta_2 + \tan\theta_3 = (a + 1)$

$\Sigma \tan\theta_1 \tan\theta_2 = (b - a)$

$\tan\theta_1 \tan\theta_2 \tan\theta_3 = b$

$$\therefore \quad \tan(\theta_1 + \theta_2 + \theta_3) = \frac{\Sigma \tan\theta_1 - \Pi \tan\theta_1}{1 - \Sigma \tan\theta_1 \tan\theta_2}$$

$$= \frac{a + 1 - b}{1 - (b - a)} = 1$$

$$\Rightarrow \quad \theta_1 + \theta_2 + \theta_3 = \frac{\pi}{4}$$

56. a. $\Sigma\alpha = 1, \Sigma\alpha\beta = 0, \alpha\beta\gamma = 1$

$$\Sigma\frac{1 + \alpha}{1 - \alpha} = -\Sigma\frac{-\alpha + 1 - 2}{1 - \alpha} = \Sigma\left(\frac{2}{1 - \alpha} - 1\right)$$

$$= 2\Sigma\frac{1}{1 - \alpha} - 3$$

Now,

$$\frac{1}{(x - \alpha)} + \frac{1}{(x - \beta)} + \frac{1}{(x - \gamma)} = \frac{3x^2 - 2x}{x^3 - x^2 - 1}$$

$$\Rightarrow \quad \frac{1}{1 - \alpha} + \frac{1}{1 - \beta} + \frac{1}{1 - \gamma} = \frac{3 - 2}{1 - 1 - 1} = -1$$

$$\Rightarrow \quad \frac{1 + \alpha}{1 - \alpha} = -5$$

57. d. Equation $8x^3 + 1001x + 2008 = 0$ has roots r, s and t.

$$r + s + r = 0,\ rst = -\frac{2008}{8} = -251$$

Now, let $r + s = A, s + t = B, t + r = C$.

$\therefore \quad A + B + C = 2(r + s + t) = 0$

Hence,

$A^3 + B^3 + C^3 = 3ABC$

$\therefore \quad (r + s)^3 + (s + t)^3 + (t + r)^3$

$= 3(r + s)(s + t)(t + r)$

$= 3(r + s + t - t)(s + t + r - r)(t + r + s - s)$

$= -3rst$ (as $r + s + t = 0$)

$= 3(251) = 753$

58. b. Let,

$$\frac{x}{x^2 - 5x + 9} = y$$

$\Rightarrow \quad yx^2 - 5yx + 9y = x$

$\Rightarrow$ $yx^2 - (5y+1)x + 9y = 0$

Now, x is real, so

$D \geq 0$

$\Rightarrow$ $(-(5y+1))^2 - 4 \cdot y \cdot (9y) \geq 0$

$\Rightarrow$ $-11y^2 + 10y + 1 \geq 0$

$\Rightarrow$ $11y^2 - 10y - 1 \leq 0$

$\Rightarrow$ $(11y+1)(y-1) \leq 0$

$\Rightarrow$ $-\frac{1}{11} \leq y \leq 1$

59. b. Let,

$$y = \frac{3x^2+9x+17}{3x^2+9x+7}$$

$\Rightarrow$ $3x^2(y-1) + 9x(y-1) + 7y - 17 = 0$

Since x is real, so,

$D \geq 0$

$\Rightarrow$ $81(y-1)^2 - 4 \times 3(y-1)(7y-17) \geq 0$

$\Rightarrow$ $(y-1)(y-41) \leq 0 \Rightarrow 1 \leq y \leq 41$

Therefore, the maximum value of y is 41.

60. a. $D = b^2 - 4a < 0 \Rightarrow a > 0$

Therefore the graph is concave upwards.

$f(x) > 0, \forall x \in R$

$\Rightarrow$ $f(-1) > 0$

$\Rightarrow$ $a + b + 1 > 0$

61. c. We know that $ax^2 + bx + c \geq 0, \forall x \in R$,

if $a > 0$ and $b^2 - 4ac \leq 0$. So,

$$mx - 1 + \frac{1}{x} \geq 0 \Rightarrow \frac{mx^2 - x + 1}{x} \geq 0$$

$\Rightarrow$ $mx^2 - x + 1 \geq 0$ as $x > 0$.

Now,

$mx^2 - x + 1 \geq 0$ if $m > 0$ and $1 - 4m \leq 0$

$\Rightarrow$ $m > 0$ and $m \geq 1/4$

Thus, the minimum value of m is 1/4.

62. b. Let $f(x) = ax^2 + bx + c$ be a quadratic expression such that $f(x) > 0$ for all $x \in R$. Then, $a > 0$ and $b^2 - 4ac < 0$. Now,

$g(x) = f(x) + f'(x) + f''(x)$

$\Rightarrow$ $g(x) = ax^2 + x(b + 2a) + (b + 2a + c)$

Discriminant of $g(x)$ is

$D = (b+2a)^2 - 4a(b + 2a + c)$

$= b^2 - 4a^2 - 4ac$

$= (b^2 - 4ac) - 4a^2$

$< 0 \quad (\because b^2 - 4ac < 0)$

Therefore, $g(x) > 0$ for all $x \in R$.

63. b. Here, $ax^2 - bx + c^2 = 0$ does not have real roots. So,

$D < 0 \Rightarrow b^2 - 4ac^2 < 0 \Rightarrow a > 0$

Therefore, $f(x)$ is always positive. So,

$f(2) > 0 \Rightarrow 4a - 2b + c^2 > 0$

64. a. $x_1(x - x_2)^2 + x_2(x - x_1)^2 = 0$

$\Rightarrow$ $x^2(x_1 + x_2) - 4x\,x_1x_2 + x_1x_2(x_1 + x_2) = 0$

$D = 16(x_1x_2)^2 - 4x_1x_2(x_1 + x_2)^2 > 0 \ (\because x_1x_2 < 0)$

The product of roots is $x_1x_2 < 0$. Thus, the roots are real and of opposite signs.

65. d. Given that $a < b < c < d$. Let

$f(x) = (x-a)(x-c) + 2(x-b)(x-d)$

$\Rightarrow$ $f(b) = (b-a)(b-c) < 0$

and

$f(d) = (d-a)(d-c) > 0$

Hence, $f(x) = 0$ has one root in (b, d). Also, $f(a)f(c) < 0$. So the other root lies in (a, c). Hence, roots of the equation are real and distinct.

66. c. $x^2 - (a-3)x + a = 0$

$\Rightarrow$ $D = (a-3)^2 - 4a$

$= a^2 - 10a + 9$

$= (a-1)(a-9)$

Case I:

Both the roots are greater than 2.

$$D \geq 0, f(2) > 0, -\frac{B}{2A} > 2$$

$\Rightarrow$ $(a-1)(a-9) \geq 0; 4 - (a-3)2 + a > 0; \frac{a-3}{2} > 2$

$\Rightarrow$ $a \in (-\infty, 1] \cup [9, \infty); a < 10; a > 7$

$\Rightarrow$ $a \in [9, 10)$ (1)

Case II:

One root is greater than 2 and the other root is less than or. equal to 2. Hence,

$f(2) \leq 0$

$\Rightarrow$ $4 - (a-3)2 + a \leq 0$

$\Rightarrow$ $a \geq 10$ (2)

From (1) and (2),

$a \in [9, 10) \cup [10, \infty) \Rightarrow a \in [9, \infty)$

67. a. Suppose $a \neq 0$. We rewrite $f(x)$ as follows:

$$f(x) = a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

$$f\left(-\frac{b}{2a} + i\right) = a\left\{\left(-\frac{b}{2a} + i + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

$$= a\left\{-1 + \frac{4ac - b^2}{4a^2}\right\}, \text{ which is a real number}$$

This is against the hypothesis. Therefore, $a = 0$.

68. c. The given equation is

$x^2 - 2mx + m^2 - 1 = 0$

$\Rightarrow$ $(x - m)^2 - 1 = 0$

$\Rightarrow$ $(x - m + 1)(x - m - 1) = 0$

$\Rightarrow$ $x = m - 1, m + 1$

From given condition,

$m - 1 > -2$ and $m + 1 < 4$

$\Rightarrow$ $m > -1$ and $m < 3$

Hence, $-1 < m < 3$.

69. b. Note that coefficient of x^2 is $(4p - p^2 - 5) < 0$. Therefore the graph is concave downward. According to the question, 1 must lie between the roots. Hence,

$$f(1) > 0$$
$$\Rightarrow \quad 4p - p^2 - 5 - 2p + 1 + 3p > 0$$
$$\Rightarrow \quad -p^2 + 5p - 4 > 0$$
$$\Rightarrow \quad p^2 - 5p + 4 < 0$$

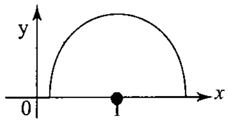


Fig. 1.70

$$\Rightarrow \quad (p-4)(p-1) < 0$$
$$\Rightarrow \quad 1 < p < 4$$
$$\Rightarrow \quad p \in \{2, 3\}$$

70. b. $\tan x = \dfrac{a - 4 - \sqrt{(a-4)^2 - 4(4-2a)}}{2}$

$$= \frac{a-4-a}{2} = a-2, -2$$

$\therefore \quad \tan x = a - 2 \qquad (\because \tan x \neq -2)$

$\because \quad x \in \left[0, \dfrac{\pi}{4}\right]$

$\therefore \quad 0 \le a - 2 \le 1$

$\Rightarrow \quad 2 \le a \le 3$

71. a. Clearly, $f(-1) > 0, f(2) < 0$. Now,

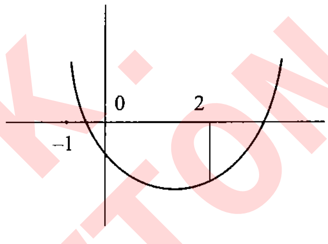


Fig. 1.71

$$f(0) = -4 < 0$$
$$\Rightarrow \quad f(-1) = 1 - a - 4 > 0 \text{ and } f(2) = 4 + 2a - 4 < 0$$
$$\Rightarrow \quad a < -3 \text{ and } a < 0$$
$$\Rightarrow \quad a \in (-\infty, -3)$$

72. b. Let,

$$f(x) = ax^2 + x + c - a$$
$$f(1) = c + 1 > 0 \qquad (\because c > -1)$$

Therefore, given expression is positive $\forall\, x \in R$. So,

$$f\left(\frac{1}{2}\right) > 0$$
$$\Rightarrow \quad \frac{a}{4} + \frac{1}{2} + c - a > 0$$
$$\Rightarrow \quad 4c - 3a + 2 > 0$$
$$\Rightarrow \quad 4c + 2 > 3a$$

73. b.

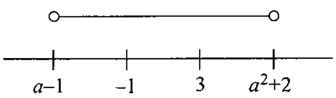


Fig. 1.72

We have,

$$a - 1 \le -1 \text{ and } a^2 + 2 \ge 3$$
$$a \le 0 \text{ and } a^2 \ge 1$$

Hence, $a \le -1$.

74. a. $x^2 + 2x - n = 0 \Rightarrow (x+1)^2 = n + 1$

$\Rightarrow \quad x = -1 \pm \sqrt{n+1}$

Thus, $n + 1$ should be a perfect square. Now,

$$n \in [5, 100] \Rightarrow n + 1 \in [6, 101]$$

Perfect square values of $n + 1$ are 9, 16, 25, 36, 49, 64, 81, 100. Hence, number of values is 8.

75. d. $x^2 - x - a = 0$, $D = 1 + 4a$ = odd

D must be perfect square of some odd integer. Let

$$D = (2\lambda + 1)^2$$
$$\Rightarrow \quad 1 + 4a = 1 + 4\lambda^2 + 4\lambda$$
$$\Rightarrow \quad a = \lambda(\lambda + 1).$$

Now,

$$a \in [6, 100]$$
$$\Rightarrow \quad a = 6, 12, 20, 30, 42, 56, 72, 90$$

Thus a can attain eight different values.

76. a. $x^2 - (a+1)x + a - 1 = 0$

$\Rightarrow \quad (x - a)(x - 1) = 1$

Now, $a \in I$ and we want x to be an integer. Hence,

$$x - a = 1, x - 1 = 1 \text{ or } x - a = -1, x - 1 = -1$$
$$\Rightarrow \quad a = 1 \text{ in both cases}$$

77. b. For given situation, $x^2 - (k-2)x + k^2 = 0$ and $x^2 + kx + 2k - 1 = 0$ should have both roots common or each should have equal roots. If both roots are common, then

$$\frac{1}{1} = \frac{-(k-2)}{k} = \frac{k^2}{2k-1}$$
$$\Rightarrow \quad k = -k + 2 \text{ and } 2k - 1 = k^2 \Rightarrow k = 1$$

If both the equations have equal roots, then

$$(k-2)^2 - 4k^2 = 0 \quad \text{and} \quad k^2 - 4(2k-1) = 0$$
$$\Rightarrow \quad (3k-2)(-k-2) = 0 \text{ and } k^2 - 8k + 4 = 0 \quad \text{(no common value)}$$

Therefore, $k = 1$ is the only possible value.

78. c. We have,

$$\alpha + \beta = -p \text{ and } \alpha\beta = q \tag{1}$$

Also, since α, β are the roots of $x^{2n} + p^n x^n + q^n = 0$, we have

$$\alpha^{2n} + p^n\alpha^n + q^n = 0 \text{ and } \beta^{2n} + p^n\beta^n + q^n = 0$$

Subtracting the above relations, we get

$$(\alpha^{2n} - \beta^{2n}) + p^n(\alpha^n - \beta^n) = 0$$
$$\therefore \quad \alpha^n + \beta^n = -p^n \tag{2}$$

Given, α/β or β/α is a root of $x^n + 1 + (x+1)^n = 0$. So,

$$(\alpha/\beta)^n + 1 + [(\alpha/\beta) + 1]^n = 0$$
$$\Rightarrow \quad (\alpha^n + \beta^n) + (\alpha + \beta)^n = 0$$
$$\Rightarrow \quad -p^n + (-p)^n = 0 \quad \text{[Using (1) and (2)]}$$

It is possible only when n is even.

79. a. Since R.H.S. is an even integer. So L.H.S. is also an even integer. So, either both x and y are even integers, or both of them are odd integers. Now,

$$x^4 - y^4 = (x-y)(x+y)(x^2+y^2)$$

$\Rightarrow$ $x - y, x + y, x^2 + y^2$ must be an even integer

Therefore, $(x - y)(x + y)(x^2 + y^2)$ must be divisible by 8. But R.H.S. is not divisible by 8. Hence, the given equation has no solution.

80. a. $xy = 2(x + y) \Rightarrow y(x - 2) = 2x$

$\therefore \quad y = \dfrac{2x}{x-2} = 2 + \dfrac{4}{x-2} \Rightarrow x = 3, 4 \ (x \neq 6 \text{ as } x < y)$

By trial, $x = 3, 4, 6$. Then $y = 6, 4, 3$. But $x \le y$. Therefore, $x = 3, 4$ and $y = 6, 4$ are two solutions.

81. a. We have,

$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\beta\gamma + \gamma\alpha + \alpha\beta)$

$\Rightarrow \quad 4 = 6 + 2(\beta\gamma + \gamma\alpha + \alpha\beta)$

$\Rightarrow \quad \beta\gamma + \gamma\alpha + \alpha\beta = -1$

Also,

$\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta)$

$\Rightarrow \quad 8 - 3\alpha\beta\gamma = 2(6 + 1)$

$\Rightarrow \quad 3\alpha\beta\gamma = 8 - 14 = -6$ or $\alpha\beta\gamma = -2$

Now,

$(\alpha^2 + \beta^2 + \gamma^2)^2 = \Sigma\alpha^4 + 2\Sigma\alpha^2\beta^2$

$= \Sigma\alpha^4 + 2[(\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\Sigma\alpha)]$

$\Rightarrow \quad \Sigma\alpha^4 = 36 - 2[(-1)^2 - 2(-2)(2)] = 18$

82. d. $(x + a)(x + 1991) + 1 = 0$

$\Rightarrow \quad (x + a)(x + 1991) = -1$

$\Rightarrow \quad (x + a) = 1$ and $x + 1991 = -1$

$\Rightarrow \quad a = 1993$

or $\quad x + a = -1$ and $x + 1991 = 1 \quad \Rightarrow \quad a = 1989$

83. b. Let $f(x) = -3 + x - x^2$. Then $f(x) < 0$ for all x, because coefficient of x^2 is less than 0 and $D < 0$. Thus, L.H.S. of the given equation is always positive whereas the R.H.S. is always less than zero. Hence, there is no solution.

84. c. Put $ab + bc + ca = t$. Now,

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2t$

$\Rightarrow \quad (a + b + c)^2 = 1 + 2t$

$\Rightarrow \quad 1 + 2t \ge 0$

$\Rightarrow \quad -\dfrac{1}{2} \le t$

Again,

$(a - b)^2 + (b - c)^2 + (c - a)^2 = 2 - 2t$

$\Rightarrow \quad 2 - 2t \ge 0$

$\Rightarrow \quad t \le 1$

$\Rightarrow \quad -\dfrac{1}{2} \le t \ \le t \le 1$

85. d. $\cot^4 x - 2(1 + \cot^2 x) + a^2 = 0$

$\Rightarrow \quad \cot^4 x - 2\cot^2 x + a^2 - 2 = 0$

$\Rightarrow \quad (\cot^2 x - 1)^2 = 3 - a^2$

Now, for at least one solution

$3 - a^2 \ge 0$

$\Rightarrow \quad a^2 - 3 \le 0$

$\therefore \quad a \in [-\sqrt{3}, \sqrt{3}]$

Integral values are $-1, 0, 1$.

$\therefore \quad$ sum $= 0$

86. b. $x = 3\cos\theta$; $y = 3\sin\theta$

$z = 2\cos\phi$; $t = 2\sin\phi$

$\therefore \quad 6\cos\theta\sin\phi - 6\sin\theta\cos\phi = 6$

$\Rightarrow \quad \sin(\phi - \theta) = 1$

$\Rightarrow \quad \phi = 90° + \theta$

$\Rightarrow \quad P = xz = -6\sin\theta\cos\theta = -3\sin 2\theta$

$\Rightarrow \quad P_{max} = 3$

87. a. The equation on simplifying gives

$$x(x - b)(x - c) + x(x - c)(x - a) + x(x - a)(x - b) - (x - a)(x - b)(x - c) = 0 \quad (1)$$

Let,

$$f(x) = x(x - b)(x - c) + x(x - c)(x - a) + x(x - a)(x - b) - (x - a)(x - b)(x - c)$$

We can assume without loss of generality that $a < b < c$. Now,

$f(a) = a(a - b)(a - c) > 0$

$f(b) = b(b - c)(b - a) < 0$

$f(c) = c(c - a)(c - b) > 0$

So, one root of (1) lies in (a, b) and one root in (b, c). Obviously the third root must also be real.

88. b. $\dfrac{(b^2 - 4ac)^2}{16a^2} < \dfrac{4}{1 + 4a^2} \quad (1)$

Now,

$\max(ax^2 + bx + c) = -\dfrac{b^2 - 4ac}{4a}$

Also,

$\dfrac{-2}{\sqrt{1 + 4a^2}} < -\dfrac{b^2 - 4ac}{4a} < \dfrac{2}{\sqrt{1 + 4a^2}} \quad$ [From (1)]

So, maximum value is always less than 2 (when $a \to 0$).

89. d.

$a = \dfrac{x^2 + 4}{|x|} - 3$

$= |x| + \dfrac{4}{|x|} - 3 = \left(\sqrt{|x|} - \dfrac{2}{\sqrt{|x|}}\right) + 1$

$\Rightarrow \quad a \ge 1$

90. d. Here, $f(x) = (2x - a)(2x - c) + (2x - b)$. So,

$f\left(\dfrac{a}{2}\right) = a - b,\ f\left(\dfrac{c}{2}\right) = c - b$

Now,

$f\left(\dfrac{a}{2}\right) f\left(\dfrac{c}{2}\right) = (a - b)(c - b) < 0 \ (a > b > c)$

Hence, exactly one of the roots lies between $c/2$ and $a/2$.

91. a. Since the equation $x^2 + ax + b = 0$ has distinct real roots and $x^2 + a|x| + b = 0$ has only one real root, so one root of the equation $x^2 + ax + b = 0$ will be zero and other root will be negative. Hence, $b = 0$ and $a > 0$.

Graph of $y = ax^2 + bx + c$ according to conditions given in question

Graph of $y = ax^2 + b|x| + c$

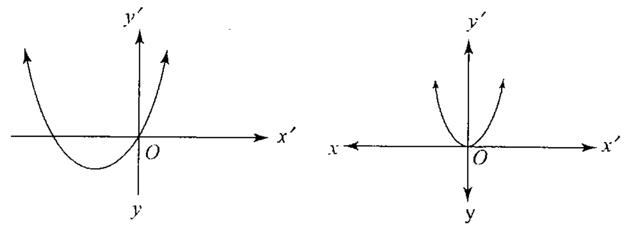


Fig. 1.73

92. c. The given equation is

$$2^{2x} + (a-1)2^{x+1} + a = 0$$

or

$$t^2 + 2(a-1)t + a = 0, \text{ where } 2^x = t$$

Now, $t = 1$ should lie between the roots of this equation.

$$\therefore \quad 1 + 2(a-1) + a < 0 \Rightarrow a < \frac{1}{3}$$

93. a.

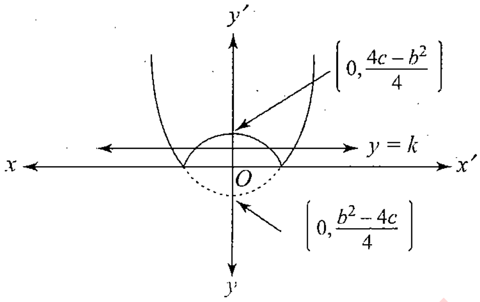


Fig. 1.74

For the equation to have four real roots, the line $y = k$ must intersect $y = |x^2 + bx + c|$ at four points.

$$\therefore \quad D > 0 \text{ and } k \in \left(0, \frac{-D}{4}\right)$$

94. d. $P(a) = P(b) = P(c) = P(d) = 3$

$\Rightarrow \quad P(x) = 3$ has a, b, c, d as its roots

$\Rightarrow \quad P(x) - 3 = (x-a)(x-b)(x-c)(x-d)\,Q(x)$

[$\because Q(x)$ has integral coefficient]

Given $P(e) = 5$, then

$$(e-a)(e-b)(e-c)(e-d)\,Q(e) = 5$$

This is possible only when at least three of the five integers $(e-a)$, $(e-b)$, $(e-c)$, $(e-d)$, $Q(e)$ are equal to 1 or -1. Hence, two of them will be equal, which is not possible. Since a, b, c, d are distinct integers, therefore $P(e) = 5$ is not possible.

95. d. $\sqrt{-x^2 + 10x - 16} < x - 2$

We must have

$$-x^2 + 10x - 16 \ge 0$$

$\Rightarrow \quad x^2 - 10x + 16 \le 0$

$\Rightarrow \quad 2 \le x \le 8 \qquad (1)$

Also,

$$-x^2 + 10x - 16 < x^2 - 4x + 4$$

$\Rightarrow \quad 2x^2 - 14x + 20 > 0$

$\Rightarrow \quad x^2 - 7x + 10 > 0$

$\Rightarrow \quad x > 5 \text{ or } x < 2 \qquad (2)$

From (1) and (2),

$$5 < x \le 8 \Rightarrow x = 6, 7, 8$$

96. c. $D > 0 \Rightarrow (a-3)^2 + 4(a+2) > 0$

$\Rightarrow \quad a^2 - 6a + 9 + 4a + 8 > 0$

$\Rightarrow \quad a^2 - 2a + 17 > 0$

$\Rightarrow \quad a \in R$

$$\therefore \quad \frac{a^2+1}{a^2+2} = 1 - \frac{1}{a^2+2} \ge \frac{1}{2}$$

97. c. Given,

$$(a-1)x^2 - (a+1)x + a - 1 \ge 0$$

$\Rightarrow \quad a(x^2 - x + 1) - (x^2 + x + 1) \ge 0$

$$\Rightarrow \quad a \ge \frac{x^2+x+1}{x^2-x+1}$$

$$= 1 + \frac{2x}{x^2-x+1}$$

$$= 1 + \frac{2}{x + \frac{1}{x} - 1} \qquad (1)$$

Let $y = x + 1/x$. Now, y is increasing in $[2, \infty)$. Hence,

$$1 + \frac{2}{x + \frac{1}{x} - 1} \in \left(1, \frac{7}{3}\right]$$

For all $x \ge 2$, Eq. (1) should be true. Hence, $a > 7/3$.

98. b. $x = 2 + \sqrt{3}$

$\Rightarrow \quad (x-2)^2 = 3$

$\Rightarrow \quad x^2 - 4x + 1 = 0 \qquad (1)$

$\Rightarrow \quad (x-2)^4 = 9$

$\Rightarrow \quad x^4 - 8x^3 + 24x^2 - 32x + 16 = 9$

$\Rightarrow \quad x^4 - 8x^3 + 18x^2 - 8x + 2 + 6(x^2 - 4x + 1) - 1 = 0$

Using (1), we get

$$x^4 - 8x^3 + 18x^2 - 8x + 2 = 1$$

Multiple Correct Answers Type

1. a, c. $2x^2 + 6xy + 5y^2 = 1 \qquad (1)$

Equation (1) can be rewritten as

$$2x^2 + (6y)x + 5y^2 - 1 = 0$$

Since x is real,

$$36y^2 - 8(5y^2 - 1) \ge 0$$

$\Rightarrow \quad y^2 \le 2$

$\Rightarrow \quad -\sqrt{2} \le y \le \sqrt{2}$

Equation (1) can also be rewritten as

$$5y^2 + (6x)y + 2x^2 - 1 = 0$$

Since y is real,

$$36x^2 - 20(2x^2 - 1) \ge 0$$

$\Rightarrow \quad 36x^2 - 40x^2 + 20 \ge 0$

$\Rightarrow \quad -4x^2 \ge -20$

$\Rightarrow \quad x^2 \le 5$

$\Rightarrow \quad -\sqrt{5} \le x \le \sqrt{5}$

2. b, c, d.

Given equation is $x^3 - ax^2 + bx - 1 = 0$. If roots of the equation be α, β, γ, then

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= a^2 - 2b$$

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$= b^2 - 2a$

$\alpha^2\beta^2\gamma^2 = 1$

So, the equation whose roots are $\alpha^2, \beta^2, \gamma^2$ is given by

$$x^3 - (a^2 - 2b)x^2 + (b^2 - 2a)x - 1 = 0$$

It is identical to

$$x^3 - ax^2 + bx - 1 = 0$$

$\Rightarrow \quad a^2 - 2b = a$ and $b^2 - 2a = b$

Eliminating b, we get

$$\frac{(a^2-a)^2}{4} - 2a = \frac{a^2-a}{2}$$

$\Rightarrow \quad a\{a(a-1)^2 - 8 - 2(a-1)\} = 0$

$\Rightarrow \quad a(a^3 - 2a^2 - a - 6) = 0$

$\Rightarrow \quad a(a-3)(a^2 + a + 2) = 0$

$\Rightarrow \quad a = 0$ or $a = 3$ or $a^2 + a + 2 = 0$

which gives $b = 0$ or $b = 3$ or $b^2 + b + 2 = 0$. So, $a = b = 0$ or $a = b = 3$ or a, b are roots of $x^2 + x + 2 = 0$.

3. a, b, c.

Since the roots of $ax^2 + bx + c = 0$ are non-real, so, $f(x) = ax^2 + bx + c$ will have same sign for every value of x. Hence,

$f(0) = c, f(1) = a + b + c, f(-1) = a - b + c$

$f(-2) = 4a - 2b + c$

$\Rightarrow \quad c(a + b + c) > 0, c(a - b + c) > 0, c(4a - 2b + c) > 0$

4. a, b.

We can write the given equation as

$$\frac{p}{2x} = \frac{(a+b)x + c(b-a)}{x^2 - c^2}$$

$\Rightarrow \quad p(x^2 - c^2) = 2(a+b)x^2 - 2c(a-b)x$

$\Rightarrow \quad (2a + 2b - p)x^2 - 2c(a-b)x + pc^2 = 0$

For this equation to have equal roots,

$c^2(a-b)^2 - pc^2(2a + 2b - p) = 0$

$\Rightarrow \quad (a-b)^2 - 2p(a+b) + p^2 = 0 \qquad [\because c^2 \neq 0]$

$\Rightarrow \quad [p - (a+b)]^2 = (a+b)^2 - (a-b)^2 = 4ab$

$\Rightarrow \quad p - (a+b) = \pm 2\sqrt{ab}$

$\Rightarrow \quad p = a + b \pm 2\sqrt{ab} = (\sqrt{a} \pm \sqrt{b})^2$

5. a, d.

Roots of $4x^2 - x - 1 = 0$ are irrational. So, one root common implies both roots are common. Therefore,

$$\frac{4}{3} = \frac{-1}{\lambda + \mu} = \frac{-1}{\lambda - \mu}$$

$$\Rightarrow \lambda = \frac{-3}{4}, \quad \mu = 0$$

6. a, b, c.

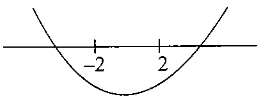


Fig. 1.75

$f(x) = ax^2 + bx + c$

$f(0) = c < 0, D > 0 \Rightarrow b^2 - 4ac > 0$

$f(1) < 0$ and $f(-1) < 0$

$\Rightarrow \quad a - |b| + c < 0$

$f(2) < 0$ and $f(-2) < 0$

$\Rightarrow \quad 4a - 2|b| + c < 0$

Nothing can be said about $f(3)$ or $f(-3)$, whether it is positive or negative.

7. a, b, d.

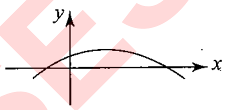


Fig. 1.76

From the graph,

$f(0) = c > 0 \qquad (1)$

Also, the graph is concave downward. Hence,

$a < 0 \qquad (2)$

Further, abscissa of the vertex,

$$\frac{b}{2a} \qquad (3)$$

From (1), (2), (3),

$ac < 0$, $ab < 0$ and $bc > 0$

8. c, d.

$$\cos x - y^2 - \sqrt{y - x^2 - 1} \geq 0 \qquad (1)$$

Now, $\sqrt{y - x^2 - 1}$ is defined when $y - x^2 - 1 \geq 0$ or $y \geq x^2 + 1$. So minimum value of y is 1. From (1),

$$\cos x - y^2 \geq \sqrt{y - x^2 - 1}$$

where $\cos x - y^2 \leq 0$ [as when $\cos x$ is maximum (=1) and y^2 is minimum (=1), so $\cos x - y^2$ is maximum]. Also,

$$\sqrt{y - x^2 - 1} \geq 0$$

Hence,

$$\cos x - y^2 = \sqrt{y - x^2 - 1} = 0$$

$\Rightarrow \quad y = 1$ and $\cos x = 1, y = x^2 + 1$

$\Rightarrow \quad x = 0, y = 1$

9. a, d.

$2^x = t$

$t^2 - 8t + 12 = 0$

$(t - 6)(t - 2) = 0$

$2^x = 6 \Rightarrow x = \log_2 6 = 1 + \dfrac{\log 3}{\log 2}$

$2^x = 2 \Rightarrow x = 1$

10. a, b, d.

Symmetric functions are those which do not change by interchanging α and β.

11. a, b, c.

$\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta . \operatorname{cosec}^2\theta$

Sum of the roots is equal to their product and the roots are real. Hence,

$$-\frac{b}{a} = \frac{c}{a}$$

$\Rightarrow \quad b + c = 0$

Also $\quad b^2 - 4ac \ge 0$

$\Rightarrow \quad c^2 - 4ac \ge 0$

$\Rightarrow \quad c(c - 4a) \ge 0$

$\Rightarrow \quad c - 4a \ge 0 \quad (\because \quad c > 0)$

Further

$b^2 + 4ab \ge 0$

$\Rightarrow \quad b + 4a \le 0 \quad (\because \quad b < 0)$

12. a, b, c.

$f(x) = Ax^2 + Bx + C$

$A = a + b - 2c = (a - c) + (b - c) > 0$

$\Rightarrow \quad A > 0$

Hence, the graph is concave upwards. Also, $x = 1$ is obvious solution; therefore, both roots are rational.

$$b + c - 2a = \underbrace{(b - a)}_{-\text{ve}} + \underbrace{(c - a)}_{-\text{ve}} < 0$$

$\Rightarrow \quad B < 0$

$\therefore \quad \text{vertex} = -\dfrac{B}{2A} > 0$

Hence, abscissa of the vertex is positive. Option (d) need not be correct as with $a = 5, b = 4, c = 2, P < 0$ and with $a = 6, b = 3, c = 2, P > 0$.

13. a, b, c.

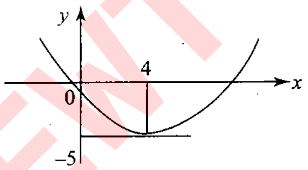


Fig. 1.77

From figure,

$a > 0$

$$-\frac{b}{2a} = 4 \Rightarrow -\frac{b}{2a} > 0$$

$\therefore \quad b < 0$

$f(0) = c < 0$

Also,

$$-\frac{b}{2a} = 4 \Rightarrow 8a + b = 0$$

14. a, c, d.

Let the roots be $a/r, a, ar$, where $a > 0, r > 1$. Now,

$a/r + a + ar = -p \qquad (1)$

$a(a/r) + a(ar) + (ar)(a/r) = q \qquad (2)$

$(a/r)(a)(ar) = 1 \qquad (3)$

$\Rightarrow \quad a^3 = 1$

$\Rightarrow \quad a = 1$

Hence, (c) is correct. From (1), putting $a = 1$, we get

$$-p - 3 > 0 \quad \left(\because r + \frac{1}{r} > 2\right)$$

$\Rightarrow \quad p < -3$

Hence, (b) is not correct. Also,

$1/r + 1 + r = -p \qquad (4)$

From (2), putting $a = 1$, we get

$1/r + r + 1 = q \qquad (5)$

From (4) and (5), we have

$-p = q \Rightarrow p + q = 0$

Hence, (a) is correct. Now, as $r > 1$

$a/r = 1/r < 1$

and

$ar = r > 1$

Hence, (d) is correct.

15. a, b.

Given, $(\sin\alpha)\, x^2 - 2x + b \ge 2$. Let $f(x) = (\sin\alpha)x^2 - 2x + b - 2$. Abscissa of the vertex is given by

$$x = \frac{1}{\sin\alpha} > 1$$

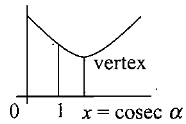


Fig. 1.78

The graph of $f(x) = (\sin\alpha)x^2 - 2x + b - 2, \forall\, x \le 1$, is shown in the figure. Therefore, minimum of $f(x) = (\sin\alpha)x^2 - 2x + b - 2$ must be greater than zero but minimum is at $x = 1$. That is,

$$\sin\alpha - 2 + b - 2 \ge 0,\ b \ge 4 - \sin\alpha,\ \alpha \in (0, \pi)$$

16. a, c.

Since each pair has common root, let the roots be α, β for Eq. (1); β, γ for Eq. (2) and γ, α for Eq. (3). Therefore,

$\alpha + \beta = -a,\ \alpha\beta = bc$

$\beta + \gamma = -b,\ \beta\gamma = ca$

$\gamma + \alpha = -c,\ \gamma\alpha = ab$

Adding, we get

$2(\alpha + \beta + \gamma) = -(a + b + c)$

$$\Rightarrow \quad \alpha + \beta + \gamma = -\frac{1}{2}(a + b + c)$$

Also by multiplying product of roots, we have

$\alpha^2\beta^2\gamma^2 = a^2b^2c^2 \Rightarrow \alpha\beta\gamma = abc$

17. a, c.

Given,

$$b^2 = ac$$

$$\Rightarrow \left(\frac{b}{a}\right)^2 = \frac{c}{a}$$

$$\Rightarrow \left(-\frac{b}{a}\right)^2 = \frac{c}{a}$$

$$\Rightarrow (\alpha+\beta)^2 = \alpha\beta$$

$$\Rightarrow \alpha^2+\beta^2+\alpha\beta = 0$$

$$\Rightarrow \left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\alpha}{\beta}\right) + 1 = 0$$

$$\Rightarrow \frac{\alpha}{\beta} = \frac{-1\pm\sqrt{3}i}{2} \text{ (where } i=\sqrt{-1})$$

18. c, d.

We have,

$$D = (b-c)^2 - 4a(a-b-c) > 0$$

$$\Rightarrow b^2+c^2-2bc-4a^2+4ab+4ac > 0$$

$$\Rightarrow c^2+(4a-2b)c-4a^2+4ab+b^2 > 0 \text{ for all } c \in R$$

Discriminant of the above expression in c must be negative. Hence,

$$(4a-2b)^2 - 4(-4a^2+4ab+b^2) < 0$$

$$\Rightarrow 4a^2-4ab+b^2+4a^2-4ab-b^2 < 0$$

$$\Rightarrow a(a-b) < 0$$

$$\Rightarrow a < 0 \text{ and } a-b > 0 \text{ or } a > 0 \text{ and } a-b < 0$$

$$\Rightarrow b < a < 0 \text{ or } b > a > 0$$

19. a, d.

Since α, β, γ, δ are in H.P., hence $1/\alpha$, $1/\beta$, $1/\gamma$, $1/\delta$ are in A.P. and they may be taken as $a-3d$, $a-d$, $a+d$, $a+3d$. Replacing x by $1/x$, we get the equation whose roots are $1/\alpha$, $1/\beta$, $1/\gamma$, $1/\delta$. Therefore, equation $x^2-4x+A=0$ has roots $a-3d$, $a+d$ and equation $x^2-6x+B=0$ has roots $a-d$, $a+3d$. Sum of the roots is

$$2(a-d)=4,\ 2(a+d)=6$$

$$\therefore a = 5/2,\ d = 1/2$$

Product of the roots is

$$(a-3d)(a+d) = A = 3$$

$$(a-d)(a+3d) = B = 8$$

20. a, b.

Equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root. Therefore,

$$(q-q')^2 = (pq'-p'q)(p'-p) \quad (1)$$

Subtracting two equations, we have

$$x = \frac{q-q'}{p'-p}$$

Also using (1),

$$x = \frac{q-q'}{p'-p} = \frac{pq'-p'q}{q-q'}$$

21. b, c.

$f(x) = x^3+3x^2-9x+c$ is of the form $(x-\alpha)^2(x-\beta)$, showing that α is a double root so that $f'(x)=0$ has also one root α, i.e., $3x^2+6x-9=0$ has one root α. Hence, $x^2+2x-3=0$ or $(x+3)(x-1)=0$ has the root α which can be either -3 or 1. If $\alpha = 1$, then $f(x)=0$ gives $c-5=0$ or $c=5$. If $\alpha=-3$, then $f(x)=0$ gives

$$-27+27+27+c=0$$

$$\therefore c = -27$$

22. a, c.

If α be the common root, then

$$\alpha^2+b\alpha-a=0 \text{ and } \alpha^2-a\alpha+b=0$$

Subtracting,

$$\alpha(b+a)-(a+b)=0$$

$$\Rightarrow (a+b)(\alpha-1)=0$$

$$\Rightarrow a+b=0 \text{ or } \alpha=1$$

When $\alpha=1$, then from any equation we have $a-b=1$.

23. a, d.

Let,

$$\frac{x^2+ax+3}{x^2+x+a} = y$$

$$\Rightarrow x^2(1-y)-x(y-a)+3-ay=0$$

$$\because x \in R$$

$$(y-a)^2-4(1-y)(3-ay) \ge 0$$

$$\Rightarrow (1-4a)y^2+(2a+12)y+a^2-12 \ge 0 \quad (1)$$

Now, (1) is true for all $y \in R$, if $1-4a>0$ and $D \le 0$. Hence,

$$a < \frac{1}{4} \text{ and } 4(a+6)^2-4(a^2-12)(1-4a) \le 0$$

$$\Rightarrow a < \frac{1}{4} \text{ and } 4a^3-36a+48 \le 0$$

$$\Rightarrow a < \frac{1}{4} \text{ and } 4a^3 \le 36a-48$$

$$\Rightarrow 4a^3 < 36\left(\frac{1}{4}\right)-48$$

$$\Rightarrow 4a^3+39 < 0 \quad \left[\because a < \frac{1}{4}\right]$$

24. a, b.

Here,

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$\Rightarrow \cos^4\theta - \sin^4\theta = \cos 2\theta$$

$$\Rightarrow (-2b)^2-4b = (-4)^2-4\times 2$$

(since L.H.S. is difference of roots of first equation and R.H.S. is difference of roots of second equation)

$$\Rightarrow 4b^2-4b = 16-8 = 8$$

$$\Rightarrow 4b^2-4b-8=0$$

$$\Rightarrow b^2-b-2=0$$

$$\Rightarrow (b+1)(b-2)=0$$

$$\Rightarrow b = 2, -1$$

25. b, d.

Let $f(x) = x^2 + ax + b$. Then,

$$x^2 + (2c + a)x + c^2 + ac + b = f(x + c)$$

Thus, the roots of $f(x + c) = 0$ will be $0, d - c$.

26. c, d.

Product of roots is

$$\frac{a}{bc} < 0 \quad [\because abc < 0]$$

Hence, roots are real and of opposite sign.

27. b, c, d.

Given equation is

$$x^2 + 2(a + 1)x + 9a - 5 = 0$$

$$D = 4(a + 1)^2 - 4(9a - 5) = 4(a - 1)(a - 6)$$

$$\therefore \quad D \ge 0 \Rightarrow a \le 1 \text{ or } a \ge 6 \Rightarrow \text{roots are real}$$

If $a < 0$, then $9a - 5 < 0$. Hence, the products of roots is less than 0. So, the roots are of opposite sign. If $a > 7$, then sum of roots is $-2(a + 1) < 0$. Product of roots is greater than 0.

28. a, c.

Since $P(x)$ divides both of them, hence $P(x)$ also divides

$$(3x^4 + 4x^2 + 28x + 5) - 3(x^4 + 6x^2 + 25)$$

$$= -14x^2 + 28x - 70$$

$$= -14(x^2 - 2x + 5)$$

which is a quadratic. Hence,

$$P(x) = x^2 - 2x + 5 \Rightarrow P(1) = 4$$

30. a., d.

On putting $x = 0$, 1 and 1/2, we get

$$-1 \le c \le 1 \qquad (1)$$

$$-1 \le a + b + c \le 1 \qquad (2)$$

$$-4 \le a + 2b + 4c \le 4 \qquad (3)$$

From (1), (2), (3), we get

$$|b| \le 8 \text{ and } |a| \le 8$$

$$\Rightarrow \quad |a| + |b| + |c| \le 17$$

31. a, b, c, d.

Let $f(x) = ax^2 + bx + c$.

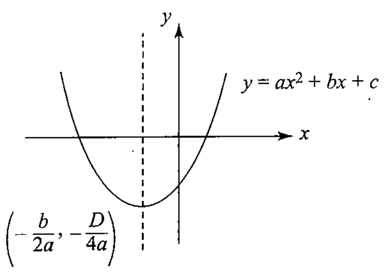


Fig. 1.79

From the diagram, we can see that $a > 0$, $c < 0$ and $-[b(2a)] < 0$. Hence, $b > 0$.

$$\therefore \quad a + b - c > 0$$

Reasoning Type

1. b. We must have

$$ax^3 + (a + b)x^2 + (b + c)x + c > 0$$

$$\Rightarrow \quad ax^2(x + 1) + bx(x + 1) + c(x + 1) > 0$$

$$\Rightarrow \quad (x + 1)(ax^2 + bx + c) > 0$$

$$\Rightarrow \quad a(x + 1)\left(x + \frac{b}{2a}\right)^2 > 0 \text{ as } b^2 = 4ac$$

$$\Rightarrow \quad x > -1 \text{ and } x \ne -\frac{b}{2a}$$

2. b. If $a > 0$, then graph of $y = ax^2 + 2bx + c$ is concave upward. Also if $b^2 - ac < 0$, then the graph always lies above x-axis; hence, $ax^2 + 2bx + c > 0$ for all real values of x. Thus, domain of function $f(x) = \sqrt{ax^2 + 2bx + c}$ is R.

If $b^2 - ac < 0$, then $ax^2 + 2bx + c = 0$ has imaginary roots. Then the graph of $y = ax^2 + 2bx + c$ never cuts x-axis, or y is either always positive or always negative. Hence, both the statements are correct but statement 2 is not correct explanation of statement 1.

3. a. $ax^2 + bx + c = 0$ has two complex conjugate roots only if all the coefficients are real. If all the coefficients are not real then it is not necessary that both the roots are imaginary. Hence, statement 2 is true.

Now, equation $x^2 - 3x + 4 = 0$ has two complex conjugate roots. If $ax^2 + bx + c = 0$ has all coefficients real, then there will be two common roots. But if there is only one root common, then at least one of a, b, c must be non-real.

Thus, both the statements are true and statement 2 is correct explanation of statement 1.

4. a. $\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1$

$$\Rightarrow \quad \cos^2\frac{\pi}{8} = \left(\frac{1}{\sqrt{2}} + 1\right)\frac{1}{2}$$

$$\Rightarrow \quad \cos^4\frac{\pi}{8} = \frac{1}{4}\left(\frac{1}{2} + 1 + \frac{2}{\sqrt{2}}\right) = \left(\frac{3}{2} + \sqrt{2}\right)\frac{1}{4}$$

$$\Rightarrow \quad \frac{1}{4}\left(\frac{3}{2} + \sqrt{2}\right) + \frac{a}{2}\left(\frac{1}{\sqrt{2}} + 1\right) + b = 0$$

$$(\because \cos^2 \pi/8 \text{ a root of equation})$$

$$\Rightarrow \quad \Rightarrow \left(\frac{3}{8} + \frac{a}{2} + b\right) + \sqrt{2}\left(\frac{1}{4} + \frac{a}{4}\right) = 0$$

Since a and b are rational, so

$$\frac{1}{4} + \frac{a}{4} = 0, \ \frac{3}{8} + \frac{a}{2} + b = 0$$

$$\Rightarrow \quad a = -1, b = \frac{1}{8}$$

Thus, both the statements are correct and statement 2 is correct explanation of statement 1.

5. a. If $a^2 + b^2 + c^2 < 0$, then all a, b, c are not real or at least one of a, b, c is imaginary number. Hence roots of equation $ax^2 + bx + c = 0$ has no complex conjugate roots, even though the roots are complex. Hence statement 1 is true. Statement 2 is obviously true (see the theory). Also, statement 2 is correct explanation of statement 1.

6. b. $ix^2 + (i-1)x - \frac{1}{2} - i = 0$

$$\Rightarrow \quad x = \frac{-(i-1) \pm \sqrt{(i-1)^2 - 4(i)\left(-\frac{i}{2} - i\right)}}{2i} = \frac{-(i-1) \pm \sqrt{-4}}{2i}$$

Thus, roots are imaginary. Also, we have $b^2 - 4ac = -4 < 0$, but this is not the correct reason for which roots are imaginary as coefficients of the equation are imaginary.

Hence, both the statements are correct but statement 2 is not correct explanation of statement 1.

7. a. $f(x) = (x - 1)(ax + b)$

$f(2) = 2a + b$

$f(4) = 3(4a + b) = 12a + 3b$

$f(2) + f(4) = 14a + 4b = 0$

$\Rightarrow \quad \frac{-b}{a} = 3.5$

Now, sum of roots is $(a-b)/a = 1 - (b/a) = 1 + 3.5 = 4.5$. Hence, the other root is 3.5.

8. c. Here, $f(x)$ is a downward parabola.

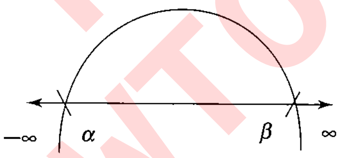


Fig. 1.80

$D = (a + 1)^2 + 20 > 0$

From the graph, clearly, statement 1 is true but statement 2 is false.

9. a. $x^2 + x + 1 = 0$

$D = -3 < 0$

Therefore, $x^2 + x + 1 = 0$ and $ax^2 + bx + c = 0$ have both the roots common. Hence,

$a = b = c$

10. d. Statement 2 is obviously true. Let,

$f(x) = (x - p)(x - r) + \lambda (x - q)(x - s) = 0$

Then,

$f(p) = \lambda (p - q)(p - s)$

$f(r) = \lambda (r - q)(r - s)$

$\Rightarrow \quad f(p) f(r) < 0$

Hence, there is a root between p and r. Thus, statement 1 is false.

11. a. Given equation is

$px^2 + qx + r = 0$

Let,

$f(x) = px^2 + qx + r$

$f(0) = r > 0$

$f(1) = p + q + r < 0$

$f(-1) = p - q + r < 0$

Hence, one root lies in $(-1, 0)$ and the other in $(0, 1)$.

$\therefore \quad [\alpha] = -1$ and $[\beta] = 0$

$\Rightarrow \quad [\alpha] + [\beta] = -1$

Therefore, statement 2 is true and is correct explanation of statement 1.

12. d. Let $f(x) = (x - \sin\alpha)(x - \cos\alpha) - 2$. Then,

$f(\sin\alpha) = -2 < 0, f(\cos\alpha) = -2 < 0$

Also, as $0 < a < \pi/4$, hence, $\sin\alpha < \cos\alpha$. Therefore, equation $f(x) = 0$ has one root in $(-\infty, \sin\alpha)$ and other in $(\cos\alpha, \infty)$.

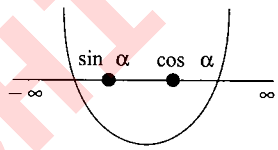


Fig. 1.81

13. b. The equation can be written as

$(2^x)^2 - (a - 3) 2^x + (a - 4) = 0$

$\Rightarrow \quad 2^x = 1$ and $2^x = a - 4$

We have,

$x \le 0$ and $2^x = a - 4 \qquad [\because x$ is non-positive$]$

$\therefore \quad 0 < a - 4 \le 1 \Rightarrow 4 < a \le 5$

$\therefore \quad a \in (4, 5]$

14. c. Clearly, Statement 1 is true but Statement 2 is false, since, $ax^2 + bx + c = 0$ is an identity when $a = b = c = 0$.

15. d. Roots of the equation $x^5 - 40x^4 + Px^3 + Qx^2 + Rx + S = 0$ are in G.P. Let the roots be a, ar, ar^2, ar^3, ar^4. Therefore,

$$a + ar + ar^2 + ar^3 + ar^4 = 40 \qquad (1)$$

and

$$\frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \frac{1}{ar^3} + \frac{1}{ar^4} = 10 \qquad (2)$$

From (1) and (2),

$$ar^2 = \pm 2 \qquad (3)$$

Now, the product of roots is $a^5 r^{10} = (ar^2)^5 = \pm 32$.

$\therefore \quad |S| = 32$

16. a. Given equations are

$$ax^2 + 2bx + c = 0 \qquad (1)$$

$$a_1x^2 + 2b_1x + c_1 = 0 \qquad (2)$$

Since Eqs. (1) and (2) have only one common root and a, b, c, a_1, b_1, c_1 are rational, therefore, common root cannot be imaginary or irrational (as irrational roots occur in conjugate pair when coefficients are rational, and complex roots always occur in conjugate pair).

Hence, the common root must be rational. Therefore, both the roots of Eqs. (1) and (2) will be rational. Therefore, $4(b^2 - ac)$ and $4(b_1^2 - a_1c_1)$ must be perfect squares (squares of rational numbers). Hence, $b^2 - ac$ and $b_1^2 - a_1c_1$ must be perfect squares.

17. a. Let $f(x) = ax^2 + bx + c$. Since coefficient are integers and one root is irrational, so both the roots are irrational. Hence, for any $\lambda \in Q$,

$$f(\lambda) \neq 0 \Rightarrow |f(\lambda)| > 0$$

$$\Rightarrow \left|\frac{ap^2}{q^2} + \frac{bp}{q} + c\right| > 0, \quad \text{where } \lambda = \frac{p}{q},\ p, q \in Z$$

$$\Rightarrow \frac{1}{q^2}\left|ap^2 + bpq + cq^2\right| > 0$$

Now, $a, b, c, p, q \in I$. Hence,

$$|ap^2 + bpq + cq^2| \geq 1$$

$$\Rightarrow |f(\lambda)| \geq \frac{1}{q^2}$$

18. d. $a^2 - 3a + 2 = 0 \Rightarrow a = 1, 2$

$a^2 - 5a + 6 = 0 \Rightarrow a = 2, 3,$

$a^2 - 4 = 0 \Rightarrow a = \pm 2$

Therefore, $a = 2$ is the only solution.

Hence, statement 1 is false. Statement 2 is true by definition.

19. b. According to statement 1, given equation is

$x^2 - bx + c = 0$

Let α, β be two roots such that

$|\alpha - \beta| = 1$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1$

$\Rightarrow b^2 - 4c = 1$

According to statement 2, given equation is $4abc\, x^2 + (b^2 - 4ac)x - b = 0$. Hence,

$D = (b^2 - 4ac)^2 + 16ab^2c$

$= (b^2 + 4ac)^2 > 0$

Hence, roots are real and unequal.

20. c. $f(x) = ax^2 + bx + c$

Given,

$f(0) + f(1) = 2$

$\Rightarrow f(x) > 0\ \forall\ x \in R$

Hence, statement 1 is true. Let,

$f(x) = x^2 - x + 1$

$a + b = 0$

Hence, statement 2 is false.

21. d. $f(x, y) = (2x - y)^2 + (x + y - 3)^2$ (1)

Therefore, statement 1 is false as it represents a point (1, 2).

22. a. $D = \underbrace{\underbrace{(2m+1)^2}_{\text{odd}} - \underbrace{4(2n+1)}_{\text{even}}}_{\text{odd}}$

For rational root, D must be a perfect square. As D is odd, let D be perfect square of $2l + 1$, where $l \in Z$.

$(2m + 1)^2 - 4(2n + 1) = (2l + 1)^2$

$\Rightarrow (2m + 1)^2 - (2l + 1)^2 = 4(2n + 1)$

$\Rightarrow [(2m + 1) + (2l + 1)]\,[2(m - l)] = 4(2n + 1)$

$\Rightarrow (m + l + 1)(m - l) = (2n + 1)$ (1)

R.H.S. of (1) is always odd but L.H.S. is always even. Hence, D cannot be a perfect square. So, the roots cannot be rational.

Hence, statement 1 is true, statement 2 is true and statement 2 is correct explanation for statement 1.

Linked Comprehension Type

For Problems 1–3

1. d, 2. c, 3. c.

Sol. Let unknown polynomial be $P(x)$. Let $Q(x)$ and $R(x)$ be the quotient and remainder, respectively, when it is divided by $(x - 3)(x - 4)$. Then,

$P(x) = (x - 3)(x - 4)Q(x) + R(x)$

Then, we have

$R(x) = ax + b$

$\Rightarrow P(x) = (x - 3)(x - 4)Q(x) + ax + b$

Given that $P(3) = 2$ and $P(4) = 1$. Hence,

$3a + b = 2$ and $4a + b = 1$

$\Rightarrow a = -1$ and $b = 5$

$\Rightarrow R(x) = 5 - x$

1. $5 - x = x^2 + ax + 1 \Rightarrow x^2 + (a + 1)x - 4 = 0$

Given that roots are real and distinct.

$\therefore D > 0 \Rightarrow (a + 1)^2 + 16 > 0$

which is true for all real x.

2. $-x + 5 = px^2 + (q - 1)x + 6 \Rightarrow px^2 + qx + 1 = 0$

Now, $p > 0$ and equation has no distinct real roots or equation has real and equal or imaginary roots. Then,

$px^2 + qx + 1 \geq 0,\ \forall\ x \in R$

$\Rightarrow f(3) \geq 0 \Rightarrow 9p + 3q + 1 \geq 0 \Rightarrow 3p + q \geq -1/3$

Hence, the least value of $3p + q$ is $-1/3$.

3. $f(x) = y = \dfrac{-x + 5}{x^2 - 3x + 2}$

$\Rightarrow yx^2 + (1 - 3y)x + 2y - 5 = 0$

Now, x is real, then

$D \geq 0$

$\Rightarrow (1 - 3y)^2 - 4y(2y - 5) \geq 0$

$\Rightarrow \quad y^2 + 14y + 1 \geq 0$

$$\Rightarrow \quad y \in \left(-\infty, \frac{-14-\sqrt{192}}{2}\right] \cup \left[\frac{-14+\sqrt{192}}{2}, \infty\right)$$

$(-\infty, -7-4\sqrt{3}] \cup [-7+4\sqrt{3}, \infty)$

For Problems 4–6

4. d, 5. b, 6. b.

Sol. $ax^2 - bx + c = 0$

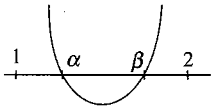


Fig. 1.82

Let $f(x) = ax^2 - bx + c$ be the corresponding quadratic expression and α, β be the roots of $f(x) = 0$. Then,

$$f(x) = a(x-\alpha)(x-\beta)$$

Now,

$$af(1) > 0,\ af(2) > 0,\ 1 < \frac{b}{2a} < 2,\ b^2 - 4ac > 0$$

$\Rightarrow \quad a(1-\alpha)(1-\beta) > 0,\ a(2-\alpha)(2-\beta) > 0,\ 2a < b < 4a,$

$b^2 - 4ac > 0$

$\Rightarrow \quad a^2(1-\alpha)(1-\beta)(2-\alpha)(2-\beta) > 0$

$\Rightarrow \quad a^2(\alpha-1)(2-\alpha)(\beta-1)(2-\beta) > 0$

As $f(1)$ and $f(2)$ both are integers and $f(1) > 0$, and $f(2) > 0$, so

$f(1)\ f(2) > 0$

$\Rightarrow \quad f(1)\ f(2) \geq 1$

$\Rightarrow \quad 1 \leq a^2(\alpha-1)(2-\alpha)(\beta-1)(2-\beta)$

Now,

$$\frac{(\alpha-1)+(2-\alpha)}{2} \geq ((\alpha-1)(2-\alpha))^{1/2}$$

$$\Rightarrow \quad (\alpha-1)(2-\alpha) \leq \frac{1}{4}$$

Similarly,

$$(\beta-1)(2-\beta) \leq \frac{1}{4}$$

$$\Rightarrow \quad (\alpha-1)(2-\alpha)(\beta-1)(2-\beta) < \frac{1}{16}$$

As $\alpha \neq \beta$, so

$a^2 > 16 \Rightarrow a \geq 5$

$\Rightarrow \quad b^2 > 20c$ and $b > 10 \Rightarrow b \geq 11$

Also,

$b^2 > 100 \Rightarrow c > 5 \Rightarrow c \geq 6$

For Problems 7–9

7. c, 8. a, 9. c.

Sol. Given equation is

$$x^4 + 2ax^3 + x^2 + 2ax + 1 = 0 \quad (1)$$

or

$$\left(x^2 + \frac{1}{x^2}\right) + 2a\left(x + \frac{1}{x}\right) + 1 = 0$$

$$\Rightarrow \quad \left(x + \frac{1}{x}\right)^2 + 2a\left(x + \frac{1}{x}\right) - 1 = 0$$

$$\Rightarrow \quad t^2 + 2at - 1 = 0 \quad (2)$$

where $t = x + (1/x)$. Now,

$$\left(x + \frac{1}{x}\right) \geq 2$$

or

$$\left(x + \frac{1}{x}\right) \leq -2$$

$\therefore \quad t \geq 2$ or $t \leq -2$

7. Now, Eq. (1) will have at least two positive roots, when at least one root of Eq. (2) will be greater than 2. From Eq. (2),

$$D = 4a^2 - 4(-1) = 4(1 + a^2) > 0,\ \forall\ a \in R \quad (3)$$

Let the roots of Eq. (2) be α, β. If $\alpha, \beta \leq 2$, then

$$\Rightarrow \quad f(2) \geq 0 \text{ and } \frac{-B}{2A} < 2$$

$$\Rightarrow \quad 4 + 4a - 1 \geq 0 \text{ and } -\frac{2a}{2} < 2$$

$$\Rightarrow \quad a \geq -\frac{3}{4} \text{ and } a > -2$$

$$\Rightarrow \quad a \geq -\frac{3}{4}$$

Therefore, at least one root will be greater than 2. Then,

$$a < -\frac{3}{4} \quad (4)$$

Combining (3) and (4), we get

$$a < -\frac{3}{4}$$

Hence, at least one root will be positive if $a \in [-\infty, -(3/4)]$.

8. Now, Eq. (1) will have at least two roots negative, when at least one root of Eq. (2) will be less than -2. If $\alpha, \beta \geq -2$, then

$$f(-2) \geq 0 \text{ and } -\frac{B}{2A} > -2$$

$$\therefore \quad 4 - 4a - 1 \geq 0 \text{ and } -\frac{2a}{2} > -2$$

$$\therefore \quad a \leq \frac{3}{4} \text{ and } a < 2$$

$$\therefore \quad a \leq \frac{3}{4} \quad (5)$$

Combining (3) and (5), at least one root will be less than -2 for Eq. (2) if

$$a > \frac{3}{4}$$

$$\therefore \quad a \in \left(\frac{3}{4}, \infty\right)$$

9. If exactly two roots are positive, then other two roots are negative. Then – 2 and 2 must lie between the roots. So,

$f(-2) < 0$ and $f(2) < 0$

$\Rightarrow$ $a > 3/4$ and $a < -3/4$

Hence, no such values of a exist.

For Problems 10–12

10. a, 11. d, 12. c.

Sol. $(\beta - \alpha) = ((\beta + h) - (\alpha + h))$

$(\beta + \alpha)^2 - 4\alpha\beta = [(\beta + h) + (\alpha + h)]^2 - 4(\beta + h)(\alpha + h)$

$(-b_1)^2 - 4c_1 = (-b_2)^2 - 4c_2$

$D_1 = D_2$ (i)

The least value of $f(x)$ is

$$-\frac{D_1}{4} = -\frac{1}{4} \Rightarrow D_1 = 1 \text{ and } D_2 = 1$$

Therefore, the least value of

$$g(x) \text{ is } -\frac{D_2}{4} = -\frac{1}{4}$$

The least value of $g(x)$ occurs at

$$-\frac{b_2}{2} = \frac{7}{2} \Rightarrow b_2 = -7$$

$\Rightarrow$ $b_2^2 - 4c_2 = D_2$

$\Rightarrow$ $49 - 4c_2 = 1 \Rightarrow \frac{48}{4} = c_2 \Rightarrow c_2 = 12$

$\Rightarrow$ $x^2 - 7x + 12 = 0 \Rightarrow x = 3, 4$

For Problems 13–15

13. d, 14. c, 15. c.

Sol.

13. $\because$ $AC = 4\sqrt{2}$

$\therefore$ $AB = BC = \dfrac{4\sqrt{2}}{\sqrt{2}} = 4$ units

$OB = \sqrt{4^2 - (2\sqrt{2})^2} = 2\sqrt{2}$

$\therefore$ $A(-2\sqrt{2}, 0), B(2\sqrt{2}, 0), C(0, -2\sqrt{2})$

Since $y = ax^2 + bx + c$ passes through A, B and C, we get

$$y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$$

14. Minimum value of $y = x^2/(2\sqrt{2}) - 2\sqrt{2}$ is $-2\sqrt{2}$ at $x = 0$.

15. $f(x) = 0$

$\Rightarrow$ $x^2/(2\sqrt{2}) - 2\sqrt{2} = 0$ or $x = \pm 2\sqrt{2}$

Therefore, number of integral values of k for which k lies in

$(-2\sqrt{2}, 2\sqrt{2})$ is 5.

For Problems 16–18

16. d, 17. c, 18. b.

Sol.

Given that $9^x - a3^x - a + 3 \le 0$

Let $t = 3^x$. Then,

$t^2 - at - a + 3 \le 0$

or

$t^2 + 3 \le a(t + 1)$ (1)

where $t \in R^+$, $\forall\, x \in R$

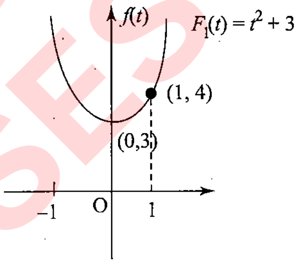


Fig. 1.83

Let $f_1(t) = t^2 + 3$ and $f_2(t) = a(t + 1)$.

16. For $x < 0$, $t \in (0, 1)$. That means (1) should have at least one solution in $t \in (0, 1)$. From (1), it is obvious that $a \in R^+$. Now $f_2(t) = a(t + 1)$ represents a straight line. It should meet the curve $f_1(t) = t^2 + 3$, at least once in $t \in (0, 1)$.

$f_1(0) = 3, f_1(1) = 4, f_2(0) = a, f_2(1) = 2a$

If $f_1(0) = f_2(0)$, Then $a = 3$; if $f_1(1) = f_2(1)$, then $a = 2$. Hence, the required range is $a \in (2, 3)$.

17. For at least one positive solution, $t \in (1, \infty)$. That means graphs of $f_1(t) = t^2 + 3$ and $f_2(t) = a(t + 1)$ should meet at least once in $t \in (1, \infty)$. If $a = 2$, both the curves touch each other at (1, 4). Hence, the required range is $a \in (2, \infty)$.

18. In this case both graphs should meet at least once in $t \in (0, \infty)$. For $a = 2$ both the curves touch, hence, the required range is $a \in [2, \infty)$.

For Problems 19–21

19. d, 20. c, 21. a.

Sol.

Let $f(x) = x^2 + x + a - 9$.

$x^2 + x + a - 9 < 0$ has at least one positive solution, then either both the roots of equation $x^2 + x + a - 9 = 0$ are non-negative or 0 lies between the roots.

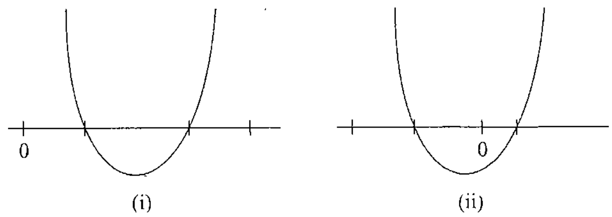


Fig. 1.84

Now sum of roots $= -\dfrac{1}{2}$, hence case I is not possible. For case II,

$f(0) < 0 \Rightarrow a - 9 < 0 \Rightarrow a < 9$

20.

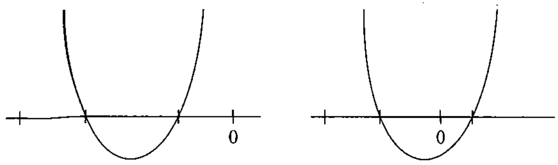


Fig. 1.85

If $x^2 + x + a - 9 < 0$ has at least one negative solution, then either both the roots of equation $x^2 + x + a - 9 = 0$ are non-positive or 0 lies between the roots.

For case **I**, sum of roots is $-1/2 < 0$. Product of roots is $a - 9 > 0 \Rightarrow a \geq 9$ and

$$D > 0 \Rightarrow 1 - 4(a - 9) > 0 \Rightarrow a < \frac{37}{4}$$

Hence, $9 \leq a < 37/4$.

For case **II** $f(0) < 0 \Rightarrow a < 9 \Rightarrow a \in \left(-\infty, \frac{37}{4}\right)$

21. If $x^2 + x + a - 9 < 0$ is true $\forall x \in (-1, 3)$, then $f(-1) < 0$ and $f(3) < 0$.

$\therefore \quad 1 - 1 + a - 9 < 0$ and $9 + 3 + a - 9 < 0$

$\Rightarrow \quad a < 9$ and $a < -3$

$\Rightarrow \quad a < -3$

For Problems 22–24

22. b, 23. a, 24. b.

Sol.

22. $\quad b^2 > (a + c)^2$

$\Rightarrow \quad (a + c - b)(a + c + b) < 0$

$\Rightarrow \quad f(-1) f(1) < 0$

So, there is exactly one root in $(-1, 1)$.

23. $\quad af(1) < 0$ and $f(0) f(1) > 0$

$\Rightarrow \quad af(1) < 0$ and $af(0) < 0$

Hence, both the numbers 0 and 1 lie between the roots.

24. $\quad f(0) f(1) < 0$ and $af(1) > 0$

$\Rightarrow \quad f(0) f(1) < 0$ and $af(0) < 0$

Hence, exactly one root lies in $(0, 1)$ and 0 lies between the roots.

For Problems 25–26

25. a, 26. b.

Sol. From the question, the real roots of $x^3 - x^2 + \beta x + \gamma = 0$ are x_1, x_2, x_3 and they are in A.P. As x_1, x_2, x_3 are in A.P., let $x_1 = a - d$, $x_2 = a$, $x_3 = a + d$. Now,

$$x_1 + x_2 + x_3 = -\frac{-1}{1} = 1$$

$\Rightarrow \quad a - d + a + a + d = 1$

$\Rightarrow \quad a = \frac{1}{3} \qquad (1)$

$$x_1x_2 + x_2x_3 + x_3x_1 = \frac{\beta}{1} = \beta$$

$\Rightarrow \quad (a - d)\,a + a\,(a + d) + (a + d)(a - d) = \beta \qquad (2)$

$$x_1x_2x_3 = -\frac{\gamma}{1} = -\gamma$$

$\Rightarrow \quad (a - d)\,a\,(a + d) = -\gamma \qquad (3)$

From (1) and (2), we get

$3a^2 - d^2 = \beta$

$\Rightarrow \quad \Rightarrow 3\frac{1}{9} - d^2 = \beta$, so $\beta = \frac{1}{3} - d^2 < \frac{1}{3}$

From (1) and (3), we get

$$\frac{1}{3}\left(\frac{1}{9} - d^2\right) = -\gamma$$

$\Rightarrow \quad \gamma = \frac{1}{3}\left(d^2 - \frac{1}{9}\right) > \frac{1}{3}\left(-\frac{1}{9}\right) = -\frac{1}{27}$

$$\gamma \in \left(-\frac{1}{27}, +\infty\right)$$

Matrix-Match Type

1. a → s; b → r; c → q; d → p.

Obviously when $a \geq 0$, we have no roots as all the terms are followed by +ve sign. Also for $a = -2$, we have

$x^2 - 2|x| + 1 = 0$

or

$|x| - 1 = 0 \Rightarrow x = \pm 1$

Hence, the equation has two roots.

Also when $a < -2$, for given equation

$$|x| = \frac{-a \pm \sqrt{a^2 - 4}}{2} > 0$$

Hence, the equation has four roots as $|-a| > \sqrt{a^2 - 4}$. Obviously, the equation has no three real roots for any value of a.

2. a → p; b → p, q, r, s; c → p, q, s; d → r, s.

a. $\quad y = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$

$\Rightarrow \quad x^2y + 2xy + 4y = x^2 - 2x + 4$

$\Rightarrow \quad (y - 1)x^2 + 2(y + 1)x + 4(y - 1) = 0$

$D \geq 0$

$\Rightarrow \quad 4(y + 1)^2 - 16(y - 1)^2 \geq 0$

$\Rightarrow \quad (y + 1)^2 - (2y - 2)^2 \geq 0$

$\Rightarrow \quad (3y - 1)(3 - y) \geq 0$

$\Rightarrow \quad (3y - 1)(y - 3) \leq 0 \Rightarrow y \in \left[\frac{1}{3}, 3\right]$

$\Rightarrow \quad \{1\} \Rightarrow P$

b. $\quad y = \dfrac{x^2 - 3x - 2}{2x - 3}$

$\Rightarrow \quad x^2 - 3x - 2 = 2xy - 3y$

$\Rightarrow \quad x^2 - (3 + 2y)x + (3y - 2) = 0$

$D \geq 0$

$\Rightarrow$ $(3+2y)^2 - 4(3y-2) \geq 0$

$\Rightarrow$ $9 + 4y^2 + 12y - 12y + 8 \geq 0$

$\Rightarrow$ $4y^2 + 17 \geq 0$

which is always true. Hence,

$y \in R \Rightarrow \{1, 4, -3, -10\} \Rightarrow p, q, r, s$

c. $y = \dfrac{2x^2 - 2x + 4}{x^2 - 4x + 3}$

$\Rightarrow$ $x^2 y - 4xy + 3y = 2x^2 - 2x + 4$

$(y-2)x^2 + 2(1-2y)x + 3y - 4 = 0$

$D \geq 0$

$4(1-2y)^2 - 4(y-2)(3y-4) \geq 0$

$\Rightarrow$ $1 + 4y^2 - 4y - (3y^2 - 10y + 8) \geq 0$

$\Rightarrow$ $y^2 + 6y - 7 \geq 0$

$\Rightarrow$ $(y+7)(y-1) \geq 0$

$\Rightarrow$ $y \geq 1$ or $y \leq -7$

$\Rightarrow$ $\{1, 4, -10\} \Rightarrow p, q, s$

d. $f(x) = x^2 - (a-3)x + 2 < 0, \forall x \in [-2, -1]$

$\Rightarrow$ $f(-2) < 0$ and $f(-1) < 0$

$\Rightarrow$ $4 + 2(a-3) + 2 < 0$ and $1 + (a-2) + 2 < 0$

$\Rightarrow$ $a < 0$ and $a < -1$

$\Rightarrow$ $a < -1$

$\Rightarrow$ $a \in \{-10, -3\}$

3. a → q, r, s; b → r; c → p; d → q.

a. $d + a - b = 0$ and $d + b - c = 0$

$d = b - a$ and $d = c - b$

$\therefore b - a = c - b \Rightarrow 2b = a + b \Rightarrow a, b, c$ are in A.P.

Also $x = 1$ satisfies the second equation. Therefore, the other root is also 1. Product of roots is 1.

$\therefore c(a-b) = a(b-c) \Rightarrow b = \dfrac{2ac}{a+c} \Rightarrow a, b, c$ are in H.P.

Therefore, a, b, c are in A.P. and a, b, c in H.P. Hence, a, b, c are in G.P.

b. $(a^2 + b^2)x^2 - 2b(a+c)x + (b^2 + c^2) = 0$

The roots are real and equal. Hence,

$4b^2(a+c)^2 - 4(a^2+b^2)(b^2+c^2) = 0$

$\Rightarrow$ $b^2(a^2 + c^2 + 2ac) - (a^2b^2 + a^2c^2 + b^4 + b^2c^2) = 0$

$\Rightarrow$ $b^2a^2 + b^2c^2 + 2ab^2c - a^2b^2 - a^2c^2 - b^4 - b^2c^2 = 0$

$\Rightarrow$ $2ab^2c - a^2c^2 - b^4 = 0 \Rightarrow (b^2 - ac)^2 = 0$

Hence, $b^2 = ac$. Thus a, b, c are in G.P.

c. $(x-1)^3 = 0 \Rightarrow x = 1$ is the common root. Hence, $a + b + c = 0$.

d. $(a+c)^2 + 4b^2 - 4b(a+c) \leq 0, \forall x \in R$

$\Rightarrow$ $((a+c) - 2b)^2 \leq 0$

$\Rightarrow$ $a + c = 2b$

$\Rightarrow$ a, b, c in A.P.

4. a → r; b → r; c → q; d → p.

a. $(m-2)x^2 - (8-2m)x - (8-3m) = 0$ has roots of opposite signs. The product of roots is

$-\dfrac{8-3m}{m-2} < 0$

$\Rightarrow$ $\dfrac{3m-8}{m-2} < 0$

$\Rightarrow$ $2 < m < 8/3$

b. Exactly one root of equation $x^2 - m(2x-8) - 15 = 0$ lies in interval (0, 1).

$f(0)f(1) < 0$

$\Rightarrow$ $(0 - m(-8) - 15)(1 - m(-6) - 15) < 0$

$\Rightarrow$ $(8m - 15)(6m - 15) < 0$

$\Rightarrow$ $15/8 < m < 15/6$

c. $x^2 + 2(m+1)x + 9m - 5 = 0$ has both roots negative. Hence, sum of roots is

$-2(m+1) < 0$ or $m > -1$ (1)

Product of roots is

$9m - 5 > 0 \Rightarrow m > 5/9$ (2)

Discriminant,

$D \geq 0 \Rightarrow 4(m+1)^2 - 4(9m-5) \geq 0$

$\Rightarrow$ $m^2 - 7m + 6 \geq 0$

$\Rightarrow$ $m \leq 1$ or $m \geq 6$ (3)

Hence, for (1), (2) and (3), we get

$m \in \left(\dfrac{5}{9}, 1\right] \cup [6, \infty)$

d. $f(x) = x^2 + 2(m-1)x + m + 5 = 0$ has one root less than 1 and the other root greater than 1. Hence,

$f(1) < 0$

$\Rightarrow$ $1 + 2(m-1) + m + 5 < 0$

$\Rightarrow$ $m < -4/3$

5. a → s; b → p; c → q; d → r.

a. $x^2 + ax + b = 0$ has root α. Hence,

$\alpha^2 + a\alpha + b = 0$ (1)

$x^2 + px + q = 0$ has roots $-\alpha, \gamma$. Hence,

$\alpha^2 - p\alpha + q = 0$ (2)

Eliminating α from (1) and (2), we get

$(q-b)^2 = (aq + bp)(-p-a)$

$\Rightarrow$ $(q-b)^2 = -(aq+bp)(p+a)$

b. $x^2 + ax + b = 0$ has root is α, β. Hence,

$\alpha^2 + a\alpha + b = 0$ (1)

$x^2 + px + q = 0$ has root $1/\alpha$. Hence,

$q\alpha^2 + p\alpha + 1 = 0$ (2)

Eliminating α from (1) and (2), we get

$(1 - bq)^2 = (a - pb)(p - aq)$

c. $x^2 + ax + b = 0$ has roots α, β. Hence,

$\alpha^2 + a\alpha + b = 0$ (1)

$x^2 + px + q = 0$ has roots $-2/\alpha, \gamma$. Hence,

$q\alpha^2 - 2p\alpha + 4 = 0$ (2)

Eliminating α from (1) and (2), we get

$(4 - bq)^2 = (4a + 2pb)(-2p - aq)$

d. $x^2 + ax + b = 0$ has roots α, β. Hence,

$\alpha^2 + a\alpha + b = 0$ (1)

$x^2 + px + q = 0$ has roots $-1/2\alpha, \gamma$. Hence,

$4q\alpha^2 - 2p\alpha + 1 = 0$ (2)

Eliminating α from (1) and (2), we get

$(1 - 4bq)^2 = (a + 2bp)(-2p - 4aq)$

Integer Type

1.(8) Let $\left(a + \frac{1}{a}\right) = t$

$\Rightarrow \quad a^3 + \frac{1}{a^3} = 18$

$t^3 - 3t - 18 = 0$ (1)

$t = 3$ satisfies (1)

hence factorizing (1)

$(t - 3)(t^2 + 3t + 6) = 0$

$t = 3$ only is the solution

$\therefore \quad a + \frac{1}{a} = 3 \Rightarrow a^2 + \frac{1}{a^2} = 7 \Rightarrow a^4 + \frac{1}{a^4} = 47$

2. (3) We have $P(x) = \frac{5}{3} - 6x - 9x^2 = -(3x + 1)^2 + \frac{8}{3}$

$\Rightarrow \quad P_{max} = \frac{8}{3}$

Similarly, $Q(y) = -4y^2 + 4y + \frac{13}{2} = -(2y - 1)^2 + \frac{15}{2}$

$\Rightarrow \quad Q_{max} = \frac{15}{2}$

Now, $P_{max} \times Q_{max} = \frac{8}{3} \times \frac{15}{2} = 20$

So $(x, y) \equiv \left(-\frac{1}{3}, \frac{1}{2}\right)$

Hence, $6x + 10y = 6\left(\frac{-1}{3}\right) + 10\left(\frac{1}{2}\right) = -2 + 5 = 3$

3. (2) We have $x_1 + x_2 + x_3 = 8$

$x_1 x_2 x_3 = d$

$x_1x_2 + x_2x_3 + x_3x_1 = c$

Possible roots 1, 2, 5 or 1, 3, 4

$\therefore \quad d = 10$ or $d = 12$

$\Rightarrow \quad c = 2 + 10 + 5 = 17$ or $3 + 12 + 4 = 19$

Hence, $d = 10$ and $c = 17$ or $d = 12$ and $c = 19$

4.(9) Let $\alpha_1 = A, \beta_1 = AR, \alpha_2 = AR^2, \beta_2 = AR^3$

we have $\alpha_1 + \beta_1 = 6 \Rightarrow A(1 + R) = 6$ (1)

$\alpha_1\beta_1 = p \Rightarrow A^2 R = p$ (2)

Also $\alpha_2 + \beta_2 = 54 \Rightarrow AR^2(1 + R) = 54$ (3)

$\alpha_2\beta_2 = q \Rightarrow A^2R^5 = q$ (4)

Now, on dividing Eq. (3) by Eq. (1), we get

$\frac{AR^2(1+R)}{A(1+R)} = \frac{54}{6} = 9 \quad \Rightarrow \quad R^2 = 9$

$\therefore \quad R = 3$ (As it is an increasing G.P.)

$\therefore$ On putting $R = 3$ in Eq. (1), we get

$A = \frac{6}{4} = \frac{3}{2}$

$\therefore \quad p = A^2 R = \frac{9}{4} \times 3 = \frac{27}{4}$ and $q = A^2R^5 = \frac{9}{4} \times 243 = \frac{2187}{4}$

Hence, $q - p = \frac{2187 - 27}{4} = \frac{2160}{4} = 540$

5.(3) $2x^2 + 4x(y - 3) + 7y^2 - 2y + t = 0$

$D = 0$ (for one solution)

$\Rightarrow 16(y - 3)^2 - 8(7y^2 - 2y + t) = 0$

$\Rightarrow 2(y - 3)^2 - (7y^2 - 2y + t) = 0$

$\Rightarrow 2(y^2 - 6y + 9) - (7y^2 - 2y + t) = 0$

$\Rightarrow -5y^2 - 10y + 18 - t = 0$

$\Rightarrow 5y^2 + 10y + t - 18 = 0$

Again $D = 0$ (for one solution)

$\Rightarrow 100 - 20(t - 18) = 0$

$\Rightarrow 5 - t + 18 = 0$

$\Rightarrow t = 23$

for $t = 23$; $5y^2 + 10y + 5 = 0$

$(y + 1)^2 = 0 \quad \Rightarrow \quad y = -1$

for $y = -1$; $2x^2 - 16x + 32 = 0$

$x^2 - 8x + 16 = 0$

$x = 4 \quad \Rightarrow \quad x + y = 3$

6. (4) As $P(x)$ is an odd function

Hence, $P(-x) = -P(x) \quad \Rightarrow \quad P(-3) = -P(3) = -6$

Let $P(x) = Q(x^2 - 9) + ax + b$

(where Q is quotient and $(ax + b) = g(x)$ = remainder)

Now $P(3) = 3a + b = 6$ (1)

$P(-3) = -3a + b = -6$ (2)

Hence, $b = 0$ and $a = 2$

Hence, $g(x) = 2x \quad \Rightarrow \quad g(2) = 4$

7. (4) $f(x) = ax^2 - (3 + 2a)x + 6$

$= (ax - 3)(x - 2)$

Here, roots of the equation $f(x) = 0$ are 2 and $3/a$, and $f(0) = 6$.

$f(x)$ should be positive for exactly three negative integral values of x.

Therefore, graph of $f(x)$ must be a downward parabola passing through $x = 2$ and $x = 3/a$ and $-4 \le \frac{3}{a} < -3$

$\therefore \quad a \in \left(-1, -\frac{3}{4}\right]$

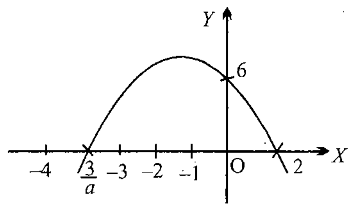


Fig. 1.86

$\therefore \quad c = -1, d = -\frac{3}{4}$

$\Rightarrow \quad c^2 + 4|d| = 1 + 3 = 4$

8. (8) Given $\alpha\beta$; $\alpha\beta(\alpha+\beta)$; $\alpha^3+\beta^3$ are in G.P.

$\alpha + \beta = 4; \quad \alpha\beta = k; \alpha\beta^2 + \alpha^2\beta = \alpha\beta(\alpha+\beta) = 4k$

$\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= 64 - 3k(4) = 4(16 - 3k)$

$\therefore \ k$; $4k$; $4(16-3k)$ are in G.P.

$16k^2 = 4k(16 - 3k)$

$4k(4k - 16 + 3k) = 0$

$k = 0; \ k = \frac{16}{7}$

9. (6)

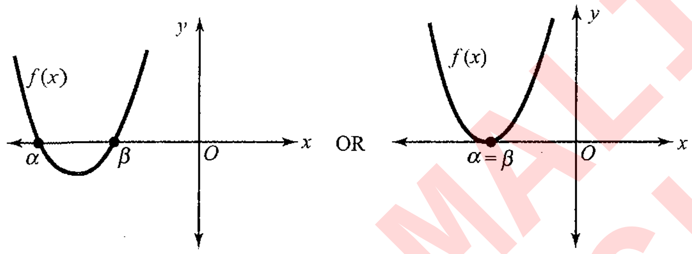


Fig. 1.87

Let $f(x) = x^2 + 2(\lambda+1)x + \lambda^2 + \lambda + 7$

If both roots of $f(x) = 0$ are negative, then

$D = b^2 - 4ac = 4(\lambda+1)^2 - 4(\lambda^2 + \lambda + 7) \geq 0 \Rightarrow \lambda - 6 \geq 0$

$\Rightarrow \ \lambda \in [6, \infty)$ (1)

Sum of roots $= -2(\lambda + 1) < 0$

$\Rightarrow \ \lambda \in (-1, \infty)$ (2)

and product of roots $= \lambda^2 + \lambda + 7 > 0 \ \forall \ \lambda \in R$ (3)

$\therefore$ From (1), (2), (3), we get $\lambda \in [6, \infty)$

(As (1), (2), (3) must be satisfied simultaneously.)

Hence, the least value of $\lambda = 6$.

10. (3) Clearly, $P(x) - x^3 = 0$ has roots 1, 2, 3, 4.

$\therefore \ P(x) - x^3 = (x-1)(x-2)(x-3)(x-4)$

$\Rightarrow \ P(x) = (x-1)(x-2)(x-3)(x-4) + x^3$

Hence, $P(5) = 1 \times 2 \times 3 \times 4 + 125 = 129$

11. (4) $x^{1/8} = (3x^4 + 4)^{1/64} \Rightarrow x^8 = 3x^4 + 4 \Rightarrow x^4 = 4$

12. (3) $f(x) = (x-1)(x^2 - 7x + 13)$

for $f(x)$ to be prime at least one of the factors must be prime.

Hence, $x - 1 = 1 \Rightarrow x = 2$ or

$x^2 - 7x + 13 = 1 \Rightarrow x^2 - 7x + 12 = 0 \Rightarrow x = 3$ or 4

$\Rightarrow x = 2, 3, 4$

13. (6) Let the roots be $a - 3d, a - d, a + d, a + 3d$

Sum of roots $= 4a = 0 \Rightarrow a = 0$

Hence, roots are $-3d, -d, d, 3d$.

Product of roots $= 9d^4 = m^2 \Rightarrow d^2 = \frac{m}{3}$ (1)

Again $\sum x_1x_2 = 3d^2 - 3d^2 - 9d^2 - d^2 - 3d^2 + 3d^2 = -10d^2$

$= -(3m + 2)$

$\Rightarrow \ \frac{10m}{3} = 3m + 2 \Rightarrow 10m = 9m + 6$

$\Rightarrow \quad m = 6$

14. (8) As $P(1) = 0$

and $p(x) \geq 0$ hence let $p(x) = k(x-1)^2, k > 0$

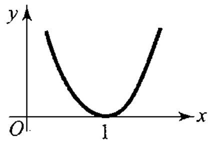


Fig. 1.88

$p(2) = k = 2 \Rightarrow k = 2$

$\therefore \ p(x) = 2(x-1)^2 \Rightarrow p(3) = 8$

15. (4) $y = \frac{3x^2 + mx + n}{x^2 + 1}$

$x^2(y - 3) - mx + y - n = 0$

$x \in R$

$D \geq 0$

$\Rightarrow \ m^2 - 4(y-3)(y-n) \geq 0$

$\Rightarrow \ m^2 - 4(y^2 - ny - 3y + 3n) \geq 0$

$4y^2 - 4y(n+3) + 12n - m^2 \leq 0$ (1)

Also given $(y+4)(y-3) \leq 0$

$y^2 + y - 12 \leq 0$ (2)

Compare (1) and (2), we get $\frac{4}{1} = -\frac{4(n+3)}{1} = \frac{12n - m^2}{-12}$

$\Rightarrow \ m = 0$ and $n = -4$

16. (6) $a^2 \geq 8b$ and $4b^2 \geq 4a$

Now $b^2 \geq a \Rightarrow b^4 \geq a^2 \geq 8b \quad (a > 0, b > 0)$

$\therefore \Rightarrow \ b^3 \geq 8 \Rightarrow b \geq 2$ (1)

Again $a^2 \geq 8b$ and $b \geq 2$

$\Rightarrow \ a^2 \geq 16$

$\Rightarrow \ a \geq 4$ (2)

From (1) and (2), $(a + b)_{\text{least}} = 6$.

17. (7) Given $a + b + c = 1$ (1)

$ab + bc + ca = 0$ (2)

$abc = 2$ (3)

Now $(a + b + c)^2 = 1$

$a^2 + b^2 + c^2 + 2\sum ab = 1$

$\therefore \ a^2 + b^2 + c^2 = 1$

Now, $a^3 + b^3 + c^3 - 3abc = (a + b + c)[\sum a^2 - \sum ab]$

$= 1(1 - 0) = 1$

$a^3 + b^3 + c^3 = 1 + 3abc = 1 + 3 \times 2 = 7$

18.(3) The given equation $x + \frac{1}{x} = 3$

$\therefore \ x^2 + \frac{1}{x^2} = 7 \Rightarrow x^4 + \frac{1}{x^4} = 47 \Rightarrow x^8 + \frac{1}{x^8} = (47)^2 - 2$

$\therefore \ x^8 + x^{-8} = 2207$ (1)

Now $E = x^9 + x^7 + x^{-9} + x^{-7}$

$= x^8\left(x + \frac{1}{x}\right) + x^{-9} + x^{-7} = x^8\left(x + \frac{1}{x}\right) + x^{-8}\left(x + \frac{1}{x}\right)$

$E = \left(x + \frac{1}{x}\right)(x^8 + x^{-8})$ (2)

Substitute the value of $x^8 + x^{-8} = 2207$ from (1) and $x + \frac{1}{x} = 3$

$E = (3)(2207) = 6621$

19.(6) $x^2 + ax + b \equiv (x+1)(x+b) \Rightarrow b + 1 = a$ (1)

also $x^2 + bx + c \equiv (x+1)(x+c) \Rightarrow c + 1 = b$

or $b + 1 = c + 2$ (2)

hence $b + 1 = a = c + 2$

also $(x+1)(x+b)(x+c) \equiv x^3 - 4x^2 + x + 6$

$\Rightarrow x^3 + (1 + b + c)x^2 + (b + bc + c)x + bc \equiv x^3 - 4x^2 + x + 6$

$\Rightarrow 1 + b + c = -4$

$\Rightarrow 2c + 2 = -4 \Rightarrow c = -3;\ b = -2$ and $a = -1$

$\Rightarrow a + b + c = -6$

20.(3) $n, n+1, n+2$

sum $= 3(n+1) = -a$

$\therefore a^2 = 9(n+1)^2$

sum of the roots taken 2 at a time $= +b$

$\therefore n(n+1) + (n+1)(n+2) + (n+2)n + 1 = b + 1$

$n^2 + n + n^2 + 3n + 2 + n^2 + 2n + 1 = b + 1$

$\therefore b + 1 = 3n^2 + 6n + 3$

$b + 1 = 3(n+1)^2 = \dfrac{a^2}{3}; \quad \therefore \dfrac{a^2}{b+1} = 3$

21.(2) $(x + y + z)^2 = 144$ (given)

$\Rightarrow \sum x^2 + 2\sum xy = 144$

$\Rightarrow 96 + 2\sum xy = 144 \quad \Rightarrow \sum xy = 24$

again $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 36 \quad \Rightarrow xyz = \dfrac{24}{36} = \dfrac{2}{3}$

now $x^3 + y^3 + z^3 - 3xyz = (x + y + z)\left(\sum x^2 - \sum xy\right)$

$\Rightarrow \sum x^3 - 2 = (12)(96 - 24) = (12)(72) = 864$

$\Rightarrow \sum x^3 = 866$

22.(6) $\alpha + \beta = 1154$ and $\alpha\beta = 1$

$(\sqrt{\alpha} + \sqrt{\beta})^2 = \alpha + \beta + 2\sqrt{\alpha\beta} = 1154 + 2 = 1156 = (34)^2$

$\Rightarrow \sqrt{\alpha} + \sqrt{\beta} = 34$

Again $(\alpha^{1/4} + \beta^{1/4})^2 = \sqrt{\alpha} + \sqrt{\beta} + 2(\alpha\beta)^{1/4} = 34 + 2 = 36$

$\alpha^{1/4} + \beta^{1/4} = 6$

23.(4) Given $a^2 - 4a + 1 = 4 \Rightarrow a^2 + 1 = 4(1 + a)$

$y = \dfrac{(a-1)(1+a^2)}{a^2 - 1} = \dfrac{a^2+1}{a+1} = \dfrac{4(a+1)}{a+1} = 4$

24.(5) Let $ax^3 + bx^2 + cx + d = 0$ has roots p, q, r

$pq + qr + rp = \dfrac{c}{a}$ (1)

but $pq + qr + rp \le p^2 + q^2 + r^2$

$= (p + q + r)^2 - 2\sum pq$

$\therefore 3(pq + qr + rp) \le (p + q + r)^2 = 16$

$\therefore 3\dfrac{c}{a} \le 16 \Rightarrow \dfrac{c}{a} \le \dfrac{16}{3} \Rightarrow$ largest possible integral value of

$\dfrac{c}{a}$ is 5

25.(3) $x^2 + x(y - a) + y^2 - ay + 1 \ge 0\ x \in R$

$\Rightarrow (y - a)^2 - 4(y^2 - ay + 1) \le 0$

$\Rightarrow -3y^2 + 2ay + a^2 - 4 \le 0$

$\therefore 3y^2 - 2ay + 4 - a^2 \ge 0 \quad y \in R$

$D \le 0$

$\Rightarrow 4a^2 - 4 \cdot 3(4 - a^2) \le 0 \Rightarrow a^2 - 3(4 - a^2) \le 0 \Rightarrow 4a^2 - 12 \le 0$

$\therefore$ range of $a \in (-\sqrt{3}, \sqrt{3}) \Rightarrow$ number of integer $\{-1, 0, 1\}$

26.(3) We have $(\alpha + \beta)^2 = \left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)(\alpha^2 + \beta^2)$

$\Rightarrow (\alpha + \beta)^2 = \left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)[(\alpha + \beta)^2 - 2\alpha\beta]$

Substituting $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$ we have

$\dfrac{b^2}{a^2} = \dfrac{-b}{c}\left(\dfrac{b^2}{a^2} - \dfrac{2c}{a}\right)$

$\Rightarrow cb^2 + b(b^2 - 2ac) = 0$

$b \ne 0, \quad \therefore bc + b^2 - 2ac = 0$

a, b, c are in AP, $\quad \therefore b = \dfrac{a+c}{2}$

$\therefore$ we have $\dfrac{(a+c)c}{2} + \left(\dfrac{a+c}{2}\right)^2 - 2ac = 0$

$\Rightarrow a^2 - 4ac + 3c^2 = 0 \quad \Rightarrow (a - c)(a - 3c) = 0$

$a \ne c \quad \therefore a = 3c \quad \therefore \dfrac{a}{c} = 3$

27.(7) For two distinct roots, $D > 0$ i.e., $k^2 + 8(k^2 + 5) > 0$ which is always true

Also let $f(x) = -2x^2 + kx + k^2 + 5 = 0$

But $f(0) > 0$ and $f(2) < 0$

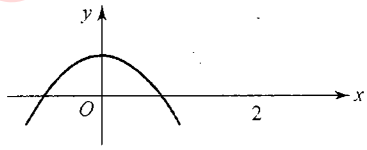


Fig. 1.89

$-8 + 2k + k^2 + 5 < 0 \quad \Rightarrow \quad k^2 + 2k - 3 < 0$

$\Rightarrow (k + 3)(k - 1) < 0$

$k \in (-3, 1) \quad \Rightarrow a = -3;\ b = 1 \quad \Rightarrow a + 10b = -3 + 10 = 7$

28.(8) $x^2 + mx + n = 0 \left\langle \begin{matrix} 2\alpha \\ 2\beta \end{matrix} \right.$ and $x^2 + px + m = 0 \left\langle \begin{matrix} \alpha \\ \beta \end{matrix} \right.$

$2(\alpha + \beta) = -m$ (1)

$4\alpha\beta = n$ (2)

and $\alpha + \beta = -p$ (3)

$\alpha\beta = m$ (4)

$\therefore$ (1) and (3) $\Rightarrow 2p = m$

and (2) and (4) $\Rightarrow 4m = n$

$\Rightarrow \dfrac{n}{p} = \dfrac{4m}{m/2} = 8$

29.(7) Let $E = \dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b}$

now $3 + E = \dfrac{a}{b+c} + 1 + \dfrac{b}{c+a} + 1 + \dfrac{c}{a+b} + 1$

$\Rightarrow 3 + E = (a + b + c)\left[\dfrac{1}{b+c} + \dfrac{1}{c+a} + \dfrac{1}{a+b}\right]$

$\Rightarrow 3 + E = 3 \times \frac{10}{3} = 10 \quad \Rightarrow E = 7$

30.(3) We have $abc + 1 = \frac{bc}{5} = \frac{-ac}{15} = \frac{ab}{3}$

$\Rightarrow \frac{a}{b} = -3$ and $\frac{c}{b} = -5$

Now $\frac{c-b}{c-a} = \frac{\frac{c}{b}-1}{\frac{c}{b}-\frac{a}{b}} = \frac{-5-1}{-5-(-3)} = 3$

31.(2) We have $\left(\frac{a^4+3a^2+1}{a^2}\right)\left(\frac{b^4+5b^2+1}{b^2}\right)\left(\frac{c^4+7c^2+1}{c^2}\right)$

$= \left(a^2+\frac{1}{a^2}+3\right)\left(b^2+\frac{1}{b^2}+5\right)\left(c^2+\frac{1}{c^2}+7\right)$

$= \left(\left(a-\frac{1}{a}\right)^2+5\right)\left(\left(b-\frac{1}{b}\right)^2+7\right)\left(\left(c-\frac{1}{c}\right)^2+9\right)$

32.(3) Let $a^2 + b^2 = x$

$1 - 2ab = (a + b)^2 - 2ab = a^2 + b^2 = x (a + b = 1)$;

also $ab = \frac{1-x}{2}$

and $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

$a^3 + b^3 = x - ab$ (1)

but $ab = \frac{1-x}{2}$

Hence $a^3 + b^3 = x - \frac{1-x}{2} = \frac{3x-1}{2}$

Hence the equation

$(1 - 2ab)(a^3 + b^3) = 12$, becomes

$x\left(\frac{3x-1}{2}\right) = 12$

$\Rightarrow 3x^2 - x - 24 = 0$

$\Rightarrow (x - 3)(3x + 8) = 0$

$\Rightarrow x = 3$ or $x = -\frac{8}{3}$ (not possible) as $x = a^2 + b^2 \not< 0$

$\therefore x = 3 \quad \Rightarrow a^2 + b^2 = 3$

33.(5) We have $2x^3 - 9x^2 + 12x + k = 0$

Let the roots are α, α, β

$2\alpha + \beta = \frac{9}{2}$ (1)

$\alpha^2 + 2\alpha\beta = \frac{12}{2} = 6$ (2)

are $\alpha^2\beta = -\frac{k}{2}$ (3)

putting $\beta = \left(\frac{9}{2} - 2\alpha\right)$ from (1) in (2)

$\alpha^2 + 2\alpha\left(\frac{9}{2} - 2\alpha\right) = 6 \quad \Rightarrow \alpha^2 + 9\alpha - 4\alpha^2 = 6$

$\Rightarrow 3\alpha^2 - 9\alpha + 6 = 0$

$\Rightarrow \alpha^2 - 3\alpha + 2 = 0$

$\Rightarrow (\alpha - 2)(\alpha - 1) = 0 \quad \Rightarrow \alpha = 2$ or 1

if $\alpha = 2$ then $\beta = \frac{1}{2}$; if $\alpha = 1$ then $\beta = \frac{5}{2}$

$\therefore k = -2(4)\frac{1}{2} = -4$ or $k = -2(1^2)\left(\frac{5}{2}\right) = -5$

34.(3) Let $\frac{1-a^3}{a} = \frac{1-b^3}{b} = \frac{1-c^3}{c} = k \Rightarrow \frac{1-x^3}{x} = k,$

where x takes 3 values a, b and c.

$\Rightarrow x^3 + kx - 1 = 0$ has roots a, b, c

Now $a + b + c = 0$ (1)

$abc = 1$ (2)

Hence $a^3 + b^3 + c^3 = 3abc = 3$

Archives

Subjective Type

1. $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$

$\Rightarrow 4^x - \frac{3^x}{\sqrt{3}} = 3^x\sqrt{3} - \frac{4^x}{2}$

$\Rightarrow \frac{3}{2}4^x = 3^x\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)$

$\Rightarrow \frac{3}{2}4^x = 3^x\frac{4}{\sqrt{3}}$

$\Rightarrow \frac{4^{x-1}}{4^{1/2}} = \frac{3^{x-1}}{\sqrt{3}}$

$\Rightarrow 4^{x-3/2} = 3^{x-3/2}$

$\Rightarrow \left(\frac{4}{3}\right)^{x-3/2} = 1$

$\Rightarrow x - \frac{3}{2} = 0$

$\Rightarrow x = 3/2$

2. We have,

$\sqrt{x+1} = 1 + \sqrt{x-1}$

Squaring both sides, we get

$x + 1 = 1 + x - 1 + 2\sqrt{x-1}$

$\Rightarrow 1 = 2\sqrt{x-1}$

$\Rightarrow 1 = 4(x - 1)$

$\Rightarrow x = 5/4$

3. Given $a > 0$. So we have two cases: $a \neq 1$ and $a = 1$. Also, it is clear that $x > 0$ and $x \neq 1$, $ax \neq 1$, $a^2x \neq 1$.

Case I: If $a > 0, \neq 1$, then given equation can be simplified as

$$\frac{2}{\log_a x} + \frac{1}{1 + \log_a x} + \frac{3}{2 + \log_a x} = 0$$

Putting $\log_a x = y$, we get

$$2(1 + y)(2 + y) + y(2 + y) + 3y(1 + y) = 0$$

$\Rightarrow$ $6y^2 + 11y + 4 = 0$

$\Rightarrow$ $y = -4/3$ and $-1/2$

$\Rightarrow$ $\log_a x = -4/3$ and $\log_a x = -1/2$

$\Rightarrow$ $x = a^{-4/3}$ and $x = a^{-1/2}$

Case II: If $a = 1$, The then equation becomes

$$2\log_x 1 + \log_x 1 + 3\log_x 1 = 5\log_x 1 = 0$$

which is true $\forall\, x > 0, \neq 1$. Hence, solution is

$$\begin{cases} x > 0, \neq 1, \text{ if } a = 1 \\ x = a^{-1/2}, a^{-4/3}, \text{ if } a > 0, \neq 1 \end{cases}$$

4. Let,

$$x = \frac{\sqrt{26 - 15\sqrt{3}}}{5\sqrt{2} - \sqrt{38 + 5\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26 - 15\sqrt{3}}{50 + 38 + 5\sqrt{3} - 10\sqrt{76 + 10\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26 - 15\sqrt{3}}{88 + 5\sqrt{3} - 10\sqrt{75 + 1 + 10\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26 - 15\sqrt{3}}{88 + 5\sqrt{3} - 10\sqrt{(5\sqrt{3})^2 + (1)^2 + 2 \times 5\sqrt{3} \times 1}}$$

$$\Rightarrow x^2 = \frac{26 - 15\sqrt{3}}{88 + 5\sqrt{3} - 10\sqrt{(5\sqrt{3} + 1)^2}}$$

$$\Rightarrow x^2 = \frac{26 - 15\sqrt{3}}{88 + 5\sqrt{3} - 10(5\sqrt{3} + 1)}$$

$$\Rightarrow x^2 = \frac{26 - 15\sqrt{3}}{78 - 45\sqrt{3}}$$

$$= \frac{26 - 15\sqrt{3}}{3(26 - 15\sqrt{3})} = \frac{1}{3},$$

which is a rational number

5. There are two parts: $(5x - 1) < (x + 1)^2$ and $(x + 1)^2 < (7x - 3)$.
Taking first part:

$(5x - 1) < (x + 1)^2$

$\Rightarrow$ $5x - 1 < x^2 + 2x + 1$

$\Rightarrow$ $x^2 - 3x + 2 > 0$

$\Rightarrow$ $(x - 1)(x - 2) > 0$

$\Rightarrow$ $x < 1$ or $x > 2$ (1)

Taking second part:

$(x + 1)^2 < (7x - 3)$

$\Rightarrow$ $x^2 - 5x + 4 < 0$

$\Rightarrow$ $(x - 1)(x - 4) < 0$

$\Rightarrow$ $1 < x < 4$ (2)

From (1) and (2), taking common values of x, we get $2 < x < 4$.

Then, integral value of x is 3 only.

6. α, β are the roots of $x^2 + px + q = 0$.

$\therefore$ $\alpha + \beta = -p$, $\alpha\beta = q$

γ, δ are the roots of $x^2 + rx + s = 0$.

$\therefore$ $\gamma + \delta = -r$, $\gamma\delta = s$

Now,

$$E = (\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$$
$$= [\alpha^2 - (\gamma + \delta)\alpha + \gamma\delta][\beta^2 - (\gamma + \delta)\beta + \gamma\delta]$$
$$= [\alpha^2 + r\alpha + s][\beta^2 + r\beta + s]$$

Also $\alpha^2 + p\alpha + q = 0$ and $\beta^2 + p\beta + q = 0$

$$\Rightarrow E = [(r - p)\alpha + (s - q)][(r - p)\beta + (s - q)]$$
$$= (r - p)^2\alpha\beta + (r - p)(s - q)(\alpha + \beta) + (s - q)^2$$
$$= q(r - p)^2 - p(r - p)(s - q) + (s - q)^2$$

Now if the equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ have a common root say α, then

$\alpha^2 + p\alpha + q = 0$

and

$\alpha^2 + r\alpha + s = 0$

$\Rightarrow$ $(q - s)^2 = (r - p)(ps - qr)$,

which is the required condition.

7. We know that for sides a, b, c of a triangle,

$(a - b)^2 \geq 0$

$\Rightarrow$ $a^2 + b^2 \geq 2ab$ (1)

Similarly,

$b^2 + c^2 \geq 2bc$ (2)

$c^2 + a^2 \geq 2ca$ (3)

Adding the three inequalities, we get

$2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$

$\Rightarrow$ $a^2 + b^2 + c^2 \geq ab + bc + ca$

Adding $2(ab + bc + ca)$ to both sides, we get

$(a + b + c)^2 \geq 3(ab + bc + ca)$

or

$3(ab + bc + ca) \leq (a + b + c)^2$ (4)

Also,

$c < a + b$ (triangle inequality)

$\Rightarrow$ $c^2 < ac + bc$ (5)

Similarly,

$b^2 < ab + bc$ (6)

$a^2 < ab + ca$ (7)

Adding (4), (5) and (6), we get

$a^2 + b^2 + c^2 < 2(ab + bc + ca)$

Adding $2(ab + bc + ca)$ to both sides, we get

$$(a + b + c)^2 < 4\,(ab + bc + ca) \qquad (8)$$

Combining (4) and (8), we get

$$3(ab + bc + ca) \le (a + b + c)^2 < 4\,(ab + bc + ca)$$

First two expressions will be equal for $a = b = c$.

8. $y = \sqrt{((x+1)(x-3))/(x-2)}$ will be real if

$$((x+1)(x-3))/(x-2) \ge 0$$

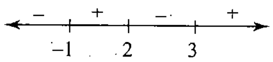


Fig. 1.90

From the sign scheme of $((x+1)(x-3))/(x-2)$, we have

$$x \in [-1, 2) \cup [3, \infty)$$

9. The given equations are

$$3x + my - m = 0$$

and

$$2x - 5y - 20 = 0$$

Solving these equations, we get

$$x = \frac{25m}{2m+15},\ y = \frac{2m-60}{2m+15}$$

For $x > 0$,

$$\frac{25m}{2m+15} > 0$$

$$\Rightarrow \quad m < -\frac{15}{2} \text{ or } m > 0 \qquad (1)$$

For $y > 0$,

$$\frac{2(m-30)}{2m+15} > 0$$

$$\Rightarrow \quad m < -\frac{15}{2} \text{ or } m > 30 \qquad (2)$$

Combining (1) and (2), we get the common values of m,

$$m < -\frac{15}{2} \text{ or } m > 30$$

$$\therefore \quad m \in \left(-\infty, \frac{-15}{2}\right) \cup (30, \infty)$$

10. The given system is

$$x + 2y + z = 1 \qquad (1)$$

$$2x - 3y - \omega = 2 \qquad (2)$$

where $x, y, z, \omega \ge 0$.

Multiplying Eq. (1) by 2 and subtracting from Eq. (2), we get

$$7y + 2z + \omega = 0$$

$$\Rightarrow \quad \omega = -(7y + 2z)$$

Now, $x, y, z, \omega \ge 0$

$$\Rightarrow \quad y = z = \omega = 0$$

$$\Rightarrow \quad x = 1$$

11. $e^{\sin x} - e^{-\sin x} - 4 = 0$

Let $e^{\sin x} = y$. Then the equation becomes

$$y - \frac{1}{y} - 4 = 0$$

$$\Rightarrow \quad y^2 - 4y - 1 = 0$$

$$\Rightarrow \quad y = 2 + \sqrt{5},\ 2 - \sqrt{5}$$

But y is real +ve number. Hence,

$$\therefore \quad y \ne 2 - \sqrt{5}$$

$$\Rightarrow \quad y = 2 + \sqrt{5}$$

$$\Rightarrow \quad e^{\sin x} = 2 + \sqrt{5}$$

$$\Rightarrow \sin x = \log_e (2 + \sqrt{5})$$

But

$$2 + \sqrt{5} > e$$

$$\Rightarrow \quad \log_e (2 + \sqrt{5}) > \log_e e$$

$$\Rightarrow \quad \log_e (2 + \sqrt{5}) > 1$$

$$\Rightarrow \quad \sin x > 1,$$

which is not possible. Therefore the given equation has no real solution.

12. For any square there can be at most four neighbouring squares.

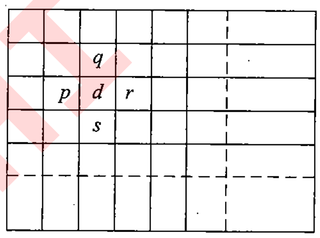


Fig. 1.91

Let for a square having largest number d, p, q, r, s be written. Then according to the question,

$$p + q + r + s = 4d$$

$$\Rightarrow \quad (d - p) + (d - q) + (d - r) + (d - s) = 0$$

Sum of four positive numbers can be zero only if these are zeros individually. Therefore,

$$d - p = d - q = d - r = d - s = 0$$

$$\Rightarrow \quad p = q = r = s = d$$

Hence, all the numbers written are same.

13. Let α, β be the roots of equation $ax^2 + bx + c = 0$. Given that $\beta = \alpha^n$. Also, $\alpha + \beta = -b/a$, $\alpha\beta = c/a$. Now,

$$\alpha\beta = \frac{c}{a} \Rightarrow \alpha\,\alpha^n = \frac{c}{a} \Rightarrow \alpha = \left(\frac{c}{a}\right)^{\frac{1}{n+1}}$$

$$\alpha + \beta = -b/a \Rightarrow \alpha + \alpha^n = \frac{-b}{a}$$

or

$$\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + \left(\frac{c}{a}\right)^{\frac{n}{n+1}} = \frac{-b}{a}$$

$$\Rightarrow \quad a\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + a\left(\frac{c}{a}\right)^{\frac{n}{n+1}} + b = 0$$

$$\Rightarrow \quad a^{\frac{n}{n+1}} c^{\frac{1}{n+1}} + a^{\frac{1}{n+1}} c^{\frac{n}{n+1}} + b = 0$$

$\Rightarrow (a^n c)^{\frac{1}{n+1}} + (a c^n)^{\frac{1}{n+1}} + b = 0$

14. $x^2 - 3x + 2 > 0, x^2 - 3x - 4 \le 0$

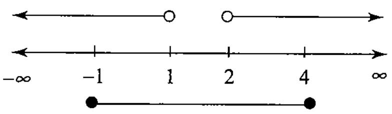


Fig. 1.92

$\Rightarrow \quad (x-1)(x-2) > 0$ and $(x-4)(x+1) \le 0$

$\Rightarrow \quad x \in (-\infty, 1) \cup (2, \infty)$ and $x \in [-1, 4]$.

Therefore, the common solution is $[-1, 1) \cup (2, 4]$

15. $(5+2\sqrt{6})(5-2\sqrt{6}) = 25 - 24 = 1$

$$\Rightarrow \quad 5 - 2\sqrt{6} = \frac{1}{5+2\sqrt{6}}$$

Hence, the given equation is

$$(5+2\sqrt{6})^{x^2-3} + \frac{1}{(5+2\sqrt{6})^{x^2-3}} = 10$$

$$\Rightarrow \quad y + \frac{1}{y} = 10, \text{ where } y = (5+2\sqrt{6})^{x^2-3}$$

$$\Rightarrow \quad y^2 - 10y + 1 = 0$$

$$\Rightarrow \quad y = \frac{10 \pm \sqrt{100-4}}{2}$$

$$\Rightarrow \quad y = 5 \pm 2\sqrt{6}$$

$$\Rightarrow \quad (5+2\sqrt{6})^{x^2-3} = 5 \pm 2\sqrt{6}$$

$$\Rightarrow \quad (5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6})$$

or

$$(5+2\sqrt{6})^{x^2-3} = \frac{1}{5+2\sqrt{6}}$$

$$\Rightarrow \quad (5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6})^1 \text{ or } (5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6})^{-1}$$

$\Rightarrow \quad x^2 - 3 = 1$ or $x^2 - 3 = -1$

$\Rightarrow \quad x^2 = 4$ or $x^2 = 2$

$\Rightarrow \quad x = \pm 2$ or $\pm \sqrt{2}$

16. The given equation is

$$x^2 - 2a|x-a| - 3a^2 = 0$$

Case I: If $x - a \ge 0$, then $|x - a| = x - a$. Hence, the equation becomes

$$x^2 - 2a(x-a) - 3a^2 = 0$$

$$\Rightarrow \quad x^2 - 2ax - a^2 = 0$$

$$\Rightarrow \quad x = \frac{2a \pm \sqrt{4a^2 + 4a^2}}{2} \Rightarrow a \pm a\sqrt{2}$$

Case II: If $x - a < 0$, then $|x - a| = -(x - a)$. Hence, the equation becomes

$$x^2 + 2a(x-a) - 3a^2 = 0$$

$$\Rightarrow \quad x^2 + 2ax - 5a^2 = 0$$

$$\Rightarrow \quad x = \frac{-2a \pm \sqrt{4a^2 + 20a^2}}{2}$$

$$\Rightarrow \quad x = \frac{-2a \pm 2a\sqrt{6}}{2}$$

$$x = -a \pm a\sqrt{6}$$

Thus, the solution set is $\{a \pm a\sqrt{2}, -a \pm a\sqrt{6}\}$.

17. We are given

$$\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$$

$$\Rightarrow \quad \frac{2x}{2x^2+5x+2} - \frac{1}{x+1} > 0$$

$$\Rightarrow \quad \frac{2x^2+2x-2x^2-5x-2}{(2x^2+5x+2)(x+1)} > 0$$

$$\Rightarrow \quad \frac{-3x-2}{(2x+1)(x+1)(x+2)} > 0$$

$$\Rightarrow \quad \frac{(3x+2)}{(x+1)(x+2)(2x+1)} < 0$$

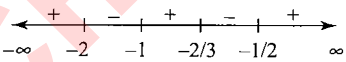


Fig. 1.93

From the sign scheme, solution is $x \in (-2, -1) \cup (-2/3, -1/2)$.

18. The given equation is

$$|x^2 + 4x + 3| + 2x + 5 = 0$$

Case I:

$$x^2 + 4x + 3 \ge 0$$

$$\Rightarrow \quad (x+1)(x+3) \ge 0$$

$$\Rightarrow \quad x \in (-\infty, -3] \cup [-1, \infty) \qquad (1)$$

Then, given equation becomes

$$x^2 + 6x + 8 = 0$$

$$\Rightarrow \quad (x+4)(x+2) = 0$$

$$\Rightarrow \quad x = -4, -2$$

But $x = -2$ does not satisfy (1); hence rejected. Therefore, $x = -4$ is the only solution.

Case II:

$$x^2 + 4x + 3 < 0$$

$$\Rightarrow \quad (x+1)(x+3) < 0$$

$$\Rightarrow \quad x \in (-3, -1) \qquad (2)$$

Then, given equation becomes

$$-(x^2 + 4x + 3) + 2x + 5 = 0$$

$$\Rightarrow \quad -x^2 - 2x + 2 = 0$$

$$\Rightarrow \quad x^2 + 2x - 2 = 0$$

$$\Rightarrow \quad x = \frac{-2 \pm \sqrt{4+8}}{2}$$

$\Rightarrow \quad x = -1 + \sqrt{3}, -1 - \sqrt{3}$

out of which $x = -1 - \sqrt{3}$ satisfies (2). Thus, $x = -4, -1 - \sqrt{3}$.

19. Let $f(x) = x^2 + (b/a)x + (c/a)$. According to the question, we have the following graph.

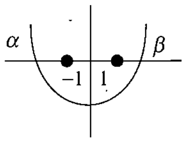


Fig. 1.94

From graph, $f(-1) < 0$ and $f(1) < 0$. So,

$$1 + \frac{c}{a} - \frac{b}{a} < 0 \text{ and } 1 + \frac{c}{a} + \frac{b}{a} < 0 \Rightarrow 1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$$

20. Refer to problems 25–26 of linked comprehension type.

21.

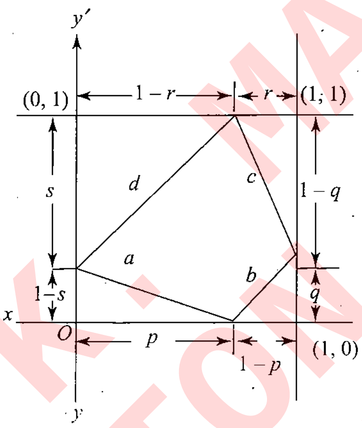


Fig. 1.95

$a^2 = p^2 + (1 - s)^2$

$b^2 = (1 - p)^2 + q^2$

$c^2 = (1 - q)^2 + r^2$

$d^2 = (1 - r)^2 + s^2$

$\therefore \quad a^2 + b^2 + c^2 + d^2 = [p^2 + (1 - p)^2] + [q^2 + (1 - q)^2] + [r^2 + (1 - r)^2] + [s^2 + (1 - s)^2]$, where $p, q, r, s \in [0, 1]$

Now consider the function

$y = x^2 + (1 - x)^2, 0 \le x \le 1$

$\Rightarrow \quad y = 2x^2 - 2x + 1$

which has vertex $(1/2, (1/2))$.

Hence, minimum value is 1/2 when $x = 1/2$ and maximum value is at $x = 1$, which is 1. Therefore, minimum value of $a^2 + b^2 + c^2 + d^2$ is $1/2 + 1/2 + 1/2 + 1/2 = 2$ and maximum value is $1 + 1 + 1 + 1 = 4$.

22. Let us consider the integral values of x as 0, 1, −1. Then $f(0)$, $f(1)$ and $f(-1)$ are all integers. Therefore, C, $A + B + C$ and $A - B + C$ are all integers.

Therefore, C is integer and hence, $A + B$ is an integer and also $A - B$ is an integer.

$2A = (A + B) + (A - B)$

Therefore, $2A$, $A + B$ and C are all integers. Conversely, let $n \in I$. Then,

$$f(n) = An^2 + Bn + C = 2A\left[\frac{n(n-1)}{2}\right] + (A + B)n + C$$

Now, A, $A + B$ and C are all integers and

$$\frac{n(n-1)}{2} = \frac{\text{Even number}}{2} = \text{integer}$$

Therefore, $f(n)$ is also an integer.

23. We know that

$(\alpha - \beta)^2 = [(\alpha + \delta) - (\beta + \delta)]^2$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = (\alpha + \delta + \beta + \delta)^2 - 4(\alpha + \delta)(\beta + \delta) \quad (1)$

Now, here

$$\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$$

and

$$(\alpha + \delta) + (\beta + \delta) = -\frac{B}{A}, (\alpha + \delta)(\beta + \delta) = \frac{C}{A}$$

$$\Rightarrow \quad \frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2} \quad \text{[From (1)]}$$

24. $\alpha + \beta = \dfrac{-b}{a}, \alpha\beta = \dfrac{c}{a}$

Roots of the equation $a^3x^2 + abcx + c^3 = 0$ are

$$x = \frac{-abc \pm \sqrt{(abc)^2 - 4a^3c^2}}{2a^3}$$

$$= \left(-\frac{b}{a}\right)\left(\frac{c}{a}\right) \pm \frac{\sqrt{\left(\frac{b}{a}\right)^2\left(\frac{c}{a}\right)^2 - 4\left(\frac{c}{a}\right)^3}}{2}$$

$$= \frac{(\alpha + \beta)(\alpha\beta) \pm \sqrt{(\alpha + \beta)^2(\alpha\beta)^2 - 4(\alpha\beta)^3}}{2}$$

$$= (\alpha\beta)\frac{[(\alpha + \beta) \pm \sqrt{(\alpha - \beta)^2}]}{2}$$

$$= \alpha\beta\frac{[(\alpha + \beta) \pm (\alpha - \beta)]}{2}$$

$$= \alpha^2\beta, \alpha\beta^2$$

25. The given equation is

$x^2 + (a - b)x + (1 - a - b) = 0, a, b \in R$

For this equation to have unequal real roots $\forall b$,

$D > 0$

$\Rightarrow \quad (a - b)^2 - 4(1 - a - b) > 0$

$\Rightarrow \quad a^2 + b^2 - 2ab - 4 + 4a + 4b > 0$

$\Rightarrow \quad b^2 + b(4 - 2a) + a^2 + 4a - 4 > 0 \quad (1)$

which is a quadratic expression in b, and it will be true $\forall b \in R$. Then its discriminant will be less than 0. Hence,

$(4 - 2a)^2 - 4(a^2 + 4a - 4) < 0$

$\Rightarrow \quad (2 - a)^2 - (a^2 + 4a - 4) < 0$

$\Rightarrow \quad 4 - 4a + a^2 - a^2 - 4a + a < 0$

$\Rightarrow \quad -8a + 8 < 0$

$\Rightarrow \quad a > 1$

26. Roots of $x^2 - 10cx - 11d = 0$ are a and b. Hence,

$a + b = 10c$ and $ab = -11d$

c and d are the roots of $x^2 - 10ax - 11b = 0$. Hence,

$$c + d = 10a \text{ and } cd = -11b$$

$\Rightarrow$ $a + b + c + d = 10(a + c)$ and $abcd = 121\, bd$

$\Rightarrow$ $b + d = 9(a + c)$ and $ac = 121$

Also, we have

$$a^2 - 10ac - 11d = 0 \text{ and } c^2 - 10ac - 11b = 0$$

$\Rightarrow$ $a^2 + c^2 - 20ac - 11(b + d) = 0$

$\Rightarrow$ $(a + c)^2 - 22 \times 121 - 99(a + c) = 0$

$\Rightarrow$ $a + c = 121$ or -22

For $a + c = -22$, we get $a = c$. Rejecting these values, we have $a + c = 121$. Therefore,

$$a + b + c + d = 10(a + c) = 1210$$

Objective Type

Fill in the blanks

1. Given polynomial is

$$(x - 1)(x - 2)(x - 3) \cdots (x - 100)$$

$$= x^{100} - (1 + 2 + 3 + \cdots + 100)\, x^{99} + (\cdots)x^{98} \cdots$$

Hence, coefficient of x^{99} is

$$-(1 + 2 + 3 + \cdots + 100) = \frac{-100 \times 101}{2}$$

$$= -5050$$

2. As p and q are real and one root is $2 + i\sqrt{3}$, so the other root must be $2 - i\sqrt{3}$. Then,

$$p = -(\text{sum of roots}) = -4$$

$$q = \text{product of roots} = (2 + i\sqrt{3})(2 - i\sqrt{3}) = 4 + 3 = 7$$

3. Given equation is

$$x^2 - 3kx + 2e^{2\ln k} - 1 = 0$$

$\Rightarrow$ $x^2 - 3kx + (2k^2 - 1) = 0$

Here, product of roots is $2k^2 - 1$.

$\therefore$ $2k^2 - 1 = 7 \Rightarrow k^2 = 4 \Rightarrow k = 2, -2$

Now for real roots, we must have

$$D \geq 0$$

$\Rightarrow$ $9k^2 - 4(2k^2 - 1) \geq 0$

$\Rightarrow$ $k^2 + 4 \geq 0$

which is true for all k. Thus, $k = 2, -2$. But for $k = -2$, $\ln k$ is not defined. Therefore, rejecting $k = -2$, we get $k = 2$.

4. By observation, one root is $x = 1$,

$\Rightarrow$ $a + b = -1$

5. Given $x < 0$, $y < 0$.

$$x + y + \frac{x}{y} = \frac{1}{2} \text{ and } (x + y)\frac{x}{y} = -\frac{1}{2}$$

Let,

$$x + y = a \text{ and } \frac{x}{y} = b \qquad (1)$$

Therefore, we get

$$a + b = \frac{1}{2},\ ab = -\frac{1}{2}$$

Solving these two, we get

$$a + \left(-\frac{1}{2a}\right) = \frac{1}{2}$$

$\Rightarrow$ $2a^2 - a - 1 = 0$

$\Rightarrow$ $a = 1, -1/2$

$\Rightarrow$ $b = -1/2, 1$

$\therefore$ (1) $\Rightarrow x + y = 1$ and $\dfrac{x}{y} = -\dfrac{1}{2}$

or

$$x + y = -\frac{1}{2} \text{ and } \frac{x}{y} = 1$$

But $x, y < 0$

$\therefore$ $x + y < 0 \Rightarrow x + y = \dfrac{-1}{2}$ and $\dfrac{x}{y} = 1$

On solving, we get $x = -1/4$ and $y = -1/4$.

6. $|x - 2|^2 + |x - 2| - 2 = 0$

$\Rightarrow$ $(|x - 2| + 2)(|x - 2| - 1) = 0$

$\Rightarrow$ $|x - 2| - 1 = 0$

$\Rightarrow$ $x - 2 = \pm 1$

$\Rightarrow$ $x = 1, 3$

Therefore, the sum of the roots is $3 + 1 = 4$.

7. $\log_7 \log_5 (\sqrt{x + 5} + \sqrt{x}) = 0$

$\Rightarrow$ $\log_5 (\sqrt{x + 5} + \sqrt{x}) = 1$

$\Rightarrow$ $(\sqrt{x + 5} + \sqrt{x}) = 5$

$\Rightarrow$ $x + 5 = 25 + x - 10\sqrt{x}$

$\Rightarrow$ $2 = \sqrt{x}$

$\Rightarrow$ $x = 4$

which satisfies the given equation

True or false

1. False.

$$2x^2 + 3x + 1 = 0$$

$\Rightarrow$ $(2x + 1)(x + 1) = 0$

$\Rightarrow$ $x = -1, -1/2$, both are rational

2. True. Given equation is

$$(x - a)(x - c) + 2(x - b)(x - d) = 0$$

Let,

$$f(x) = (x - a)(x - c) + 2(x - b)(x - d)$$

$$f(b) = (b - a)(b - c) < 0$$

$$f(d) = (d - a)(d - c) > 0$$

Thus,

$$f(b)\, f(d) < 0$$

Therefore, one root lies between b and d; hence the roots are real.

3. False. Consider $N = n_1 + n_2 + n_3 + \cdots + n_p$, where N is an even number. Let k numbers among these p numbers be odd, then $p - k$ are even numbers.

Now, sum of $p - k$ even numbers is even and for N to be an even number, sum of k odd numbers must be even, which is possible only when k is even.

4. True. We have $P(x) = ax^2 + bx + c$, for which

$$D_1 = b^2 - 4ac \qquad (1)$$

and $Q(x) = -ax^2 + dx + c$, for which

$$D_2 = d^2 + 4ac \qquad (2)$$

Given that $ac \neq 0$. Following two cases are possible.

If $ac > 0$, then from Eq. (2), D_2 is +ve $\Rightarrow Q(x)$ has real roots.

If $ac < 0$, then from Eq. (1), D_1 is +ve $\Rightarrow P(x)$ has real roots.
Thus, $P(x)\,Q(x) = 0$ has at least two real roots.

Multiple choice questions with one correct answer

1. c. l, m, n are real and $l \neq m$. Given equation is

$$(l-m)x^2 - 5(l+m)x - 2(l-m) = 0$$
$$D = 25(l+m)^2 + 8(l-m)^2 > 0,\ l, m \in R$$

Therefore, the roots are real and unequal.

2. a. $u = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$

$$= \frac{1}{2}\left[2x^2 + 8y^2 + 18z^2 - 12yz - 6zx - 4xy\right]$$
$$= \frac{1}{2}\left[(x^2 - 4xy + 4y^2) + (4y^2 + 9z^2 - 12yz) + (x^2 + 9z^2 - 6zx)\right]$$
$$= \frac{1}{2}\left[(x-2y)^2 + (2y-3z)^2 + (3z-x)^2\right] \geq 0$$

Hence, u is always non-negative.

3. c. As $a, b, c > 0$, so a, b, c should be real (note that other relation is not defined in the set of complex numbers). Therefore, the roots of equation are either real or complex conjugate.

Let α, β be the roots of $ax^2 + bx + c = 0$. Then,

$$\alpha + \beta = -\frac{b}{a} = -\text{ve and } \alpha\beta = \frac{c}{a} = +\text{ve}$$

Hence, either both α, β are –ve (if roots are real) or both α, β have –ve real part (if roots are complex conjugate).

4. b. The given equation is

$$(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b) = 0$$
$$\Rightarrow 3x^2 - 2(a+b+c)x + (ab+bc+ca) = 0$$
$$D = 4(a+b+c)^2 - 12(ab+bc+ca)$$
$$= 4[a^2 + b^2 + c^2 - ab - bc - ca]$$
$$= 2[(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0,\ \forall\ a, b, c$$

Therefore, the roots of the given equation are always real.

5. c.

$$\begin{array}{r|l} & ax - ap \\ \hline x^2 + px + 1 & ax^3 + bx + c \\ & ax^3 + apx^2 + ax \\ \hline & -apx^2 + (b-a)x + c \\ & -apx^2 - ap^2x - ap \\ \hline & (b - a + ap^2)x + c + ap \end{array}$$

Now, remainder must be zero. Hence,

$$b - a + ap^2 = 0 \text{ and } c + ap = 0$$
$$\Rightarrow p = -\frac{c}{a} \text{ and } p^2 = \frac{a-b}{a}$$
$$\Rightarrow \left(\frac{-c}{a}\right)^2 = \frac{a-b}{a}$$
$$\Rightarrow c^2 = a^2 - ab$$
$$\Rightarrow a^2 - c^2 = ab$$

6. a. $|x|^2 - 3|x| + 2 = 0$

$$\Rightarrow (|x| - 2)(|x| - 1) = 0$$
$$\Rightarrow |x| = 1 \text{ or } 2$$
$$\Rightarrow x = \pm 1, \pm 2$$

Hence, there are four real solutions.

7. c. Let the distance of the school from A be x. Therefore, the distance of the school from B is $60 - x$. The total distance covered by 200 students is

$$[150x + 50(60 - x)] = [100x + 3000]$$

This is minimum when $x = 0$. Hence, the school should be at town A.

8. d. Given expression is

$$x^{12} - x^9 + x^4 - x + 1 = f(x)$$

For $x < 0$, put $x = -y$, where $y > 0$. Thus, we get

$$f(x) = y^{12} + y^9 + y^4 + y + 1 > 0 \text{ for } y > 0$$

For $0 < x < 1$,

$$x^9 < x^4 \Rightarrow -x^9 + x^4 > 0$$

Also,

$$1 - x > 0 \text{ and } x^{12} > 0$$
$$\Rightarrow x^{12} - x^9 + x^4 + 1 - x > 0 \Rightarrow f(x) > 0$$

For $x > 1$,

$$f(x) = x(x^3 - 1)(x^8 + 1) + 1 > 0$$

So $f(x) > 0$ for $-\infty < x < \infty$.

9. a. Given equation is

$$x - \frac{2}{x-1} = 1 - \frac{2}{x-1}$$

Clearly, $x \neq 1$ for the given equation to be defined if $x - 1 \neq 0$. We can cancel the common term $-2/(x-1)$ on both sides to get $x = 1$, but it is not possible. So, given equation has no roots.

10. c. Given that

$$a^2 + b^2 + c^2 = 1 \qquad (1)$$

We know that

$$(a+b+c)^2 \geq 0$$
$$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \geq 0$$
$$\Rightarrow 2(ab + bc + ca) \geq -1 \quad \text{[Using (1)]}$$
$$\Rightarrow ab + bc + ca \geq -1/2 \qquad (2)$$

Also, we know that

$$\frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] \geq 0$$
$$\Rightarrow a^2 + b^2 + c^2 - ab\ \ bc - ca \geq 0$$
$$\Rightarrow ab + bc + ca \leq 1 \quad \text{[Using (1)]} \qquad (3)$$

Combining (2) and (3), we get

$$-1/2 \leq ab + bc + ca \leq 1$$
$$\Rightarrow ab + bc + ca \in [-1/2, 1]$$

11. a. α, β are roots of $x^2 + px + q = 0$. Hence,

$$\alpha + \beta = -p \text{ and } \alpha\beta = q$$

α^4, β^4 are roots of $x^2 - rx + s = 0$. Hence,

$$\alpha^4 + \beta^4 = r,\ \alpha^4\beta^4 = q$$

Now for equation $x^2 - 4qx + 2q^2 - r = 0$, the product of roots is

$$2q^2 - r = 2(\alpha\beta)^2 - (\alpha^4 + \beta^4)$$
$$= -(\alpha^2 - \beta^2)^2$$
$$< 0$$

Therefore, the product of roots is negative. So, the roots must be real and of opposite signs.

12. d. We know that if $f(\alpha)$ and $f(\beta)$ are of opposite signs then there must be a value γ between α and β such that $f(\gamma) = 0$. Hence, a, b, c are real numbers and $a \neq 0$. As α is a root of $a^2x^2 + bx + c = 0$, so

$a^2\alpha^2 + b\alpha + c = 0$ (1)

Also, β is a root of $a^2x^2 - bx - c = 0$, so

$a^2\beta^2 - b\beta - c = 0$ (2)

Now, let $f(x) = a^2x^2 + 2bx + 2c$. Then,

$$\begin{aligned} f(\alpha) &= a^2\alpha^2 + 2b\alpha + 2c \\ &= a^2\alpha^2 + 2(b\alpha + c) \\ &= a^2\alpha^2 + 2(-a^2\alpha^2) \quad \text{[Using (1)]} \\ &= -a^2\alpha^2 < 0 \end{aligned}$$

and

$$\begin{aligned} f(\beta) &= a^2\beta^2 + 2b\beta + 2c \\ &= a^2\beta^2 + 2(b\beta + c) \\ &= a^2\beta^2 + 2(a^2\beta^2) \quad \text{[Using (2)]} \\ &= 3a^2\beta^2 > 0 \end{aligned}$$

Since $f(\alpha)$ and $f(\beta)$ are of opposite signs and γ is a root of equation $f(x) = 0$, therefore, γ must lie between α and β. Thus, $\alpha < \gamma < \beta$.

13. a. The given equation is $\sin(e^x) = 5^x + 5^{-x}$. We know that 5^x and 5^{-x} both are +ve real numbers.

Now, $5^x + 5^{-x} = \left(\sqrt{5^x} - \sqrt{5^{-x}}\right)^2 + 2 \ge 2$

But L.H.S. = $\sin(e^x) \le 1$

Hence, no solution.

14. c. α, β are roots of the equation $(x-a)(x-b) = c, c \ne 0$.

$\therefore \quad (x-a)(x-b) - c = (x-\alpha)(x-\beta)$

$\Rightarrow \quad (x-\alpha)(x-\beta) + c = (x-a)(x-b)$

Hence, the roots of $(x-\alpha)(x-\beta) + c = 0$ are a and b.

15. a. Minimum value of $5x^2 + 2x + 3$ is

$$-\frac{D}{4a} = -\frac{(2)^2 - 4(5)(3)}{4(5)} > 2$$

where maximum value of $2\sin x$ is 2. Therefore, the two curves do not meet at all.

16. b. For real roots,

$q^2 - 4pr \ge 0$

$\Rightarrow \left(\dfrac{p+r}{2}\right)^2 - 4pr \ge 0$ ($\because$ p, q, r are in A.P.)

$\Rightarrow p^2 + r^2 - 14pr \ge 0$

$\Rightarrow \dfrac{p^2}{r^2} - 14\dfrac{p}{r} + 1 \ge 0$

$\Rightarrow \left(\dfrac{p}{r} - 7\right)^2 - 48 \ge 0$

$\Rightarrow \left|\dfrac{p}{r} - 7\right| \ge 4\sqrt{3}$

17. a. The given equation is

$$\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$$

Squaring both sides, we get

$$x + 1 + x - 1 - 2\sqrt{x^2-1} = 4x - 1$$

$\Rightarrow \quad -2\sqrt{x^2-1} = 2x - 1$

Again squaring both sides, we get

$\Rightarrow \quad 4(x^2-1) = 4x^2 - 4x + 1$

$\Rightarrow \quad -4x = -5$

$\Rightarrow \quad x = 5/4$

Substituting this value of x in given equation, we get

$$\sqrt{\frac{5}{4}+1} - \sqrt{\frac{5}{4}-1} = \sqrt{4 \times \frac{5}{4} - 1}$$

$\Rightarrow \quad \dfrac{3}{2} - \dfrac{1}{2} = 2$ (not satisfied)

Therefore, 5/4 is not a solution of given equation. Hence, the given equation has no solution.

18. a. If both the roots of a quadratic equation $ax^2 + bx + c = 0$ are less than k, then $af(k) > 0$, $-b/2a < k$ and $D \ge 0$. Now,

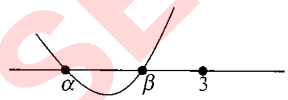


Fig. 1.96

$f(x) = x^2 - 2ax + a^2 + a - 3$

$\Rightarrow \quad f(3) > 0, a < 3, -4a + 12 \ge 0$

$\Rightarrow \quad a^2 - 5a + 6 > 0, a < 3, -4a + 12 \ge 0$

$\Rightarrow \quad a < 2$ or $a > 3, a < 3, a \le 3$

$\Rightarrow \quad a < 2$

19. b. Here $D = b^2 - 4c > 0$ because $c < 0 < b$. So, roots are real and unequal. Now,

$\alpha + \beta = -b < 0$ and $\alpha\beta = c < 0$

Therefore, one root is positive and the other root is negative, the negative root being numerically bigger. As $\alpha < \beta$, so α is the negative root while β is the positive root. So, $|\alpha| > \beta$ and $\alpha < 0 < \beta < |\alpha|$.

20. d. Given equation is

$(x-a)(x-b) - 1 = 0$

Let $f(x) = (x-a)(x-b) - 1$. Then,

$f(a) = -1$ and $f(b) = -1$

Also, graph of $f(x)$ is concave upward; hence, a and b lie between the roots. Also, if $b > a$, then one root lies in $(-\infty, a)$ and the other root lies in $(b, +\infty)$.

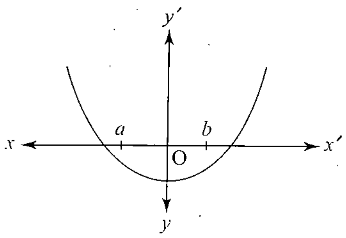


Fig. 1.97

21. c. Let α, α^2 be the roots of $3x^2 + px + 3 = 0$. Now,

$S = \alpha + \alpha^2 = -p/3, p = \alpha^3 = 1$

$\Rightarrow \quad \alpha = 1, \omega, \omega^2 \quad \left(\text{where } \omega = \dfrac{-1+\sqrt{3}i}{2}\right)$

$\alpha + \alpha^2 = -p/3 \Rightarrow \omega + \omega^2 = -p/3$

$\Rightarrow \quad -1 = -p/3 \Rightarrow p = 3$

22. d. Minimum value of $f(x) = (1 + b^2)x^2 + 2bx + 1$ is

$$m(b) = -\frac{(2b)^2 - 4(1+b^2)}{4(1+b^2)} = \frac{1}{1+b^2}$$

Clearly, $m(b)$ has range (0, 1].

23. a. Clearly, $\alpha + \beta = 1$, $\alpha\beta = p$, $\gamma + \delta = 4$, $\gamma\delta = q$ ($p, q \in I$).

Since $\alpha, \beta, \gamma, \delta$ are in G.P. (with common ratio r), so

$\alpha + \alpha r = 1$, $\alpha(r^2 + r^3) = 4$

$\Rightarrow$ $\alpha(1 + r) = 1$, $\alpha r^2 (1 + r) = 4$

$\Rightarrow$ $r^2 \times 1 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2, -2$

If $r = 2$,

$\alpha + 2\alpha = 1 \Rightarrow \alpha = \frac{1}{3}$

If $r = -2$,

$\alpha - 2\alpha = 1 \Rightarrow \alpha = -1$

But $p = \alpha\beta \in I$

$\therefore$ $r = -2$ and $\alpha = -1$

$\Rightarrow$ $p = -2$,

$q = \alpha^2 r^5 = 1\,(-2)^5 = -32$

24. b. $x^2 - |x + 2| + x > 0$

$\Rightarrow$ $x^2 + 2 > |x + 2|$

Let us draw the graphs of $y = x^2 + 2$ and $y = |x + 2|$.

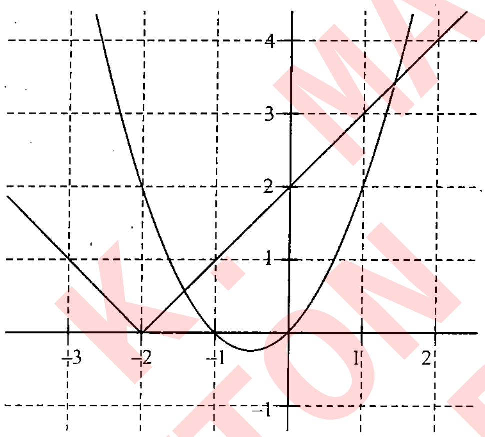


Fig. 1.98

Solving $y = |x + 2|$ and $y = x^2 + x$ for their points of intersection, we have

$$x + 2 = x^2 + x \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

Hence, solution of $x^2 + 2 > |x + 2|$ is $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

25. d. $f(x) = x^2 + 2bx + 2c^2$

$= (x + b)^2 + 2c^2 - b^2$

$g(x) = -x^2 - 2cx + b^2$

$= -(x + c)^2 + b^2 + c^2$

Given that

$\min f(x) > \max g(x)$

$\Rightarrow$ $2c^2 - b^2 > b^2 + c^2$

$\Rightarrow$ $c^2 > 2b^2$

$\Rightarrow$ $|c| > |b|\sqrt{2}$

26. b. $x^2 + 2ax + 10 - 3a > 0$, $\forall\, x \in R$

$\Rightarrow$ $D < 0$

$\Rightarrow$ $4a^2 - 4(10 - 3a) < 0$

$\Rightarrow$ $a^2 + 3a - 10 < 0$

$\Rightarrow$ $(a + 5)(a - 2) < 0$

$\Rightarrow$ $a \in (-5, 2)$

27. a. α and α^2 are the roots of the equation $x^2 + px + q = 0$. Hence,

$\alpha + \alpha^2 = -p$ (1)

and

$\alpha\alpha^2 = q \Rightarrow \alpha^3 = q$ (2)

Cubing (1),

$\alpha^3 + \alpha^6 + 3\alpha\,\alpha^2(\alpha + \alpha^2) = -p^3$

$\Rightarrow$ $q + q^2 + 3q(-p) = -p^3$

$\Rightarrow$ $p^3 + q^2 - q\,(3p - 1) = 0$

28. d. $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$ are in G.P. Hence,

$(\alpha^2 + \beta^2)^2 = (\alpha + \beta)(\alpha^3 + \beta^3)$

$\Rightarrow$ $\alpha\beta(\alpha - \beta)^2 = 0$

$\Rightarrow$ $c\Delta = 0$

29. a. a, b, c are sides of a triangle and $a \ne b \ne c$.

$\therefore$ $|a - b| < |c| \Rightarrow c^2 + b^2 - 2ab < c^2$

Similarly, we have

$b^2 + c^2 - 2bc < a^2$

and

$c^2 + a^2 - 2ca < b^2$

On adding, we get

$a^2 + b^2 + c^2 < 2(ab + bc + ca)$

$\Rightarrow$ $\dfrac{a^2 + b^2 + c^2}{ab + bc + ca} < 2$ (1)

Since the roots of the given equation are real, therefore

$(a + b + c)^2 - 3\lambda(ab + bc + ca) \ge 0$

$\Rightarrow$ $\dfrac{a^2 + b^2 + c^2}{ab + bc + ca} > 3\lambda - 2$ (2)

From (1) and (2), we get

$3\lambda - 2 < 2 \Rightarrow \lambda < \frac{4}{3}$

30. d. α, β are the roots of $x^2 - px + r = 0$. Hence,

$\alpha + \beta = p$ (1)

and

$\alpha\beta = r$ (2)

Also, $\alpha/2$, 2β are the roots of $x^2 - qx + r = 0$. Hence,

$\dfrac{\alpha}{2} + 2\beta = q$ (3)

or

$\alpha + 4\beta = 2q$

Solving (1) and (3) for α and β, we get

$\beta = \frac{1}{3}(2q - p)$ and $\alpha = \frac{2}{3}(2p - q)$

Substituting values of α and β, in Eq. (2), we get

$\frac{2}{9}(2p - q)(2q - p) = r$

31. b. $\alpha^3 + \beta^3 = q$

$\Rightarrow$ $(\alpha + \beta) - 3\alpha\beta(\alpha + \beta) = q$

$\Rightarrow$ $-p^3 + 3p\alpha\beta = q \Rightarrow \alpha\beta = \dfrac{q + p^3}{3p}$

Required equation is

$$x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=0$$

$$x^2-\frac{(\alpha^2+\beta^2)}{\alpha\beta}x+1=0$$

$$\Rightarrow\quad x^2-\left(\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}\right)x+1=0$$

$$\Rightarrow\quad x^2-\frac{p^2-2\left(\dfrac{p^3+q}{3p}\right)}{\dfrac{p^3+q}{3p}}x+1=0$$

$$\Rightarrow\quad (p^3+q)x^2-(3p^3-2p^3-2q)x+(p^3+q)=0$$

$$\Rightarrow\quad (p^3+q)x^2-(p^3-2q)x+(p^3+q)=0.$$

32. b. $x^2+bx-1=0$

$x^2+x+b=0$ (1)

Common root is $(b-1)x-1-b=0$

$$\Rightarrow\quad x=\frac{b+1}{b-1}$$

This value of x satisfies equation (1)

$$\Rightarrow\quad \frac{(b+1)^2}{(b-1)^2}+\frac{b+1}{b-1}+b=0$$

$$\Rightarrow\quad b=\sqrt{3}\,i,-\sqrt{3}\,i,0$$

33. c. $a_n=\alpha^n-\beta^n$

Also $\alpha^2-6\alpha-2=0$

Multiply with α^8 on both sides

$\Rightarrow\quad \alpha^{10}-6\alpha^9-2\alpha^8=0$ (1)

similarly $\beta^{10}-6\beta^9-2\beta^8=0$ (2)

Subtracting (2) from (1) we have

$\alpha^{10}-\beta^{10}-6(\alpha^9-\beta^9)=2(\alpha^8-\beta^8)$

$$\Rightarrow a_{10}-6a_9=2a_8\Rightarrow\frac{a_{10}-2a_8}{2a_9}=3.$$

Multiple choice questions with one or more than one correct answer

1. c, d. Let,

$$y=\frac{(x-a)(x-b)}{(x-c)}$$

$\Rightarrow\quad (x-c)y=x^2-(a+b)x+ab$

$\Rightarrow\quad x^2-(a+b+y)x+ab+cy=0$

Since x is real, so

$D\ge 0$

$\Rightarrow\quad (a+b+y)^2-4(ab+cy)\ge 0,\ \forall\, x\in R$

$\Rightarrow\quad y^2+2y(a+b-2c)+(a-b)^2\ge 0,\ \forall\, x\in R$

$\Rightarrow\quad 4(a+b-2c)^2-4(a-b)^2<0$

$\Rightarrow\quad (a+b-2c+a-b)(a+b-2c-a+b)<0$

$\Rightarrow\quad 4(a-c)(b-c)<0$

$\Rightarrow\quad a-c<0$ and $b-c>0$ or $a-c>0$ and $b-c<0$

$\Rightarrow\quad a<c<b$ or $a>c>b$

2. a, d. We have,

$$f(x)=\frac{2x-1}{2x^3+3x^2+x}=\frac{2x-1}{x(2x+1)(x+1)}$$

Critical points are $x=1/2, 0, -1/2, -1$.

On number line by sign scheme method, we have

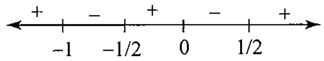


Fig. 1.99

For $f(x)>0$, $x\in(-\infty,-1)\cup(-1/2,0)\cup(1/2,\infty)$. Clearly, S contains $(-\infty,-3/2)$ and $(1/2,3)$.

3. a, b, c.

$$x^{\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}}=\sqrt{2}$$

$$\Rightarrow\quad \left(\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}\right)\log_2 x=\log_2\sqrt{2}$$

(taking logarithm both sides on base 2)

$$\Rightarrow\quad \left(\frac{3}{4}t^2+t-\frac{5}{4}\right)t=\frac{1}{2}\ \ \text{(putting } \log_2 x=t\text{)}$$

$\Rightarrow\quad 3t^3+4t^2-5t-2=0$

$\Rightarrow\quad 3t^3-3t^2+7t^2-7t+2t-2=0$

$\Rightarrow\quad (3t^2+7t+2)(t-1)=0$

$\Rightarrow\quad (3t+1)(t+2)(t-1)=0$

$$\Rightarrow\quad t=\log_2 x=1,-2,-\frac{1}{3}$$

$$\Rightarrow\quad x=2,2^{-2},2^{-\frac{1}{3}}$$

Assertion and Reasoning

1. b. Suppose the roots are imaginary. Then

$$\beta=\bar{\alpha}\ \text{ and }\ \frac{1}{\beta}=\bar{\alpha}\Rightarrow\beta=\frac{1}{\beta}$$

which is not possible. The roots are real, so

$(p^2-q)(b^2-ac)\ge 0$

Hence, statement 1 is correct.

Also, $-2b/a=\alpha+\beta$ and $\alpha/\beta=c/a$, $\alpha+\beta=-2p$, $\alpha\beta=q$. If $\beta=1$, then

$\alpha=q\Rightarrow c=qa$ (which is not possible)

Also,

$$s+1=\frac{-2b}{a}\Rightarrow -2p=\frac{-2b}{a}\Rightarrow b=ap\quad\text{(which is not possible)}$$

Hence, statement 2 is correct, but it is not correct explanation of statement 1.

Integer type

1. b. Let $f(x)=x^4-4x^3+12x^2+x-1=0$

$f'(x)=4x^3-12x^2+24x+1$

$f''(x)=12x^2-24x+24=12(x^2-2x+2)>0$

$\Rightarrow\quad f''(x)=0$ has imaginary roots

$\Rightarrow\quad f(x)=0$ has maximum 2 distinct real roots.

CHAPTER
2

Complex Numbers

INTRODUCTION

If a, b are natural numbers such that $a > b$, then the equation $x + a = b$ is not solvable in N, the set of natural numbers, i.e. there is no natural number satisfying the equation $x + a = b$. So, the set of natural numbers is extended to form the set I of integers in which every equation of the form $x + a = b$ such that $a, b \in N$ is solvable.

But equations of the form $xa = b$, where $a, b \in I, a \neq 0$ are not solvable in I also. Therefore the set I of integers is extended to obtain the set Q of all rational numbers in which every equation of the form $xa = b$, $a \neq 0$, $a, b \in I$ is uniquely solvable.

The equations of the form $x^2 = 2$, $x^2 = 3$ etc. are not solvable in Q because there is no rational number whose square is 2. Such numbers are known as irrational numbers. The set Q of all rational numbers is extended to obtain the set R which included both rational and irrational numbers. This set is known as the set of real numbers.

The equations of the form $x^2 + 1 = 0$, $x^2 + 4 = 0$, etc. are not solvable in R, i.e. there is no real number whose square is a negative real number. Euler was the first mathematician to introduce the symbol i (iota) for the square root of -1 with the property $i^2 = -1$. He also called this symbol as the imaginary number.

DEFINITION OF COMPLEX NUMBERS

A number of the form $x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$ is called a complex number and 'i' is called iota. A complex number is usually denoted by z and the set of complex numbers is denoted by C.

$$C = \{x + iy; x \in R, y \in R, i = \sqrt{-1}\ \}$$

For example, $5 + 3i$, $-1 + i$, $0 + 4i$, $4 + 0i$, etc. are complex numbers.

Here 'x' is called the real part of z and 'y' is known as the imaginary part of z. The real part of z is denoted by Re(z) and the imaginary part by Im(z). If $z = 3 - 4i$, then Re(z) = 3 and Im(z) = – 4.

Note that the sign '+' does not indicate addition as normally understood, nor does the symbol i denote a number. These things are parts of the scheme used to express numbers of a new class and they signify the pair of real numbers (x, y) to form a single complex number.

A complex number z is purely real if its imaginary part is zero, i.e. Im(z) = 0 and purely imaginary if its real part is zero, i.e. Re(z) = 0.

Note:

1. *For any positive real number a, we have* $\sqrt{-a} = \sqrt{-1 \times a}$ $= \sqrt{-1}\sqrt{a} = i\sqrt{a}$.
2. *The property* $\sqrt{a}\sqrt{b} = \sqrt{ab}$ *is valid only if at least one of a and b is non-negative. If a and b are both negative, then* $\sqrt{a}\sqrt{b} = -\sqrt{|a||b|}$.
3. *Inequality in complex numbers are never talked. If* $a + ib > c + id$ *has to be meaningful then* $b = d = 0$. *Equalities however in complex numbers are meaningful. Two complex numbers* $z_1 = a_1 + ib_1$ *and* $z_2 = a_2 + ib_2$ *are equal if* $a_1 = a_2$ *and* $b_1 = b_2$ *i.e.* $Re(z_1) = Re(z_2)$ *and* $Im(z_1) = Im(z_2)$.
4. *In real number system* $a^2 + b^2 = 0 \Rightarrow a = 0 = b$. *But if* z_1 *and* z_2 *are complex numbers then* $z_1^2 + z_2^2 = 0$ *does not imply* $z_1 = z_2 = 0$, *e.g.* $z_1 = 1 + i$ *and* $z_2 = 1 - i$. *However, if the product of two complex numbers is zero then at least one of them must be zero, same as in case of real numbers.*

Integral Power of Iota (i)

Since $i = \sqrt{-1}$, we have $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$.
To find the value of i^n $(n > 4)$, first divide n by 4.
Let q be the quotient and r be the remainder,
i.e. $n = 4q + r$ where $0 \le r \le 3$
$i^n = i^{4q+r} = (i^4)^q\,(i)^r = (1)^q(i)^r = i^r$
In general, we have the following results: $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, where n is any integer.

Algebraic Operations with Complex Numbers

Let two complex number be $z_1 = a + ib$ and $z_2 = c + id$

Addition $(z_1 + z_2)$:

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

Subtraction $(z_1 - z_2)$:

$$(a + ib) - (c + id) = (a - c) + i(b - d)$$

Multiplication $(z_1 z_2)$:

$$(a + ib)\,(c + id) = (ac - bd) + i(ad + bc)$$

Division (z_1/z_2):

$$\frac{a+ib}{c+id} = \frac{(a+ib)}{(c+id)}\,\frac{(c-id)}{(c-id)} \quad \text{(Rationalization)}$$

(where at least one of c and d is non-zero)

$$\Rightarrow \frac{a+ib}{c+id} = \frac{(ac+bd)}{c^2+d^2} + \frac{i(bc-ad)}{c^2+d^2}$$

Properties of Algebraic Operations on Complex Numbers

Let z_1, z_2 and z_3 are any three complex numbers. Then their algebraic operations satisfy the following properties:

1. Addition of complex numbers satisfies the commutative and associative properties, i.e.
   $z_1 + z_2 = z_2 + z_1$ and $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
2. Multiplication of complex numbers satisfies the commutative and associative properties, i.e.
   $z_1 z_2 = z_2 z_1$ and $(z_1 z_2)z_3 = z_1(z_2 z_3)$
3. Multiplication of complex numbers is distributive over addition, i.e.
   $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ and $(z_2 + z_3)z_1 = z_2 z_1 + z_3 z_1$

Example 2.1 **Evaluate:**

a. i^{135}

b. $(-\sqrt{-1})^{4n+3}, n \in N$

c. $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9}$

Sol. a. 135 leaves remainder as 3 when it is divided by 4

$\therefore \quad i^{135} = i^3 = -i$

b.
$$(-\sqrt{-1})^{4n+3} = (-i)^{4n+3} = (-i)^{4n}(-i)^3 = \{(-i)^4\}^n(-i)^3 = 1 \times (-i)^3 = i$$

c.
$$\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9} = 5i + 6i + 6i = 17i$$

Example 2.2 **Find the value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ for all $n \in N$.**

Sol.
$$i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n[1 + i + i^2 + i^3] = i^n[1 + i - 1 - i] = i^n(0) = 0$$

Example 2.3 **Find the value of $1 + i^2 + i^4 + i^6 + \cdots + i^{2n}$**

Sol. $S = 1 + i^2 + i^4 + \cdots + i^{2n} = 1 - 1 + 1 - 1 + \cdots + (-1)^n$

Obviously it depends on n. Hence it cannot be determined unless n is known.

Example 2.4 **Show that the polynomial $x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$ is divisible by $x^3 + x^2 + x + 1$ where $p, q, r, s \in N$.**

Sol. Let $f(x) = x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$. Now,

$$x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1) = (x + i)(x - i)(x + 1)$$

$$f(i) = i^{4p} + i^{4q+1} + i^{4r+2} + i^{4s+3} = 1 + i^1 + i^2 + i^3 = 1 + i - 1 - i = 0$$

$$f(-i) = (-i)^{4p} + (-i)^{4q+1} + (-i)^{4r+2} + (-i)^{4s+3} = 1 + (-i)^1 + (-i)^2 + (-i)^3 = 1 - i - 1 + i = 0$$

$$f(-1) = (-1)^{4p} + (-1)^{4q+1} + (-1)^{4r+2} + (-1)^{4s+3} = 0$$

Thus by division theorem $f(x)$ is divisible by $x^3 + x^2 + x + 1$.

Example 2.5 **If $z \neq 0$ is a complex number, then prove that $\text{Re}(z) = 0 \Rightarrow \text{Im}(z^2) = 0$.**

Sol. If $z \neq 0$, let $z = x + iy$. Then,

$z^2 = x^2 - y^2 + i(2xy)$

$\therefore \quad \text{Re}(z) = 0 \Rightarrow x = 0$

$\Rightarrow \quad \text{Im}(z^2) = 2xy = 0$

Thus,

$\text{Re}(z) = 0 \Rightarrow \text{Im}(z^2) = 0$

Example 2.6 **Express each one of the following in the standard form $a + ib$.**

a. $\dfrac{5+4i}{4+5i}$ **b.** $\dfrac{(1+i)^2}{3-i}$ **c.** $\dfrac{1}{1-\cos\theta + 2i\sin\theta}$

Sol. a.
$$\frac{5+4i}{4+5i} = \frac{5+4i}{4+5i} \times \frac{4-5i}{4-5i} = \frac{(20+20)+i(16-25)}{16-25i^2} = \frac{40-9i}{41} = \frac{40}{41} - \frac{9}{41}i$$

b.
$$\frac{(1+i)^2}{3-i} = \frac{1+2i+i^2}{3-i} = \frac{2i}{3-i} = \frac{2i}{3-i}\,\frac{3+i}{3+i} = \frac{6i+2i^2}{9-i^2} = \frac{-2+6i}{10} = -\frac{1}{5} + \frac{3}{5}i$$

c.
$$\frac{1}{1-\cos\theta + 2i\sin\theta} = \frac{1}{1-\cos\theta+2i\sin\theta}\,\frac{1-\cos\theta-2i\sin\theta}{1-\cos\theta-2i\sin\theta}$$

$$= \frac{1-\cos\theta-2i\sin\theta}{(1-\cos\theta)^2+4\sin^2\theta}$$

$$= \frac{1-\cos\theta-2i\sin\theta}{1-2\cos\theta+\cos^2\theta+4\sin^2\theta}$$

$$= \frac{1-\cos\theta-2i\sin\theta}{2-2\cos\theta+3\sin^2\theta}$$

$$= \left(\frac{1-\cos\theta}{2-2\cos\theta+3\sin^2\theta}\right) + i\left(\frac{-2\sin\theta}{2-2\cos\theta+3\sin^2\theta}\right)$$

Example 2.7 **Solve**

(i) $ix^2 - 3x - 2i = 0$, (ii) $2(1+i)x^2 - 4(2-i)x - 5 - 3i = 0$

Sol. i. $ix^2 - 3x - 2i = 0$,

$\Rightarrow \quad x^2 + 3ix - 2 = 0$ (dividing by i)

$$\Rightarrow \quad x = \frac{-3i \pm \sqrt{-9+4.1.2}}{2.1} = \frac{-3i \pm \sqrt{-1}}{2} = \frac{-3i \pm i}{2}$$

$\Rightarrow \quad x = -i$, or $x = -2i$

(ii) $2(1+i)x^2 - 4(2-i)x - 5 - 3i = 0$

$$\Rightarrow \quad x = \frac{4(2-i) \pm \sqrt{16(2-i)^2 + 4.2.(1+i).(5+3i)}}{2.(1+i)}$$

$$= \frac{4(2-i) \pm \sqrt{16(4-4i-1) + 8(5+3i+5i-3)}}{4(1+i)}$$

$$= \frac{4(2-i) \pm \sqrt{48-64i+16+64i}}{4(1+i)}$$

$$= \frac{4(2-i)\pm 8}{4(1+i)}$$

$$= \frac{(2-i)\pm 2}{1+i}$$

$$= \frac{2-i+2}{1+i}; \frac{2-i-2}{1+i}$$

$$= \frac{(4-i)(1-i)}{2}; \frac{-i(1-i)}{2}$$

$$= \frac{3-5i}{2}; -\left(\frac{1+i}{2}\right)$$

Example 2.8 **If $z = 4 + i\sqrt{7}$, then find the value of $z^3 - 4z^2 - 9z + 91$.**

Sol. $z = 4 + i\sqrt{7}$

$\Rightarrow$ $z - 4 = i\sqrt{7}$

$\Rightarrow$ $z^2 - 8z + 16 = -7$

$\Rightarrow$ $z^2 - 8z + 23 = 0$

Now dividing $z^3 - 4z^2 - 9z + 91$ by $z^2 - 8z + 23$

We get $z^3 - 4z^2 - 9z + 91 = (z^2 - 8z + 23)(z + 4) - 1 = -1$

Equality of Complex Numbers

Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal if $a_1 = a_2$ and $b_1 = b_2$, i.e., $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$. Thus, $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

Example 2.9 **If $(a + b) - i(3a + 2b) = 5 + 2i$, then find a and b.**

Sol. We have,

$(a + b) - i(3a + 2b) = 5 + 2i$

$\Rightarrow$ $a + b = 5$ and $-(3a + 2b) = 2$

$\Rightarrow$ $a = -12, b = 17$

Example 2.10 **Given that $x, y \in R$, solve:**

$$\frac{x}{1+2i} + \frac{y}{3+2i} = \frac{5+6i}{8i-1}$$

Sol. $\dfrac{x}{1+2i} + \dfrac{y}{3+2i} = \dfrac{5+6i}{8i-1}$

$$\Rightarrow \frac{x(1-2i)}{1-4i^2} + \frac{y(3-2i)}{9-4i^2} = \frac{(5+6i)(8i+1)}{(8i)^2-1^2}$$

$$\Rightarrow \frac{x-2xi}{5} + \frac{3y-2yi}{13} = \frac{40i+5-48+6i}{-64-1}$$

$$\Rightarrow \frac{13x-26xi+15y-10yi}{65} = \frac{-43+46i}{-65}$$

$\Rightarrow$ $(13x + 15y) - i(26x + 10y) = 43 - 46i$

equating real and imaginary parts,

$13x + 15y = 43$ (1)

$13x + 5y = 23$ (2)

Solving for x and y we get $x = 1$ and $y = 2$

Example 2.11 **Find the ordered pair (x, y) for which $x^2 - y^2 - i(2x + y) = 2i$**

Sol. We have

$x^2 - y^2 = 0$ or $x = \pm y$ (1)

and $-(2x + y) = 2$

or $2x + y = -2$ (2)

Solving $x = y$ with (2) we have $x = y = -\dfrac{2}{3}$

Solving $x = -y$ with (2) we have $x = -2$ and $y = 2$

Example 2.12 **If $\sqrt{x+iy} = \pm(a + ib)$, then find $\sqrt{-x-iy}$.**

Sol. $\sqrt{x+iy} = \pm(a+bi)$

$\Rightarrow$ $x + iy = a^2 - b^2 + 2iab$

$\Rightarrow$ $x = a^2 - b^2, y = 2ab$

$\therefore$ $\sqrt{-x-iy} = \sqrt{-(a^2-b^2)-2iab}$

$= \sqrt{b^2+(ia)^2-2iab}$

$= \sqrt{(b-ia)^2}$

$= \pm(b - ia)$

Example 2.13 **If sum of square of roots of the equation $x^2 + (p + iq)x + 3i = 0$ is 8 then find the value of p and q, where p and q are real.**

Sol. Let the roots are α, β

We have $\alpha + \beta = -(p + iq)$; $\alpha\beta = 3i$

Given: $\alpha^2 + \beta^2 = 8$

$\Rightarrow$ $(\alpha + \beta)^2 - 2\alpha\beta = 8$

$\Rightarrow$ $(p + iq)^2 - 6i = 8$

$\Rightarrow$ $p^2 - q^2 + i(2pq - 6) = 8$

$\Rightarrow$ $p^2 - q^2 = 8$ and $pq = 3$

$\Rightarrow$ $p = 3$ and $q = 1$ or $p = -3$ and $q = -1$

Example 2.14 **If $z = x + iy, z^{1/3} = a - ib$ and $x/a - y/b = k(a^2 - b^2)$, then find the value of k.**

Sol. $(x + iy)^{1/3} = a - ib$

$\Rightarrow$ $x + iy = (a - ib)^3$

$= (a^3 - 3ab^2) + i(b^3 - 3a^2b)$

$\Rightarrow$ $x = a^3 - 3ab^2, y = b^3 - 3a^2b$

$\Rightarrow$ $\dfrac{x}{a} = a^2 - 3b^2$ and $\dfrac{y}{b} = b^2 - 3a^2$

$\Rightarrow$ $\dfrac{x}{a} - \dfrac{y}{b} = a^2 - 3b^2 - b^2 + 3a^2 = 4(a^2 - b^2)$

$\therefore$ $k = 4$

Example 2.15 **Let z be a complex number satisfying the equation $z^2 - (3 + i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root. Then find the non-real root.**

Sol. Let α be the real root. Then,

$\alpha^2 - (3+i)\alpha + m + 2i = 0$

$\Rightarrow (\alpha^2 - 3\alpha + m) + i(2-\alpha) = 0$

$\Rightarrow \alpha = 2 \Rightarrow 4 - 6 + m = 0 \Rightarrow m = 2$

Product of the roots is $2(1+i)$ with one root as 2. Hence the non-real root is $1+i$.

Square Root of a Complex Number

Let $a + ib$ be a complex number such that $\sqrt{a+ib} = x + iy$, where x and y are real numbers. Then,

$\sqrt{a+ib} = x + iy$

$\Rightarrow (a+ib) = (x+iy)^2$

$\Rightarrow a + ib = (x^2 - y^2) + 2ixy$

On equating real and imaginary parts, we get

$$x^2 - y^2 = a \qquad \text{(i)}$$

$$2xy = b$$

Now,

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$\Rightarrow (x^2+y^2)^2 = a^2 + b^2$

$\Rightarrow (x^2+y^2) = \sqrt{a^2+b^2} \quad [\because x^2+y^2 \geq 0] \qquad \text{(ii)}$

Solving equations (i) and (ii), we get

$$x^2 = \left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2}+a\right] \text{ and } y^2 = \left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2}-a\right]$$

$$\Rightarrow x = \pm\sqrt{\left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2}+a\right]} \text{ and } y = \pm\sqrt{\left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2}-a\right]}$$

If b is positive, then by the relation $2xy = b$, x and y are of the same sign. Hence,

$$\sqrt{a+ib} = \pm\left\{\sqrt{\frac{1}{2}\left[\sqrt{a^2+b^2}+a\right]} + i\sqrt{\frac{1}{2}\left[\sqrt{a^2+b^2}-a\right]}\right\}$$

If b is negative, then by the relation $2xy = b$, x and y are of different signs. Hence,

$$\sqrt{a+ib} = \pm\left\{\sqrt{\frac{1}{2}\left[\sqrt{a^2+b^2}+a\right]} - i\sqrt{\frac{1}{2}\left[\sqrt{a^2+b^2}-a\right]}\right\}$$

Example 2.16 **Find the square roots of the following:**

a. $7 - 24i$ **b. $5 + 12i$** **c. $-15 - 8i$**

Sol. a. Let $\sqrt{7-24i} = x + iy$. Then,

$\sqrt{7-24i} = x + iy$

$\Rightarrow 7 - 24i = (x+iy)^2$

$\Rightarrow 7 - 24i = (x^2-y^2) + 2ixy$

$\Rightarrow x^2 - y^2 = 7 \qquad \text{(i)}$

and

$2xy = -24 \qquad \text{(ii)}$

Now,

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$\Rightarrow (x^2+y^2)^2 = 49 + 576 = 625$

$\Rightarrow x^2 + y^2 = 25 \quad [\because x^2+y^2 > 0] \qquad \text{(iii)}$

On solving (i) and (iii), we get

$x^2 = 16$ and $y^2 = 9 \Rightarrow x = \pm 4$ and $y = \pm 3$

From (ii), $2xy$ is negative. So, x and y are of opposite signs. Hence, $x = 4$ and $y = -3$ or $x = -4$ and $y = 3$.

Hence, $\sqrt{7-24i} = \pm(4-3i)$.

b. Let $\sqrt{5+12i} = x + iy$. Then,

$\sqrt{5+12i} = x + iy$

$\Rightarrow 5 + 12i = (x+iy)^2$

$\Rightarrow 5 + 12i = (x^2-y^2) + 2ixy$

$\Rightarrow x^2 - y^2 = 5 \qquad \text{(i)}$

and

$2xy = 12 \qquad \text{(ii)}$

Now,

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$\Rightarrow (x^2+y^2)^2 = 5^2 + 12^2 = 169$

$\Rightarrow x^2 + y^2 = 13 \quad (\because x^2+y^2 > 0) \qquad \text{(iii)}$

On solving (i) and (iii), we get

$x^2 = 9$ and $y^2 = 4 \Rightarrow x = \pm 3$ and $y = \pm 2$

From (ii), $2xy$ is positive. So, x and y are of the same sign.

$\Rightarrow x = 3$ and $y = 2$ or $x = -3$ and $y = -2$

Hence, $\sqrt{5+12i} = \pm(3+2i)$.

c. Let $\sqrt{-15-8i} = x + iy$. Then,

$\sqrt{-15-8i} = x + iy$

$\Rightarrow -15 - 8i = (x+iy)^2$

$\Rightarrow -15 - 8i = (x^2-y^2) + 2ixy$

$\Rightarrow -15 = x^2 - y^2 \qquad \text{(i)}$

and

$2xy = -8 \qquad \text{(ii)}$

Now,

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$\Rightarrow (x^2+y^2)^2 = (-15)^2 + 64 = 289$

$\Rightarrow x^2 + y^2 = 17$

On solving (i) and (iii), we get

$x^2 = 1$ and $y^2 = 16 \Rightarrow x = \pm 1$ and $y = \pm 4$

From (ii), $2xy$ is negative. So, x and y are of opposite signs. Hence, $x = 1$ and $y = -4$ or $x = -1$ and $y = 4$

Hence, $\sqrt{-15-8i} = \pm(1-4i)$.

Example 2.17 **Find all possible values of $\sqrt{i} + \sqrt{-i}$.**

Sol. $\sqrt{i} + \sqrt{-i} = \sqrt{0+1.i} + \sqrt{0-1.i}$

$$\text{Now } \sqrt{a+i.b} = \pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right\}$$

$$\text{and } \sqrt{a-i.b} = \pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} - i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right\}$$

$$\Rightarrow \sqrt{0+1.i} = \pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{0+1^2}+0\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{0+1^2}-0\right\}}\right\}$$

$$= \pm \frac{1}{\sqrt{2}}(1+i)$$

and $\sqrt{0-1.i} = \pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{0+1^2}+0\right\}} - i\sqrt{\frac{1}{2}\left\{\sqrt{0+1^2}-0\right\}}\right\}$

$$= \pm \frac{1}{\sqrt{2}}(1-i)$$

Now, $\sqrt{i}+\sqrt{-i} = \pm\frac{1}{\sqrt{2}}(1+i) \pm \frac{1}{\sqrt{2}}(1-i)$

or $\quad \sqrt{i}+\sqrt{-i} = \pm\sqrt{2}+0.i$ or $0 \pm \sqrt{2}i$

Example 2.18 **Solve for z: $z^2-(3-2i)z=(5i-5)$.**

Sol. $z^2-(3-2i)z=(5i-5)$

$\Rightarrow \quad z^2-(3-2i)z-(5i-5)=0$

$$\Rightarrow \quad z = \frac{(3-2i)\pm\sqrt{(3-2i)^2+4(5i-5)}}{2}$$

$$= \frac{(3-2i)\pm\sqrt{9-4-12i+20i-20}}{2}$$

$$= \frac{(3-2i)\pm\sqrt{8i-15}}{2}$$

Now $\sqrt{-15+8i}$

$$= \pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{(-15)^2+8^2}+(-15)\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{(-15)^2+8^2}-(-15)\right\}}\right\}$$

$$= \pm\left\{\sqrt{\frac{1}{2}\left(\sqrt{289}-15\right)} + \sqrt{\frac{1}{2}\left(\sqrt{289}+15\right)}\right\}$$

$$= \pm(1+i4)$$

$$\Rightarrow \quad z = \frac{3-2i\pm(1+4i)}{2}$$

$\Rightarrow \quad z=(2+i)$ and $(1-3i)$

Concept Application Exercise 2.1

1. Is the following computation correct? If not give the correct computation:
   $\sqrt{(-2)}\sqrt{(-3)} = \sqrt{(-2)(-3)} = \sqrt{6}$
2. Find the value of
   a. $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1$
   b. $(1+i)^6+(1-i)^6$
3. Find the value of $x^4+9x^3+35x^2-x+4$ for $x=-5+2\sqrt{-4}$.
4. The value of $i^{1+3+5+\cdots+(2n+1)}$ is ——.
5. If one root of the equation $z^2-az+a-1=0$ is $(1+i)$, where a is a complex number, then find the other root.
6. If the number $(1-i)^n/(1+i)^{n-2}$ is real and positive, then n is
   a. any integer b. any even integer
   c. any odd integer d. none of these
7. If $(x+iy)(p+iq)=(x^2+y^2)i$, prove that $x=q$, $y=p$.
8. Simplify:
   $$\frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$$
9. Find square root of $9+40i$.

GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

A complex number $z=x+iy$ can be represented by a point (x,y) on the plane which is known as the Argand plane. To represent $z=x+iy$ geometrically we take two mutually perpendicular straight lines $X'OX$ and $Y'OY$. Now plot a point whose x and y coordinates are respectively the real and the imaginary parts of z. This $P(x,y)$ represents the complex number $z=x+iy$.

If a complex number is purely real, then its imaginary part is zero. Therefore, a purely real number is represented by a point on x-axis. A purely imaginary complex number is represented by a point on y-axis. That is why x-axis is known as the real axis and y-axis, as the imaginary axis.

Conversely, if $P(x,y)$ is a point in the plane, then the point $P(x,y)$ represents a complex number $z=x+iy$. The complex number $z=x+iy$ is known as the affix of the point P.

The plane in which we represent a complex number geometrically is known as the complex plane, the **Argand plane** or the **Gaussian plane**.

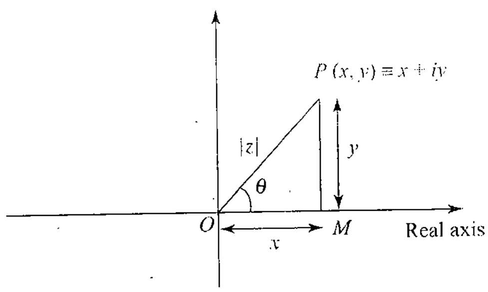


Fig. 2.1

Modulus of Complex Number

The length of the line segment OP is called the modulus of z and is denoted by $|z|$. From Fig. 2.1, we have

$$OP^2 = OM^2 + MP^2$$

$$\Rightarrow \quad OP^2 = x^2+y^2 \Rightarrow OP = \sqrt{x^2+y^2}.$$

Thus,

$$|z| = \sqrt{x^2+y^2} = \sqrt{\{\mathrm{Re}(z)\}^2+\{\mathrm{Im}(z)\}^2}$$

Clearly, $|z|\ge 0$ for all $z \in C$. If $z_1=3-4i$, $z_2=-5+2i$ and $z_3 = 1+\sqrt{-3}$, then $|z_1| = \sqrt{3^2+(-4)^2}=5$, $|z_2| = \sqrt{(-5)^2+2^2}=\sqrt{29}$ and $|z_3| = |1+i\sqrt{3}| = \sqrt{1^2+(\sqrt{3})^2}=2$.

Remark

In the set C of all complex numbers, the order relation is not defined. As such $z_1 > z_2$ or $z_1 < z_2$ has no meaning but $|z_1| > |z_2|$ or $|z_1| < |z_2|$ has got its meaning since $|z_1|$ and $|z_2|$ are real numbers.
Infinite complex numbers having same $|z|$ lying on circle which has center origin and radius $|z|$

Argument of Complex Number

The angle θ which OP makes with x-axis is called the **argument** or **amplitude** of z and is denoted by arg(z) or amp(z). From Fig. 2.1, we have

$$\tan\theta = \frac{PM}{OM} = \frac{y}{x} = \frac{\text{Im}(z)}{\text{Re}(z)} \Rightarrow \theta = \tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$$

This angle θ has infinitely many values differing by multiples of 2π.

The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the amplitude or **principal argument**. This formula for determining the argument of $z = x + iy$ has severe drawback, because $z_1 = 1 + i\sqrt{3}$ and $z_2 = -1 - i\sqrt{3}$ are two distinct complex numbers represented by two distinct points in the Argand plane but their arguments seem to be $\tan^{-1}\sqrt{3} = \pi/3$ or $4\pi/3$ which is not correct. In fact the argument is the common solution of the simultaneous trigonometric equations

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

The argument of z depends upon the quadrant in which the point P lies as discussed below:

(i) $z = x + iy$, z lies in first quadrant ($x > 0$ and $y > 0$)
From the figure $\tan\alpha = |y/x|$ and $\theta = \alpha$.
Then $\arg(z) = \tan^{-1}|y/x|$.

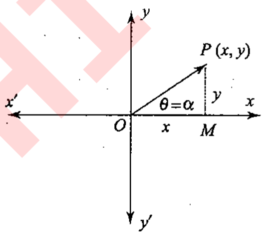


Fig. 2.2

(ii) $z = x + iy$, z lies in second quadrant ($x < 0$ and $y > 0$)
From the figure $\tan\alpha = |y/x|$ and $\theta = \pi - \alpha$.
Then $\arg(z) = \pi - \alpha$, where α is the acute angle given by $\tan^{-1}|y/x|$.

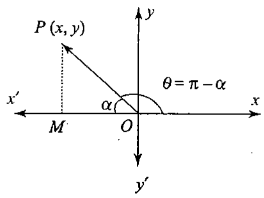


Fig. 2.3

(iii) $z = x + iy$, z lies in third quadrant ($x < 0$ and $y < 0$)
From the figure $\tan\alpha = |y/x|$ and $\theta = -(\pi - \alpha) = -\pi + \alpha$.
Then, $\arg(z) = \alpha - \pi$ where α is the acute angle given by $\tan\alpha = |y/x|$.

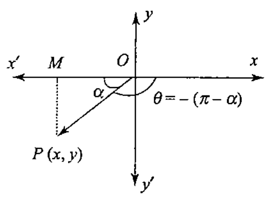


Fig. 2.4

(iv) $z = x + iy$, z lies in fourth quadrant ($x > 0$ and $y < 0$)
From the figure $\tan\alpha = |y/x|$ and $\theta = -\alpha$.
Then $\arg(z) = -\alpha$, where α is the acute angle given by $\tan\alpha = |y/x|$.

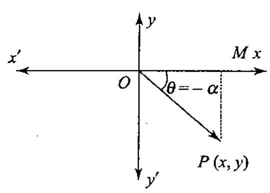


Fig. 2.5

Polar Form of Complex Number

We have,

$$z = x + iy$$

$$= \sqrt{x^2+y^2}\left[\frac{x}{\sqrt{x^2+y^2}} + i\frac{x}{\sqrt{x^2+y^2}}\right] = |z|[\cos\theta + i\sin\theta]$$

where $|z|$ is the modulus of complex number, i.e. the distance of z from origin and θ is the argument or amplitude of the complex number.

Here we should take the principal value of θ. For general values of argument $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$ (where n is an integer). This is polar form of the complex number.

Euler's Form of Complex Number

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This form makes the study of complex numbers and its properties simple. Any complex number can be expressed as

$$z = x + iy \quad \text{(Cartesian form)}$$
$$= |z|[\cos\theta + i\sin\theta] \quad \text{(polar form)}$$
$$= |z|e^{i\theta}$$

Product of Two Complex Numbers

Let two complex numbers be $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$. Now,

$$z_1z_2 = |z_1|e^{i\theta_1} \times |z_2|e^{i\theta_2}$$
$$= |z_1||z_2|e^{i(\theta_1+\theta_2)}$$
$$= |z_1||z_2|[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

Thus,

$$|z_1z_2| = |z_1||z_2|$$
$$\arg(z_1z_2) = \theta_1 + \theta_2$$
$$= \arg(z_1) + \arg(z_2)$$

Division of Two Complex Numbers

$$\frac{z_1}{z_2} = \frac{|z_1|e^{i\theta_1}}{|z_2|e^{i\theta_2}} = \frac{|z_1|}{|z_2|}e^{i(\theta_1-\theta_2)}$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\text{and } \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$
$$= \arg(z_1) - \arg(z_2)$$

Logarithm of a complex number is given by

$$\log_e(x+iy) = \log_e(|z|e^{i\theta})$$
$$= \log_e|z| + \log_e e^{i\theta}$$
$$= \log_e|z| + i\theta$$
$$= \log_e\sqrt{(x^2+y^2)} + i\arg(z)$$
$$\therefore \log_e(z) = \log_e|z| + i\arg(z)$$

Example 2.19 **Prove that the triangle formed by the points 1, $\frac{1+i}{\sqrt{2}}$ and i as vertices in the Argand diagram is isosceles.**

Sol. The vertices of the triangle are $A(1, 0)$, $B(1/\sqrt{2}, 1/\sqrt{2})$ and $C(0, 1)$,

$$\therefore AB^2 = 2 - \sqrt{2},\ BC^2 = 2 - \sqrt{2},\ AC^2 = 1 + 1 = 2$$
$$\therefore AB = BC \Rightarrow \text{Triangle is isosceles}$$

Example 2.20 **Write the following complex numbers in polar form:**

a. $-3\sqrt{2} + 3\sqrt{2}i$ **b.** $1 + i$ **c.** $-1 - i$ **d.** $1 - i$ **e.** $\frac{(1+7i)}{(2-i)^2}$

Sol. a. Let $z = -3\sqrt{2} + 3\sqrt{2}\,i$. Then,

$$r = |z| = \sqrt{(-3\sqrt{2})^2 + (3\sqrt{2})^2} = 6$$

Let

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = 1 \Rightarrow \alpha = \pi/4$$

Since the point representing z lies in the second quadrant, therefore, the argument of z is given by

$$\theta = \pi - \alpha = \pi - \left(\frac{\pi}{4}\right) = \left(\frac{3\pi}{4}\right)$$

So, the polar form of $z = -3\sqrt{2} + 3\sqrt{2}\,i$ is

$$z = r(\cos\theta + i\sin\theta) = 6\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

b. Let $z = 1 + i$. Then, $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Let,

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$$

Then,

$$\tan\alpha = \left|\frac{1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

Since the point (1, 1) representing z lies in first quadrant, therefore, the argument of z is given by $\theta = \alpha = \pi/4$. So, the polar form of $z = 1 + i$ is

$$z = r(\cos\theta + i\sin\theta) = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

c. Let $z = -1 - i$. Then, $r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$. Let,

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|.$$

Then,

$$\tan\alpha = \left|\frac{-1}{-1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

Since the point $(-1, -1)$ representing z lies in the third quadrant, therefore, the argument of z is given by

$$\theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{4}\right) = \frac{-3\pi}{4}$$

So, the polar form of $z = -1 - i$ is

$$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left\{\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right\}$$

d. Let $z = 1 - i$. Then, $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Let

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$$

Then,

$$\tan\alpha = \left|\frac{-1}{1}\right| = 1 \Rightarrow \alpha = \pi/4$$

Since the point $(1, -1)$ lies in the fourth quadrant, therefore the argument of z is given by $\theta = -\alpha = -\pi/4$. So, the polar form of $z = 1 - i$ is

$$r(\cos\theta + i\sin\theta) = \sqrt{2}\left\{\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right\}$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

e. Let $z = (1+7i)/[(2-i)^2]$ Then,

$$z = \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{3-4i} = \left(\frac{1+7i}{3-4i}\right)\left(\frac{3+4i}{3+4i}\right) = \frac{-25+25i}{25} = -1+i$$

$$\therefore \quad r = |z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

Let α be the acute angle given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|-\frac{1}{1}\right| = 1$$

Then $\alpha = \pi/4$. Since the point $(-1, 1)$ representing z lies in the second quadrant, therefore $\theta = \arg(z) = \pi - \alpha = \pi - \pi/4 = 3\pi/4$. Hence, z in the polar form is given by

$$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Example 2.21 **If $z = re^{i\theta}$, then prove that $|e^{iz}| = e^{-r\sin\theta}$.**

Sol. $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$\Rightarrow iz = ir(\cos\theta + i\sin\theta)$

$= -r\sin\theta + ir\cos\theta$

$\Rightarrow e^{iz} = e^{(-r\sin\theta + ir\cos\theta)}$

$= e^{-r\sin\theta}\, e^{ri\cos\theta}$

$\Rightarrow |e^{iz}| = |e^{-r\sin\theta}|\, |e^{ri\cos\theta}|$

$= e^{-r\sin\theta}|e^{i\alpha}|$, where $\alpha = r\cos\theta$

$= e^{-r\sin\theta}[\cos^2\alpha + \sin^2\alpha]^{1/2}$

$= e^{-r\sin\theta}$

Example 2.22 **Prove that**

$$\tan\left(i\log_e\left(\frac{a-ib}{a+ib}\right)\right) = \frac{2ab}{a^2-b^2} \text{ (where } a, b \in R^+\text{)}$$

Sol. Let

$a + ib = re^{i\theta}$

$\Rightarrow a - ib = re^{-i\theta}$

$\Rightarrow \dfrac{a-ib}{a+ib} = e^{-i2\theta}$

$\Rightarrow \log_e\left(\dfrac{a-ib}{a+ib}\right) = -i2\theta$

$\Rightarrow \tan\left(i\log_e\left(\dfrac{a-ib}{a+ib}\right)\right) = \tan 2\theta$

$= \dfrac{2\tan\theta}{1-\tan^2\theta}$

$= \dfrac{2b/a}{1-b^2/a^2}$

$= \dfrac{2ab}{a^2-b^2}$

Example 2.23 **Find the real part of $(1-i)^{-i}$.**

Sol. Let $z = (1-i)^{-i}$. Taking log on both sides,

$\log z = -i\log_e(1-i)$

$= -i\log_e\sqrt{2}\left(\cos\dfrac{\pi}{4} - i\sin\dfrac{\pi}{4}\right)$

$= -i\log_e(\sqrt{2}e^{-i(\pi/4)})$

$= -i\left[\dfrac{1}{2}\log_e 2 + \log_e e^{-i\pi/4}\right]$

$= -i\left[\dfrac{1}{2}\log_e 2 - \dfrac{i\pi}{4}\right]$

$= -\dfrac{i}{2}\log_e 2 - \dfrac{\pi}{4}$

$\Rightarrow z = e^{-\pi/4}\, e^{-i(\log 2)/2}$

$\Rightarrow \text{Re}(z) = e^{-\pi/4}\cos\left(\dfrac{1}{2}\log 2\right)$

Example 2.24 **Show that $e^{2mi\theta}\left(\dfrac{i\cot\theta+1}{i\cot\theta-1}\right)^m = 1$.**

Sol. Let $\cot^{-1} p = \theta$. Then $\cot\theta = p$. Now,

$$\text{L.H.S.} = e^{2mi\theta}\left(\frac{i\cot\theta+1}{i\cot\theta-1}\right)^m$$

$$= e^{2mi\theta}\left[\frac{i(\cot\theta-i)}{i(\cot\theta+i)}\right]^m$$

$$= e^{2mi\theta}\left(\frac{\cot\theta-i}{\cot\theta+i}\right)^m$$

$$= e^{2m i\theta}\left(\frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}\right)^m$$

$$= e^{2m i\theta}\left(\frac{e^{-i\theta}}{e^{i\theta}}\right)^m$$

$$= e^{2m i\theta}(e^{-2i\theta})^m$$

$$= e^{2m i\theta}e^{-2m i\theta} = e^0 = 1 = \text{R.H.S.}$$

Conjugate of a Complex Number

For complex number $z = x + iy$, $(x, y) \in R$, its conjugate is defined as $\bar{z} = x - iy$. Clearly, $z = x + iy$ is represented by a point $P(x, y)$ in the Argand plane. Now,

$$z = x + iy \Rightarrow \bar{z} = x - iy = x + i(-y)$$

So, $\bar{z}$ is represented by a point $Q(x, -y)$ in the Argand plane. Clearly, Q is the image of point P on the real axis.

Thus, if a point P represents a complex number z, then its conjugate $\bar{z}$ is represented by the image of P on the real axis.

It is evident from the following figure that $|z| = |\bar{z}|$ and $\arg(\bar{z}) = -\arg(z)$.

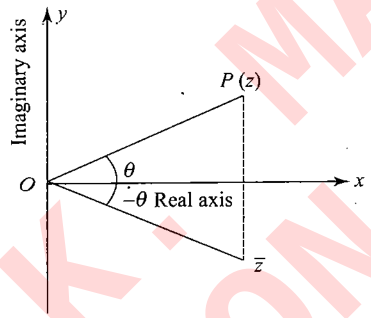


Fig. 2.6

Also, we have $\text{Re}(z) = (z + \bar{z})/2$ and $\text{Im}(z) = (z - \bar{z})/2i$. Thus if $z = |z|e^{i\theta}$, then $\bar{z} = |z|e^{-i\theta}$.

Properties of Conjugate

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\text{Re}(z)$

(iii) $z - \bar{z} = 2i\,\text{Im}(z)$

(iv) $z = \bar{z} \Leftrightarrow z$ is purely real

(v) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary

(vi) $z\bar{z} = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2 = |z|^2$

(vii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(viii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(ix) $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$

From this we can say that $\overline{z^n} = \bar{z}^n$, where $n \in N$.

(x) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

Example 2.25 **Find real values of x and y for which the complex numbers $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugate of each other.**

Sol. Given,

$$-3 + ix^2y = \overline{x^2 + y + 4i}$$

$$\Rightarrow -3 + ix^2y = x^2 + y - 4i$$

$$\Rightarrow -3 = x^2 + y \quad \text{(i)}$$

and

$$x^2y = -4 \quad \text{(ii)}$$

$$\therefore -3 = x^2 - \frac{4}{x^2} \quad [\text{Putting } y = -4/x^2 \text{ from (ii) in (i)}]$$

$$\Rightarrow x^4 + 3x^2 - 4 = 0$$

$$\Rightarrow (x^2 + 4)(x^2 - 1) = 0$$

$$\Rightarrow x^2 - 1 = 0$$

$$\Rightarrow x = \pm 1$$

From (ii), $y = -4$, when $x = \pm 1$. Hence, $x = 1$, $y = -4$ or $x = -1$, $y = -4$.

Example 2.26 **If $(x + iy)^5 = p + iq$, then prove that $(y + ix)^5 = q + ip$.**

Sol. $(x + iy)^5 = p + iq$

$$\Rightarrow \overline{(x+iy)^5} = \overline{p + iq}$$

$$\Rightarrow \overline{(x+iy)}^5 = p - iq$$

$$\Rightarrow (x - iy)^5 = p - iq$$

$$\Rightarrow i^5(x - iy)^5 = pi^5 - i^6q$$

$$\Rightarrow (xi - i^2y)^5 = pi + q$$

$$\Rightarrow (y + ix)^5 = pi + q$$

Example 2.27 **Find the values of θ if $(3 + 2i\sin\theta)/(1 - 2i\sin\theta)$ is purely real or purely imaginary.**

Sol. $z = \dfrac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$

Multiplying numerator and denominator by conjugate,

$$z = \frac{(3 + 2i\sin\theta)(1 + 2i\sin\theta)}{1 + 4\sin^2\theta}$$

$$= \frac{3 - 4\sin^2\theta + 8i\sin\theta}{1 + 4\sin^2\theta}$$

Now z is purely real if $\sin\theta = 0$ or $\theta = n\pi$, $n \in Z$. z is purely imaginary if

$$3 - 4\sin^2\theta = 0$$

$$\Rightarrow \sin\theta = \pm\frac{\sqrt{3}}{2} = \pm\sin\frac{\pi}{3}$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in Z$$

Example 2.28 **If z is a complex number such that $z^2 = (\bar{z})^2$, then find the location of z on the Argand plane.**

Sol. Let, $z = x + iy \Rightarrow \bar{z} = x - iy$

Given that

$$z^2 = (\bar{z})^2$$

$$\Rightarrow x^2 - y^2 + 2ixy = x^2 - y^2 - 2ixy$$

$$\Rightarrow 4ixy = 0$$

If $x \neq 0$, then $y = 0$ and if $y \neq 0$, then $x = 0$.

Example 2.29 **Find the least positive integer n which will reduce $\left(\frac{(i-1)}{(i+1)}\right)^n$ to a real number.**

Sol. $$\left(\frac{i-1}{i+1}\right)^n = \left(\frac{i-1}{i+1} \times \frac{i+1}{i+1}\right)^n$$

$$= \left(\frac{i^2-1}{(i+1)^2}\right)^n$$

$$= \left(\frac{-2}{i^2+2i+1}\right)^n$$

$$= \left(\frac{-2}{2i}\right)^n$$

$$= \left(\frac{-1}{i}\right)^n$$

Hence, the required positive integer is 2.

Example 2.30 **Consider two complex numbers α and β as $\alpha = [(a + bi)/(a - bi)]^2 + [(a - bi)/(a + bi)]^2$, where $a, b \in R$ and $\beta = (z - 1)/(z + 1)$, where $|z| = 1$, then find the correct statement:**

a. **both α and β are purely real**
b. **both α and β are purely imaginary**
c. **α is purely real and β is purely imaginary**
d. **β is purely real and α is purely imaginary**

Sol. Note that $\alpha = \bar{\alpha} \Rightarrow \alpha$ is real.

$$\beta + \bar{\beta} = \frac{z-1}{z+1} + \frac{\bar{z}-1}{\bar{z}+1}$$

$$= \frac{(z-1)(\bar{z}+1) + (z+1)(\bar{z}-1)}{(z+1)(\bar{z}+1)}$$

$$= \frac{2z\bar{z} - 2}{(z+1)(\bar{z}+1)}$$

$$= 0 \qquad [\text{as } z\bar{z} = |z|^2 = 1 \text{ (given)}]$$

Hence the correct statement is (c).

Expressing Complex Numbers in $a + ib$ Form

The following example illustrates how a complex number can be expressed in the standard $a + ib$ form.

Solving Complex Equations

Simple equations in z may be solved by putting $z = x + iy$ in the equation and equating the real part on the L.H.S. with the real part on the R.H.S. and the imaginary part on the L.H.S. with the imaginary part on the R.H.S.

Example 2.31 **Find the complex number 'z' satisfying $\text{Re}(z^2) = 0$, $|z| = \sqrt{3}$.**

Sol. $z = x + iy$

$$\Rightarrow z^2 = x^2 - y^2 + 2ixy$$

$$\Rightarrow \text{Re}(z^2) = x^2 - y^2$$

Also,

$$|z| = \sqrt{x^2 + y^2}$$

$$\Rightarrow x^2 - y^2 = 0, x^2 + y^2 = 3$$

$$\Rightarrow x^2 = y^2 = \frac{3}{2}$$

$$\Rightarrow x = \pm\sqrt{\frac{3}{2}}, y = \pm\sqrt{\frac{3}{2}} \Rightarrow z = \pm\sqrt{\frac{3}{2}} \pm \sqrt{\frac{3}{2}}i$$

Thus there are four complex numbers.

Example 2.32 **Solve the equation $|z| = z + 1 + 2i$.**

Sol. $|z| = z + 1 + 2i$

$$\Rightarrow \sqrt{x^2 + y^2} = x + iy + 1 + 2i$$
$$= x + 1 + (2 + y)i$$

$$\Rightarrow \sqrt{x^2 + y^2} = x + 1 \text{ and } 0 = 2 + y \text{ or } y = -2$$

$$\Rightarrow \sqrt{x^2 + 4} = x + 1$$

$$\Rightarrow x^2 + 4 = x^2 + 2x + 1$$

$$\Rightarrow 2x = 3$$

$$\Rightarrow x = 3/2$$

$$\Rightarrow x + iy = \frac{3}{2} - 2i$$

Concept Application Exercise 2.2

1. Express the following complex numbers in $a + ib$ form:
   a. $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ **b.** $\frac{2-\sqrt{-25}}{1-\sqrt{-16}}$
2. If $z_1 = 9y^2 - 4 - 10ix$, $z_2 = 8y^2 - 20i$, where $z_1 = \bar{z}_2$, then find $z = x + iy$.
3. Find the least positive integer n such that $\left(\frac{2i}{1+i}\right)^n$ is a positive integer.
4. Find the real part of $e^{e^{i\theta}}$.
5. Prove that $z = i^i$, where $i = \sqrt{-1}$, is purely real.
6. Solve: $z^2 + |z| = 0$.

Geometric Presentation of Various Algebraic Operations

Addition:

Let z_1 and z_2 be two complex numbers represented by points P and Q.

Now $OP = |z_1|$ and $OQ = |z_2|$. Let R represent complex number $z_1 + z_2$. Now,

PR = distance between complex numbers $z_1 + z_2$ and z_1

$= |(z_1 + z_2) - z_1|$

$= |z_2| = OQ$

(For distance formula, see properties of modulus on page 2.10.)

Similarly $\quad QR = |(z_1 + z_2) - z_2| = |z_1| = OP$

Hence points O, P, R, Q complete the parallelogram.

Also in triangle OPR, we have

$OP + PR \geq OR \Rightarrow |z_1| + |z_2| \geq |z_1 + z_2|$

(however, equality sign holds when origin, z_1 and z_2 are collinear)

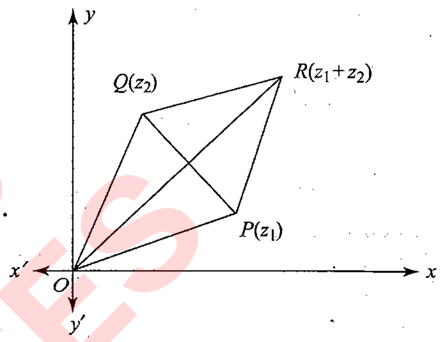


Fig. 2.7

Subtraction

In the adjacent figure $OPRQ'$ is parallelogram.

$OP = Q'R = |z_1|$ and $OQ = OQ' = PR = |z_2|$

Also in triangle OPR, we have

$|OP - PR| \leq OR \geq ||z_1| - |z_2|| \leq |z_1 - z_2|$

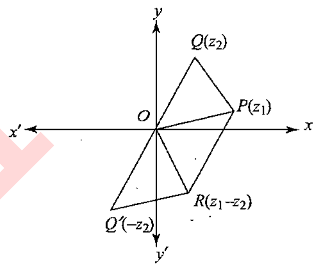


Fig. 2.8

Multiplication

Let $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$. Then we have

$|z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$.

So, to find the point R representing the complex number $z_1 z_2$, we have to construct the point with polar coordinates $(r_1 r_2, \theta_1 + \theta_2)$. For this take a point L on real axis such that $OL = 1$ and draw a triangle OQR similar to the triangle OLP. Since triangles OQR and OLP are similar, therefore

$$\frac{OR}{OQ} = \frac{OP}{OL} \Rightarrow \frac{OR}{r_2} = \frac{r_1}{1} \Rightarrow OR = r_1 r_2$$

and $\quad \angle QOR = \angle LOP = \theta_1$

so that $\quad \angle XOR = \angle XOQ + \angle QOR = \theta_2 + \theta_1$

Hence, R has the polar coordinates $(r_1 r_2, \theta_1 + \theta_2)$. Consequently, it represents the complex number $z_1 z_2$.

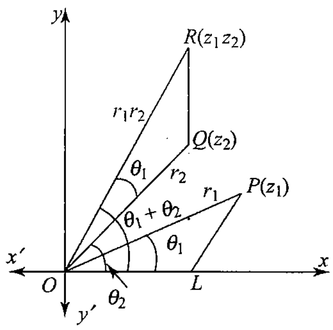


Fig. 2.9

Division

We have $|z_1/z_2| = r_1/r_2$ and $\arg(z_1/z_2) = \theta_1 - \theta_2$. So, to find a point R representing the complex number z_1/z_2, we have to construct the point with polar coordinates $(r_1/r_2, \theta_1 - \theta_2)$. For this take a point L on real axis OX such that $OL = 1$ and draw a triangle OPR similar to OQL. Since triangle OPR and OQL are similar, therefore

$$\frac{OP}{OQ} = \frac{OR}{OL} \Rightarrow \frac{OR}{1} = \frac{r_1}{r_2} \Rightarrow OR = \frac{r_1}{r_2}$$

and

$$\angle XOR = \angle XOP - \angle ROP = \theta_1 - \theta_2$$

Hence, R has the polar coordinates $(r_1/r_2, \theta_1 - \theta_2)$. Consequently, it represents the complex number z_1/z_2.

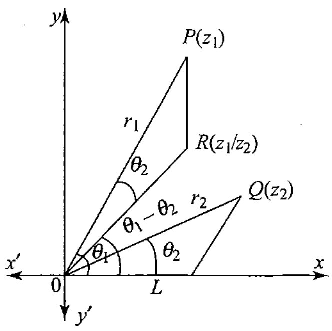


Fig. 2.10

Properties of Modulus

If $z, z_1, z_2 \in C$, then

1. $|z| = 0 \Leftrightarrow z = 0$, i.e., $\text{Re}(z) = \text{Im}(z) = 0$
2. $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
3. $-|z| \le \text{Re}(z) \le |z|$; $-|z| \le \text{Im}(z) \le |z|$
4. $z\bar{z} = |z|^2$
5. $|z_1z_2| = |z_1|\,|z_2|$, in general $|z_1 z_2 z_3 \cdots z_n| = |z_1|\,|z_2||z_3| \cdots |z_n|$
6. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}; z_2 \ne 0$
7. $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$
   $= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$
   $= |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\bar{z}_2)$
8. $|z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$
   $= |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - z_2\bar{z}_1$
   $= |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\bar{z}_2)$
9. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
10. $|z_1 - z_2| = |(a_1 + ib_1) - (a_2 + ib_2)|$
    $= |(a_1 - a_2) + i(b_1 - b_2)|$
    $= \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$

    This is the distance between the points (a_1, b_1) and (a_2, b_2) which is the distance between z_1 and z_2.
11. $|z^n| = |z|^n$, where $n \in Q$
12. $|z_1 \pm z_2| \le |z_1| + |z_2|$, in general $|z_1 + z_2 + z_3 + \cdots + z_n| \le |z_1| + |z_2| + |z_3| + \cdots + |z_n|$
13. $|z_1 \pm z_2| \ge ||z_1| - |z_2||$

Example 2.33 **If $(1+i)(1+2i)(1+3i) \cdots (1+ni) = (x+iy)$, then show that $2 \times 5 \times 10 \times \cdots \times (1+n^2) = x^2 + y^2$.**

Sol. We have,

$(1+i)(1+2i)(1+3i) \cdots (1+ni) = x + iy$

$\Rightarrow |(1+i)(1+2i) \cdots (1+ni)| = |x+iy|$

$\Rightarrow |1+i|\;|1+2i| \cdots |1+ni| = |x+iy|$

$[\because |z_1 z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|]$

$\Rightarrow \sqrt{1+1}\,\sqrt{1+4} \cdots \sqrt{1+n^2} = \sqrt{x^2+y^2}$

$\Rightarrow 2 \times 5 \times 10 \cdots (1+n^2) = (x^2+y^2)$ [On squaring both sides]

Example 2.34 **If $z = x + iy$ and $w = (1-iz)/(z-i)$, then show that $|w| = 1 \Rightarrow z$ is purely real.**

Sol. We have,

$|w| = 1$

$\Rightarrow \left|\frac{1-iz}{z-i}\right| = 1$

$\Rightarrow \frac{|1-iz|}{|z-i|} = 1$

$\Rightarrow |1 - iz| = |z - i|$ (i)

$\Rightarrow |1 - i(x+iy)| = |x + iy - i|$, where $z = x + iy$

$\Rightarrow |1 + y - ix| = |x + i(y-1)|$

$\Rightarrow \sqrt{(1+y)^2 + (-x)^2} = \sqrt{x^2 + (y-1)^2}$

$\Rightarrow (1+y)^2 + x^2 = x^2 + (y-1)^2$

$\Rightarrow y = 0$

$\Rightarrow z = x + i0 = x$, which is purely real

Example 2.35 **If α and β are different complex numbers with $|\beta| = 1$, then find the value of $|(\beta - \alpha)/(1 - \bar{\alpha}\beta)|$.**

Sol. Given,

$|\beta| = 1 \Rightarrow \beta\bar{\beta} = 1 \Rightarrow \beta = \frac{1}{\bar{\beta}}$ (1)

Now,

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = \left|\frac{\frac{1}{\bar{\beta}} - \alpha}{1-\bar{\alpha}\beta}\right|$$

$$= \left|\frac{1}{\bar{\beta}}\right|\left|\frac{1-\alpha\bar{\beta}}{1-\bar{\alpha}\beta}\right|$$

$$= \frac{1}{|\beta|}\left|\frac{\overline{1-\bar{\alpha}\beta}}{1-\bar{\alpha}\beta}\right|$$

$$= \left|\frac{1-\bar{\alpha}\beta}{1-\bar{\alpha}\beta}\right| = 1$$

Example 2.36 **Solve the equation $z^3 = \bar{z}\ (z \ne 0)$.**

Sol. As $z \ne 0$,

$|z^3| = |\bar{z}|$

$\Rightarrow |z| = 1$

$\therefore z^3 = \bar{z} \Rightarrow z^4 = z\bar{z} = |z|^2 = 1$

$\Rightarrow z = \pm 1, \pm i$

Example 2.37 **If z_1, z_2, z_3, z_4 are the affixes of four points in the Argand plane, z is the affix of a point such that $|z - z_1| = |z - z_2| = |z - z_3| = |z - z_4|$, then prove that z_1, z_2, z_3, z_4 are concyclic.**

Sol. We have,

$|z - z_1| = |z - z_2| = |z - z_3| = |z - z_4|$

Therefore, the point having affix z is equidistant from the four points having affixes z_1, z_2, z_3, z_4. Thus, z is the affix of either the centre of a circle or the point of intersection of diagonals of a square. Therefore z_1, z_2, z_3, z_4 are either concyclic.

Example 2.38 **If $|z| = 1$ and let $\omega = \frac{(1-z)^2}{1-z^2}$, then prove that the locus of ω is equivalent to $|z - 2| = |z + 2|$.**

Sol. Given $\omega = \frac{(1-z)^2}{1-z^2} = \frac{1-z}{1+z}$

$$\Rightarrow \quad = \frac{z\bar{z} - z}{z\bar{z} + z}$$

$$= \frac{\bar{z}-1}{\bar{z}+1} = -\overline{\left(\frac{1-z}{1+z}\right)} = -\bar{\omega}$$

$\therefore \quad \omega + \bar{\omega} = 0 \Rightarrow \omega$ is purely imaginary. Hence ω lies on y axis.

Also $|z - 2| = |z + 2| \Rightarrow z$ lies on perpendicular bisector of 2 and -2, which is imaginary axis.

Example 2.39 **Let $z = x + iy$ then find the locus of $P(z)$ such that $\dfrac{1+\bar{z}}{z} \in R$.**

Sol. Given $\dfrac{1+\bar{z}}{z}$ is real

$\Rightarrow \quad \dfrac{1+\bar{z}}{z} = \dfrac{1+z}{\bar{z}}$

$\Rightarrow \quad \bar{z} + \bar{z}^2 = z + z^2$

$\Rightarrow \quad (\bar{z} - z) + (\bar{z} - z)(\bar{z} + z) = 0$

$\Rightarrow \quad (\bar{z} - z)(1 + \bar{z} + z) = 0$

$\Rightarrow$ either $\bar{z} = z$ ($z \neq 0$) or $z + \bar{z} + 1 = 0$

$\Rightarrow \quad y = 0$ or $x = \dfrac{-1}{2}$ but excluding origin.

Example 2.40 **Identify the locus z if $\mathrm{Re}(z + 1) = |z - 1|$.**

Sol. $\mathrm{Re}(z + 1) = |z - 1|$

$\Rightarrow \quad x + 1 = \sqrt{(x-1)^2 + y^2}$

$\Rightarrow \quad (x + 1)^2 = (x - 1)^2 + y^2$

$\Rightarrow \quad y^2 = 4x$

Example 2.41 **If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, then find the value of $|z_1 + z_2 + z_3|$.**

Sol. $|z_1| = 1 \Rightarrow z_1\bar{z}_1 = 1$, $|z_2| = 2 \Rightarrow z_2\bar{z}_2 = 4$, $|z_3| = 3 \Rightarrow z_3\bar{z}_3 = 9$

Also,

$|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$

$\Rightarrow \quad |z_1z_2z_3\bar{z}_3 + z_1z_2z_3\bar{z}_2 + z_1\bar{z}_1z_2z_3| = 12$

$\Rightarrow \quad |z_1z_2z_3|\,|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad |z_1|\,|z_2|\,|z_3|\,|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad 6\,|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 2$

$\Rightarrow \quad |z_1 + z_2 + z_3| = 2$

Example 2.42 **If $|z - i\,\mathrm{Re}(z)| = |z - \mathrm{Im}(z)|$, then prove that z lies on the bisectors of the quadrants.**

Sol. $z = x + iy \Rightarrow \mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$

$|z - i\mathrm{Re}(z)| = |z - \mathrm{Im}(z)|$

$\Rightarrow \quad |x + iy - ix| = |x + iy - y|$

$\Rightarrow \quad x^2 + (x - y)^2 = (x - y)^2 + y^2$

$\Rightarrow \quad x^2 = y^2$

$\Rightarrow \quad |x| = |y|$

$\Rightarrow \quad z$ lies on the bisectors the quadrants.

Example 2.43 **Show that $(x^2 + y^2)^4 = (x^4 - 6x^2y^2 + y^4)^2 + (4x^3y - 4xy^3)^2$.**

Sol. $(x^2 + y^2)^4 = |x + iy|^8$

$= |(x + iy)^2|^4$

$= |(x^2 - y^2) + 2ixy|^4$

$= |[(x^2 - y^2) + 2ixy)]^2|^2$

$= |(x^2 - y^2)^2 + (2ixy)^2 + 2(x^2 - y^2)(2ixy)|^2$

$= |x^4 + y^4 - 2x^2y^2 - 4x^2y^2 + i(4x^3y - 4xy^3)|^2$

$= |x^4 + y^4 - 6x^2y^2 + i(4x^3y - 4xy^3)|^2$

$= (x^4 - 6x^2y^2 + y^4)^2 + (4x^3y - 4xy^3)^2$

Example 2.44 **Let $|(\bar{z}_1 - 2\bar{z}_2)/(2 - z_1\bar{z}_2)| = 1$ and $|z_2| \neq 1$, where z_1 and z_2 are complex numbers. Show that $|z_1| = 2$.**

Sol. $\left|\dfrac{\bar{z}_1 - 2\bar{z}_2}{2 - z_1\bar{z}_2}\right| = 1$

$\Rightarrow \quad |\bar{z}_1 - 2\bar{z}_2|^2 = |2 - z_1\bar{z}_2|^2$

$\Rightarrow \quad (\bar{z}_1 - 2\bar{z}_2)(\overline{\bar{z}_1 - 2\bar{z}_2}) = (2 - z_1\bar{z}_2)(\overline{2 - z_1\bar{z}_2})$

$\Rightarrow \quad (\bar{z}_1 - 2\bar{z}_2)(z_1 - 2z_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1z_2)$

$\Rightarrow \quad z_1\bar{z}_1 - 2\bar{z}_1z_2 - 2z_1\bar{z}_2 + 4z_2\bar{z}_2 = 4 - 2\bar{z}_1z_2 - 2z_1\bar{z}_2 + z_1\bar{z}_1z_2\bar{z}_2$

$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2|z_2|^2$

$\Rightarrow \quad |z_1|^2 - |z_1|^2|z_2|^2 + 4|z_2|^2 - 4 = 0$

$\Rightarrow \quad |z_1|^2(1 - |z_2|^2) + 4(|z_2|^2 - 1) = 0$

$\Rightarrow \quad (|z_2|^2 - 1)(|z_1|^2 - 4) = 0$

$\Rightarrow \quad |z_1| = 2$ (as $|z_2| \neq 1$)

Example 2.45 **If z_1 and z_2 are complex numbers and $u = \sqrt{z_1z_2}$, then prove that**

$$|z_1| + |z_2| = \left|\frac{z_1 + z_2}{2} + u\right| + \left|\frac{z_1 + z_2}{2} - u\right|$$

Sol. $\left|\dfrac{z_1 + z_2}{2} + u\right| + \left|\dfrac{z_1 + z_2}{2} - u\right|$

$= \left|\dfrac{z_1 + z_2}{2} + \sqrt{z_1z_2}\right| + \left|\dfrac{z_1 + z_2}{2} - \sqrt{z_1z_2}\right|$

$= \left|\dfrac{(\sqrt{z_1} + \sqrt{z_2})^2}{2}\right| + \left|\dfrac{(\sqrt{z_1} - \sqrt{z_2})^2}{2}\right|$

$= \left|\dfrac{(p+q)^2}{2}\right| + \left|\dfrac{(p-q)^2}{2}\right|$ (where $p = \sqrt{z_1}$ and $q = \sqrt{z_2}$)

$= \dfrac{1}{2}\left[|p+q|^2 + |p-q|^2\right]$

$= \dfrac{1}{2}\left[2|p|^2 + 2|q|^2\right]$

$= |p|^2 + |q|^2$

$= |p^2| + |q^2|$

$= |z_1| + |z_2|$

Example 2.46 If $(\sqrt{8}+i)^{50} = 3^{49}(a+ib)$, then find the value of $a^2 + b^2$.

Sol. Given that

$(\sqrt{8}+i)^{50} = 3^{49}(a+ib)$

Taking modulus and squaring both sides, we get

$(8+1)^{50} = 3^{98}(a^2+b^2)$

$\Rightarrow$ $9^{50} = 3^{98}(a^2+b^2)$

$\Rightarrow$ $3^{100} = 3^{98}(a^2+b^2)$

$\Rightarrow$ $(a^2+b^2) = 9$

Example 2.47 Find complex number satisfying the system of equations $z^3 + \bar{\omega}^7 = 0$ and $z^5\omega^{11} = 1$.

Sol. $z^3 + \bar{\omega}^7 = 0$

$\Rightarrow$ $z^3 = -\bar{\omega}^7$

$\Rightarrow$ $|z|^3 = |\bar{\omega}|^7 = |\omega|^7$

$\Rightarrow$ $|z|^{15} = |\omega|^{35}$ (1)

Again,

$z^5\omega^{11} = 1$

$\Rightarrow$ $|z|^5\,|\omega|^{11} = 1$

$\Rightarrow$ $|z|^{15}\,|\omega|^{33} = 1$ (2)

From (1) and (2), we have

$|z| = |\omega| = 1$

Again,

$\bar{\omega}^7 = -z^3$ and $\omega^{11} = z^{-5}$

$\Rightarrow$ $\bar{\omega}^{77}\,.\,\omega^{77} = -z^{33}\,.\,z^{-35}$

$\Rightarrow$ $z^2 = -1 = i^2$

$\Rightarrow$ $z = \pm i$

Example 2.48 If $|z_1 - 1| \le 1$, $|z_2 - 2| \le 2$, $|z_3 - 3| \le 3$, then find the greatest value of $|z_1 + z_2 + z_3|$.

Sol. $|z_1 + z_2 + z_3| = |(z_1 - 1) + (z_2 - 2) + (z_3 - 3) + 6|$

$\le |z_1 - 1| + |z_2 - 2| + |z_3 - 3| + 6$

$\le 1 + 2 + 3 + 6 = 12$

Hence the greatest value is 12.

Example 2.49 For any complex number z, find the minimum value of $|z| + |z - 2i|$.

Sol. We have, for $z \in C$

$|2i| = |z + (2i - z)|$

$\le |z| + |2i - z|$

$\Rightarrow$ $2 \le |z| + |z - 2i|$

Thus, minimum value of $|z| + |z - 2i|$ is 2.

Example 2.50 If z is any complex number such that $|z + 4| \le 3$, then find the greatest value of $|z + 1|$.

Sol. $|z + 1| = |z + 4 - 3|$

$= |(z + 4) + (-3)|$

$\le |z + 4| + |-3|$

$= |z + 4| + 3$

$\le 3 + 3 = 6$ $[\because |z + 4| \le 3]$

Hence, the greatest value of $|z + 1|$ is 6.

Example 2.51 Prove that the distance of the roots of the equation $|\sin\theta_1|\,z^3 + |\sin\theta_2|z^2 + |\sin\theta_3|\,z + |\sin\theta_4| = 3$ from $z = 0$ is greater than 2/3.

Sol. We know that $|\sin\theta_k| < 1$. Given,

$|\sin\theta_1|\,z^3 + |\sin\theta_2|z^2 + |\sin\theta_3|\,z + |\sin\theta_4| = 3$

$\Rightarrow$ $|3| = ||\sin\theta_1|z^3 + |\sin\theta_2|\,z^2 + |\sin\theta_3|z + |\sin\theta_4||$

$\le 1|z^3 + z^2 + z + 1|$

$< |z|^3 + |z|^2 + |z| + 1$

$< 1 + |z| + |z|^2 + |z|^3 + |z|^4 + \cdots\infty$ $(\because |z| < 1)$

$\Rightarrow$ $3 < \dfrac{1}{1-|z|}$

$\Rightarrow$ $3 - 3|z| < 1$

$\Rightarrow$ $2 < 3|z|$

$\Rightarrow$ $|z| > \dfrac{2}{3}$

Example 2.52 If z_1 and z_2 are two complex numbers and $c > 0$, then prove that $|z_1 + z_2|^2 \le (1 + c)\,|z_1|^2 + (1 + c^{-1})\,|z_2|^2$.

Sol. We have to prove

$|z_1 + z_2|^2 \le (1 + c)\,|z_1|^2 + (1 + c^{-1})\,|z_2|^2$

$\Rightarrow$ $|z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_2 z_2 \le (1 + c)\,|z_1|^2 + (1 + c^{-1})\,|z_2|^2$

$\Rightarrow$ $z_1\bar{z}_2 + \bar{z}_2 z_2 \le c|z_1|^2 + c^{-1}|z_2|^2$

$\Rightarrow$ $c|z_1|^2 + \dfrac{1}{c}|z_2|^2 - z_1\bar{z}_2 - \bar{z}_2 z_2 \ge 0$ [using Re $(z_1\bar{z}_2) \le |z_1\bar{z}_2|$]

$\Rightarrow$ $\left(\sqrt{c}\,|z_1| - \dfrac{1}{\sqrt{c}}|z_2|\right)^2 \ge 0$

which is always true.

Example 2.53 Find the greatest and the least value of $|z_1 + z_2|$ if $z_1 = 24 + 7i$ and $|z_2| = 6$.

Sol. $|z_1 + z_2| \le |z_1| + |z_2|$

$= |24 + 7i| + 6$

$= 25 + 6 = 31$

Also,

$|z_1 + z_2| = |z_1 - (-z_2)| \ge ||z_1| - |z_2||$

$\Rightarrow$ $|z_1 + z_2| \ge |25 - 6| = 19$

Hence the least value of $|z_1 + z_2|$ is 19 and the greatest value is 25.

Properties of Arguments

1. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.
   In general,
   $\arg(z_1 z_2 z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \cdots + \arg(z_n)$
2. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
3. $\arg \bar{z} = -\arg z$
4. $\arg\left(\frac{1}{z}\right) = \arg\left(\frac{z\bar{z}}{z}\right)$
   $= \arg\left(\frac{|z|^2}{z}\right)$
   $= \arg(|z|^2) - \arg z = 0 - \arg z$
5. $\arg\left(\frac{z}{\bar{z}}\right) = \arg(z) - \arg(\bar{z}) = \theta - (-\theta) = 2\theta = 2\arg(z)$,
   where $\theta = \arg(z)$.
6. $\arg(z^n) = n \arg z$, when $n \in \theta$.
7. $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.
8. If z is purely imaginary, then $\arg(z) = \pm\pi/2$.
9. If z is purely real, then $\arg(z) = 0$ or π.
10. Locus of z, if $\arg(z) = \theta$ (= constant) is ray excluding origin

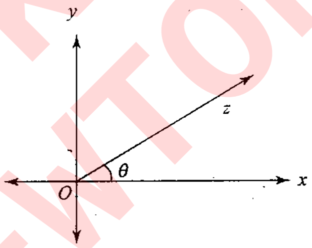


Fig. 2.11

11. Locus of z, if $\arg(z - a) = \theta$ (= constant) and $a > 0$

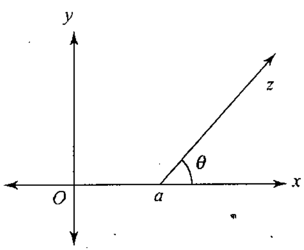


Fig. 2.12

12. Angle between two lines,
    $= \alpha - \beta$
    $= \arg(z_3 - z_1) - \arg(z_2 - z_1)$
    $= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

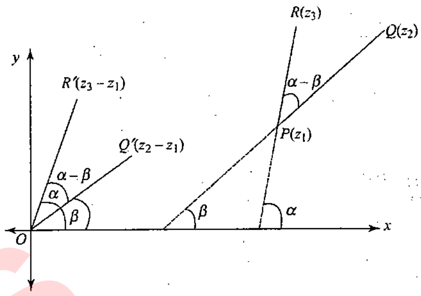


Fig. 2.13

13. Angle between lines joining z_1, z_2 and z_3, z_4:
    $\theta = \arg\left(\frac{z_4 - z_3}{z_1 - z_2}\right)$

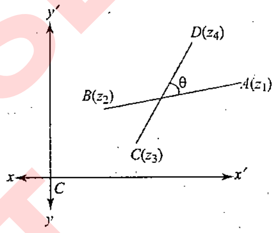


Fig. 2.14

Note: *By specifying the modulus and argument, a complex number is completely defined. However, for the complex number 0 + 0i the argument is not defined and this is the only complex number which is completely defined by taking in terms of its modulus.*

Important Results

1. If z_1, z_2, z_3 be the vertices of an equilateral triangle, then
   $$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$
   or
   $$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

Proof:

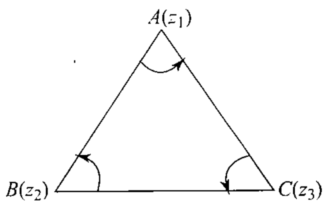


Fig. 2.15

Since ΔABC is equilateral, therefore
$AB = BC = CA$
$\therefore \quad |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$
$$\therefore \quad \left|\frac{z_1 - z_2}{z_3 - z_2}\right| = \left|\frac{z_2 - z_3}{z_1 - z_3}\right| \qquad \text{(i)}$$
Also,
$\angle CBA = \angle ACB$

$\therefore \quad \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$ (ii)

From (i) and (ii), it follows that

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{z_2 - z_3}{z_1 - z_3}$$

$\Rightarrow \quad (z_1 - z_2)(z_1 - z_3) = -(z_2 - z_3)^2$

$\Rightarrow \quad z_1^2 - z_1z_2 - z_1z_3 + z_2z_3 = -(z_2^2 + z_3^2 - 2z_2z_3)$

$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

2. If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that they are similar, then $\begin{vmatrix} a & u & 1 \\ b & v & 1 \\ c & w & 1 \end{vmatrix} = 0$ or $\frac{a-c}{a-b} = \frac{u-w}{u-v}$

Proof:

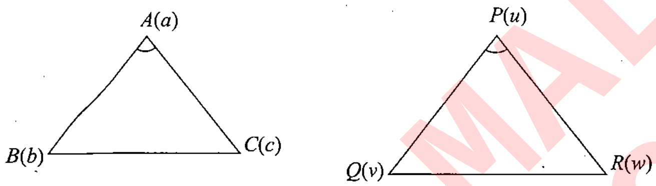


Fig. 2.16

In the figure triangles are similar. Hence

$$\frac{AC}{PR} = \frac{AB}{PQ}$$

$\Rightarrow \quad \frac{AC}{AB} = \frac{PR}{PQ}$

$\Rightarrow \quad \left|\frac{a-c}{a-b}\right| = \left|\frac{u-w}{u-v}\right|$ (1)

Also,

$$\angle BAC = \angle QPR$$

$\Rightarrow \quad \arg\left(\frac{a-c}{a-b}\right) = \arg\left(\frac{u-w}{u-v}\right)$ (2)

From (1) and (2), we have

$$\frac{a-c}{a-b} = \frac{u-w}{u-v}$$

Simplifying, we get

$$\begin{vmatrix} a & u & 1 \\ b & v & 1 \\ c & w & 1 \end{vmatrix} = 0$$

Example 2.54 **Find the amplitude of**

a. $\frac{1+\sqrt{3}i}{\sqrt{3}+i}$ **b.** $-1 - i\sqrt{3}$ **c.** $\sin\alpha + i(1 - \cos\alpha)$, $0 < \alpha < \pi$

Sol.

a. $\text{amp}\left(\frac{1+\sqrt{3}\,i}{\sqrt{3}+i}\right) = \text{amp}(1+\sqrt{3}\,i) - \text{amp}(\sqrt{3}+i)$

$$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

b. Let, $z = -1 - i\sqrt{3}$.

Then $\alpha = \tan^{-1}|b/a| = \tan^{-1}|\sqrt{3}/1| = \pi/3$

Here, z is in third quadrant. Therefore argument is $\theta = -(\pi - \alpha) = -(\pi - \pi/3) = -2\pi/3$

c. $z = \sin\alpha + i(1 - \cos\alpha)$, $0 < \alpha < \pi = \sin\alpha + i(1 - \cos\alpha)$

$\Rightarrow \quad \text{amp}(z) = \tan^{-1}\left(\frac{1-\cos\alpha}{\sin\alpha}\right)$ ($\because$ z lies in first quadrant)

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right) = \tan^{-1}\tan\left(\frac{\alpha}{2}\right) = \frac{\alpha}{2}$$

Example 2.55 **Find the modulus, argument and the principal argument of the complex numbers.**

(i) $(\tan 1 - i)^2$

(ii) $\dfrac{i-1}{i\left(1-\cos\frac{2\pi}{5}\right)+\sin\frac{2\pi}{5}}$

Sol. (i) $z = (\tan 1 - i)^2 = (\tan^2 1 - 1) - (2\tan 1)i$

$$|z| = \sqrt{(\tan^2 1 - 1)^2 + 4\tan^2 1}$$

$$= \sqrt{(\tan^2 1 + 1)^2} = \tan^2 1 + 1$$

Since $\tan^2 1 - 1 < 0$ and $-2\tan 1 < 0$

$\Rightarrow$ z lies on third quadrant

$\Rightarrow \arg(z) = -\pi + \tan^{-1}\left|\frac{2\tan 1}{1-\tan^2 1}\right|$

$$= -\pi + \tan^{-1}|\tan 2|$$

$$= 2 - \pi$$

(ii) $z = \dfrac{i-1}{i\left(1-\cos\frac{2\pi}{5}\right)+\sin\frac{2\pi}{5}}$

$$= \frac{i-1}{i2\sin^2\frac{\pi}{5} + 2\sin\frac{\pi}{5}\cos\frac{\pi}{5}}$$

$$= \frac{i-1}{\left(2\sin\frac{\pi}{5}\right)\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)}$$

$$|z| = \frac{|i-1|}{\left(2\sin\frac{\pi}{5}\right)\left|\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)\right|}$$

$$= \frac{\sqrt{2}}{\left(2\sin\frac{\pi}{5}\right)\left|\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)\right|}$$

$$= \frac{1}{\sqrt{2}}\operatorname{cosec}\frac{\pi}{5}$$

$$\arg z = \arg\left[\frac{i-1}{\left(2\sin\frac{\pi}{5}\right)\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)}\right]$$

$$= \arg(-1+i) - \arg\left(2\sin\frac{\pi}{5}\right) - \arg\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)$$

$$= \frac{3\pi}{4} - 0 - \frac{\pi}{5} = \frac{11\pi}{20}$$

Example 2.56 **If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find the value of $\arg(z_1/z_4) + \arg(z_2/z_3)$.**

Sol. We have $z_2 = \bar{z}_1$ and $z_4 = \bar{z}_3$. Therefore,

$z_1 z_2 = |z_1|^2$ and $z_3 z_4 = |z_3|^2$

Now,

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg\left(\frac{z_1 z_2}{z_4 z_3}\right)$$

$$= \arg\left(\frac{|z_1|^2}{|z_3|^2}\right)$$

$$= \arg\left(\left|\frac{z_1}{z_3}\right|^2\right) = 0$$

Example 2.57 **Prove that $|z_1 + z_2| = |z_1| + |z_2| \Rightarrow \arg(z_1) = \arg(z_2)$.**

Sol. $|z_1 + z_2| = |z_1| + |z_2|$

$\Rightarrow$ $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow$ $|z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow$ $2\,\mathrm{Re}(z_1\bar{z}_2) = 2|z_1||z_2|$

$\Rightarrow$ $\cos(\theta_1 - \theta_2) = 1$

$\Rightarrow$ $\theta_1 - \theta_2 = 0$

$\Rightarrow$ $\arg(z_1) = \arg(z_2)$

Example 2.58 **If $\arg(z_1) = 170°$ and $\arg(z_2) = 70°$, then find the principal argument of $z_1 z_2$.**

Sol. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = 170° + 70° = 240°$

Thus $z_1 z_2$ lies in third quadrant. Hence its principal argument is $-120°$.

Example 2.59 **If z_1 and z_2 are conjugate to each other, then find $\arg(-z_1 z_2)$.**

Sol. z_1 and z_2 are conjugate to each other, i.e., $z_2 = \bar{z}_1$. Therefore,

$$\arg(-z_1 z_2) = \arg(-z_1\bar{z}_1)$$

$$= \arg(-|z_1|^2)$$

$$= \arg(\text{negative real number})$$

$$= \pi$$

Example 2.60 **If $\pi/2 < \alpha < 3\pi/2$, then find the modulus and argument of $(1 + \cos 2\alpha) + i\sin 2\alpha$.**

Sol. $z = (1 + \cos 2\alpha) + i\sin 2\alpha$

$= 2\cos^2\alpha + 2i\sin\alpha\cos\alpha$

$= 2\cos\alpha\,[\cos\alpha + i\sin\alpha]$

$= -2\cos\alpha\,[-\cos\alpha - i\sin\alpha]$

$= -2\cos\alpha\,[\cos(\alpha - \pi) + i\sin(\alpha - \pi)]$

$[\because \pi/2 < \alpha < 3\pi/2]$

Thus, $|z| = -2\cos\alpha$ and $\arg(z) = \alpha - \pi$.

Example 2.61 **Find the point of intersection of the curves $\arg(z - 3i) = 3\pi/4$ and $\arg(2z + 1 - 2i) = \pi/4$.**

Sol. Given loci are as follows:

$$\arg(z - 3i) = \frac{3\pi}{4}$$

which is a ray that starts from $3i$ and makes an angle $3\pi/4$ with positive real axis as shown in the figure.

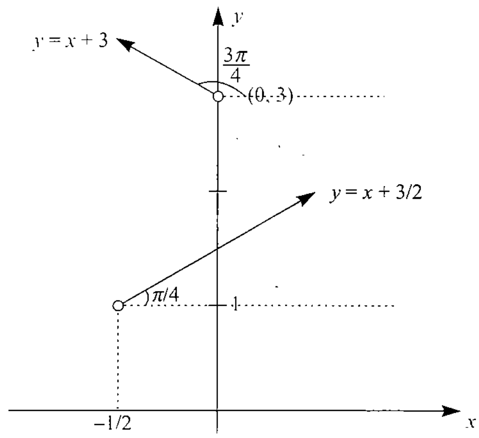


Fig. 2.17

$$\arg(2z + 1 - 2i) = \frac{\pi}{4}$$

$$\Rightarrow \arg\left[2\left(z + \frac{1}{2} - i\right)\right] = \frac{\pi}{4}$$

$$\Rightarrow \arg 2 + \arg\left[z - \left(-\frac{1}{2} + i\right)\right] = \frac{\pi}{4}$$

$$\Rightarrow 0 + \arg\left[z - \left(-\frac{1}{2} + i\right)\right] = \frac{\pi}{4}$$

$$\Rightarrow \arg\left[z-\left(-\frac{1}{2}+i\right)\right]=\frac{\pi}{4}$$

This is a ray that starts from point $-1/2 + i$ and makes an angle $\pi/4$ with positive real axis as shown in the figure. From the figure, it is obvious that the system of equations has no solution.

Example 2.62 **Let z and w be two non-zero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$. Then prove that $z = -\bar{w}$.**

Sol. Let $\arg(w) = \theta$. Then $\arg(z) = \pi - \theta$. Therefore,

$$w = |w| (\cos\theta + i \sin\theta)$$

and

$$z = |z| (\cos(\pi - \theta) + i \sin(\pi - \theta))$$

$$= |w|(-\cos\theta + i \sin\theta) \qquad [\because |z| = |w|]$$

$$= -|w| (\cos\theta - i \sin\theta) = -\bar{w}.$$

Example 2.63 **Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by $A = \{z : |z| \le 2\}$ and $B = \{z : (1 - i)z + (1 + i)\bar{z} \ge 4\}$. Find the area of region $A \cap B$.**

Sol. $z = x + iy$

$A = \{z : |z| \le 2\}$

$\Rightarrow \sqrt{x^2+y^2} \le 2$

$\Rightarrow x^2 + y^2 \le 4$

$\Rightarrow$ z lies on or inside the circle $x^2 + y^2 = 4$

$B = \{z : (1 - i)z + (1 + i)\bar{z} \ge 4\}$

$\Rightarrow (1 - i)(x + iy) + (1 + i)(x - iy) \ge 4$

$\Rightarrow x + iy - ix + y + x - iy + ix + y \ge 4$

$\Rightarrow x + y \ge 2$

Area of region $A \cap B$ is shaded region the diagram.

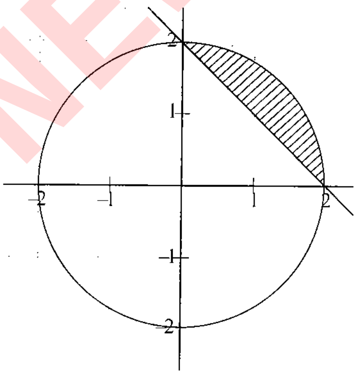


Fig. 2.18

$$\text{Area} = \frac{\pi(2)^2}{4} - \frac{1}{2}\times 2\times 2 = \pi - 2$$

Example 2.64 **Find the area bounded by $\arg|z| \le \pi/4$ and $|z - 1| < |z - 3|$.**

Sol. $|\arg z| < \pi/4$

$-\pi/4 < \arg z < \pi/4$ (i)

Which represents the region given in the following diagram.

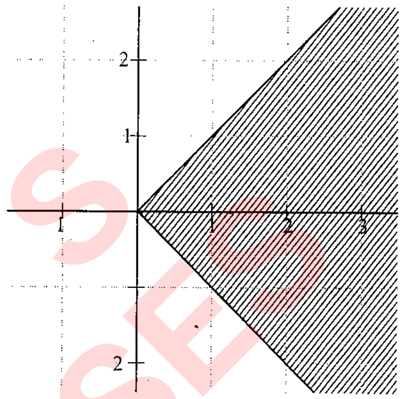


Fig. 2.19

$|z - 1| < |z - 3|$

$\Rightarrow (x - 1)^2 + y^2 < (x - 3)^2 + y^2$

$\Rightarrow x < 2$ (ii)

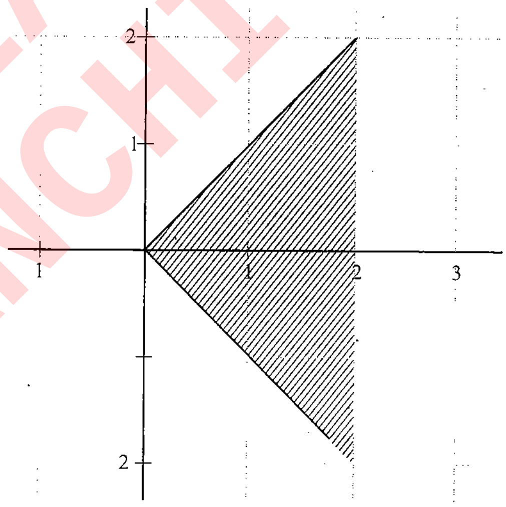


Fig. 2.20

Common region of (i) and (ii) is shown in above figure.

Area of the shaded region is $\frac{1}{2}(4)(2) = 4$ square units.

Example 2.65 **If $z + 1/z = 2\cos\theta$, prove that**

$$\left|(z^{2n} - 1)/(z^{2n} + 1)\right| = |\tan n\theta|$$

Sol. $z + \frac{1}{z} = 2\cos\theta$

$\Rightarrow z^2 - 2\cos\theta\, z + 1 = 0$

$$\Rightarrow z = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$= \cos\theta + i\sin\theta$

Taking positive sign,

$$z = \cos\theta + i\sin\theta,\ \frac{1}{z} = (\cos\theta - i\sin\theta)$$

$$\therefore \quad \frac{z^{2n}-1}{z^{2n}+1}=\frac{z^n-\frac{1}{z^n}}{z^n+\frac{1}{z^n}}$$

$$=\frac{(\cos\theta+i\sin\theta)^n-(\cos\theta-i\sin\theta)^n}{(\cos\theta+i\sin\theta)^n+(\cos\theta-i\sin\theta)^n}$$

$$=\frac{2i\sin n\theta}{2\cos n\theta}$$

$$=i\tan n\theta$$

Taking negative sign, we get

$$\frac{z^{2n}-1}{z^{2n}+1}=\frac{-2i\sin n\theta}{2\cos n\theta}=-\tan n\theta$$

$$\Rightarrow \quad \left|\frac{z^{2n}-1}{z^{2n}+1}\right|=|\pm i\tan\theta|=\tan n\theta$$

Concept Application Exercise 2.3

1. For $z_1=\sqrt[6]{(1-i)/(1+i\sqrt{3})}$, $z_2=\sqrt[6]{(1-i)/(\sqrt{3}+i)}$, $z_3=\sqrt[6]{(1+i)/(\sqrt{3}-i)}$, prove that $|z_1|=|z_2|=|z_3|$.
2. Let z be a complex number satisfying the equation $(z^3+3)^2=-16$, then find the value of $|z|$.
3. Show that a real value of x will satisfy the equation $(1-ix)/(1+ix)=a-ib$ if $a^2+b^2=1$, where a, b are real.
4. Find non-zero integral solutions of $|1-i|^x=2^x$.
5. Let z is not a real number such that $(1+z+z^2)/(1-z+z^2)\in R$, then prove that $|z|=1$.
6. If a, b, c are non-zero complex numbers of equal moduli and satisfy $az^2+bz+c=0$, then prove that $(\sqrt{5}-1)/2\le |z|\le(\sqrt{5}+1)/2$.
7. If z_1, z_2, z_3 are distinct non-zero complex numbers and a, b, $c\in R^+$ such that
$$\frac{a}{|z_1-z_2|}=\frac{b}{|z_2-z_3|}=\frac{c}{|z_3-z_1|}$$
then find the value of
$$\frac{a^2}{z_1-z_2}+\frac{b^2}{z_2-z_3}+\frac{c^2}{z_3-z_1}$$
8. Prove that $|z_1+z_2|^2=|z_1|^2+|z_2|^2$, if z_1/z_2 is purely imaginary.
9. If θ is real and z_1, z_2 are connected by $z_1^2+z_2^2+2z_1z_2\cos\theta=0$, then prove that the triangle formed by vertices O, z_1 and z_2 is isosceles.
10. If $2z_1/3z_2$ is a purely imaginary number, then find the value of $|(z_1-z_2)/(z_1+z_2)|$.
11. If $|z|\le 4$ then find the maximum value of $|iz+3-4i|$.
12. If $\sqrt{3}+i=(a+ib)(c+id)$, then find the value of $\tan^{-1}(b/a)+\tan^{-1}(d/c)$.
13. Let z be any non-zero complex number, then find $\arg(z)+\arg(\bar{z})$.
14. Find the area bounded by the curves $\arg z=\pi/3$, $\arg z=2\pi/3$ and $\arg(z-2-2\sqrt{3}\,i)=\pi$ on the complex plane.
15. If $\sqrt{ab}=\sqrt{a}\sqrt{b}$ and $a<0$, then find $\arg(b+ai)$.
16. If $|z_1-z_0|=|z_2-z_0|=a$ and amp $((z_2-z_0)/(z_0-z_1))=\pi/2$, then find z_0.
17. Let $z_1, z_2, z_3, \ldots, z_n$ are the complex numbers such that $|z_1|=|z_2|=\cdots=|z_n|=1$. If
$$z=\left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n}\frac{1}{z_k}\right)$$
then prove that
   a. z is a real number
   b. $0<z\le n^2$

DE MOIVRE'S THEOREM

Statement:

(i) If $n\in Z$ (the set of integers), then
$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$$

(ii) If $n\in Q$ (the set of rational numbers), then $\cos n\theta+i\sin n\theta$ is one of the values of $(\cos\theta+i\sin\theta)^n$.

Proof:

(i) When $n\in Z$, we know that

$e^{i\theta}=\cos\theta+i\sin\theta$

$\Rightarrow\ (e^{i\theta})^n=(\cos\theta+i\sin\theta)^n$

$\Rightarrow\ e^{i(n\theta)}=(\cos\theta+i\sin\theta)^n$

$\Rightarrow\ \cos(n\theta)+i\sin(n\theta)=(\cos\theta+i\sin\theta)^n$

(ii) Let n be rational number. Let $n=p/q$, where p, q are integers and $q\neq 0$. From part (i), we have

$$\left(\cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}\right)^q=\cos\left(\left(\frac{p\theta}{q}\right)q\right)+i\sin\left(\left(\frac{p\theta}{q}\right)q\right)$$
$$=\cos p\theta+i\sin p\theta$$

$\Rightarrow\ \cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}$ is one of the values of $(\cos p\theta+i\sin p\theta)^{1/q}$

$\Rightarrow\ \cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}$ is one of the values of $[(\cos\theta+i\sin\theta)^p]^{1/q}$

$\Rightarrow\ \cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}$ is one of the values of $(\cos\theta+i\sin\theta)^{p/q}$

Note:

1. *De Moivre's theorem is also true for* $(\cos\theta - i\sin\theta)$, *i.e.,*
$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$, *because*
$$(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta.$$
2. $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$
3. $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta.$
4. $(\sin\theta + i\cos\theta)^n = [\cos(\pi/2 - \theta) + i\sin(\pi/2 - \theta)]^n$
$= \cos(n\pi/2 - n\theta) + i\sin(n\pi/2 - n\theta)$
5. $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$

Important Result

Writing the binomial expression of $(\cos\theta + i\sin\theta)^n$ and equating the real part to $\cos n\theta$ and the imaginary part to $\sin n\theta$, we get

$$\cos(n\theta) = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \cdots$$

$$\sin(n\theta) = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \cdots$$

$$\tan(n\theta) = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - {}^nC_7\tan^7\theta + \cdots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - {}^nC_6\tan^6\theta + \cdots}$$

Example 2.66 **Express the following in $a + ib$ form:**

a. $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4$

b. $\dfrac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^{-2}(\cos 3\theta - i\sin 3\theta)^{-9}}$

c. $\dfrac{(\sin\pi/8 + i\cos\pi/8)^8}{(\sin\pi/8 - i\cos\pi/8)^8}$

Sol. a. $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4$

$$= \frac{(\cos\theta + i\sin\theta)^4}{i^4(\cos\theta - i\sin\theta)^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta + i\sin\theta)^{-4}}$$

$= (\cos 8\theta + i\sin\theta)^8$

$= \cos 8\theta + i\sin 8\theta$

b. $\dfrac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^{-2}(\cos 3\theta - i\sin 3\theta)^{-9}}$

$$= \frac{\left[(\cos\theta + i\sin\theta)^{-2}\right]^4\left[(\cos\theta + i\sin\theta)^4\right]^{-5}}{\left[(\cos\theta + i\sin\theta)^3\right]^{-2}\left[(\cos\theta + i\sin\theta)^{-3}\right]^{-9}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{-8}(\cos\theta + i\sin\theta)^{-20}}{(\cos\theta + i\sin\theta)^{-6}(\cos\theta + i\sin\theta)^{27}}$$

$= (\cos\theta + i\sin\theta)^{-8-20+6-27}$

$= (\cos\theta + i\sin\theta)^{-49}$

$= \cos 49\theta - i\sin 49\theta$

c. $\dfrac{(\sin\pi/8 + i\cos\pi/8)^8}{(\sin\pi/8 - i\cos\pi/8)^8}$

$$\frac{i^8(\cos\pi/8 - i\sin\pi/8)^8}{(-i)^8(\cos\pi/8 + i\sin\pi/8)^8} = \frac{\cos\pi - i\sin\pi}{\cos\pi + i\sin\pi} = 1$$

Example 2.67 **If $z = \left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^5 + \left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)^5$, then prove that Im(z) = 0.**

Sol. Given that

$$z = \left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)^5$$

$$= \left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]^5 + \left[\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right]^5$$

$$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} + \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}$$

Hence Im(z) = 0.

Example 2.68 **Find the value of the expression**

$$\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\ldots \text{ to } \infty$$

Sol. $\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)\left(\cos\dfrac{\pi}{2^2} + i\sin\dfrac{\pi}{2^2}\right)\ldots$ to ∞

$$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \cdots\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \cdots\right)$$

$$= \cos\left[\frac{\pi}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)\right] + i\sin\left[\frac{\pi}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)\right]$$

$$= \cos\left[\frac{\pi}{2}\left(\frac{1}{1 - \frac{1}{2}}\right)\right] + i\sin\left[\frac{\pi}{2}\left(\frac{1}{1 - \frac{1}{2}}\right)\right] = \cos\pi + i\sin\pi = -1$$

Example 2.69 **If $z = (\sqrt{3} + i)^{17}/(1 - i)^{50}$, then find amp(z).**

Sol.
$$z = \frac{(\sqrt{3} + i)^{17}}{(1 - i)^{50}} = \frac{\left[2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\right]^{17}}{\left[\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]^{50}}$$

$$= \frac{2^{17}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{17}}{2^{25}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{50}}$$

$$= \frac{\left(\cos\frac{17\pi}{6} + i\sin\frac{17\pi}{6}\right)}{2^8\left(\cos\frac{50\pi}{4} - i\sin\frac{50\pi}{4}\right)}$$

$$= \frac{\left[\cos\left(2\pi + \frac{5\pi}{6}\right) + i\sin\left(2\pi + \frac{5\pi}{6}\right)\right]}{2^8\left[\cos\left(12\pi + \frac{\pi}{2}\right) - i\sin\left(12\pi + \frac{\pi}{2}\right)\right]}$$

$$= \frac{\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}{2^8\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)}$$

$$= \frac{\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}{2^8\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]}$$

Hence,

$$\arg(z) = \frac{5\pi}{6} - \left(-\frac{\pi}{2}\right) = \frac{4\pi}{3}$$

Thus z lies in the third quadrant and principal argument is $-2\pi/3$.

Example 2.70 **If $z = x + iy$ is a complex number with x, $y \in Q$ and $|z| = 1$, then show that $|z^{2n} - 1|$ is a rational number for every $n \in N$.**

Sol. $|z| = 1 \Rightarrow z = e^{i\theta} = x + iy$

$\Rightarrow \quad x = \cos\theta,\ y = \sin\theta$

Now $\cos\theta$ and $\sin\theta \in Q$. Also,

$$|z^{2n} - 1|^2 = (z^{2n} - 1)(\bar{z}^{2n} - 1)$$
$$= (z\bar{z})^{2n} - z^{2n} - \bar{z}^{2n} + 1$$
$$= 2 - (z^{2n} + \bar{z}^{2n})$$
$$= 2 - 2\cos 2n\theta = 4\sin^2 n\theta$$

$\Rightarrow \quad |z^{2n} - 1| = 2|\sin n\theta|$

Now,

$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \cdots$

= Rational number $\quad (\because \sin\theta, \cos\theta$ are rationals)

$\Rightarrow \quad |z^{2n} - 1|$ = Rational number

Example 2.71 **If $z = \cos\theta + i\sin\theta$ be a root of the equation $a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0$, then prove that**

(i) $a_0 + a_1\cos\theta + a_2\cos 2\theta + \cdots + a_n\cos n\theta = 0$

(ii) $a_1\sin\theta + a_2\sin 2\theta + \cdots + a_n\sin n\theta = 0$

Sol. Dividing the given equation by z^n, we get

$$a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{n-1} z^{1-n} + a_n z^{-n} = 0$$

Now, $z = \cos\theta + i\sin\theta = e^{i\theta}$ satisfies the above equation. Hence,

$$a_0 + a_1 e^{-i\theta} + a_2 e^{-2i\theta} + \cdots + a_{n-1} e^{-i(n-1)\theta} + a_n e^{-in\theta} = 0$$

$\Rightarrow \quad (a_0 + a_1\cos\theta + a_2\cos 2\theta + \cdots + a_n\cos n\theta)$
$+ i(a_1\sin\theta + a_2\sin 2\theta + \cdots + a_n\sin n\theta) = 0$

$\Rightarrow a_0 + a_1\cos\theta + a_2\cos 2\theta + \cdots + a_n\cos n\theta = 0$

and

$$a_1\sin\theta + \theta_2\sin 2\theta + \cdots + a_n\sin n\theta = 0$$

Concept Application Exercise 2.4

1. Express the following in $a + ib$ form
   a. $\dfrac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^5}$
   b. $\left(\dfrac{1+\cos\phi + i\sin\phi}{1+\cos\phi - i\sin\phi}\right)^n$
   c. $\dfrac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{(\cos\gamma + i\sin\gamma)(\cos\delta + i\sin\delta)}$
2. If $(1/x) + x = 2\cos\theta$, then prove that $x^n + 1/x^n = 2\cos n\theta$.
3. Find the value of the following expression:
   $$\left[\frac{1 - \cos\frac{\pi}{10} + i\sin\frac{\pi}{10}}{1 - \cos\frac{\pi}{10} - i\sin\frac{\pi}{10}}\right]^{10}$$
4. If $iz^4 + 1 = 0$, then prove that z can take the value $\cos\pi/8 + i\sin\pi/8$.
5. If n is a positive integer, then prove that
   $(1 + i)^n + (1 - i)^n = (\sqrt{2})^{n+2}\cos\left(\frac{n\pi}{4}\right)$
6. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and also $\sin\alpha + \sin\beta + \sin\gamma = 0$, then prove that
   a. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$
   b. $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$
   c. $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

CUBE ROOTS OF UNITY

Let $z = 1^{1/3}$. Then,

$$z^3 = 1$$
$\Rightarrow \quad z^3 - 1 = 0$
$\Rightarrow \quad (z - 1)(z^2 + z + 1) = 0$
$\Rightarrow \quad z - 1 = 0$, or $z^2 + z + 1 = 0$
$\Rightarrow \quad z = 1$ or $z = \dfrac{-1 \pm \sqrt{1-4}}{2}$
$\Rightarrow \quad z = 1$ or $z = \dfrac{-1 \pm i\sqrt{3}}{2}$

So, the cube roots of unity are 1, $(-1 + i\sqrt{3})/2$ and $(-1 - i\sqrt{3})/2$. Clearly, one of the roots of unity is real and the other two are complex.

Properties of Cube Roots of Unity

1. Each complex cube root of unity is the square of the other.

Proof:

Complex cube roots of unity are $(-1 + i\sqrt{3})/2$ and $(-1 - i\sqrt{3})/2$.

Now,

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^2=\frac{1-2i\sqrt{3}+3i^2}{4}=\frac{1-2i\sqrt{3}-3}{4}=\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$\left(\frac{-1-i\sqrt{3}}{2}\right)^2=\frac{1+2i\sqrt{3}+3i^2}{4}=\frac{1+2i\sqrt{3}-3}{4}=\left(\frac{-1+i\sqrt{3}}{2}\right)$$

Hence, each complex cube root of unity is the square of the other.

It follows from the above discussion that if we denote one of the complex cube roots of unity by ω (omega), then the other complex cube root of unity is ω^2. Let,

$$\omega = (-1+i\sqrt{3})/2$$

Then

$$\omega^2 = (-1-i\sqrt{3})/2$$

Clearly, $\bar{\omega} = \omega^2$ and $\overline{\omega^2} = \omega$.

2. Integral powers of ω

Since ω is a root of the equation $z^3 - 1 = 0$, so, it satisfies the equation $z^3 - 1 = 0$. Therefore,

$$\omega^3 - 1 = 0 \Rightarrow \omega^3 = 1$$

Since $\omega^3 = 1$, therefore $\omega^n = \omega^r$, where r is the least non-negative remainder obtained by dividing n by 3. For example, $\omega^{18} = (\omega^3)^6 = 1^6 = 1$, $\omega^{20} = (\omega^3)^6\omega^2 = 1^6\omega^2 = \omega^2$, $\omega^{-30} = (\omega^3)^{-10} = 1^{-10} = 1$, $\omega^{28} = (\omega^{27})\,\omega = \omega$.

3. The sum of three cube roots of unity is zero, i.e., $1 + \omega + \omega^2 = 0$.

Proof:

We have,

$$1+\omega+\omega^2 = 1+\left(\frac{-1+i\sqrt{3}}{2}\right)+\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$=\frac{2-1+i\sqrt{3}-1-i\sqrt{3}}{2}=0$$

4. The product of three cube roots of unity is 1.

Proof:

Three cube roots of unity are 1, ω and ω^2. So, product of cube roots of unity is $1\times\omega\times\omega^2 = \omega^3 = 1$.

5. Each complex cube root of unity is the reciprocal of the other.

Proof:

We have,

$$\omega\times\omega^2 = \omega^3 = 1 \Rightarrow \omega = \frac{1}{\omega^2} \text{ and } \omega^2 = \frac{1}{\omega}$$

6. Cube roots of -1 are -1, $-\omega$ and $-\omega^2$.

Proof:

$$z = (-1)^{1/3}$$

$$\Rightarrow z^3 = -1$$

$$\Rightarrow (-z)^3 = 1$$

$$\Rightarrow -z = 1, \omega, \omega^2$$

$$\Rightarrow z = -1, -\omega, -\omega^2$$

The idea of finding cube roots of 1 and -1 can be extended to find cube roots of any real number. If a is any positive real number then $a^{1/3}$ has values $a^{1/3}$, $a^{1/3}\,\omega$ and $a^{1/3}\,\omega^2$. If a is a negative real number, then $a^{1/3}$ has values $-|a|^{1/3}$, $-|a|^{1/3}\,\omega$ and $-|a|^{1/3}\,\omega^2$. For example, $8^{1/3}$ has values 2, 2ω and $2\omega^2$ whereas $(-8)^{1/3}$ attains values -2, -2ω and $-2\omega^2$.

7. If 1, ω, ω^2 be cube roots of unity and n is a positive integer, then

$$1+\omega^n+\omega^{2n} = \begin{cases} 3, & \text{when } n \text{ is a multiple of 3} \\ 0, & \text{when } n \text{ is not a multiple of 3} \end{cases}$$

Proof:

Case I: n is a multiple of 3.

In this case, $n = 3m$ for some positive integer m.

$$\therefore\quad 1+\omega^n+\omega^{2n} = 1+\omega^{3m}+\omega^{6m} = 1+(\omega^3)^m+(\omega^3)^{2m}$$

$$= 1+1+1 = 3\quad [\because (\omega^3)^m = 1^m = 1 \text{ and } (\omega^3)^{2m} = 1^{2m} = 1]$$

Case II: n is not a multiple of 3.

In this case, $n = 3m+1$ or $n = 3m+2$ for some positive integer m. When $n = 3m+1$,

$$1+\omega^n+\omega^{2n} = 1+\omega^{3m+1}+\omega^{6m+2}$$
$$= 1+\omega^{3m}\omega+\omega^{6m}\,\omega^2$$
$$= 1+(\omega^3)^m\omega+(\omega^3)^{2m}\,\omega^2$$
$$= 1+\omega+\omega^2 = 0$$

When $n = 3m+2$,

$$1+\omega^n+\omega^{2n} = 1+\omega^{3m+2}+\omega^{6m+4}$$
$$= 1+\omega^{3m}\omega^2+\omega^{6m}\,\omega^4$$
$$= 1+(\omega^3)^m\,\omega^2+(\omega^3)^{2m}\omega^3\omega$$
$$= 1+\omega^2+\omega = 0$$

8. Factorization of a^3+b^3 and a^3-b^3

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$
$$= (a+b)(a+b\omega)(a+b\omega^2)$$

and

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$
$$= (a-b)(a-b\omega)(a-b\omega^2)$$

9. Factorization of $a^3+b^3+c^3-3abc$

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = (a+b+c)\,(a+b\omega+c\omega^2)\,(a+b\omega^2+c\omega)$$

10. Cube roots of unity represent vertices of equilateral triangle on the Argand plane.

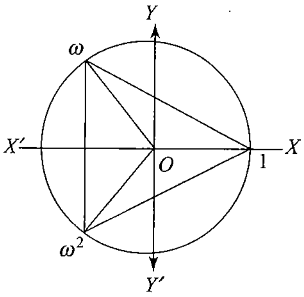


Fig. 2.21

Example 2.72 **If ω is a cube root of unity, then find the value of the following:**

(i) $(1+\omega-\omega^2)(1-\omega+\omega^2)$

(ii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

(iii) $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$

Sol.

(i) If ω is a complex cube root of unity, then $\omega^3=1$ and $1+\omega+\omega^2=0$.

$\therefore$ $(1+\omega-\omega^2)(1-\omega+\omega^2)=(-2\omega^2)(-2\omega)=4$

(ii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$
$=(1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$
$=(1-\omega)^2(1-\omega^2)^2$
$=(1-2\omega+\omega^2)(1-2\omega^2+\omega^4)$
$=(1-2\omega+\omega^2)(1-2\omega^2+\omega)$
$=(-3\omega)(-3\omega^2)=9\omega^3=9$

(iii) Multiplying the numerator and denominator of expressions I and II by ω and ω^2, respectively, we have

$$\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$$
$$=\frac{\omega(a+b\omega+c\omega^2)}{(b\omega+c\omega^2+a)}+\frac{\omega^2(a+b\omega+c\omega^2)}{(c\omega^2+a+b\omega)}$$
$$=\omega+\omega^2=-1$$

Example 2.73 **Solve the equation $(x-1)^3+8=0$ in the set C of all complex numbers.**

Sol. We have,

$(x-1)^3+8=0 \Rightarrow (x-1)^3=-8 \Rightarrow x-1=(-8)^{1/3}$

$\Rightarrow x-1=2(-1)^{1/3}$

$\Rightarrow x-1=2(-1)$ or $x-1=2(-\omega)$ or $x-1=2(-\omega^2)$

$[\because (-1)^{1/3}=-1$ or $-\omega$ or $-\omega^2]$

$\Rightarrow x-1=-2$ or $x-1=-2\omega$ or $x-1=-2\omega^2$

$\Rightarrow x=-1$ or $x=1-2\omega$ or $x=1-2\omega^2$

Hence, the solutions of the equation $(x-1)^3+8=0$ are -1, $1-2\omega$ and $1-2\omega^2$.

Example 2.74 **If n is an odd integer that is greater than or equal to 3 but not a multiple of 3, then prove that $(x+1)^n-x^n-1$ is divisible by x^3+x^2+x.**

Sol. Let

$f(x)=(x+1)^n-x^n-1$

$x^3+x^2+x=x(x^2+x+1)=x(x-\omega)(x-\omega^2)$

$f(0)=(0+1)^n-0^n-1=0$

$f(\omega)=(\omega+1)^n-\omega^n-1=(-\omega^2)^n-\omega^n-1=-(\omega^{2n}+\omega^n+1)=0$
when n is not a multiple of 3.

$f(\omega^2)=(\omega^2+1)^n-\omega^{2n}-1=(-\omega)^n-\omega^{2n}-1=-(\omega^n+\omega^{2n}+1)=0$
when n is not a multiple of 3.

Example 2.75 **ω is an imaginary root of unity. Prove that**

(i) $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3=(2a-b-c)(2b-a-c)(2c-a-b)$,

(ii) If $a+b+c=0$, then prove that $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3=27abc$.

Sol.

(i) Let $a+b\omega+c\omega^2=x$ and $a+b\omega^2+c\omega=y$.

$\therefore$ $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3=x^3+y^3$
$=(x+y)(x+\omega y)(x+\omega^2 y)$

Now,

$x+y=(a+b\omega+c\omega^2)+(a+b\omega^2+c\omega)$
$=2a+b(\omega+\omega^2)+c(\omega+\omega^2)$
$=2a-b-c$

$x+\omega y=(a+b\omega+c\omega^2)+\omega(a+b\omega^2+c\omega)$
$=(1+\omega)a+(1+\omega)b+2\omega^2c$
$=\omega^2(2c-a-b)$

Similarly,

$x+\omega^2y=\omega(2b-a-c)$

$\Rightarrow$ $(x+y)(x+\omega y)(x+\omega^2y)$
$=\omega^3(2a-b-c)(2c-a-b)(2b-a-c)$
$=(2a-b-c)(2c-a-b)(2b-a-c)$

(ii) $a+b+c=0 \Rightarrow b+c=-a,\ c+a=-b$ and $a+b=-c$

Putting these values on the R.H.S. of result (i), we get

$(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3=27abc$

Example 2.76 **Find the complex number ω satisfying the equation $z^3=8i$ and lying in the second quadrant on the complex plane.**

Sol. $z^3=8i$

$\Rightarrow z^3=-8i^3$

$\Rightarrow \left(\dfrac{z}{-2i}\right)^3=1$

$\Rightarrow \dfrac{z}{-2i}=1$ or ω or ω^2

$\Rightarrow z=-2i$ or $-2i\omega$ or $-2i\omega^2$

$\Rightarrow z=-2i$ or $-2i\left(\dfrac{-1+\sqrt{3}i}{2}\right)$ or $-2i\left(\dfrac{-1-\sqrt{3}i}{2}\right)$

Hence, $z=-2i$ or $i+\sqrt{3}$ or $i-\sqrt{3}$ out of which $i-\sqrt{3}$ lies in second quadrant.

Example 2.77 **When the polynomial $5x^3+Mx+N$ is divided by x^2+x+1, the remainder is 0. Then find the value of $M+N$.**

Sol. Let $f(x)=5x^3+Mx+N$.

Also, $x^2+x+1=(x-\omega)(x-\omega^2)$

$f(x)$ is divisible by x^2+x+1. Hence, $f(\omega)=5+M\omega+N=0$

and

$$f(\omega^2) = 5 + M\omega^2 + N = 0$$

$\Rightarrow$ $M = 0; N = -5$

$\Rightarrow$ $M + N = -5$

Example 2.78 **If ω and ω^2 are the non-real cube roots of unity and $[1/(a+\omega)]+[1/(b+\omega)]+[1/(c+\omega)] = 2\omega^2$ and $[1/(a+\omega)^2]+[1/(b+\omega)^2]+[1/(c+\omega)^2] = 2\omega$, then find the value of $[1/(a+1)]+[1/(b+1)]+[1/(c+1)]$.**

Sol. The given relations can be rewritten as

$$\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}=\frac{2}{\omega}$$

and

$$\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}=\frac{2}{\omega^2}$$

$\Rightarrow$ ω and ω^2 are roots of $\dfrac{1}{a+x}+\dfrac{1}{b+x}+\dfrac{1}{c+x}=\dfrac{2}{x}$

$$\Rightarrow \frac{3x^2+2(a+b+c)x+bc+ca+ab}{(a+x)(b+x)(c+x)}=\frac{2}{x}$$

$$\Rightarrow x^3+(bc+ca+ab)x-2abc=0 \quad (1)$$

Two roots of the Eq. (1) are ω and ω^2. Let the third root be α. Then,

$$\alpha+\omega+\omega^2=0 \Rightarrow \alpha=-\omega-\omega^2=1$$

Therefore, $\alpha = 1$ will satisfy Eq. (1). Hence,

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=2$$

Concept Application Exercise 2.5

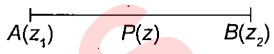
1. If α and β are imaginary cube roots of unity, then find the value of $\alpha^4+\beta^4+1/(\alpha\beta)$.
2. If ω is a complex cube root of unity, then find the value of $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\cdots$ to $2n$ factors.
3. Write the complex number in $a+ib$ form using cube roots of unity:
   a. $\left(-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)^{1000}$
   b. If $z=\dfrac{(\sqrt{3}+i)^{17}}{(1-i)^{50}}$
   c. $(i+\sqrt{3})^{100}+(i-\sqrt{3})^{100}+2^{100}$
4. If $z+z^{-1}=1$, then find the value of $z^{100}+z^{-100}$.
5. Find the common roots of $x^{12}-1=0$, $x^4+x^2+1=0$.
6. If $\omega\ (\neq 1)$ is a cube root of unity, then find the value of
$$\begin{vmatrix} 1 & 1+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -1+\omega & -1 \end{vmatrix}$$

GEOMETRY WITH COMPLEX NUMBERS

Section Formula

If $P(z)$ divides the line segment joining $A(z_1)$ and $B(z_2)$ internally in the ratio $m:n$, then

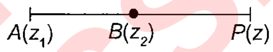
$$z=\frac{mz_2+nz_1}{m+n}$$



$$\frac{AP}{PB}=\frac{m}{n}$$

If division is external, then

$$z=\frac{mz_2-nz_1}{m-n}$$



Explanation:

Let $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$. Then, $A\equiv(x_1, y_1)$ and $B\equiv(x_2, y_2)$. Let $z = x+iy$. Then $P\equiv(x, y)$. We have from coordinate geometry.

$$x=\frac{mx_2+nx_1}{m+n} \text{ and } y=\frac{my_2+ny_1}{m+n}$$

Hence complex number of P is

$$z=\frac{mx_2+nx_1}{m+n}+i\frac{my_2+ny_1}{m+n}$$

$$=\frac{m(x_2+iy_2)+n(x_1+iy_1)}{m+n}$$

$$=\frac{mz_2+nz_1}{m+n}$$

Note:

1. *In acute triangle, orthocentre (H), centroid (G) and circumcentre (O) are collinear and $HG:GO\equiv 2:1$.*
2. *Centroid of the triangle formed by points $A(z_1)$, $B(z_2)$ and $C(z_3)$ is $(z_1+z_2+z_3)/3$.*
3. *If circumcentre of the triangle is origin, then its orthocentre is $z_1+z_2+z_3$ (using 1).*

Example 2.79 **Find the relation if z_1, z_2, z_3, z_4 are the affixes of the vertices of a parallelogram taken in order.**

Sol. As the diagonals of a parallelogram bisect each other, affix of the mid-point of AC is same as the affix of the mid-point of BD, i.e.,

$$\frac{z_1+z_3}{2}=\frac{z_2+z_4}{2}$$

$$\Rightarrow z_1+z_3=z_2+z_4$$

Example 2.80 **If z_1, z_2, z_3 are three non-zero complex numbers such that $z_3=(1-\lambda)z_1+\lambda z_2$ where $\lambda\in R-\{0\}$, then points corresponding to z_1, z_2 and z_3.**

a. **lie on a circle** **b.** **are vertices of a triangle**
c. **are collinear** **d.** **none of these**

Sol. $z_3 = (1-\lambda)z_1 + z_2$

$$= \frac{(1-\lambda)z_1 + \lambda z_2}{1-\lambda+\lambda}$$

Hence, z_3 divides the line joining $A(z_1)$ and $B(z_2)$ in the ratio $\lambda : (1-\lambda)$. Thus the given points are collinear.

Example 2.81 **In ΔABC, $A(z_1)$, $B(z_2)$ and $C(z_3)$ are inscribed in the circle $|z| = 5$. If $H(z_H)$ be the orthocentre of triangle ABC, then find z_H.**

Sol. Circumcentre of ΔABC is clearly origin. Let $G(z_G)$ be its centroid. Then,

$$z_G = \frac{1}{3}(z_1+z_2+z_3)$$

Now we know that

$OG:GH = 1:2$

$$\Rightarrow \quad z_G = \frac{2\times 0 + 1\times z_H}{3}$$

$$\Rightarrow \quad z_H = 3z_G = z_1+z_2+z_3$$

Example 2.82 **Let z_1, z_2, z_3 be three complex numbers and a, b, c be real numbers not all zero, such that $a+b+c = 0$ and $az_1 + bz_2 + cz_3 = 0$. Show that z_1, z_2, z_3 are collinear.**

Sol. Given,

$$a+b+c = 0 \quad \text{(i)}$$

and

$$az_1 + bz_2 + cz_3 = 0 \quad \text{(ii)}$$

Since a, b, c are not all zero, from (ii), we have

$$az_1 + bz_2 - (a+b)z_3 = 0 \quad \text{[From (i), } c = -(a+b)\text{]}$$

$$\Rightarrow \quad az_1 + bz_2 = (a+b)z_3$$

$$\Rightarrow \quad z_3 = \frac{az_1+bz_2}{a+b} \quad \text{(iii)}$$

From (iii), it follows that z_3 divides the line segment joining z_1 and z_2 internally in the ratio $b:a$.

If a and b are of the same sign, then division is in fact internal and if a and b are of opposite sign, then division is external in the ratio $|b|:|a|$.

Equation of the line passing through the points z_1 and z_2

Such equations are given by

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

or

$$\frac{z-z_1}{\bar{z}-\bar{z}_1} = \frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}$$

Explanation:

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Let A and B be the points representing z_1 and z_2, respectively.

Let $P(z)$ be any point on the line joining A and B. Let $z = x + iy$. Then $P \equiv (x, y)$, $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$. Points P, A, B are collinear.

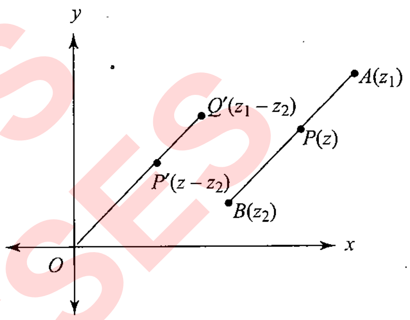


Fig. 2.22

From the diagram, points A, P and B are collinear.

Shifting the line AB at origin as shown in the figure, points O, P' and Q' are collinear. Hence,

$$\arg(z-z_2) = \arg(z_1-z_2)$$

$$\Rightarrow \quad \arg\left(\frac{z-z_2}{z_1-z_2}\right) = 0$$

$$\Rightarrow \quad \frac{z-z_2}{z_1-z_2} \text{ is purely real}$$

$$\Rightarrow \quad \frac{z-z_2}{z_1-z_2} = \overline{\left(\frac{z-z_2}{z_1-z_2}\right)}$$

$$\Rightarrow \quad \frac{z-z_2}{z_1-z_2} = \frac{\bar{z}-\bar{z}_2}{\bar{z}_1-\bar{z}_2} \quad (1)$$

$$\Rightarrow \quad \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \quad (2)$$

From (2), if points z_1, z_2, z_3 are collinear, then

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

Equation (2) can also be written as

$$(\bar{z}_1 - \bar{z}_2)z - (z_1 - z_2)\bar{z} + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

$$\Rightarrow \quad i(\bar{z}_1 - \bar{z}_2)z - i(z_1-z_2)\bar{z} + i(z_1\bar{z}_2 - z_2\bar{z}_1) = 0$$

$$\Rightarrow \quad \bar{a}z + a\bar{z} + b = 0 \quad (3)$$

where

$$a = -i(z_1 - z_2)$$

and

$$b = i(z_1\bar{z}_2 - z_2\bar{z}_1)$$
$$= i\,2i \times \text{Im}(z_1\bar{z}_2)$$
$$= -2 \times \text{Im}(z_1\bar{z}_2)$$
$$= \text{a real number}$$

Slope of the Given Line

In Eq. (3), replacing z by $x + iy$, we get

$$(x + iy)\,\bar{a} + (x - iy)\,a + b = 0$$

$$\Rightarrow \quad (a + \bar{a})\,x + iy\,(\bar{a} - a) + b = 0$$

$$\therefore \quad \text{Slope} = \frac{a + \bar{a}}{i(a - \bar{a})} = \frac{2\,\text{Re}(a)}{2i^2 \times \text{Im}(a)} = -\frac{\text{Re}(a)}{\text{Im}(a)}$$

Equation of a line parallel to the line $z\bar{a} + \bar{z}a + b = 0$ is $z\bar{a} + \bar{z}a + \lambda = 0$ (where λ is a real number).

Equation of a line perpendicular to the line $z\bar{a} + \bar{z}a + b = 0$ is $z\bar{a} + \bar{z}a + i\lambda = 0$ (where λ is a real number).

Equation of Perpendicular Bisector

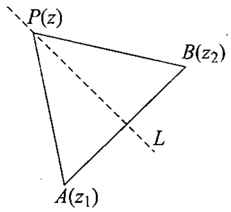


Fig. 2.23

Consider a line segment joining $A(z_1)$ and $B(z_2)$. Let the line 'L' be its perpendicular bisector. If $P(z)$ be any point on 'L', then we have

$$PA = PB \Rightarrow |z - z_1| = |z - z_2|$$

$$\Rightarrow \quad |z - z_1|^2 = |z - z_2|^2$$

$$\Rightarrow \quad (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$$

$$\Rightarrow \quad z\bar{z} - z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1 = z\bar{z} - z\bar{z}_2 - z_2\bar{z} + z_2\bar{z}_2$$

$$\Rightarrow \quad z(\bar{z}_2 - \bar{z}_1) + \bar{z}(z_2 - z_1) + z_1\bar{z}_1 - z_2\bar{z}_2 = 0$$

Here, $a = z_2 - z_1$ and $b = z_1\bar{z}_1 - z_2\bar{z}_2$.

Distance of a Given Point from a Given Line

Let the given line be $z\bar{a} + \bar{z}a + b = 0$ and the given point be z_c

$z_c = x_c + iy_c$

Replacing z by $x + iy$ in the given equation, we get

$$x(a + \bar{a}) + iy(\bar{a} - a) + b = 0$$

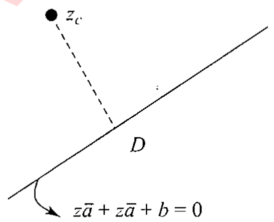


Fig. 2.24

Distance of (x_c, y_c) from this line is

$$\frac{|x_c(a + \bar{a}) + iy_c(\bar{a} - a) + b|}{\sqrt{(a + \bar{a})^2 - (a - \bar{a})^2}} = \frac{|z_c\bar{a} + \bar{z}_c a + b|}{\sqrt{4(\text{Re}(a))^2 + 4(\text{Im}(a))^2}}$$

$$= \frac{|z_c\,\bar{a} + \bar{z}_c\,a + b|}{2\,|a|}$$

Example 2.83 **Let $A(z_1)$ and (z_2) represent two complex numbers on the complex plane. Suppose the complex slope of the line joining A and B is defined as $(z_1 - z_2)/(\bar{z}_1 - \bar{z}_2)$. If the line l_1 with complex slope ω_1 and l_2 with complex slope ω_2 on the complex plane are perpendicular then prove that $\omega_1 + \omega_2 = 0$.**

Sol. l_1 is perpendicular to l_2. Hence, $(z_1 - z_2)/(\bar{z}_1 - \bar{z}_2)$ is purely imaginary.

$$\frac{z_1 - z_2}{z_3 - z_4} + \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_4} = 0$$

$$\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} + \frac{z_3 - z_4}{\bar{z}_3 - \bar{z}_4} = 0$$

$$\Rightarrow \quad \omega_1 + \omega_2 = 0$$

Note:

If l_1 is parallel to l_2, then

$$\frac{z_1 - z_2}{z_3 - z_4} = \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_4}$$

$$\omega_1 = \omega_2$$

Example 2.84 **If z_1, z_2, z_3 are three complex numbers such that $5z_1 - 13z_2 + 8z_3 = 0$, then prove that**

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

Sol. $\quad 5z_1 - 13z_2 + 8z_3 = 0$

$$\Rightarrow \quad \frac{5z_1 + 8z_2}{5 + 8} = z_3$$

$\Rightarrow \quad z_1, z_2, z_3$ are collinear

$$\Rightarrow \quad \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0 \quad \text{(condition of collinear points)}$$

Example 2.85 **If $\bar{z} = \bar{z}_0 + A(z - z_0)$, where A is a constant, then prove that locus of z is a straight line.**

Sol. $\quad \bar{z} = \bar{z}_0 + A(z - z_0)$

$$Az - \bar{z} - Az_0 + \bar{z}_0 = 0 \qquad (1)$$

$$\bar{A}\bar{z} - z - \bar{A}\bar{z}_0 + z_0 = 0 \qquad (2)$$

Adding (1) and (2),

$$(A - 1)z + (\bar{A} - 1)\bar{z} - (Az_0 + \bar{A}\bar{z}_0) + z_0 + \bar{z}_0 = 0$$

This is of the form $\bar{a}z + a\bar{z} + b = 0$, where $a = \bar{A} - 1$ and $b = -(Az_0 + \bar{A}\bar{z}_0) + z_0 + \bar{z}_0 \in R$. Hence locus of z is a straight line.

Equation of a Circle

Consider a fixed complex number z_0 and let z be any complex number which moves in such a way that its distance from z_0 is always to 'r'. This implies z would lie on a circle whose centre is z_0 and radius r. And its equation would be

$$|z - z_0| = r$$

$$\Rightarrow \quad |z - z_0|^2 = r^2$$

$\Rightarrow \quad (z - z_0)(\bar{z} - \bar{z}_0) = r^2$

$\Rightarrow \quad z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 - r^2 = 0$

Let $-a = z_0$ and $z_0\bar{z}_0 - r^2 = b$. Then,

$z\bar{z} + a\bar{z} + \bar{a}z + b = 0$

It represents the general equation of a circle in the complex plane.

Remark

$z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ represents a circle whose centre is $-a$ and radius is $\sqrt{a\bar{a} - b}$. Thus $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ $(b \in R)$ represents a real circle if and only if $a\bar{a} - b \geq 0$.

Now, let us consider a circle described on a line segment AB $(A(z_1), B(z_2))$ as its diameter. Let $P(z)$ be any point on the circle. As the angle in the semicircle is $\pi/2$, so

$\angle APB = \pi/2$

$\Rightarrow \quad \arg\left(\dfrac{z_1 - z}{z_2 - z}\right) = \pm\pi/2$

$\Rightarrow \quad \dfrac{z - z_1}{z - z_2}$ is purely imaginary

$\Rightarrow \quad \dfrac{z - z_1}{z - z_2} + \dfrac{\bar{z} - \bar{z}_1}{\bar{z} - \bar{z}_2} = 0$

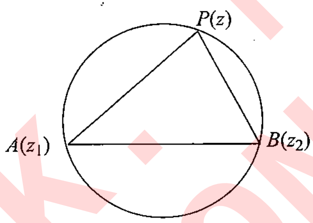


Fig. 2.25

$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

Example 2.86 **Let vertices of acute angled triangle are $A(z_1)$, $B(z_2)$ and $C(z_3)$. If the origin 'O' is the orthocentre of the triangle, then prove that $z_1\bar{z}_2 + \bar{z}_1 z_2 =_2 \bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1$.**

Sol.

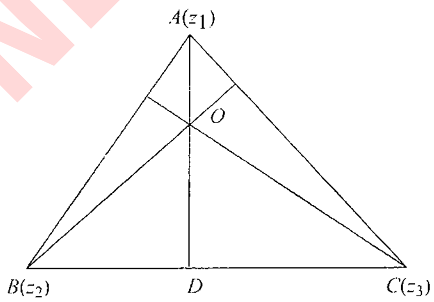


Fig. 2.26

Here O is orthocenter then

(AD=)OA $\perp$ BC

$\therefore \quad \arg\left(\dfrac{z_1 - 0}{z_2 - z_3}\right) = \dfrac{\pi}{2}$

$\therefore \quad \dfrac{z_1 - 0}{z_2 - z_3}$ is purely imaginary.

$\Rightarrow \quad \dfrac{z_1 - 0}{z_2 - z_3} + \overline{\left(\dfrac{z_1 - 0}{z_2 - z_3}\right)} = 0$

$\Rightarrow \quad \dfrac{z_1}{z_2 - z_3} + \dfrac{\bar{z}_1}{\bar{z}_2 - \bar{z}_3} = 0$

$\Rightarrow \quad z_1(\bar{z}_2 - \bar{z}_3) + \bar{z}_1(z_2 - z_3) = 0$

$\Rightarrow \quad z_1\bar{z}_2 + \bar{z}_1 z_2 = z_1\bar{z}_3 + \bar{z}_1 z_3 \qquad (1)$

similarly OB $\perp$ AC

$\Rightarrow \quad z_1\bar{z}_2 + \bar{z}_1\bar{z}_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 \qquad (2)$

From (1) and (2) $z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_1\bar{z}_3 + \bar{z}_1 z_3$

Example 2.87 **Show that the equation of a circle passing through the origin and having intercepts and on real and imaginary axis respectively on the argand plane is given by = (Re) + (Im).**

Sol. From figure,

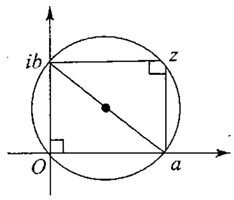


Fig. 2.27

$\arg\left(\dfrac{z - a}{z - ib}\right) = \pm\dfrac{\pi}{2}$

$\Rightarrow \quad \dfrac{z - a}{z - ib} + \dfrac{\bar{z} - a}{\bar{z} + ib} = 0$

$\Rightarrow \quad z\bar{z} - a\left(\dfrac{z + \bar{z}}{2}\right) - b\left(\dfrac{z - \bar{z}}{2i}\right) = 0$

$\Rightarrow \quad z\bar{z} = a(\text{Re } z) + b(\text{Im } z)$

Example 2.88 **Intercept made by the circle $z\bar{z} + \bar{\alpha} + \alpha\bar{z} + r = 0$ on the real axis on complex plane is**

a. $\sqrt{(a + \bar{a}) - r}$ **b.** $\sqrt{(a + \bar{a})^2 - 2r}$

c. $\sqrt{(a + \bar{a})^2 - 4r}$ **d.** $\sqrt{(a + \bar{a})^2 - 4r}$

Sol. Points where the circle cuts the x-axis $z = \bar{z}$.

Hence substituting $z = \bar{z}$ in the equation of circle, we get

$z^2 + \bar{\alpha}z + \alpha z + r = 0$

$\Rightarrow \quad z^2 + (\alpha + \bar{\alpha})z + r = 0$

$\Rightarrow \quad AB = |z_1 - z_2|$ (where A and B are points of intersection of circle with x-axis)

$= \sqrt{(z_1 + z_2)^2 - 4z_1 z_2}$

$= \sqrt{(\alpha + \bar{\alpha})^2 - 4r}$

Example 2.89 **Prove that $|Z - Z_1|^2 + |Z - Z_2|^2 = a$ will represent a real circle [with centre $(Z_1 + Z_2)/2$] on the Argand plane if $2a \geq |Z_1 - Z_2|^2$.**

Sol. $|Z - Z_1|^2 + |Z - Z_2|^2 = a$

$\Rightarrow \quad (Z - Z_1)(\bar{Z} - \bar{Z}_1) + (Z - Z_2)(\bar{Z} - \bar{Z}_2) = a$

$\Rightarrow \quad 2Z\bar{Z} - Z(\bar{Z}_1 + \bar{Z}_2) - \bar{Z}(Z_1 + Z_2) + Z_1\bar{Z}_1 + Z_2\bar{Z}_2 = a$

$$\Rightarrow \quad z\bar{z} - \left(\frac{\bar{z}_1 + \bar{z}_2}{2}\right) z - \left(\frac{z_1 + z_2}{2}\right)\bar{z} + \frac{z_1\bar{z}_1 + z_2\bar{z}_2 - a}{2} = 0 \qquad (1)$$

Equation (1) is of the form of $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$. Hence centre $= -$ coefficient of $\bar{z}$; which is given by $(z_1 + z_2)/2$. Also, Eq. (1) will represent a real circle if $\alpha\bar{\alpha} - r > 0$

$$\Rightarrow \quad \frac{(z_1 + z_2)(\bar{z}_1 + \bar{z}_2)}{4} \geq \frac{z_1\bar{z}_1 + z_2\bar{z}_2 - a}{2}$$

$$\Rightarrow \quad z_1\bar{z}_1 + z_1\bar{z}_2 + \bar{z}_1 z_2 + z_2\bar{z}_2 > 2(z_1\bar{z}_1 + z_2\bar{z}_2) - 2a$$

$$\Rightarrow \quad 2a \geq z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 - z_2\bar{z}_1$$

$$= z_1(\bar{z}_1 - \bar{z}_2) - z_2(\bar{z}_1 - \bar{z}_2)$$

$$= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$= (z_1 - z_2)\overline{(z_1 - z_2)}$$

$$\Rightarrow \quad 2a \geq |z_1 - z_2|^2$$

Example 2.90 **Two different non-parallel lines cut the circle $|z| = r$ at points a, b, c and d, respectively. Prove that these lines meet at the point given by**

$$\frac{a^{-1} + b^{-1} - c^{-1} - d^{-1}}{a^{-1}b^{-1} - c^{-1}d^{-1}}$$

Sol.

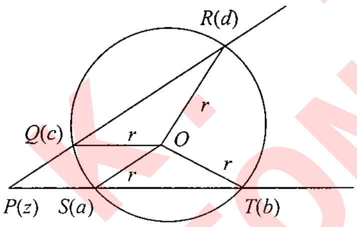


Fig. 2.28

Since P, Q, R are collinear, so

$$\begin{vmatrix} z & \bar{z} & 1 \\ c & \bar{c} & 1 \\ d & \bar{d} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad z(\bar{c} - \bar{d}) - \bar{z}(c - d) + (c\bar{d} - \bar{c}d) = 0 \qquad (1)$$

Similarly,

$$\bar{z}(\bar{a} - \bar{b}) - z(a - b) + (a\bar{b} - \bar{a}b) = 0 \qquad (2)$$

From $\{(1) \times (a - b)\} - \{(2) \times (c - d)\}$,

$$z[(\bar{c} - \bar{d}))(a - b) - (\bar{a} - \bar{b})(c - d)]$$

$$= (a\bar{b} - b\bar{a})(c - d) - (c\bar{d} - \bar{c}d)(a - b) \qquad (3)$$

Now,

$$|a|^2 = a\bar{a} = r^2 \Rightarrow \bar{a} = \frac{r^2}{a}$$

Similarly,

$$\bar{b} = \frac{r^2}{b}, \bar{c} = \frac{r^2}{c}, \bar{d} = \frac{r^2}{d}$$

From (3),

$$z\left[\left(\frac{r^2}{c} - \frac{r^2}{d}\right)(a - b) - \left(\frac{r^2}{a} - \frac{r^2}{b}\right)(c - d)\right]$$

$$= \left(\frac{ar^2}{b} - \frac{br^2}{a}\right)(c - d) - \left(\frac{cr^2}{d} - \frac{dr^2}{c}\right)(a - b)$$

$$\Rightarrow \quad z\left[-\frac{1}{cd} + \frac{1}{ab}\right] = \frac{(a + b)}{ab} - \frac{c + d}{cd}$$

$$\Rightarrow \quad z = \frac{a^{-1} + b^{-1} - c^{-1} - d^{-1}}{a^{-1}b^{-1} - c^{-1}d^{-1}}$$

Example 2.91 **Prove that the circles $z\bar{z} + z\bar{a}_1 + z\bar{a}_1 + b_1 = 0$, $b_1 \in R$ and $z\bar{z} + z\bar{a}_2 + \bar{z}a_2 + b_2 = 0$, $b_2 \in R$ will intersect orthogonally if $2\,\text{Re}(a_1\bar{a}_2) = b_1 + b_2$.**

Sol. Centre and radius of $z\bar{z} + z\bar{a}_1 + \bar{z}a_1 + b_1 = 0$ are $-a_1$ and $\sqrt{a_1\bar{a}_1 - b_1}$, respectively, and that for other circle are $-a_2$ and $\sqrt{a_2\bar{a}_2 - b_2}$, respectively. These circles will intersect orthogonally, if sum of squares of radii is equal to square of distance between their centres. Therefore,

$$|a_1 - a_2|^2 = a_1\bar{a}_1 - b_1 + a_2\bar{a}_2 - b_2$$

$$\Rightarrow \quad a_1\bar{a}_1 + a_2\bar{a}_2 - a_1\bar{a}_2 - \bar{a}_1 a_2 = a_1\bar{a}_1 + a_2\bar{a}_2 - b_1 - b_2$$

$$\Rightarrow \quad a_1\bar{a}_2 + \bar{a}_1 a_2 = b_1 + b_2$$

$$\Rightarrow \quad 2\text{Re}(a_1\bar{a}_2) = b_1 + b_2$$

Condition for Four Points to be Concyclic

Let $ABCD$ be a cyclic quadrilateral such that $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$ lie on a circle. Clearly,

$$\arg\left(\frac{z_4 - z_1}{z_2 - z_1}\right) + \arg\left(\frac{z_2 - z_3}{z_4 - z_3}\right) = \pi$$

$$\Rightarrow \quad \arg\left(\frac{z_4 - z_1}{z_2 - z_1}\right)\left(\frac{z_2 - z_3}{z_4 - z_3}\right) = \pi$$

$$\Rightarrow \quad \left(\frac{z_4 - z_1}{z_2 - z_1}\right)\left(\frac{z_2 - z_3}{z_4 - z_3}\right) \text{ is purely real}$$

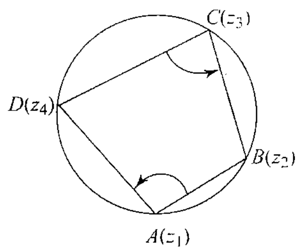


Fig. 2.29

Thus points $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$ (taken in order) would be concyclic if $[(z_4 - z_1)(z_2 - z_3)]/[(z_2 - z_1)(z_4 - z_3)]$ is purely real.

Example 2.92 **If z_1, z_2, z_3 are complex numbers such that $(2/z_1) = (1/z_2) + (1/z_3)$, then show that the points represented by z_1, z_2, z_3 lie on a circle passing through the origin.**

Sol.

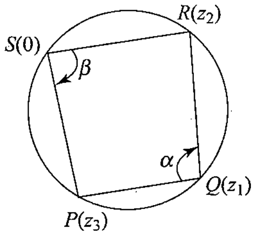


Fig. 2.30

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$$

$$\Rightarrow \quad \frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1}$$

$$\Rightarrow \quad \frac{z_2 - z_1}{z_1 z_2} = \frac{z_1 - z_3}{z_3 z_1}$$

$$\Rightarrow \quad \frac{z_2 - z_1}{z_3 - z_1} = -\frac{z_2}{z_3}$$

$$\Rightarrow \quad \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(-\frac{z_2}{z_3}\right)$$

$$\Rightarrow \quad \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \pi + \arg\left(\frac{z_2}{z_3}\right)$$

$$\Rightarrow \quad \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \pi - \arg\left(\frac{z_3}{z_2}\right)$$

$\Rightarrow \quad \alpha = \pi - \beta$

$\Rightarrow \quad \alpha + \beta = \pi$

Hence, the said points are concyclic.

Concept of Rotation

If z and z' are two complex numbers, then argument of z/z' is the angle through which Oz' must be turned in order that it may lie along Oz.

$$\frac{z}{z'} = \frac{|z|e^{i\theta}}{|z'|e^{i\theta}} = \frac{|z|}{|z'|}e^{i(\theta - \theta')} = \frac{|z|}{|z'|}e^{i\alpha}$$

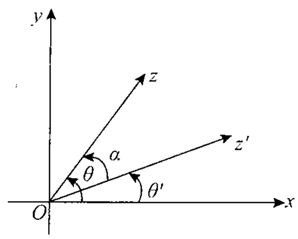


Fig. 2.31

In general, let z_1, z_2, z_3 be the three vertices of a triangle ABC described in the counterclockwise sense. Draw OP and OQ parallel and equal to AB and AC, respectively. Then the point P is $z_2 - z_1$ and Q is $z_3 - z_1$ and

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{OQ}{OP}(\cos\alpha + i\sin\alpha) = \frac{CA}{BA}e^{i\alpha} = \frac{|z_3 - z_1|}{|z_2 - z_1|}e^{i\alpha}$$

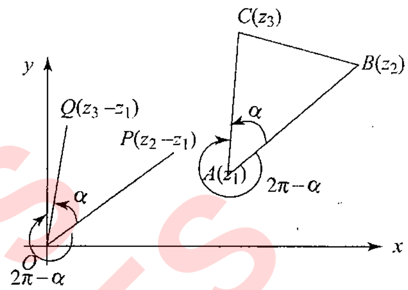


Fig. 2.32

Note that $\arg(z_3 - z_1) - \arg(z_2 - z_1) = \alpha$ the angle through which OP must be rotated in the anticlockwise direction so that it becomes parallel to OQ.

Also in this case we are rotating OP in clockwise direction by an angle $2\pi - \alpha$. Since the rotation is in clockwise direction, we are taking negative sign with angle $2\pi - \alpha$. Here, we can write

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|}e^{-i(2\pi - \alpha)}$$

Example 2.93 **$A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of the triangle ABC (in anticlockwise order). If $\angle ABC = \pi/4$ and $AB = \sqrt{2}(BC)$, then prove that $z_2 = z_3 + i(z_1 - z_3)$.**

Sol.

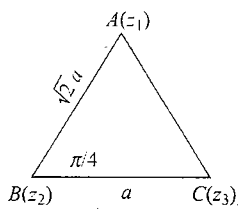


Fig. 2.33

Rotating about the point 'B', we get

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{a\sqrt{2}}{a}e^{i\pi/4} = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = (1 + i)$$

$\Rightarrow \quad z_1 - z_2 = (z_3 - z_2)(1 + i)$

$\Rightarrow \quad z_2 = z_1 - (z_3 - z_2)(1 + i)$

$\Rightarrow \quad z_2(1 - (1 + i)) = z_1 - z_3(1 + i)$

$$\Rightarrow \quad z_2 = \frac{z_1}{-i} - \frac{z_3}{-i}(1 + i) = (iz_1 - iz_3(1 + i))$$

$= z_3 + i(z_1 - z_3)$

Example 2.94 **If one vertex of a square whose diagonals intersect at the origin is $3(\cos\theta + i\sin\theta)$, then find the two adjacent vertices.**

Sol.

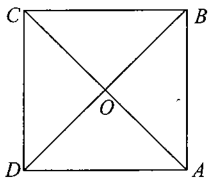


Fig. 2.34

Let the vertex A be $3(\cos\theta + i\sin\theta)$, then OB and OD can be obtained by rotating OA through $\pi/2$ and $-\pi/2$. Thus,

$$\overrightarrow{OB} = (\overrightarrow{OA})\, e^{i\pi/2} \text{ and } \overrightarrow{OD} = \overrightarrow{OA}\, e^{-i\pi/2}$$

$$\Rightarrow \quad \overrightarrow{OB} = 3(\cos\theta + i\sin\theta)i \text{ and } \overrightarrow{OD} = 3(\cos\theta + i\sin)(-i)$$

$$\Rightarrow \quad \overrightarrow{OB} = 3(-\sin\theta + i\cos\theta) \text{ and } \overrightarrow{OD} = 3(\sin\theta - i\cos\theta)$$

Thus, vertices B and D are represented by $\pm 3(\sin\theta - i\cos\theta)$.

Example 2.95 **Find the centre of the arc represented by $\arg[(z-3i)/(z-2i+4)] = \pi/4$.**

Sol.

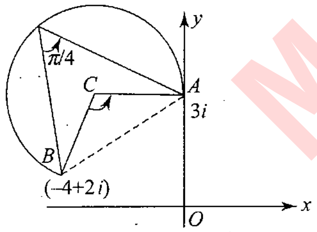


Fig. 2.35

If 'C' be the centre of the arc, then $\angle BCA = \pi/2$. Let C be z_c. Then,

$$\frac{z_c - 3i}{z_c - 2i + 4} = e^{i\pi/2} = i$$

$$\Rightarrow \quad z_c = 3i + i(z_c - 2i + 4)$$

$$\Rightarrow \quad z_c = \frac{7i+2}{(1-i)} = \frac{1}{2}(9i-5)$$

Example 2.96 **z_1 and z_2 are the roots of $3z^2 + 3z + b = 0$. If $O(0)$, (z_1), (z_2) form an equilateral triangle, then find the value of b.**

Sol. $z_1 + z_2 = -1,\ z_1 z_2 = \dfrac{b}{3}$

Triangle OAB is equilateral. So,

$$0^2 + z_1^2 + z_2^2 = 0 \times z_1 + 0 \times z_2 + z_1 z_2$$

$$\Rightarrow \quad (z_1 + z_2)^2 - 2z_1 z_2 = z_1 z_2$$

$$\Rightarrow \quad 1 = 3z_1 z_2 = 3\frac{b}{3}$$

$$\Rightarrow \quad b = 1$$

Example 2.97 **Let z_1, z_2 and z_3 represent the vertices A, B and C of the triangle ABC in the Argand plane, such that $|z_1| = |z_2| = |z_3| = 5$. Prove that $z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C = 0$.**

Sol.

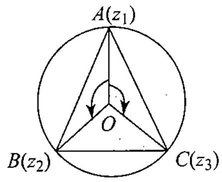


Fig. 2.36

$|z_1| = |z_2| = |z_3| = 5$

$\Rightarrow$ $|z| = 5$ is the circumcircle of triangle ABC

$\Rightarrow$ $\angle AOB = 2C$, $\angle BOC = 2A$ and $\angle COA = 2B$

We have,

$$\frac{z_2}{z_1} = \frac{OB}{OA} e^{i2C} = e^{i2C}$$

Similarly,

$$\frac{z_3}{z_1} = \frac{OC}{OA} e^{-i2B} = e^{-i2B}$$

Now,

$$z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C$$

$$= z_1\left(\sin 2A + \frac{z_2}{z_1}\sin 2B + \frac{z_3}{z_1}\sin 2C\right)$$

$$= z_1(\sin 2A + \sin 2B e^{i2C} + \sin 2C\, e^{-2iB})$$

$$= z_1(\sin 2A + \sin 2B \cos 2C + i \sin 2B \sin 2C + \sin 2C \cos 2B - i \sin 2C \sin 2B)$$

$$= z_1(\sin 2A + \sin(2B + 2C))$$

$$= z_1(\sin 2A + \sin(2\pi - 2A))$$

$$= 0$$

Example 2.98 **On the Argand plane z_1, z_2 and z_3 are, respectively, the vertices of an isosceles triangle ABC with $AC = BC$ and equal angles are θ. If z_4 is the incentre of the triangle, then prove that $(z_2 - z_1)(z_3 - z_1) = (1 + \sec\theta)(z_4 - z_1)^2$.**

Sol.

$$\frac{z_2 - z_1}{|z_2 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|} e^{-i\theta/2} \text{ (clockwise)} \qquad (1)$$

$$\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|} e^{i\theta/2} \text{ (anticlowkwise)} \qquad (2)$$

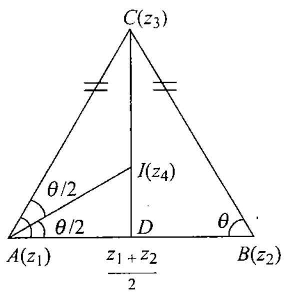


Fig. 2.37

Multiplying (1) and (2), we get

$$\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = \frac{|(z_2 - z_1)||(z_3 - z_1)|}{|z_4 - z_1|^2}$$

$$= \frac{(AB)(AC)}{(AI)^2}$$

$$= \frac{2(AD)(AC)}{(AI)^2}$$

$$= \frac{2(AD)^2}{(AI)^2} \cdot \frac{AC}{AD}$$

$$= 2\cos^2\frac{\theta}{2}\sec\theta = (1 + \cos\theta)\sec\theta$$

Concept Application Exercise 2.6

1. The centre of a regular polygon of n sides is located at the point $z = 0$, and one of its vertices z_1 is known. If z_2 be the vertex adjacent to z_1, then find z_2.
2. Let z_1 and z_2 be two complex numbers such that $z_1/z_2 + z_2/z_1 = 1$.

   Prove that z_1, z_2 are the origin form an equilateral triangle.
3. If one vertex of the triangle having maximum area that can be inscribed in the circle $|z - i| = 5$ is $3 - 3i$, then find the other vertices of the triangle.
4. Consider the circle $|z| = r$ in the Argand plane, which is in fact the incircle of triangle ABC. If contact points opposite to the vertices A, B, C are $A_1(z_1)$, $B_1(z_2)$ and $C_1(z_3)$, obtain the complex numbers associated with the vertices A, B, C in terms of z_1, z_2 and z_3.
5. P is a point on the Argnad plane. On the circle with OP as diameter, two points Q and R are taken such that $\angle POQ = \angle QOR = \theta$. If O is the origin and P, Q and R are represented by the complex numbers z_1, z_2 and z_3, respectively, show that $z_2^2 \cos 2\theta = z_1 z_3 \cos^2\theta$.

Standard Loci in the Argand Plane

If $P(z)$ is a variable point and $A(z_1)$, $B(z_2)$ are two fixed points in the Argand plane, then

1. $|z - z_1| = |z - z_2|$

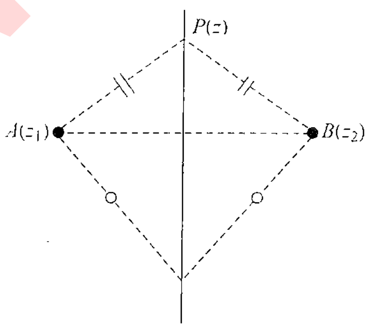
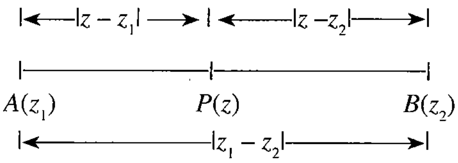


Fig. 2.38

$\Rightarrow AP = BP$

That is, the distance of z from two fixed points z_1 and z_2 is same. Hence, locus of z is the perpendicular bisector of the line segment joining z_1 and z_2.

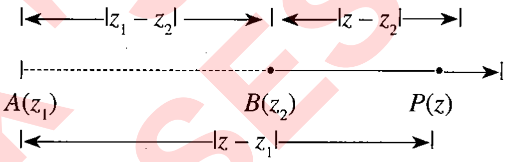
2. $|z - z_1| + |z - z_2| = |z_1 - z_2|$



$AP + BP = AB$

Hence the locus of z is the line segment joining z_1 and z_2.

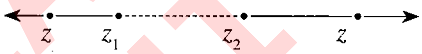
3. $|z - z_1| - |z - z_2| = |z_1 - z_2|$



$\Rightarrow AP - BP = AB$

Hence, the locus of z is a ray as shown in the figure.

4. $||z - z_1| - |z - z_2|| = |z_1 - z_2|$



$\Rightarrow$ Locus of z is a straight line joining z_1 and z_2 but z does not lie between z_1 and z_2 or locus of z is two rays.

5. $|z - z_1| + |z - z_2| = k$ (= constant $> |z_1 - z_2|$)

$\Rightarrow PA + PB =$ constant

Hence, the locus of z is an ellipse

(as in ellipse $SP + S'P = 2a$, where S, S' are foci, P is any point on ellipse and a is semi-major axis)

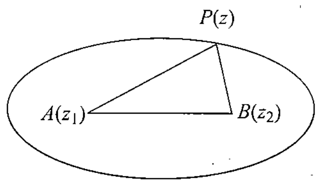


Fig. 2.39

Eccentricity of ellipse is

$$\frac{AB}{AP + BP} = \frac{|z_1 - z_2|}{|z - z_1| + |z - z_2|} = \frac{|z_1 - z_2|}{k}$$

6. $||z - z_1| - |z - z_2|| = k$ (= constant $< |z_1 - z_2|$)

$\Rightarrow |AP - BP| =$ constant

Hence, the locus of z is a hyperbola.

(as in hyperbola $S'P - SP = 2a$, where S, S' are foci, P is any point on the hyperbola and a is semi-transverse axis)

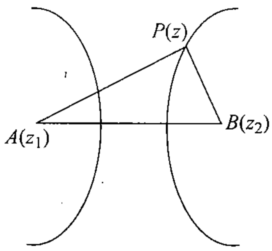


Fig. 2.40

Eccentricity of hyperbola is

$$\frac{AB}{|AP - BP|} = \frac{|z_1 - z_2|}{||z - z_1| - |z - z_2||} = \frac{|z_1 - z_2|}{k}$$

7. $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

$\Rightarrow$ $AP^2 + BP^2 = AB^2$

Hence, the locus of z is a circle with z_1 and z_2 as the extremities of diameter.

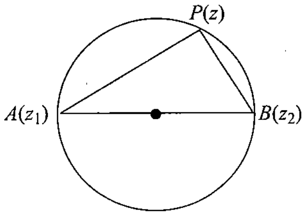


Fig. 2.41

8. $|z - z_1| = k|z - z_2|$ $(\neq 1)$

$\Rightarrow$ Locus of z is circle

9. $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ **(fixed)**

Hence, the locus of z is a segment of circle.

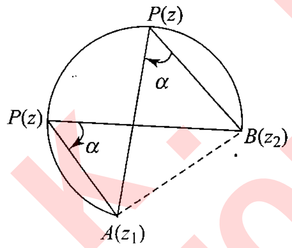


Fig. 2.42

10. $\arg\left(\frac{z - z_1}{z - z_2}\right) = \pm\pi/2$

Hence, the locus of z is a circle with z_1 and z_2 as the vertices of diameter.

11. $\arg\left(\frac{z - z_1}{z - z_2}\right) = 0$ **or** π

Hence, the locus of z is a straight line passing through z_1 and z_2

12. $|z - z_0| = \left|\frac{\bar{\alpha} z_0 + \alpha \bar{z}_0 + r}{2|\alpha|}\right|$

Hence, the locus of z is a parabola whose focus is z_0 and directrix is the line $\bar{\alpha} z + \alpha \bar{z} + r = 0$

Example 2.99 **Find the locus of the points representing the complex number z for which $|z + 5|^2 - |z - 5|^2 = 10$.**

Sol. $|z + 5|^2 - |z - 5|^2 = 10$

$\Rightarrow$ $(z + 5)(\bar{z} + 5) - (z - 5)(\bar{z} - 5) = 10$

$\Rightarrow$ $5(z + \bar{z}) + 25 + 5(z + \bar{z}) - 25 = 10$

$\Rightarrow$ $2 \times 2x = 2$

$\Rightarrow$ $x = \frac{1}{2}$,

which is the equation of a straight line.

Example 2.100 **Identify the locus of z if $\bar{z} = \bar{a} + \frac{r^2}{z - a}$, > 0.**

Sol. $\bar{z} = \bar{a} + \frac{r^2}{z - a}$

$\Rightarrow$ $\bar{z} - \bar{a} = \frac{r^2}{z - a}$

$\Rightarrow$ $(z - a)(\bar{z} - \bar{a}) = r^2$

$\Rightarrow$ $|z - a|^2 = r^2$

$\Rightarrow$ $|z - a| = r$

Hence, locus of z is circle having center a and radius r.

Example 2.101 **Let z be a complex number having the argument θ, $0 < \theta < \pi/2$ and satisfying the equation $|z - 3i| = 3$. Then find the value of $\cot\theta - 6/z$.**

Sol. Let $z = r(\cos\theta + i\sin\theta)$. Now,

$r = OA \sin\theta = 6\sin\theta$

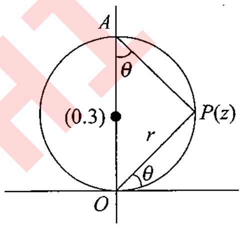


Fig. 2.43

$\Rightarrow$ $z = 6\sin\theta(\cos\theta + i\sin\theta)$

$\Rightarrow$ $\frac{6}{z} = \frac{1}{\sin\theta(\cos\theta + i\sin\theta)}$

$= \frac{\cos\theta - i\sin\theta}{\sin\theta}$

$= -i + \cot\theta$

$\Rightarrow$ $\cot\theta - \frac{6}{z} = i$

Example 2.102 **If 'z' be any complex number such that $|3 - 2| + |3 + 2| = 4$, then identify the locus of 'z'.**

Sol. $|3z - 2| + |3z + 2| = 4$

$\Rightarrow$ $\left|z - \frac{2}{3}\right| + \left|z + \frac{2}{3}\right| = \frac{4}{3}$ (1)

If $P(z)$ be any point $A \equiv (2/3, 0)$, $B \equiv (-2/3, 0)$, then (i) represents

$PA + PB = 4$

Clearly, $AB = 4/3 \Rightarrow PA + PB = AB \Rightarrow$ 'P' is any point on the line segment AB.

Example 2.103 **$|z - 2 - 3|^2 + |z - 4 - 3|^2 = \lambda$ represents the equation of a circle with least radius. Find the value of 'λ'.**

Sol.

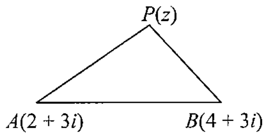


Fig. 2.44

If $P(z)$ lies on a circle and $PB^2 + PA^2 = \lambda$ (constant), then $\lambda = AB^2 \Rightarrow \lambda = 4$.

Example 2.104 **Find the number of complex numbers which satisfy both the equations $|z - 1 - i| = \sqrt{2}$ and $|z + 1 + i| = 2$.**

Sol. The given equations represent circles

$$(x-1)^2 + (y-1)^2 = 2 \text{ and } (x+1)^2 + (y+1)^2 = 4$$

$$\Rightarrow \quad C_1(1, 1),\ r_1 = \sqrt{2},\ C_2(-1, -1),\ r_2 = 2$$

$$C_1C_2 = \sqrt{8} = 2.8,\ r_1 + r_2 = 2 + \sqrt{2} = 3.41$$

$C_1C_2 < r_1 + r_2$ and also $C_1C_2 > r_2 - r_1$ and hence the two circles are intersecting at two points. The common two points will satisfy both.

Example 2.105 **If the imaginary part of $(2z+1)/(iz+1)$ is -2, then find the locus of the point representing in the complex plane.**

Sol. Let,

$$z = x + iy$$

$$\Rightarrow \quad \frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1}$$

$$= \frac{(2x+1)+i2y}{1-y+ix}$$

$$= \frac{(2x+1)+i2y}{(1-y)+ix} \cdot \frac{(1-y)-ix}{(1-y)-ix}$$

$$= \frac{(2x+1)(1-y)+2xy+i\left[-x(2x+1)+2y(1-y)\right]}{(1-y)^2+x^2}$$

Since imaginary part of $(2z+1)/(iz+1) = -2$, hence

$$\frac{-x(2x+1)+2y(1-y)}{(1-y)^2+x^2} = -2$$

$$\Rightarrow \quad -2x^2 - x + 2y - 2y^2 = -2[1 + y^2 - 2y + x^2]$$

$$\Rightarrow \quad x + 2y - 2 = 0, \text{ which is a straight line}$$

Example 2.106 **If $|(z-2)/(z-3)| = 2$ represents a circle, then find its radius.**

Sol. $$\left|\frac{z-2}{z-3}\right| = 2$$

$$\Rightarrow \quad |z-2|^2 = 4|z-3|^2$$

$$\Rightarrow \quad |x - 2 + iy|^2 = 4|x - 3 + iy|^2$$

$$\Rightarrow \quad (x-2)^2 + y^2 = 4[(x-3)^2 + y^2]$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 24x + 4x + 36 - 4 = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

This represents a circle with centre $[(10/3, 0)]$ and radius $\sqrt{(100/9)-(32/3)} = \sqrt{(4/9)} = 2/3$.

Example 2.107 **If $z_1 + z_2 + z_3 + z_4 = 0$ where $b_i \in R$ such that the sum of no two values being zero and $b_1z_1 + b_2z_2 + b_3z_3 + b_4z_4 = 0$ where z_1, z_2, z_3, z_4 are arbitrary complex numbers such that no three of them are collinear, prove that the four complex numbers would be concyclic if $|b_1b_2||z_1 - z_2|^2 = |b_3b_4||z_3 - z_4|^2$.**

Sol.

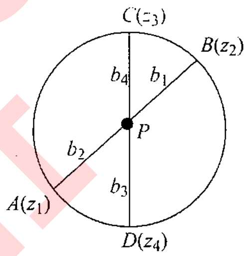


Fig. 2.45

$$b_1 + b_2 = -(b_3 + b_4),\ b_1z_1 + b_2z_2 = -(b_3z_3 + b_4z_4)$$

$$\Rightarrow \quad \frac{b_1z_1 + b_2z_2}{b_1 + b_2} = \frac{b_3z_3 + b_4z_4}{b_3 + b_4}$$

Hence, the line joining the complex numbers $A(z_1)$, $B(z_2)$ and $C(z_3)$, $D(z_4)$ meet.

Let $P(z)$ be the point of intersection. These points will be concyclic, if $PA \times PB = PC \times PD$. Now,

$$PA = \left|\frac{b_2}{b_1+b_2}\right| |z_1 - z_2|,\ PB = \left|\frac{b_1}{b_1+b_2}\right| |z_1 - z_2|$$

$$PC = \left|\frac{b_4}{b_3+b_4}\right| |z_3 - z_4|,\ PD = \left|\frac{b_3}{b_3+b_4}\right| |z_3 - z_4|$$

$$\Rightarrow \quad |b_1b_2|\,|z_1 - z_2|^2 = |b_3\,b_4|\,|z_3 - z_4|^2$$

$$(\because |b_1 + b_2| = |b_3 + b_4|)$$

Example 2.108 **Consider an ellipse having its foci at $A(z_1)$ and $B(z_2)$ in the Argand plane. If the eccentricity of the ellipse be 'e' and it is known that origin is an interior point of the ellipse, then prove that**

$$\in \left(0, \frac{|z_1 - z_2|}{|z_1| + |z_2|}\right)$$

Sol. Let $P(z)$ be any point on the ellipse. Then equation of the ellipse is

$$|z - z_1| + |z - z_2| = \frac{|z_1 - z_2|}{e} \qquad (1)$$

If we replace z by z_1 or z_2, L.H.S. of (1) becomes $|z_1 - z_2|$. Thus for any interior point of the ellipse, we have

$$|z - z_1| + |z - z_2| < \frac{|z_1 - z_2|}{a}$$

It is given that origin is an interior point of the ellipse

$$|0 - z_1| + |0 - z_2| < \frac{|z_1 - z_2|}{e}$$

$$\Rightarrow \quad e \in \frac{|z_1 - z_2|}{|z_1| + |z_2|}$$

Example 2.109 **Find the locus of point z if z, i and iz are collinear.**

Sol. If z_1, z_2, z_3 are collinear, then

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

Given z, i and iz are collinear. Hence,

$$\begin{vmatrix} z & \bar{z} & 1 \\ i & -i & 1 \\ iz & -i\bar{z} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad -iz + iz\bar{z} + \bar{z} - z - i\bar{z} + iz\bar{z} = 0$$

$$\Rightarrow \quad z - 2z\bar{z} + i\bar{z} - iz + \bar{z} = 0 \qquad \text{(multiplying with } i)$$

$$\Rightarrow \quad 2z\bar{z} - (z + \bar{z}) + i(z - \bar{z}) = 0$$

$$\Rightarrow \quad x^2 + y^2 - 2x - 2y = 0 \quad (\text{putting } z = x + iy)$$

Example 2.110 **If the equation $|z - a| + |z - b| = 3$ represents an ellipse, and $a, b \in C$, where a is fixed, then find the locus of b.**

Sol. $|z - a| + |z - b| = 3$ represents an ellipse. Now,

$$|a - b| < 3$$

$$\Rightarrow \quad |b - a| < 3$$

Hence, b lies inside the circle having centre a and radius 3.

Concept Application Exercise 2.7

1. If $\log_{\sqrt{3}}\left(\frac{|z|^2 - |z| + 1}{2 + |z|}\right) > 2$, then locate the region in the Argand plane which represent z.
2. Identify the region of the Argand diagram defined by $|z - 1| + |z + 1| \le 4$.
3. If $w = z/[z - (1/3)i]$ and $|w| = 1$, then find the locus of z.
4. Let $z\ (\neq 2)$ be a complex number such that $\log_{1/2}|z - 2| > \log_{1/2}|z|$. Then prove that $\text{Re}(z) > 1$.
5. Locate the region in the Argand plane determined by $z^2 + \bar{z}^2 + 2|z|^2 < (8i(\bar{z} - z))$.
6. If $|z - 1| + |z + 3| \le 8$, then find the range of values of $|z - 4|$.

THE n^{th} ROOT OF UNITY

Let x be n^{th} root of unity. Then,

$$x^n = 1$$
$$= 1 + i(0)$$
$$= \cos 0 + i \sin 0$$
$$= \cos(2k\pi + 0) + i\sin(2k\pi + 0)$$
$$= \cos 2k\pi + i \sin 2k\pi \qquad (\text{where } k \text{ is an integer})$$

$$\Rightarrow \quad x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},\ k = 0, 1, 2, \ldots, n-1$$

Let $\alpha = \cos(2\pi/n) + i\sin(2\pi/n)$. Then the n^{th} roots of unity are α^t $(t = 0, 1, 2, \ldots, n-1)$, i.e., the n^{th} roots of unity are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$.

Sum of the Roots

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} = \frac{1 - \alpha^n}{1 - \alpha}$$

$$\frac{1 - \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n}{1 - \alpha} = \frac{1 - (\cos 2\pi + i\sin 2\pi)}{1 - \alpha} = 0$$

$$\Rightarrow \sum_{k=0}^{n-1}\cos\frac{2k\pi}{n} = 0 \text{ and } \sum_{k=0}^{n-1}\sin\frac{2k\pi}{n} = 0$$

Thus the sum of the roots of unity is zero.

Product of the Roots

$$1 \times \alpha \times \alpha^2 \times \cdots \times \alpha^{n-1} = \alpha^{\frac{n(n-1)}{2}} = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{n\left(\frac{n-1}{2}\right)}$$

$$= (\cos\pi + i\sin\pi)^{n-1}$$

If n is even, the product is $(-1)^{n-1}$. If n is odd, the product is 1.

Note: *The points represented by the n^{th} roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle having centre at the origin, one vertex being on the positive real axis (geometrically represented as shown).*

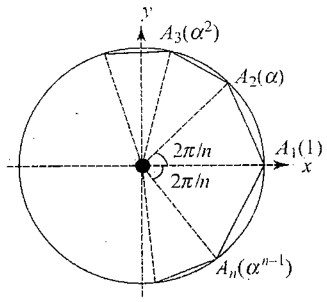


Fig. 2.46

Example 2.111 **If $a = \cos(2\pi/7) + i\sin(2\pi/7)$, then find the quadratic equation whose roots are $\alpha = a + a^2 + a^4$ and $\beta = a^3 + a^5 + a^7$.**

Sol. $a = \cos(2\pi/7) + i\sin(2\pi/7)$

$\Rightarrow \quad a^7 = [\cos(2\pi/7) + i\sin(2\pi/7)]^7$

$= \cos 2\pi + i\sin 2\pi = 1 \qquad \text{(i)}$

$S = \alpha + \beta = (a + a^2 + a^4) + (a^3 + a^5 + a^6)$

$= a + a^2 + a^3 + a^4 + a^5 + a^6 = \dfrac{a(1-a^6)}{1-a}$

$= \dfrac{a-a^7}{1-a} = \dfrac{a-1}{1-a} = -1 \qquad \text{(ii)}$

$P = \alpha\beta = (a + a^2 + a^4)(a^3 + a^5 + a^6)$

$= a^4 + a^6 + a^7 + a^5 + a^7 + a^8 + a^7 + a^9 + a^{10}$

$= a^4 + a^6 + 1 + a^5 + 1 + a + 1 + a^2 + a^3$ [From Eq. (i)]

$= 3 + (a + a^2 + a^3 + a^4 + a^5 + a^6)$

$= 3 + S$

$= 3 - 1 = 2$ [From Eq. (ii)]

Therefore the required equation is

$x^2 - Sx + P = 0$

$\Rightarrow \quad x^2 + x + 2 = 0$

Example 2.112 **If ω is an imaginary fifth root of unity, then find the value of $\log_2 |1 + \omega + \omega^2 + \omega^3 - 1/\omega|$.**

Sol. Here $\omega^5 = 1 \therefore \omega^{-1} = \omega^4$

Also,

$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

$\therefore \quad \log_2 \left|1 + \omega + \omega^2 + \omega^3 - \dfrac{1}{\omega}\right|$

$= \log_2 |1 + \omega + \omega^2 + \omega^3 - \omega^4| \qquad (\because |\omega| = 1)$

$= \log_2 |-2\omega^4| = \log_2 2 = 1$

Example 2.113 **If $1, z_1, z_2, z_3, \ldots, z_{n-1}$ are the n^{th} roots of unity, then prove that $(1 - z_1)(1 - z_2) \cdots (1 - z_{n-1}) = n$.**

Sol. $z^n - 1 = (z - 1)(z - z_1)(z - z_2) \cdots (z - z_{n-1})$

$\Rightarrow \dfrac{z^n - 1}{z - 1} = (z - z_1)(z - z_2) \cdots (z - z_{n-1})$

$\Rightarrow 1 + z + z^2 + \cdots + z^{n-1} = (z - z_1)(z - z_2) \cdots (z - z_{n-1})$

Putting $z = 1$, we get

$(1 - z_1)(1 - z_2) \cdots (1 - z_{n-1}) = 1 + 1 + \cdots + 1 = n$

Example 2.114 **If $\alpha = e^{i2\pi/7}$ and $f(x) = a_0 + \sum\limits_{k=1}^{20} a_k x^k$, then prove that the value of $f(x) + f(\alpha x) + \cdots + f(\alpha^6 x)$ is independent of α.**

Sol. $\alpha = e^{i2\pi/7} = 7^{\text{th}}$ root of unity or root of the equation $z^7 - 1 = 0$. Its other roots are $1, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$. Now,

$f(x) + f(\alpha x) + \cdots + f(\alpha^6 x)$

$= 7a_0 + \sum\limits_{k=1}^{20} a_k x^k + \sum\limits_{k=1}^{20} a_k (\alpha x)^k + \cdots + \sum\limits_{k=1}^{20} a_k (\alpha^6 x)^k$

$= 7a_0 + \sum\limits_{k=1}^{20} a_k (x^k + \alpha^k x^k + \alpha^{2k} x^k + \cdots + \alpha^{6k} x^k)$

$= 7a_0 + \sum\limits_{k=1}^{20} a_k (x^k + \alpha^k x^k + \alpha^{2k} x^k + \cdots + \alpha^{6k} x^k)$

$= 7a_0 + \sum\limits_{k=1}^{20} a_k \left(x^k \dfrac{(\alpha^k)^7 - 1}{\alpha^k - 1}\right)$

$= 7a_0 + \sum\limits_{k=1}^{20} a_k \left(x^k \dfrac{(\alpha^7)^k - 1}{\alpha^k - 1}\right)$

$= 7a_0 + \sum\limits_{k=1}^{20} a_k \left(x^k \dfrac{1-1}{\alpha^k - 1}\right)$

$(\because \alpha \text{ is a root of } z^7 - 1 = 0 \Rightarrow \alpha^7 - 1 = 0)$

$= 7a_0$

Example 2.115 **If $n \ge 3$ and $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are n^{th} roots of unity, then find the sum $\sum\limits_{1 \le i < j \le n-1} \alpha_i \alpha_j$.**

Sol. We have,

$z^n - 1 = (z - 1)(z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_{n-1})$

Now sum of products taken two at a time,

$S = \sum\limits_{1 \le i < j \le n-1} \alpha_i \alpha_j + \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$

Now $S = 0$ as coefficient of z^{n-1} is zero and

$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = -1$

$\therefore \quad 0 = \sum\limits_{1 \le i < j \le n-1} \alpha_i \alpha_j - 1$

$\Rightarrow \quad \sum\limits_{1 \le i < j \le n-1} \alpha_i \alpha_j = 1$

Concept Application Exercise 2.8

1. Given α, β, respectively, the fifth and the fourth non-real roots of unity, then find the value of $(1 + \alpha)(1 + \beta)(1 + \alpha^2)(1 + \beta^2)(1 + \alpha^4)(1 + \beta^4)$
2. If the six solutions of $x^6 = -64$ are written in the form $a + bi$, where a and b are real, then find the product of those solutions with $a > 0$.
3. If z_r: $r = 1, 2, 3, \ldots, 50$ are the roots of the equation $\sum\limits_{r=0}^{50} z^r = 0$, then find the value of $\sum\limits_{r=1}^{50} 1/(z_r - 1)$.
4. If $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the n^{th} roots of unity, prove that $(1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_{n-1}) = n$ Deduce that
$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$$
5. If $n > 1$, show that the roots of the equation $z^n = (z + 1)^n$ are collinear.

EXERCISES

Subjective Type

Solutions on page 2.52

1. If $\left|z/|\bar{z}|-\bar{z}\right| = 1+|z|$, then prove that z is a purely imaginary number.

2. z_1, z_2 and z_3 are the vertices of an isosceles triangle in anticlockwise direction with origin as in centre then prove that z_2, z_1 and kz_3 are in G.P. where $k \in R^+$.

3. For $|z-1| = 1$, show that $\tan\{[\arg(z-1)]/2\}-(2i/z) = -i$.

4. The altitude from the vertices A, B and C of the triangle ABC meet its circumcircle at D, E and F, respectively. The complex numbers representing the points D, E and F are z_1, z_2 and z_3, respectively. If $(z_3-z_1)/(z_2-z_1)$ is purely real, then show that triangle ABC is right angled at A.

5. Let A, B, C, D be four concyclic points in order in which AD:AB = CD:CB. If A, B, C are represented by complex numbers a, b, c, find the complex number associated with point D.

6. For $x \in (0, 1)$, prove that
$$i^{2i+3}\ln\left(\frac{i^3x^2+2x+i}{ix^2+2x+i^3}\right) = \frac{1}{e^{\pi}}(\pi-4\tan^{-1}x)$$

7. If a, b are complex numbers and one of the roots of the equation $x^2+ax+b=0$ is purely real whereas the other is purely imaginary, prove that $a^2-\bar{a}^2 = 4b$.

8. Solve for z, i.e. find all complex numbers z which satisfy $|z| - 2iz + 2c(1+i) = 0$ where c is real.

9. If 'a' is a complex number such that $|a| = 1$, find arg(a), so that equation $az^2+z+1=0$ has one purely imaginary root.

10. Prove the following inequalities:

 a. $\left|\frac{z}{|z|}-1\right| \le |\arg z|$ b. $|z-1| \le |z|\,|\arg z| + \big||z|-1\big|$

11. If n is a positive integer, prove that $|\text{Im}(z^n)| \le n|\text{Im}(z)|\,|z|^{n-1}$.

12. Let z and z_0 be two complex numbers. It is given that $|z| = 1$ and the numbers z, z_0, $z\bar{z}_0$, 1 and 0 are represented in an Argand diagram by the points P, P_0, Q, A and the origin, respectively. Show that the triangles POP_0 and AOQ are congruent. Hence, or otherwise, prove that $|z-z_0| = |z\bar{z}_0-1|$.

13. Show that the equation $az^3+bz^2+\bar{b}z+\bar{a}=0$ has a root α, such that $|\alpha| = 1$. a, b, z and α belong to the set of complex numbers.

14. Let $z = t^2-1+\sqrt{t^4-t^2}$, where $t \in R$ is a parameter. Find the locus of 'z' depending upon t, and draw the locus of 'z' in the Argand plane.

15. If $|z| = 1$, then prove that points represented by $\sqrt{(1+z)/(1-z)}$ lie on one or other of two fixed perpendicular straight lines.

16. If $\alpha = (z-i)/(z+i)$, show that, when z lies above the real axis, α will lie within the unit circle which has centre at the origin. Find the locus of α as z travels on the real axis from $-\infty$ to $+\infty$.

17. Let x_1, x_2 are the roots of the quadratic equation $x^2+ax+b=0$ where a, b are complex numbers and y_1, y_2 are the roots of the quadratic equation $y^2+|a|y+|b|=0$. If $|x_1| = |x_2| = 1$ then prove that $|y_1| = |y_2| = 1$.

18. Plot the region represented by $\pi/3 \le \arg[(z+1)/(z-1)] \le 2\pi/3$ in the Argand plane.

19. Consider an equilateral triangle having vertices at the points
$$A\left(\frac{2}{\sqrt{3}}e^{i\frac{\pi}{2}}\right),\ B\left(\frac{2}{\sqrt{3}}e^{-i\frac{\pi}{6}}\right),\ C\left(\frac{2}{\sqrt{3}}e^{-i\frac{5\pi}{6}}\right).$$
Let P be any point on its incircle. Prove that $AP^2+BP^2+CP^2=5$.

20. Prove that the locus of mid-point of line segment intercepted between real and imaginary axes by the line $a\bar{z}+\bar{a}z+b=0$, where b is a real parameter and a is a fixed complex number with non-zero real and imaginary parts, is $az+\overline{az}=0$.

Objective Type

Solutions on page 2.56

Each question has four choices a, b, c and d, out of which *only one* is correct.

1. If $a<0$, $b>0$ then $\sqrt{a}\sqrt{b}$ is equal to

 a. $-\sqrt{|a|b}$ b. $\sqrt{|a|b}\,i$ c. $\sqrt{|a|b}$ d. none of these

2. If $x = 9^{1/3}9^{1/9}9^{1/27}\cdots\infty$, $y = 4^{1/3}4^{-1/9}4^{1/27}\cdots\infty$, and $z = \sum_{r=1}^{\infty}(1+i)^{-r}$, then arg$(x+yz)$ is equal to

 a. 0 b. $\pi-\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$ c. $-\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$ d. $-\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

3. Consider the equation $10z^2-3iz-k=0$, where z is a complex variable and $i^2=-1$. Which of the following statements is true?

 a. For real positive numbers k, both roots are purely imaginary.

 b. For all complex numbers k, neither root is real.

 c. For all purely imaginary numbers k, both roots are real and irrational.

 d. For real negative numbers k, both roots are purely imaginary.

4. The number of solutions of the equation $z^2+\bar{z}=0$ is

 a. 1 b. 2 c. 3 d. 4

5. If $a^2+b^2=1$, then $(1+b+ia)/(1+b-ia)=$

 a. 1 b. 2 c. $b+ia$ d. $a+ib$

6. The expression $\left[\dfrac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right]^8=$

 a. 1 b. -1 c. i d. $-i$

7. If z_1 and z_2 be complex numbers such that $z_1 \ne z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $[(z_1+z_2)/(z_1-z_2)]$ may be

a. Purely imaginary b. Real and positive
c. Real and negative d. None of these

8. If $|z_1| = |z_2|$ and arg $(z_1/z_2) = \pi$, then $z_1 + z_2$ is equal to
a. 0 b. purely imaginary
c. purely real d. none of these

9. If $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, then the value of $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma$ is
a. $\sin(a+b+\gamma)$ b. $3\sin(\alpha+\beta+\gamma)$
c. $18\sin(\alpha+\beta+\gamma)$ d. $\sin(\alpha+2\beta+3)$

10. If centre of a regular hexagon is at origin and one of the vertices on Argand diagram is $1 + 2i$, then its perimeter is
a. $2\sqrt{5}$ b. $6\sqrt{2}$ c. $4\sqrt{5}$ d. $6\sqrt{5}$

11. If $z(1 + a) = b + ic$ and $a^2 + b^2 + c^2 = 1$, then $[(1 + iz)/(1 - iz) =$
a. $\dfrac{a+ib}{1+c}$ b. $\dfrac{b-ic}{1+a}$ c. $\dfrac{a+ic}{1+b}$ d. none of these

12. If z_1, z_2, z_3 are three complex numbers and

$$A = \begin{vmatrix} \arg z_1 & \arg z_2 & \arg z_3 \\ \arg z_2 & \arg z_3 & \arg z_1 \\ \arg z_3 & \arg z_1 & \arg z_2 \end{vmatrix}$$

then A is divisible by
a. arg $(z_1 + z_2 + z_3)$ b. arg $(z_1z_2z_3)$
c. all numbers d. cannot say

13. Let z, w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and arg $zw = \pi$. Then arg z equals
a. $\dfrac{\pi}{4}$ b. $\dfrac{\pi}{2}$ c. $\dfrac{3\pi}{4}$ d. $\dfrac{5\pi}{4}$

14. If for complex numbers z_1 and z_2, $\arg(z_1) - \arg(z_2) = 0$, then $|z_1 - z_2|$ is equal to
a. $|z_1| + |z_2|$ b. $|z_1| - |z_2|$ c. $||z_1| - |z_2||$ d. 0

15. If $k > 0$, $|z| = |w| = k$ and $\alpha = \dfrac{z-\bar{w}}{k^2+z\bar{w}}$, then Re($\alpha$) equals
a. 0 b. $k/2$ c. k d. none of these

16. If $z = x + iy$ and $x^2 + y^2 = 16$, then the range of $||x| - |y||$ is
a. [0, 4] b. [0, 2] c. [2, 4] d. none of these

17. If $k + |k + z^2| = |z|^2$ $(k \in R^-)$, then possible argument of z is
a. 0 b. π c. $\pi/2$ d. none of these

18. If $z = x + iy$ $(x, y \in R, x \neq -1/2)$, the number of values of z satisfying $|z|^n = z^2|z|^{n-2} + z|z|^{n-2} + 1\cdot$ $(n \in N, n > 1)$ is
a. 0 b. 1 c. 2 d. 3

19. If x and y are complex numbers, then the system of equations $(1 + i)x + (1 - i)y = 1$, $2ix + 2y = 1 + i$ has
a. unique solution b. no solution
c. infinite number of solutions d. none of these

20 Number of solutions of the equation $z^3 + [3(\bar{z})^2]/|z| = 0$ where z is a complex number is
a. 2 b. 3 c. 6 d. 5

21. The principal argument of the complex number $[(1+i)^5(1+\sqrt{3}i)^2]/[-2i(-\sqrt{3}+i)]$ is
a. $\dfrac{19\pi}{12}$ b. $-\dfrac{7\pi}{12}$ c. $-\dfrac{5\pi}{12}$ d. $\dfrac{5\pi}{12}$

22. The polynomial $x^6 + 4x^5 + 3x^4 + 2x^3 + x + 1$ is divisible by where w is cube root of units
a. $x + \omega$ b. $x + \omega^2$
c. $(x + \omega)(x + \omega^2)$ d. $(x - \omega)(x - \omega^2)$
where ω is one of the imaginary cube roots of unity.

23. If $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\cdots(\cos n\theta + i\sin n\theta) = 1$, then the value of θ is, $m \in N$
a. $4m\pi$ b. $\dfrac{2m\pi}{n(n+1)}$ c. $\dfrac{4m\pi}{n(n+1)}$ d. $\dfrac{m\pi}{n(n+1)}$

24. Given $z = (1 + i\sqrt{3})^{100}$, then [RE($z$)/IM($z$)] equals
a. 2^{100} b. 2^{50} c. $\dfrac{1}{\sqrt{3}}$ d. $\sqrt{3}$

25. If $z = (i)^{(i)^{(i)}}$ where $i = \sqrt{-1}$, then $|z|$ is equal to
a. 1 b. $e^{-\pi/2}$ c. $e^{-\pi}$ d. none of these

26. If $z = i\log(2 - \sqrt{-3})$, then $\cos z =$
a. -1 b. $-1/2$ c. 1 d. 1/2

27. If the equation $z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$, where a_1, a_2, a_3, a_4 are real coefficients different from zero, has a purely imaginary root, then the expression $a_3/(a_1a_2) + (a_1a_4)/(a_2a_3)$ has the value equal to
a. 0 b. 1 c. -2 d. 2

28. Suppose A is a complex number and $n \in N$, such that $A^n = (A + 1)^n = 1$, then the least value of n is
a. 3 b. 6 c. 9 d. 12

29. Number of complex numbers z such that $|z| = 1$ and $|z/\bar{z} + \bar{z}/z| = 1$ is $(\arg(z) \in [0, 2\pi))$
a. 4 b. 6 c. 8 d. more than 8

30. If α, β be the roots of the equation $u^2 - 2u + 2 = 0$ and if $\cot\theta = x + 1$, then $[(x + \alpha)^n - (x + \beta)^n]/[\alpha - \beta]$ is equal to
a. $\dfrac{\sin n\theta}{\sin^n\theta}$ b. $\dfrac{\cos n\theta}{\cos^n\theta}$ c. $\dfrac{\sin n\theta}{\cos^n\theta}$ d. $\dfrac{\cos n\theta}{\sin^n\theta}$

31. Dividing $f(z)$ by $z - i$, we obtain the remainder i and dividing it by $z + i$, we get the remainder $1 + i$, then remainder upon the division of $f(z)$ by $z^2 + 1$ is
a. $\dfrac{1}{2}(z+1)+i$ b. $\dfrac{1}{2}(iz+1)+i$
c. $\dfrac{1}{2}(iz-1)+i$ d. $\dfrac{1}{2}(z+i)+1$

32. If $z_1, z_2 \in C$, $z_1^2 + z_2^2 \in R$, $z_1(z_1^2 - 3z_2^2) = 2$ and $z_2(3z_1^2 - z_2^2) = 11$, then the value of $z_1^2 + z_2^2$ is
a. 10 b. 12 c. 5 d. 8

33. z_1 and z_2 are two distinct points in an Argand plane. If $a|z_1| = b|z_2|$ (where $a, b \in R$), then the point $(az_1/bz_2) + (bz_2/az_1)$ is a point on the
a. line segment [– 2, 2] of the real axis
b. line segment [– 2, 2] of the imaginary axis
c. unit circle $|z| = 1$
d. the line with arg $z = \tan^{-1} 2$

34. If $x^2 + x + 1 = 0$, then the value of $(x + 1/x)^2 + (x^2 + 1/x^2)^2 + \cdots + (x^{27} + 1/x^{27})^2$ is

a. 27 **b.** 72 **c.** 45 **d.** 54

35. If ω be a complex n^{th} root of unity, then $\sum_{r=1}^{n} (ar + b)\,\omega^{r-1}$ is equal to

a. $\frac{n(n+1)a}{2}$ **b.** $\frac{nb}{1-n}$ **c.** $\frac{na}{\omega - 1}$ **d.** none of these

36. The roots of the cubic equation $(z + ab)^3 = a^3$, such that $a \neq 0$, represent the vertices of a triangle of sides of length

a. $\frac{1}{\sqrt{3}}|ab|$ **b.** $\sqrt{3}|a|$ **c.** $\sqrt{3}|b|$ **d.** $|a|$

37. Sum of common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ is

a. -1 **b.** 1 **c.** 0 **d.** 1

38. If $\left|\frac{z_1}{z_2}\right| = 1$ and $\arg(z_1 z_2) = 0$, then

a. $z_1 = z_2$ **b.** $|z_2|^2 = z_1 z_2$ **c.** $z_1 z_2 = 1$ **d.** none of these

39. If z_1 and z_2 are the complex roots of the equation $(x - 3)^3 + 1 = 0$, then $z_1 + z_2$ equals to

a. 1 **b.** 3 **c.** 5 **d.** 7

40. Which of the following is equal to $\sqrt[3]{-1}$?

a. $\frac{\sqrt{3} + \sqrt{-1}}{2}$ **b.** $\frac{-\sqrt{3} + \sqrt{-1}}{\sqrt{-4}}$

c. $\frac{\sqrt{3} - \sqrt{-1}}{\sqrt{-4}}$ **d.** $-\sqrt{-1}$

41. If $|z - 1| \leq 2$ and $|\omega z - 1 - \omega^2| = a$ (where ω is a cube root of unity) then complete set of values of a is

a. $0 \leq a \leq 2$ **b.** $\frac{1}{2} \leq a \leq \frac{\sqrt{3}}{2}$

c. $\frac{\sqrt{3}}{2} - \frac{1}{2} \leq a \leq \frac{1}{2} + \frac{\sqrt{3}}{2}$ **d.** $0 \leq a \leq 4$

42. If $|z^2 - 3| = 3|z|$ then the maximum value of $|z|$ is

a. 1 **b.** $\frac{3 + \sqrt{21}}{2}$ **c.** $\frac{\sqrt{21} - 3}{2}$ **d.** none of these

43. If $|2z - 1| = |z - 2|$ and z_1, z_2, z_3 are complex numbers such that $|z_1 - \alpha| < \alpha, |z_2 - \beta| < \beta$, then $\left|\frac{z_1 + z_2}{\alpha + \beta}\right|$

a. $<|z|$ **b.** $<2|z|$ **c.** $>|z|$ **d.** $>2|z|$

44. If z_1 is a root of the equation $a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 3$, where $|a_i| < 2$ for $i = 0, 1, \ldots, n$. Then

a. $|z_1| > \frac{1}{3}$ **b.** $|z_1| < \frac{1}{4}$ **c.** $|z_1| > \frac{1}{4}$ **d.** $|z| < \frac{1}{3}$

45. If $8iz^3 + 12z^2 - 18z + 27i = 0$, then

a. $|z| = \frac{3}{2}$ **b.** $|z| = \frac{2}{3}$ **c.** $|z| = 1$ **d.** $|z| = \frac{3}{4}$

46. If $|z| < \sqrt{2} - 1$, then $|z^2 + 2z\cos\alpha|$ is

a. less than 1 **b.** $\sqrt{2} + 1$

c. $\sqrt{2} - 1$ **d.** none of these

47. If the complex number z satisfies the condition $|z| \geq 3$, then the least value of $[z + (1/z)]$ is equal to

a. 5/3 **b.** 8/3 **c.** 11/3 **d.** none of these

48. Let $|z_r - r| \leq r, \forall r = 1, 2, 3, \ldots, n$. Then $\left|\sum_{r=1}^{n} z_r\right|$ is less than

a. n **b.** $2n$ **c.** $n(n + 1)$ **d.** $\frac{n(n+1)}{2}$

49. If $|z^2 - 1| = |z|^2 + 1$, then z lies on

a. a circle **b.** a parabola **c.** an ellipse **d.** none of these

50. If $|z| = 1$ then the point representing the complex number $-1 + 3z$ will lie on

a. a circle **b.** a straight line **c.** a parabola **d.** a hyperbola

51. If $z = (\lambda + 3) - i\sqrt{5 - \lambda^2}$, then the locus of z is

a. ellipse **b.** semicircle **c.** parabola **d.** straight line

52. If $A(z_1), B(z_2), C(z_3)$ are the vertices of the triangle ABC such that $(z_1 - z_2)/(z_3 - z_2) = (1/\sqrt{2}) + (i/\sqrt{2})$, the triangle ABC is

a. equilateral **b.** right angled

c. isosceles **d.** obtuse angled

53. If z_1, z_2, z_3 are the vertices of an equilateral triangle ABC such that $|z_1 - i| = |z_2 - i| = |z_3 - i|$, then $|z_1 + z_2 + z_3|$ equals to

a. $3\sqrt{3}$ **b.** $\sqrt{3}$ **c.** 3 **d.** $\frac{1}{3\sqrt{3}}$

54. The greatest positive argument of complex number satisfying $|z - 4| = \text{Re}(z)$ is

a. $\frac{\pi}{3}$ **b.** $\frac{2\pi}{3}$ **c.** $\frac{\pi}{2}$ **d.** $\frac{\pi}{4}$

55. The complex number associated with the vertices A, B, C of ΔABC are $e^{i\theta}, \omega, \bar{\omega}$, respectively [where $\omega, \bar{\omega}$ are the complex cube roots of unity and $\cos\theta > \text{Re}(\omega)$], then the complex number of the point where angle bisector of A meets the circumcircle of the triangle, is

a. $e^{i\theta}$ **b.** $e^{-i\theta}$ **c.** $\omega\bar{\omega}$ **d.** $\omega + \bar{\omega}$

56. The maximum area of the triangle formed by the complex coordinates z, z_1, z_2 which satisfy the relations $|z - z_1| = |z - z_2|$ and $|z - (z_1 + z_2)/2| \leq r$, where $r > |z_1 - z_2|$ is

a. $\frac{1}{2}|z_1 - z_2|^2$ **b.** $\frac{1}{2}|z_1 - z_2|r$

c. $\frac{1}{2}|z_1 - z_2|^2 r^2$ **d.** $\frac{1}{2}|z_1 - z_2|r^2$

57. Locus of z if $\arg[z - (1 + i)] = \begin{cases} \frac{3\pi}{4} & \text{when } |z| \leq |z - 2| \\ \frac{-\pi}{4} & \text{when } |z| > |z - 4| \end{cases}$ is

a. straight lines passing through (2, 0)
b. straight lines passing through (2, 0), (1, 1)
c. a line segment
d. a set of two rays

58. If z is a complex number such that $-\pi/2 \le \arg z \le \pi/2$, then which of the following inequality is true?
a. $|z-\bar{z}| \le |z|(\arg z - \arg \bar{z})$ b. $|z-\bar{z}| \ge |z|(\arg z - \arg \bar{z})$
c. $|z-\bar{z}| < (\arg z - \arg \bar{z})$ d. none of these

59. If z is a complex number lying in the fourth quadrant of Argand plane and $|[kz/(k+1)] + 2i| > \sqrt{2}$ for all real value of k $(k \ne -1)$, then range of arg(z) is
a. $\left(-\frac{\pi}{8}, 0\right)$ b. $\left(-\frac{\pi}{6}, 0\right)$
c. $\left(-\frac{\pi}{4}, 0\right)$ d. none of these

60 If 'z' is complex number then the locus of 'z' satisfying the condition $|2z-1| = |z-1|$ is
a. perpendicular bisector of line segment joining 1/2 and 1
b. circle
c. parabola
d. none of the above curves

61. If $|z_2 + iz_1| = |z_1| + |z_2|$ and $|z_1| = 3$ and $|z_2| = 4$, then area of ΔABC, if affixes of A, B and C are z_1, z_2 and $[(z_2 - iz_1)/(1-i)]$ respectively, is,
a. $\frac{5}{2}$ b. 0 c. $\frac{25}{2}$ d. $\frac{25}{4}$

62. If a complex number z satisfies $|2z + 10 + 10i| \le 5\sqrt{3} - 5$, then the least principal argument of z is
a. $-\frac{5\pi}{6}$ b. $-\frac{11\pi}{12}$ c. $-\frac{3\pi}{4}$ d. $-\frac{2\pi}{3}$

63. If t and c are two complex numbers such that $|t| \ne |c|$, $|t| = 1$ and $z = (at+b)/(t-c)$, $z = x + iy$. Locus of z is (where a, b are complex numbers)
a. line segment b. straight line
c. circle d. none of these

64. The number of complex numbers z satisfying $|z-3-i| = |z-9-i|$ and $|z-3+3i| = 3$ are
a. one b. two c. four d. none of these

65. Let a be a complex number such that $|a| < 1$ and $z_1, z_2, z_3, \ldots$ be the vertices of a polygon such that $z_k = 1 + a + a^2 + \cdots + a^{k-1}$ for all $k = 1, 2, 3, \ldots$ then $z_1, z_2, \ldots$ lie within the circle
a. $\left|z - \frac{1}{1-a}\right| = \frac{1}{|a-1|}$ b. $\left|z + \frac{1}{a+1}\right| = \frac{1}{|a+1|}$
c. $\left|z - \frac{1}{1-a}\right| = |a-1|$ d. $\left|z + \frac{1}{a+1}\right| = |a+1|$

66. Let $\lambda \in R$, the origin and the non-real roots of $2z^2 + 2z + \lambda = 0$ form the three vertices of an equilateral triangle in the Argand plane then λ is
a. 1 b. $\frac{2}{3}$ c. 2 d. -1

67. Let $z = 1 - t + i\sqrt{t^2+t+2}$, where t is a real parameter. The locus of z in the Argand plane is
a. a hyperbola b. an ellipse
c. a straight line d. none of these

68. If $z^2 + z|z| + |z^2| = 0$, then the locus of z is
a. a circle b. a straight line
c. a pair of straight lines d. none of these

69. The roots of the equation $t^3 + 3at^2 + 3bt + c = 0$ are z_1, z_2, z_3 which represent the vertices of an equilateral triangle, then
a. $a^2 = 3b$ b. $b^2 = a$
c. $a^2 = b$ d. $b^2 = 3a$

70. If 'z' lies on the circle $|z - 2i| = 2\sqrt{2}$ then the value of $\arg[(z-2)/(z+2)]$ is equal to
a. $\frac{\pi}{3}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{6}$ d. $\frac{\pi}{2}$

71. $P(z)$ be a variable point in the Argand plane such that $|z| = \text{minimum } \{|z-1|, |z+1|\}$ then $z + \bar{z}$ will be equal to
a. -1 or 1 b. 1 but not equal to -1
c. -1 but not equal to 1 d. none of these

72. The locus of point z satisfying $\text{Re}\left(\frac{1}{z}\right) = k$, where k is a non-zero real number, is
a. a straight line b. a circle
c. an ellipse d. a hyperbola

73. z_1 and z_2 lie on a circle with centre at the origin. The point of intersection z_3 of the tangents at z_1 and z_2 is given by
a. $\frac{1}{2}(\bar{z}_1 + \bar{z}_2)$ b. $\frac{2z_1z_2}{z_1+z_2}$
c. $\frac{1}{2}\left(\frac{1}{z_1} + \frac{1}{z_2}\right)$ d. $\frac{z_1+z_2}{\bar{z}_1\bar{z}_2}$

74. If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$, then area of the triangle whose vertices are z_1, z_2, z_3 is
a. $3\sqrt{3}/4$ b. $\sqrt{3}/4$ c. 1 d. 2

75. z_1, z_2, z_3, z_4 are distinct complex numbers representing the vertices of a quadrilateral $ABCD$ taken in order. If $z_1 - z_4 = z_2 - z_3$ and $\arg[(z_4 - z_1)/(z_2 - z_1)] = \pi/2$, then the quadrilateral is
a. rectangle b. rhombus c. square d. trapezium

76. If $\arg\left(\frac{z_1 - \frac{z}{|z|}}{\frac{z}{|z|}}\right) = \frac{\pi}{2}$ and $\left|\frac{z}{|z|} - z_1\right| = 3$ then $|z_1|$ equals to
a. $\sqrt{26}$ b. $\sqrt{10}$ c. $\sqrt{3}$ d. $2\sqrt{2}$

77. The points $z_1 = 3 + \sqrt{3}i$ and $z_2 = 2\sqrt{3} + 6i$ are given on a complex plane. The complex number lying on the bisector of the angle formed by the vectors z_1 and z_2 is

a. $z = \frac{(3+2\sqrt{3})}{2} + \frac{\sqrt{3}+2}{2}i$ b. $z = 5 + 5i$
c. $z = -1 - i$ d. none of these

78. Let C_1 and C_2 are concentric circles of radius 1 and 8/3, respectively, having centre at (3, 0) on the Argand plane. If the complex number z satisfies the inequality

$$\log_{1/3}\left(\frac{|z-3|^2+2}{11|z-3|-2}\right) > 1 \text{ then}$$

a. z lies outside C_1 but inside C_2
b. z lies inside of both C_1 and C_2
c. z lies outside both of C_1 and C_2
d. none of these

79. If $|z-2-i| = |z|\left|\sin\left(\frac{\pi}{4} - \arg z\right)\right|$, then locus of z is

a. a pair of straight lines b. circle
c. parabola d. ellipse

80. If z is a complex number having least absolute value and $|z-2+2i| = 1$, then $z =$

a. $(2-1/\sqrt{2})(1-i)$ b. $(2-1/\sqrt{2})(1+i)$
c. $(2+1/\sqrt{2})(1-i)$ d. $(2+1/\sqrt{2})(1+i)$

81. If $z = 3/(2+\cos\theta+i\sin\theta)$, then locus of z is

a. a straight line
b. a circle having centre on y-axis
c. a parabola
d. a circle having centre on x-axis

82. If 'p' and 'q' are distinct prime numbers, then the number of distinct imaginary numbers which are p^{th} as well as q^{th} roots of unity are

a. min (p, q) b. max (p, q) c. 1 d. zero

83. If α is the n^{th} root of unity, then $1 + 2\alpha + 3\alpha^2 + \cdots$ to n terms equal to

a. $\frac{-n}{(1-\alpha)^2}$ b. $\frac{-n}{1-\alpha}$
c. $\frac{-2n}{1-\alpha}$ d. $\frac{-2n}{(1-\alpha)^2}$

84. Given z is a complex number with modulus 1. Then the equation $[(1+ia)/(1-ia)]^4 = z$ has

a. all roots real and distinct
b. two real and two imaginary
c. three roots real and one imaginary
d. one root real and three imaginary

85. Roots of the equations are $(z+1)^5 = (z-1)^5$ are

a. $\pm i\tan\left(\frac{\pi}{5}\right), \pm i\tan\left(\frac{2\pi}{5}\right)$ b. $\pm i\cot\left(\frac{\pi}{5}\right), \pm i\cot\left(\frac{2\pi}{5}\right)$
c. $\pm i\cot\left(\frac{\pi}{5}\right), \pm i\tan\left(\frac{2\pi}{5}\right)$ d. none of these

86. The value of z satisfying the equation $\log z + \log z^2 + \cdots + \log z^n = 0$ is

a. $\cos\frac{4m\pi}{n(n+1)} + i\sin\frac{4m\pi}{n(n+1)}$, $m = 1, 2, \ldots$
b. $\cos\frac{4m\pi}{n(n+1)} - i\sin\frac{4m\pi}{n(n+1)}$, $m = 1, 2, \ldots$
c. $\sin\frac{4m\pi}{n} + i\cos\frac{4m\pi}{n}$, $m = 1, 2, \ldots$
d. 0

87. If $n \in N > 1$ then sum of real part of roots of $z^n = (z+1)^n$ is equal to

a. $\frac{n}{2}$ b. $\frac{(n-1)}{2}$ c. $-\frac{n}{2}$ d. $\frac{(1-n)}{2}$

88. Which of the following represents a point in an Argand plane, equidistant from the roots of the equation $(z+1)^4 = 16z^4$?

a. (0, 0) b. $\left(-\frac{1}{3}, 0\right)$ c. $\left(\frac{1}{3}, 0\right)$ d. $\left(0, \frac{2}{\sqrt{5}}\right)$

89. $1, z_1, z_2, z_3, \ldots, z_{n-1}$ are the n^{th} roots of unity, then the value of $1/(3-z_1) + 1/(3-z_2) + \cdots + 1/(3-z_{n-1})$ is equal to

a. $\frac{n3^{n-1}}{3^n-1} + \frac{1}{2}$ b. $\frac{n3^{n-1}}{3^n-1} - 1$
c. $\frac{n3^{n-1}}{3^n-1} + 1$ d. none of these

Multiple Correct Answers Type

Solutions on page 2.66

Each question has 4 choices a, b, c and d, out of which *one or more* answers are correct.

1. If $z = \omega, \omega^2$, where ω is a non-real complex cube root of unity, are two vertices of an equilateral triangle in the Argand plane then the third vertex may be represented by

a. $z = 1$ b. $z = 0$ c. $z = -2$ d. $z = -1$

2. $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$ are four complex numbers representing the vertices of a rhombus taken in order on the complex plane, then which one of the following is/are correct?

a. $\frac{z_1 - z_4}{z_2 - z_3}$ is purely real
b. amp $\frac{z_1 - z_4}{z_2 - z_4} =$ amp $\frac{z_2 - z_4}{z_3 - z_4}$
c. $\frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary
d. it is not necessary that $|z_1 - z_3| \neq |z_2 - z_4|$

3. If $z^3 + (3+2i)z + (-1+ia) = 0$ has one real root, then the value of 'a' lies in the interval $(a \in R)$

a. (– 2, 1) b. (–1, 0) c. (0, 1) d. (–2, 3)

4. A rectangle of maximum area is inscribed in the circle $|z-3-4i| = 1$. If one vertex of the rectangle is $4+4i$, then another adjacent vertex of this rectangle can be

a. $2+4i$ b. $3+5i$ c. $3+3i$ d. $3-3i$

5. If $|z_1| = 15$ and $|z_2 - 3 - 4i| = 5$, then
   a. $|z_1 - z_2|_{min} = 5$
   b. $|z_1 - z_2|_{min} = 10$
   c. $|z_1 - z_2|_{max} = 20$
   d. $|z_1 - z_2|_{max} = 25$

6. If the points $A(z)$, $B(-z)$ and $C(1 - z)$ are the vertices of an equilateral triangle ABC, then
   a. sum of possible z is 1/2
   b. sum of possible z is 1
   c. product of possible z is 1/4
   d. product of possible z is 1/2

7. If $\left|(z-z_1)/(z-z_2)\right| = 3$, where z_1 and z_2 are fixed complex numbers and z is a variable complex number, then 'z' lies on a
   a. Circle with 'z_1' as its interior point
   b. Circle with 'z_2' as its interior point
   c. Circle with 'z_1' as its exterior point
   d. Circle with 'z_2' as its exterior point

8. If $\arg(z + a) = \pi/6$ and $\arg(z - a) = 2\pi/3$ $(a \in R^+)$, then
   a. $|z| = a$
   b. $|z| = 2a$
   c. $\arg(z) = \dfrac{\pi}{2}$
   d. $\arg(z) = \dfrac{\pi}{3}$

9. Value(s) $(-i)^{1/3}$ is/are
   a. $\dfrac{\sqrt{3}-i}{2}$
   b. $\dfrac{\sqrt{3}+i}{2}$
   c. $\dfrac{-\sqrt{3}-i}{2}$
   d. $\dfrac{-\sqrt{3}+i}{2}$

10. If $1, z_1, z_2, z_3, \ldots, z_{n-1}$ be the n^{th} roots of unity and ω be a non-real complex cube root of unity, then the product $\prod_{r=1}^{n-1}(\omega - z_r)$ can be equal to
   a. 0
   b. 1
   c. -1
   d. $1 + \omega$

11. If the equation, $z^3 + (3 + i)z^2 - 3z - (m + i) = 0$, where $m \in R$, has at least one real root, then m can have the value equal to
   a. 1
   b. 2
   c. 3
   d. 5

12. Let $P(x)$ and $Q(x)$ be two polynomials. Suppose that $f(x) = P(x^3) + xQ(x^3)$ is divisible by $x^2 + x + 1$, then
   a. $P(x)$ is divisible by $(x - 1)$ but $Q(x)$ is not divisible by $x - 1$
   b. $Q(x)$ is divisible by $(x - 1)$ but $P(x)$ is not divisible by $x - 1$
   c. Both $P(x)$ and $Q(x)$ are divisible by $x - 1$
   d. $f(x)$ is divisible by $x - 1$

13. If amp $(z_1z_2) = 0$ and $|z_1| = |z_2| = 1$, then
   a. $z_1 + z_2 = 0$
   b. $z_1z_2 = 1$
   c. $z_1 = \bar{z}_2$
   d. none of these

14. If $|z - (1/z)| = 1$ then
   a. $|z|_{max} = \dfrac{1+\sqrt{5}}{2}$
   b. $|z|_{min} = \dfrac{\sqrt{5}-1}{2}$
   c. $|z|_{max} = \dfrac{\sqrt{5}-2}{2}$
   d. $|z|_{min} = \dfrac{\sqrt{5}-1}{\sqrt{2}}$

15. z_0 is a root of the equation $z^n \cos\theta_0 + z^{n-1}\cos\theta_1 + \cdots + z\cos\theta_{n-1} + \cos\theta_n = 2$, where $\theta_i \in R$, then
   a. $|z_0| > 1$
   b. $|z_0| > \dfrac{1}{2}$
   c. $|z_0| > \dfrac{1}{4}$
   d. $|z_0| > \dfrac{3}{2}$

16. If from a point P representing the complex number z_1 on the curve $|z| = 2$, two tangents are drawn from P to the curve $|z| = 1$, meeting at points $Q(z_2)$ and $R(z_3)$, then
   a. complex number $(z_1 + z_2 + z_3)/3$ will be on the curve $|z| = 1$
   b. $\left(\dfrac{4}{\bar{z}_1} + \dfrac{1}{\bar{z}_2} + \dfrac{1}{\bar{z}_3}\right)\left(\dfrac{4}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right) = 9$
   c. $\arg\left(\dfrac{z_2}{z_3}\right) = \dfrac{2\pi}{3}$
   d. orthocentre and circumcnetre of ΔPQR will coincide

17. If z_1, z_2 be two complex numbers $(z_1 \neq z_2)$ satisfying $|z_1^2 - z_2^2| = \left|\bar{z}_1^2 + \bar{z}_2^2 - 2\bar{z}_1\bar{z}_2\right|$, then
   a. $\dfrac{z_1}{z_2}$ is purely imaginary
   b. $\dfrac{z_1}{z_2}$ is purely real
   c. $|\arg z_1 - \arg z_2| = \pi$
   d. $|\arg z_1 - \arg z_2| = \dfrac{\pi}{2}$

18. A complex number z is rotated in anticlockwise direction by an angle α and we get z' and if the same complex number z is rotated by an angle α in clockwise direction and we get z'' then
   a. z', z, z'' are in G.P.
   b. z', z, z'' are in H.P.
   c. $z' + z'' = 2z\cos\alpha$
   d. $z'^2 + z''^2 = 2z^2\cos 2\alpha$

19. Let z be a complex number satisfying equation $z^p = \bar{z}^q$, where $p, q \in N$, then
   a. if $p = q$, then number of solutions of equation will be infinite
   b. if $p = q$, then number of solutions of equation will be finite
   c. if $p \neq q$, then number of solutions of equation will be $p + q + 1$.
   d. if $p \neq q$, then number of solutions of equation will be $p + q$

20. If $z_1 = 5 + 12i$ and $|z_2| = 4$ then
   a. maximum $(|z_1 + iz_2|) = 17$
   b. minimum $(|z_1 + (1 + i)z_2|) = 13 - 4\sqrt{2}$
   c. minimum $\left|\dfrac{z_1}{z_2 + \dfrac{4}{z_2}}\right| = \dfrac{13}{4}$
   d. maximum $\left|\dfrac{z_1}{z_2 + \dfrac{4}{z_2}}\right| = \dfrac{13}{3}$

21. If $p = a + b\omega + c\omega^2$, $q = b + c\omega + a\omega^2$ and $r = c + a\omega + b\omega^2$ where $a, b, c \neq 0$ and ω is the complex cube root of unity, then

a. If p, q, r lie on the circle $|z| = 2$, the triangle formed by these points is equilateral.

b. $p^2 + q^2 + r^2 = a^2 + b^2 + c^2$

c. $p^2 + q^2 + r^2 = 2(pq + qr + rp)$

d. none of these

22. z_1, z_2, z_3 and z'_1, z'_2, z'_3 are non-zero complex numbers such that $z_3 = (1 - \lambda)z_1 + \lambda z_2$ and $z'_3 = (1 - \mu)z'_1 + \mu z'_2$ then which of the following statements is/are true?

a. If $\lambda, \mu \in R - \{0\}$, then z_1, z_2 and z_3 are collinear and z'_1, z'_2, z'_3 are collinear separately.

b. If λ, μ are complex numbers, where $\lambda = \mu$ then triangles formed by points z_1, z_2, z_3 and z'_1, z'_2, z'_3 are similar.

c. If λ, μ are distinct complex numbers, then points z_1, z_2, z_3 and z'_1, z'_2, z'_3 are not connected by any well defined geometry.

d. If $0 < \lambda < 1$, then z_3 divides the line joining z_1 and z_2 internally and if $\mu > 1$ then z'_3 divides the line joining of z'_1, z'_2 externally.

23. If $|z - 3| = \min\{|z - 1|, |z - 5|\}$, then Re($z$) equals to

a. 2 **b.** $\frac{5}{2}$ **c.** $\frac{7}{2}$ **d.** 4

24. If n is a natural number ≥ 2, such that $z^n = (z + 1)^n$, then

a. roots of equation lie on a straight line parallel to y-axis

b. roots of equation lie on a straight line parallel to x-axis

c. sum of the real parts of the roots is $-[(n-1)/2]$

d. none of these

25. If $|z - 1| = 1$, then

a. $\arg((z-1-i)/z)$ can be equal to $-\pi/4$

b. $(z-2)/z$ is purely imaginary number

c. $(z-2)/z$ is purely real number

d. if $\arg(z) = \theta$, where $z \neq 0$ and θ is acute, then $1 - 2/z = i\tan\theta$

26. If $z = x + iy$, then the equation $|(2z-i)/(z+1)| = m$ represents a circle then m can be

a. 1/2 **b.** 1 **c.** 2 **d.** $3 < r < 2\sqrt{2}$

27. Given that the two curves $\arg(z) = \pi/6$ and $|z - 2\sqrt{3}i| = r$ intersect in two distinct points, then

a. $[r] \neq 2$ **b.** $0 < r < 3$ **c.** $r = 6$ **d.** $3 < r < 2\sqrt{3}$

([r] represents integral part of r)

28. If P and Q are represented by the complex numbers z_1 and z_2, such that $|1/z_2 + 1/z_1| = |1/z_2 - 1/z_1|$, then

a. ΔOPQ (where O is the origin) is equilateral

b. ΔOPQ is right angled

c. the circumcentre of ΔOPQ is $\frac{1}{2}(z_1 + z_2)$

d. the circiucentre of ΔOPQ is $\frac{1}{3}(z_1 + z_2)$

29. Given $z = f(x) + i\,g(x)$ where $f, g:(0, 1) \to (0, 1)$ are real valued functions. Then, which of the following does not hold good?

a. $z = \frac{1}{1-ix} + i\left(\frac{1}{1+ix}\right)$ **b.** $z = \frac{1}{1+ix} + i\left(\frac{1}{1-ix}\right)$

c. $z = \frac{1}{1+ix} + i\left(\frac{1}{1+ix}\right)$ **d.** $z = \frac{1}{1-ix} + i\left(\frac{1}{1-ix}\right)$

30. Given that the complex numbers which satisfy the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ form a rectangle in the Argand plane with the length of its diagonal having an integral number of units, then

a. area of rectangle is 48 sq. units

b. if z_1, z_2, z_3, z_4 are vertices of rectangle then $z_1+z_2+z_3+z_4 = 0$

c. rectangle is symmetrical about real axis

d. $\arg(z_1 - z_3) = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

31. Equation of tangent drawn to circle $|z| = r$ at the point $A(z_0)$ is

a. $\mathrm{Re}\left(\frac{z}{z_0}\right) = 1$ **b.** $z\bar{z}_0 + z_0\bar{z} = 2r^2$

c. $\mathrm{Im}\left(\frac{z}{z_0}\right) = 1$ **d.** $\mathrm{Im}\left(\frac{z_0}{z}\right) = 1$

32. z_1 and z_2 are the roots of the equation $z^2 - az + b = 0$, where $|z_1| = |z_2| = 1$ and a, b are non-zero complex numbers, then

a. $|a| \leq 1$ **b.** $|a| \leq 2$

c. $\arg(a^2) = \arg(b)$ **d.** $\arg a = \arg(b^2)$

33. Let z_1, z_2, z_3 be the three non-zero complex numbers such that $z_2 \neq 1$, $a = |z_1|$, $b = |z_2|$ and $c = |z_3|$. Let,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

Then

a. $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2$

b. orthocentre of triangle formed by z_1, z_2, z_3 is $z_1 + z_2 + z_3$

c. if triangle formed by z_1, z_2, z_3 is equilateral, then its area is $\frac{3\sqrt{3}}{2}|z_1|^2$

d. if triangle formed by z_1, z_2, z_3 is equilateral then $z_1+z_2+z_3 = 0$

34. Locus of complex number satisfying $\arg[(z-5+4i)/(z+3-2i)] = -\pi/4$ is the arc of a circle

a. whose radius is $5\sqrt{2}$

b. whose radius is 5

c. whose lengh (of arc) is $\frac{15\pi}{\sqrt{2}}$

d. whose centre is $-2-5i$

35. If α is a complex constant such that $\alpha z^2 + z + \bar{\alpha} = 0$ has a real root, then

a. $\alpha + \bar{\alpha} = 1$ **b.** $\alpha + \bar{\alpha} = 0$

c. $\alpha + \bar{\alpha} = -1$ **d.** the absolute value of the real root is 1

36. If $\sqrt{5-12i}+\sqrt{-5-12i}=z$, then principal value of arg z can be

a. $-\frac{\pi}{4}$ b. $\frac{\pi}{4}$ c. $\frac{3\pi}{4}$ d. $-\frac{3\pi}{4}$

Reasoning Type

Solutions on page 2.72

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.

b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.

c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** If $\arg(z_1z_2)=2\pi$, then both z_1 and z_2 are purely real (z_1 and z_2 have principal arguments).
Statement 2: Principal argument of complex number lies in $(-\pi, \pi)$

2. **Statement 1:** If n is an odd integer greater than 3 but not a multiple of 3, then $(x+1)^n-x^n-1$ is divisible by x^3+x^2+x.
Statement 2: If n is an odd integer greater than 3 but not a multiple of 3, we have $1+\omega^n+\omega^{2n}=3$.

3. **Statement 1:** If $z_1+z_2=a$ and $z_1z_2=b$, where $a=\bar{a}$ and $b=\bar{b}$, then $\arg(z_1z_2)=0$.
Statement 2: The sum and product of two complex numbers are real if and only if they are conjugate of each other.

4. **Statement 1:** If $x+(1/x)=1$ and $p=x^{4000}+(1/x^{4000})$ and q be the digit at unit place in the number $2^{2^n}+1$, $n\in N$ and $n>1$, then the value of $p+q=8$.
Statement 2: If ω, ω^2 are the roots of $x+1/x=-1$, then $x^2+1/x^2=-1$, $x^3+(1/x^3)=2$.

5. **Statement 1:** Let z_1 and z_2 are two complex numbers such that $|z_1-z_2|=|z_1+z_2|$ then the orthocentre of ΔAOB is $[(z_1+z_2)/2]$. (where O is origin).
Statement 2: In case of right angled triangle, orthocentre is that point at which the triangle is right angled.

6. **Statement 1:** Locus of z, satisfying the equation $|z-1|+|z-8|=5$ is an ellipse.
Statement 2: Sum of focal distances of any point on ellipse is constant.

7. **Statement 1:** $|z_1-a|<a$, $|z_2-b|<b$, $|z_3-c|<c$, where a, b, c are positive real numbers, then $|z_1+z_2+z_3|$ is greater than $2|a+b+c|$.
Statement 2: $|z_1\pm z_2|\le|z_1|+|z_2|$.

8. Let fourth roots of unity are z_1, z_2, z_3 and z_4 respectively.
Statement 1: $z_1^2+z_2^2+z_3^2+z_4^2=0$.
Statement 2: $z_1+z_2+z_3+z_4=0$.

9. **Statement 1:** If the equation $ax^2+bx+c=0$, $0<a<b<c$, has non-real complex roots z_1 and z_2, then $|z_1|>1$, $|z_2|>1$.
Statement 2: Complex roots always occur in conjugate pairs.

10. **Statement 1:** If z_1, z_2 are the roots of the quadratic equation $az^2+bz+c=0$ such that $\text{Im}(z_1z_2)\neq 0$, then at least one of a, b, c is imaginary.
Statement 2: If quadratic equation having real coefficients has complex roots, then roots are always conjugate to each other.

11. **Statement 1:** The product of all values of $(\cos\alpha+i\sin\alpha)^{3/5}$ is $\cos 3\alpha+i\sin 3\alpha$.
Statement 2: The product of fifth roots of unity is 1.

12. **Statement 1:** If $|z_1|=|z_2|=|z_3|$, $z_1+z_2+z_3=0$ and (z_1), $B(z_2)$, $C(z_3)$ are the vertices of ΔABC, then one of the values of arg $\left((z_2+z_3-2z_1)/(z_3-z_2)\right)$ is $\pi/2$.
Statement 2: In equilateral triangle orthocentre coincides with centroid.

13. **Statement 1:** Let z be a complex number, then the equation $z^4+z+2=0$ cannot have a root, such that $|z|<1$.
Statement 2: $|z_1+z_2|\le|z_1|+|z_2|$

14. If $z_1\neq -z_2$ and $|z_1+z_2|=\left|(1/z_1)+(1/z_2)\right|$ then
Statement 1: z_1z_2 is unimodular.
Statement 2: z_1 and z_2 both are unimodular.

Linked Comprehension Type

Solutions on page 2.73

Based upon each paragraph, the relevant multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–4

Consider the complex numbers z_1 and z_2 satisfying the relation $|z_1+z_2|^2=|z_1|^2+|z_2|^2$.

1. Complex number $z_1\bar{z}_2$ is
a. purely real **b.** purely imaginary
c. zero **d.** none of these

2. Complex number z_1/z_2 is
a. purely real **b.** purely imaginary
c. zero **d.** none of these

3. One of the possible argument of complex number $i(z_1/z_2)$
a. $\frac{\pi}{2}$ **b.** $-\frac{\pi}{2}$
c. 0 **d.** none of these

4. Possible difference between the argument of z_1 and z_2 is
a. 0 **b.** π
c. $-\frac{\pi}{2}$ **d.** none of these

For Problems 5–8

Consider the complex numbers $z=(1-i\sin\theta)/(1+i\cos\theta)$.

5. The value of θ for which z is purely real are
a. $n\pi-\frac{\pi}{4}$, $n\in I$ **b.** $n\pi+\frac{\pi}{4}$, $n\in I$
c. $n\pi$, $n\in I$ **d.** none of these

6. The value of θ for which z is purely imaginary are
   a. $n\pi - \frac{\pi}{4}, n \in I$ b. $n\pi + \frac{\pi}{4}, n \in I$
   c. $n\pi,\ n \in I$ d. no real values of θ

7. The value of θ for which z is unimodular is given by
   a. $n\pi \pm \frac{\pi}{6}, n \in I$ b. $n\pi \pm \frac{\pi}{3}, n \in I$
   c. $n\pi \pm \frac{\pi}{4}, n \in I$ d. no real values of θ

8. If argument of z is $\pi/4$, then
   a. $\theta = n\pi, n \in I$ only
   b. $\theta = (2n + 1), n \in I$ only
   c. both $\theta = n\pi$ and $\theta = (2n + 1)\frac{\pi}{2}, n \in$ I
   d. none of these

For Problems 9–11

Consider a quadratic equation $az^2 + bz + c = 0$ where a, b, c are complex numbers.

9. The condition that the equation has one purely imaginary root is
   a. $(c\bar{a} - a\bar{c})^2 = -(b\bar{c} + c\bar{b})(a\bar{b} + \bar{a}b)$
   b. $(c\bar{a} + a\bar{c})^2 = (b\bar{c} + c\bar{b})(a\bar{b} + \bar{a}b)$
   c. $(c\bar{a} - a\bar{c})^2 = (b\bar{c} - c\bar{b})(a\bar{b} - \bar{a}b)$
   d. none of these

10. If equation has two purely imaginary roots, then which of the following is not true
    a. $a\bar{b}$ is purely imaginary
    b. $b\bar{c}$ is purely imaginary
    c. $c\bar{a}$ is purely real
    d. none of these

11. The condition that the equation has one purely real root is
    a. $(c\bar{a} - a\bar{c})^2 = (b\bar{c} + c\bar{b})(a\bar{b} - \bar{a}b)$
    b. $(c\bar{a} - a\bar{c})^2 = (b\bar{c} - c\bar{b})(a\bar{b} + \bar{a}b)$
    c. $(c\bar{a} - a\bar{c})^2 = (b\bar{c} + c\bar{b})(a\bar{b} + \bar{a}b)$
    d. $(c\bar{a} - a\bar{c})^2 = (b\bar{c} - c\bar{b})(a\bar{b} - \bar{a}b)$

For Problems 12–14

Consider the equation $az + b\bar{z} + c = 0$, where $a, b, c \in Z$.

12. If $|a| \neq |b|$, then z represents
    a. circle b. straight line
    c. one point d. ellipse

13. If $|a| = |b|$ and $\bar{a}c \neq b\bar{c}$, then z has
    a. infinite solutions b. no solutions
    c. finite solutions d. cannot say anything

14. If $|a| = |b| \neq 0$ and $\bar{a}c = b\bar{c}$, then $az + b\bar{z} + c = 0$ represents
    a. an ellipse b. a circle
    c. a point d. a straight line

For Problems 15–17

Let z be a complex number satisfying $z^2 + 2z\lambda + 1 = 0$, where λ is a parameter which can take any real value.

15. The roots of this equation lie on a certain circle if
    a. $-1 < \lambda < 1$ b. $\lambda > 1$
    c. $\lambda < 1$ d. none of these

16. One root lies inside the unit circle and one outside if
    a. $-1 < \lambda < 1$ b. $\lambda > 1$
    c. $\lambda < 1$ d. none of these

17. For every large value of λ, the roots are approximately
    a. $-2\lambda, 1/\lambda$ b. $-\lambda, -1/\lambda$
    c. $-2\lambda, -\frac{1}{2\lambda}$ d. none of these

For Problems 18–20

Consider the equation $az^2 + z + 1 = 0$ having purely imaginary root where $a = \cos\theta + i\sin\theta$, $i = \sqrt{-1}$ and function $f(x) = x^3 - 3x^2 + 3(1 + \cos\theta)x + 5$, then answer the following questions.

18. Which of the following is true about $f(x)$?
    a. $f(x)$ decreases for $x \in [2n\pi, (2n + 1)\pi], n \in Z$
    b. $f(x)$ decreases for $x \in \left[(2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right], n \in Z$
    c. $f(x)$ is non-monotonic function
    d. $f(x)$ increases for $x \in R$.

19. Which of the following is true?
    a. $f(x) = 0$ has three real distinct roots
    b. $f(x) = 0$ has one positive real root
    c. $f(x) = 0$ has one negative real root
    d. $f(x) = 0$ has three but not distinct roots

20. Number of roots of the equation $\cos 2\theta = \cos\theta$ in $[0, 4\pi]$ are
    a. 2 b. 3
    c. 4 d. 6

For Problems 21–23

Complex numbers z satisfy the equation $|z - (4/z)| = 2$.

21. The difference between the least and the greatest moduli of complex numbers is
    a. 2 b. 4
    c. 1 d. 3

22. The value of arg (z_1/z_2), where z_1 and z_2 are complex numbers with the greatest and the least moduli can be
    a. 2π b. π c. $\pi/2$ d. none of these

23. Locus of z if $|z - z_1| = |z - z_2|$, where z_1 and z_2 are complex numbers with the greatest and the least moduli is
    a. line parallel to real axis
    b. line parallel to imaginary axis
    c. line having positive slope
    d. line having negative slope

For Problems 24–26

Consider ΔABC in Argand plane. Let $A(0)$, $B(1)$ and $C(1 + i)$ be its vertices and M be the mid-point of CA. Let z be a variable complex number on the line BM. Let u be another variable complex number defined as $u = z^2 + 1$.

24. Locus of u is

a. parabola **b.** ellipse
c. hyperbola **d.** none of these

25. Axis of locus of u is

a. imaginary axis **b.** real axis
c. $z + \bar{z} = 2$ **d.** none of these

26. Directrix of locus of u is

a. imaginary axis **b.** $z - \bar{z} = 2i$
c. real axis **d.** none of these

For Problems 27–29

In an Argand plane z_1, z_2 and z_3 are, respectively, the vertices of an isosceles triangle ABC with $AC = BC$ and $\angle CAB = \theta$. If z_4 is the centre of triangle, then

27. the value of $AB \times AC/(IA)^2$ is

a. $\dfrac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2}$ **b.** $\dfrac{(z_2 - z_1)(z_1 - z_3)}{(z_4 - z_1)^2}$

c. $\dfrac{(z_4 - z_1)}{(z_2 - z_1)(z_3 - z_1)}$ **d.** none of these

28. the value of $(z_4 - z_1)^2 (\cos\theta + 1) \sec\theta$ is

a. $\dfrac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)}$ **b.** $(z_2 - z_1)(z_3 - z_1)$

c. $(z_2 - z_1)(z_3 - z_1)^2$ **d.** $\dfrac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2}$

29. the value of $(z_2 - z_1)^2 \tan\theta \tan\theta/2$ is

a. $(z_1 + z_2 - 2z_3)$ **b.** $(z_1 + z_2 - z_3)(z_1 + z_2 - z_4)$

c. $-(z_1 + z_2 - 2z_3)(z_1 + z_2 - 2z_4)$ **d.** none of these

For Problems 30–32

$A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of a triangle ABC inscribed in the circle $|z| = 2$. Internal angle bisector of the angle A meets the circumcircle again at $D(z_4)$.

30. Complex number representing point D is

a. $z_4 = \dfrac{1}{z_2} + \dfrac{1}{z_3}$ **b.** $\sqrt{\dfrac{z_2 + z_3}{z_1}}$

c. $\sqrt{\dfrac{z_2 z_3}{z_1}}$ **d.** $z_4 = \sqrt{z_2 z_3}$

31. $\arg[z_4/(z_2 - z_3)]$ is equal to

a. $\dfrac{\pi}{4}$ **b.** $\dfrac{\pi}{3}$

c. $\dfrac{\pi}{2}$ **d.** $\dfrac{2\pi}{3}$

32. For fixed positions of $B(z_2)$ and $C(z_3)$ all the bisectors (internal) of $\angle A$ will pass through a fixed point which is

a. H.M. of z_2 and z_3 **b.** A.M. of z_2 and z_3
c. G.M. of z_2 and z_3 **d.** none of these

Matrix-Match Type

Solutions on page 2.76

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct match is a→p, a→s, b→r, c→p, c→q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1.

Column I	Column II: possible argument of $z = a + ib$
a. $ab > 0$	**p.** $-\tan^{-1}\left\|\dfrac{b}{a}\right\|$
b. $ab < 0$	**q.** $\pi - \tan^{-1}\left\|\dfrac{b}{a}\right\|$
c. $a^2 + b^2 = 0$	**r.** $\tan^{-1}\dfrac{b}{a}$
d. $ab = 0$	**s.** $-\pi + \tan^{-1}\dfrac{b}{a}$
	t. not defined
	u. 0 or $\dfrac{\pi}{2}$

2.

Column I	Column II (one of the values of z)
a. $z^4 - 1 = 0$	**p.** $z = \cos\dfrac{\pi}{8} + i\sin\dfrac{\pi}{8}$
b. $z^4 + 1 = 0$	**q.** $z = \cos\dfrac{\pi}{8} - i\sin\dfrac{\pi}{8}$
c. $iz^4 + 1 = 0$	**r.** $z = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}$
d. $iz^4 - 1 = 0$	**s.** $z = \cos 0 + i\sin 0$

3.

Column I	Column II (Locus)
a. $\|z - 1\| = \|z - i\|$	**p.** pair of straight lines
b. $\|z + \bar{z}\| + \|z - \bar{z}\| = 2$	**q.** a line through the origin
c. $\|z + \bar{z}\| = \|z - \bar{z}\|$	**r.** circle
d. If $\|z\| = 1$, then $2/z$ lies on	**s.** square

4. Which of the condition/conditions in column II are satisfied by the quadrilateral formed by z_1, z_2, z_3, z_4 in order given in column I?

Column I	Column II
a. parallelogram	p. $z_1 - z_4 = z_2 - z_3$
b. rectangle	q. $\lvert z_1 - z_3\rvert = \lvert z_2 - z_4\rvert$
c. rhombus	r. $\frac{z_1 - z_2}{z_3 - z_4}$ is purely real
d. square	s. $\frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary
	t. $\frac{z_1 - z_2}{z_3 - z_2}$ is purely imaginary

5.

Column I	Column II
a. If $\lvert z - 2i\rvert + \lvert z - 7i\rvert = k$, then locus of z is an ellipse if $k =$	p. 7
b. If $\lvert(2z - 3)/(3z - 2)\rvert = k$, then locus of z is a circle if 2/3 is a point inside circle and 3/2 is outside the circle if $k =$	q. 8
c. If $\lvert z - 3\rvert - \lvert z - 4i\rvert = k$, then locus of z is a hyperbola if k is	r. 2
d. If $\lvert z - (3 + 4i)\rvert = (k/50)\lvert a\bar{z} + \bar{a}z + b\rvert$, where $a = 3 + 4i$, then locus of z is a hyperbola with $k =$	s. 4
	t. 5

6.

Column I	Column II
a. The value of $\sum_{n=1}^{5}(x^n + 1/x^n)^2$ when $x^2 - x + 1 = 0$ is	p. 2
b. If $\left[\frac{1+\cos\theta + i\sin\theta}{\sin\theta + i(1+\cos\theta)}\right]^4 = \cos n\theta + i\sin n\theta$, then $n =$	q. 4
c. The adjacent vertices of a regular polygon of n sides having centre at origin are the points z and $\bar{z}$. If Im(z)/Re(z) = $\sqrt{2} - 1$, then the value of $n/4$ is	r. 9
d. $(1/50)\left\{\sum_{r=1}^{10}(r-\omega)(r-\omega^2)\right\}$ = (where ω is cube root of unity)	s. 8

Integer Type

Solutions on page 2.78

1. If $x = a + bi$ is a complex number such that $x^2 = 3 + 4i$ and $x^3 = 2 + 11i$ where $i = \sqrt{-1}$, then $(a + b)$ equal to.

2. If the complex numbers x and y satisfy $x^3 - y^3 = 98i$ and $x - y = 7i$ then $xy = a + ib$ where $a, b \in R$. The value of $(a + b)/3$ equals.

3. If $x = \omega - \omega^2 - 2$, then the value of $x^4 + 3x^3 + 2x^2 - 11x - 6$ is (where ω is cube root of unity).

4. Let $z = 9 + bi$ where b is non zero real and $i^2 = -1$. If the imaginary part of z^2 and z^3 are equal, then $b/3$ is.

5. Modulus of non zero complex number z, satisfying $\bar{z} + z = 0$ and $\lvert z\rvert^2 - 4zi = z^2$ is.

6. If the expression $(1 + ir)^3$ is of the form of $s(1 + i)$ for some real 's' where 'r' is also real and, then the sum of all possible values of r is.

7. If complex number $z(z \neq 2)$ satisfies the equation $z^2 = 4z + \lvert z\rvert^2 + \frac{16}{\lvert z\rvert^3}$ then the value of $\lvert z\rvert^4$ is.

8. The complex number z satisfies $z + \lvert z\rvert = 2 + 8i$. The value of $(\lvert z\rvert - 8)$ is.

9. Let $\lvert z\rvert = 2$ and $w = \frac{z+1}{z-1}$ where $z, w \in C$ (where C is the set of complex numbers).
Then product of least and greatest value of modulus of w is.

10. If $\left[\frac{1+\cos\theta + i\sin\theta}{\sin\theta + i(1+\cos\theta)}\right]^4 = \cos n\theta + i\sin n\theta$, then n is.

11. If z be a complex number satisfying $z^4 + z^3 + 2z^2 + z + 1 = 0$ then $\lvert z\rvert$ is equal to.

12. Let 1, w, w^2 be the cube root of unity. The least possible degree of a polynomial with real coefficients having roots $2w$, $(2 + 3w)$, $(2 + 3w^2)$, $(2 - w - w^2)$, is

13. If ω is the imaginary cube root of unity, then find the number of pairs of integers (a, b) such that $\lvert a\omega + b\rvert = 1$.

14. Suppose that z is a complex number that satisfies $\lvert z - 2 - 2i\rvert \leq 1$. The maximum value of $\lvert 2iz + 4\rvert$ is equal to.

15. If $\lvert z + 2 - i\rvert = 5$ and maximum value of $\lvert 3z + 9 - 7i\rvert$ is M then the value of $M/4$ is.

16. Let $Z_1 = (8 + i)\sin\theta + (7 + 4i)\cos\theta$ and $Z_2 = (1 + 8i)\sin\theta + (4 + 7i)\cos\theta$ are two complex numbers. If $Z_1 \cdot Z_2 = a + ib$ where $a, b \in R$. If M is the greatest value of $(a + b)$ $\forall\theta \in R$, then the value of $M^{1/3}$ is.

17. Let $A = \{a \in R \mid$ the equation $(1 + 2i)x^3 - 2(3 + i)x^2 + (5 - 4i)x + 2a^2 = 0\}$ has at least one real root. Then the value of $\frac{\sum a^2}{2}$ is.

18. The minimum value of the expression $E = \lvert z\rvert^2 + \lvert z - 3\rvert^2 + \lvert z - 6i\rvert^2$ is m then the value of $m/5$ is.

Archives

Solutions on page 2.80

Subjective Type

1. Express $1/(1 - \cos\theta + 2i\sin\theta)$ in the form $x + iy$.
(IIT-JEE, 1978)

2. If $x = a + b$, $y = a\beta + b\gamma$, $z = a\gamma + b\beta$ where γ and β are the complex cube roots of unity, show that $xyz = a^3 + b^3$ **(IIT-JEE, 1978)**

3. If $x + iy = \sqrt{(a+ib)/(c+id)}$, then prove that $(x^2 + y^2)^2 = (a^2+b^2)/(c^2+d^2)$. **(IIT-JEE, 1979)**

4. It is given that n is an odd integer greater than 3, but n is not a multiple of 3. Prove that $x^3 + x^2 + x$ is a factor of $(x + 1)^n - x^n - 1$. **(IIT-JEE, 1980)**

5. Find the real values of x and y for which of the following equation is satisfied:
$$\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$$
(IIT-JEE, 1980)

6. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$. **(IIT-JEE, 1981)**

7. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2 + z_2^2 - z_1z_2 = 0$.

8. Show that the area of the triangle on the Argand diagram formed by the complex numbers z, iz and $z + iz$ is $1/2|z|^2$. **(IIT-JEE, 1986)**

9. Complex numbers z_1, z_2, z_3 are the vertices A, B, C, respectively, of an isosceles right-angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$. **(IIT-JEE, 1986)**

10. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $(z - z_1)/(z - z_2)$ is $\pi/4$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$. **(IIT-JEE, 1990)**

11. If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$. **(IIT-JEE, 1995)**

12. If $|z| \le 1$, $|w| \le 1$, then show that $|z - w|^2 \le (|z| - |w|)^2 + (\arg z - \arg w)^2$. **(IIT-JEE, 1995)**

13. Find all non-zero complex numbers z satisfying $\bar{z} = iz^2$. **(IIT-JEE, 1996)**

14. Let $\bar{b}z + b\bar{z} = c$, $b \ne 0$, be a line in the complex plane, where $\bar{b}$ is the complex conjugate of b. If a point z_1 is the reflection of a point z_2 through the line, then show that $c = \bar{z}_1 b + z_2\bar{b}$. **(IIT-JEE, 1997)**

15. Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \theta \ne 0$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q\cos^2(\theta/2)$. **(IIT-JEE, 1997)**

16. For complex numbers z and w, prove that $|z|^2 w - |w|^2 z = z - w$ if and only if $z = w$ or $z\bar{w} = 1$. **(IIT-JEE, 1999)**

17. Let a complex number α, $\alpha \ne 1$, be a root of the equation $z^{p+q} - z^p - z^q + 1 = 0$, where p, q are distinct primes. Show that either $1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + \cdots + \alpha^{q-1} = 0$, but not both together. **(IIT-JEE, 2002)**

18. If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$, then prove that $|(1 - z_1\bar{z}_2)/(z_1 - z_2)| < 1$. **(IIT-JEE, 2003)**

19. Prove that there exists no complex number z such that $|z| < 1/3$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$. **(IIT-JEE, 2003)**

20. Find the centre and radius of the circle given by $|(z-\alpha)/(z-\beta)| = k$, $k \ne 1$, where $z = x + iy$, $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$. **(IIT-JEE, 2004)**

21. If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}i$, find the other vertices of the square. **(IIT-JEE, 2005)**

22. The maximum value of $\left|Arg\left(\frac{1}{1-z}\right)\right|$ for $|z| = 1$, $z \ne 1$ is given by. (This question is part of matrix match question) **(IIT-JEE, 2011)**

23. The set $\mathrm{Re}\left(\frac{2iz}{1-z^2}\right)$: z is a complex number, $|z| = 1$, $z = \pm 1\}$ is (This question is part of matrix match question) **(IIT-JEE, 2011)**

Objective Type

Fill in the blanks

1. If the expression $\dfrac{\left[\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)+i\tan(x)\right]}{\left[1+2i\sin\left(\frac{x}{2}\right)\right]}$ is real, then the set of all possible values of x is ______________. **(IIT-JEE, 1987)**

2. For any two complex numbers z_1, z_2 and any real numbers a and b, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 =$ ______________. **(IIT-JEE, 1988)**

3. If a, b, c are the numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then $a =$ ___________ and $b =$ ______________. **(IIT-JEE, 1989)**

4. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M represent the complex numbers $1 + i$ and $2 - i$, respectively, then A represents the complex number ___________ or ___________. **(IIT-JEE, 1993)**

5. Suppose z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$ then z_2 = ____________, z_3 = ____________. **(IIT-JEE, 1994)**

6. The value of the expression $1 \times (2-\omega) \times (2-\omega^2) + 2 \times (3-\omega) \times (3-\omega^2) + \cdots + (n-1) \times (n-\omega) \times (n-\omega^2)$, where ω is an imaginary cube root of unity, is ____________. **(IIT-JEE, 1996)**

True or false

1. For complex number $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$, if $x_1 \le x_2$ and $y_1 \le y_2$. Then for all complex numbers z with $1 \cap z$, we have $((1-z)/(1+z)) \cap 0$. **(IIT-JEE, 1984)**

2. If the complex numbers z_1, z_2 and z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3 = 0$. **(IIT-JEE, 1984)**

3. If three complex numbers are in A.P. then they lie on a circle in the complex plane. **(IIT-JEE, 1985)**

4. The cube roots of unity when represented on Argand diagram form the vertices of an equilateral triangle. **(IIT-JEE, 1988)**

Multiple choice questions with one correct answer

1. If the cube roots of unity are $1, \omega, \omega^2$, then the roots of the equation $(x-1)^3 + 8 = 0$ are
   a. $-1,\ 1+2\omega,\ 1+2\omega^2$ **b.** $-1,\ 1-2\omega,\ 1-2\omega^2$
   c. $-1, -1, -1$ **d.** none of these
   (IIT-JEE, 1979)

2. The smallest positive integer n for which $[(1+i)/(1-i)]^n = 1$ is
   a. $n = 8$ **b.** $n = 16$
   c. $n = 12$ **d.** none of these
   (IIT-JEE, 1980)

3. The complex numbers $z = x + iy$ which satisfy the equation $|(z-5i)/(z+5i)| = 1$ lie on
   a. the x-axis
   b. the straight line $y = 5$
   c. a circle passing through the origin
   d. none of these **(IIT-JEE, 1981)**

4. If $z = [(\sqrt{3}/2) + i/2]^5 + [(\sqrt{3}/2) - i/2]^5$, then
   a. $\text{Re}(z) = 0$ **b.** $\text{Im}(z) = 0$
   c. $\text{Re}(z) > 0, \text{Im}(z) > 0$ **d.** $\text{Re}(z) > 0, \text{Im}(z) < 0$
   (IIT-JEE, 1982)

5. The inequality $|z-4| < |z-2|$ represents the region given by
   a. $\text{Re}(z) \ge 0$ **b.** $\text{Re}(z) < 0$
   c. $\text{Re}(z) > 0$ **d.** none of these
   (IIT-JEE, 1982)

6. If $z = x + iy$ and $\omega = (1 - iz)/(z - i)$, then $|\omega| = 1$ implies that, in the complex plane
   a. z lies on the imaginary axis **b.** z lies on the real axis
   c. z lies on the unit circle **d.** none of these
   (IIT-JEE, 1983)

7. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if
   a. $z_1 + z_4 = z_2 + z_3$ **b.** $z_1 + z_3 = z_2 + z_4$
   c. $z_1 + z_2 = z_3 + z_4$ **d.** none of these
   (IIT-JEE, 1983)

8. The complex numbers $\sin x + i \cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other for
   a. $x = n\pi$ **b.** $x = 0$
   c. $x = (n + 1/2)\pi$ **d.** no value of x
   (IIT-JEE, 1988)

9. If $\omega\ (\ne 1)$ is a cube root of unity and $(1+\omega)^7 = A + B\omega$ then A and B are respectively
   a. 0, 1 **b.** 1, 1
   c. 1, 0 **d.** -1, 1 **(IIT-JEE, 1995)**

10. Let z and ω be two non-zero complex numbers such that $|z| = |\omega|$ and $\arg z = \pi - \arg \omega$, then z equals
    a. ω **b.** $-\omega$
    c. $\bar{\omega}$ **d.** $-\bar{\omega}$ **(IIT-JEE, 1995)**

11. Let z and ω be two complex numbers such that $|z| \le 1$, $|\omega| \le 1$ and $|z - i\omega| = |z - i\bar{\omega}| = 2$ then z equals
    a. 1 or i **b.** i or $-i$ **c.** 1 or -1 **d.** i or -1

12. For positive integers n_1, n_2 the value of the expression $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$, where $i = \sqrt{-1}$ is a real number if and only if
    a. $n_1 = n_2 + 1$ **b.** $n_1 = n_2 - 1$
    c. $n_1 = n_2$ **d.** $n_1 > 0, n_2 > 0$
    (IIT-JEE, 1996)

13. If $i = \sqrt{-1}$, then $4 + 5[(-1/2) + i\sqrt{3}/2]^{334} + 3[(-1/2) + (i\sqrt{3}/2)]^{365}$ is equal to
    a. $1 - i\sqrt{3}$ **b.** $-1 + i\sqrt{3}$
    c. $i\sqrt{3}$ **d.** $-i\sqrt{3}$ **(IIT-JEE, 1999)**

14. If $\arg(z) < 0$, then $\arg(-z) - \arg(z) =$
    a. π **b.** $-\pi$
    c. $-\dfrac{\pi}{2}$ **d.** $\dfrac{\pi}{2}$ **(IIT-JEE, 2000)**

15. If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = |(1/z_1) + (1/z_2) + (1/z_2)| = 1$, then $|z_1 + z_2 + z_3|$ is
    a. equal to 1 **b.** less than 1
    c. greater than 3 **d.** equal to 3

16. Let z_1 and z_2 be n^{th} roots of unity which subtend a right angle at the origin. Then n must be of the form

a. $4k + 1$ **b.** $4k + 2$
c. $4k + 3$ **d.** $4k$ **(IIT-JEE, 2001)**

17. The complex numbers z_1, z_2 and z_3 satisfying $[(z_1 - z_3)/(z_2 - z_3)] = [(1 - i\sqrt{3})/2]$ are the vertices of a triangle which is

a. of area zero **b.** right-angled isosceles
c. equilateral **d.** obtuse-angled isosceles

18. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is

a. 0 **b.** 2 **c.** 7 **d.** 17

19. If $|z| = 1$ and $\omega = (z - 1)/(z + 1)$ (where $z \neq -1$), then Re(ω) is

a. 0 **b.** $\dfrac{1}{|z+1|^2}$
c. $\left|\dfrac{z}{z+1}\right| \dfrac{1}{|z+1|^2}$ **d.** $\dfrac{\sqrt{2}}{|z+1|^2}$
(IIT-JEE, 2003)

20. If ω ($\neq 1$) be a cube root of unity and $(1 + \omega^2)^n = (1 + \omega^4)^n$, then the least positive value of n is

a. 2 **b.** 3 **c.** 5 **d.** 6

21. The locus of z which lies in shaded region (excluding the boundaries) is best represented by

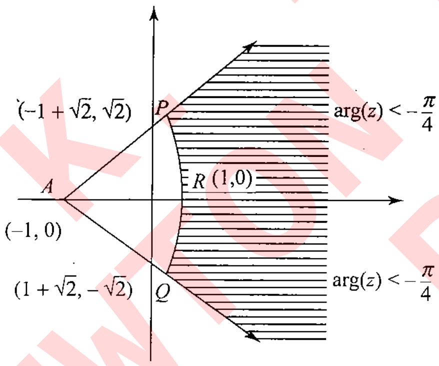


Fig. 2.47

a. z: $|z + 1| > 2$ and $|\arg(z + 1)| < \pi/4$
b. z: $|z - 1| > 2$ and $|\arg(z - 1)| < \pi/4$
c. z: $|z + 1| < 2$ and $|\arg(z + 1)| < \pi/2$
d. z: $|z - 1| < 2$ and $|\arg(z + 1)| < \pi/2$ **(IIT-JEE, 2005)**

22. a, b, c are integers, not all simultaneously equal and ω is cube root of unity ($\omega \neq 1$), then minimum value of $|a + b\omega + c\omega^2|$ is

a. 0 **b.** 1 **c.** $\dfrac{\sqrt{3}}{2}$ **d.** $\dfrac{1}{2}$
(IIT-JEE, 2005)

23. If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that $c = (1 - r)a + rb$ and $w = (1 - r)u + rv$, where r is a complex number, then the two triangles

a. have the same area **b.** are similar
c. are congruent **d.** none of these
(IIT-JEE, 1985)

24. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg z_1 - \arg z_2$ is equal to

a. $-\pi$ **b.** $-\dfrac{\pi}{2}$ **c.** 0
d. $\dfrac{\pi}{2}$ **e.** π **(IIT-JEE, 1987)**

25. The value of $\sum_{k=1}^{6} (\sin(2\pi k/7) - i\cos(2\pi k/7))$ is

a. -1 **b.** 0 **c.** $-i$
d. i **e.** none **(IIT-JEE, 1987)**

26. If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ equals

a. 128ω **b.** -128ω
c. $128\omega^2$ **d.** $-128\omega^2$ **(IIT-JEE, 1998)**

27. The value of the sum $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals

a. i **b.** $i - 1$
c. $-i$ **d.** 0 **(IIT-JEE, 1998)**

28. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then

a. $x = 3, y = 0$ **b.** $x = 1, y = 3$
c. $x = 0, y = 3$ **d.** $x = 0, y = 0$ **(IIT-JEE, 1998)**

29. Let $\omega = (-1/2) + i(\sqrt{3}/2)$. Then the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is

a. 3ω **b.** $3\omega(\omega - 1)$
c. $3\omega^2$ **d.** $3\omega(1 - \omega)$ **(IIT-JEE, 2002)**

30. If $(w - \bar{w}z)/(1 - z)$ is purely real where $w = \alpha + i\beta$, $\beta \neq 0$ and $z \neq 1$, then the set of the values of z is

a. $\{z : |z| = 1\}$ **b.** $\{z : z = \bar{z}\}$
c. $\{z : z \neq 1\}$ **d.** $\{z : |z| = 1, z \neq 1\}$
(IIT-JEE, 2006)

31. A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. Form there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P, Then the position of P in the Argand plane is

a. $3e^{i\pi/4} + 4i$ **b.** $(3 - 4i)e^{i\pi/4}$
c. $(4 + 3i)e^{i\pi/4}$ **d.** $(3 + 4i)e^{i\pi/4}$ **(IIT-JEE, 2007)**

32. If $|z| = 1$ and $z \neq = 1$, then all the values of $z/(1 - z^2)$ lie on

a. a line not passing through the origin
b. $|z| = \sqrt{2}$
c. the x-axis
d. the y-axis **(IIT-JEE, 2007)**

33. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away form origin by 5 units and then vertically away form origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $i + j$ and then it moves through an angle $\pi/2$ in anticlockwise direction on a circle with centre at origin to reach a point z_2. Then point z_2 is given by

a. $6 + 7i$ **b.** $-7 + 6i$
c. $7 + 6i$ **d.** $-6 + 7i$

34. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $\bar{z}z^3 + z\bar{z}^3 = 350$ is

a. 48 **b.** 32
c. 40 **d.** 80 **(IIT-JEE, 2009)**

Multiple choice questions with one or more than one correct answer

1. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and Re $(z_1\bar{z}_2) = 0$, then the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$ satisfies

a. $|\omega_1| = 1$ **b.** $|\omega_2| = 1$
c. $\text{Re}(\omega_1\bar{\omega}_2) = 0$ **d.** $\omega_1\bar{\omega}_2 = 0$

2. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $(z_1 + z_2)/(z_1 - z_2)$ may be

a. zero **b.** real and positive
c. real and negative **d.** purely imaginary
(IIT-JEE, 1986)

3. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If arg (w) denotes the principal argument of a non–zero complex number w, then

a. $|z - z_1| + |z - z_2| = |z_1 - z_2|$ **b.** $(z - z_1) = (z - z_2)$

c. $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$ **d.** arg $(z - z_1) = $ arg $(z_2 - z_1)$

(IIT-JEE, 2010)

Comprehension

For Problems 1–3

Let A, B, C be three sets of complex numbers as defined below:

$A = \{z: \text{Im } z \geq 1\}$

$B = \{z: |z - 2 - i| = 3\}$

$C = \{z: \text{Re}((1 - i)z) = \sqrt{2}\}$ **(IIT-JEE, 2008)**

1. The number of elements in the set $A \cap B \cap C$ is

a. 0 **b.** 1 **c.** 2 **d.** ∞

2. Let z be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

a. 25 and 29 **b.** 30 and 34
c. 35 and 39 **d.** 40 and 44

3. Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between

a. -6 and 3 **b.** -3 and 6
c. -6 and 6 **d.** -3 and 9

Matrix-match type

1. Match the statements in column-I with those I column-II [Note: Here z takes the values in the complex plane and Im(z) and Re(z) denote, respectively, the imaginary part and the real part of z]

Column I	Column II:
a. The set of points z satisfying $\|z - i\|z\|\| - \|z + i\|z\|\| = 0$ is contained in or equal to	**p.** an ellipse with eccentricity 4/5
b. The set of points z satisfying $\|z + 4\| + \|z - 4\| = 10$ is contained in or equal to	**q.** the set of points z satisfying Im $z = 0$
c. If $\|\omega\| = 2$, then the set of points $z = \omega - (1/\omega)$ is contained in or equal to	**r.** the set of points z satisfying $\|\text{Im } z\| \leq 1$
d. If $\|\omega\| = 1$, then the set of points $z = \omega + 1/\omega$ is contained in or equal to	**s.** the set of points z satisfying $\|\text{Re } z\| \leq 1$
	t. the set of points z satisfying $\|z\| \leq 3$

(IIT-JEE, 2010)

Integer type

1. Let ω be the complex number $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0 \text{ is equal to.}$$ **(IIT-JEE 2010)**

2. If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is. **(IIT-JEE 2011)**

3. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non–zero complex numbers such that

$a + b + c = x$
$a + b\omega + c\omega^2 = y$
$a + b\omega^2 + c\omega = z$

Then the value of $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is. **(IIT-JEE 2011)**

ANSWERS AND SOLUTIONS

Subjective Type

1. Given that

$$\left|\frac{z}{|\bar{z}|} - \bar{z}\right| = 1 + |z|$$

Putting $z = re^{i\theta} \Rightarrow \bar{z} = re^{-i\theta}$, we have

$$\Rightarrow \left|\frac{z}{|\bar{z}|} - \bar{z}\right| = |e^{i\theta} - re^{-i\theta}| = 1 + r$$

$\Rightarrow$ $(1 - r)^2 \cos^2\theta + (1 + r)^2 \sin^2\theta = (1 + r)^2$

$\Rightarrow$ $(1 - r)^2 \cos^2\theta - (1 + r)^2 \cos^2\theta = 0$

$\Rightarrow$ $\cos^2\theta = 0 \Rightarrow \text{Re}(z) = 0$

Hence, z is a purely imaginary number.

2.

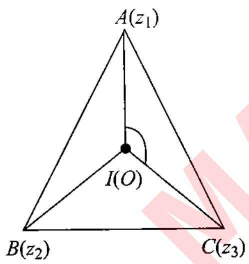


Fig. 2.48

Let z_1, z_2 and z_3 represent the vertices A, B and C, respectively, of triangle ABC. Now, the triangle is isosceles.

$\therefore$ $\angle B = \angle C$

$$\angle AIC = \frac{\pi}{2} + \frac{B}{2}$$

$$\angle BIA = \frac{\pi}{2} + \frac{C}{2}$$

$$\Rightarrow \frac{z_1}{z_3} = \left|\frac{z_1}{z_3}\right| e^{i\left(\frac{\pi}{2}+\frac{B}{2}\right)} \quad (1)$$

and

$$\frac{z_2}{z_1} = \left|\frac{z_2}{z_1}\right| e^{i\left(\frac{\pi}{2}+\frac{C}{2}\right)} \quad (2)$$

On dividing Eq. (1) by Eq. (2), we get

$$\frac{z_1^2}{z_2 z_3} = \frac{|z_1|^2}{|z_2||z_3|} e^{i\left(\frac{B-C}{2}\right)}$$

$$\Rightarrow \frac{z_1^2}{z_2 z_3} = \frac{|z_1|^2}{|z_2||z_3|} = \text{positive real number} \quad (\because \angle B = \angle C)$$

$\Rightarrow$ $z_1^2 = kz_2z_3$ (where $k \in R^+$)

Hence z_2, z_1 and kz_3 are in G.P.

3. Here $z - 1 = e^{i\theta}$ so that

$z = 1 + \cos\theta + i \sin\theta$

$$\Rightarrow z = 2\cos^2\frac{\theta}{2} + i\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\Rightarrow z = 2\cos\frac{\theta}{2}\,(e^{i\theta/2})$$

Hence

$$\tan\left(\frac{\arg(z-1)}{2}\right) - \frac{2i}{z} = \tan\frac{\theta}{2} - \frac{i}{\cos\theta/2} e^{-i\theta/2}$$

$$\Rightarrow \tan\frac{\theta}{2} - i\frac{\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}} = -i$$

4.

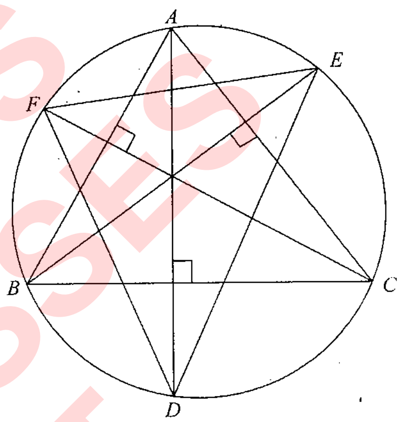


Fig. 2.49

$\angle FDA = \angle FCA = 90° - A$

$\angle ADE = \angle ABF = 90° - A$

$\Rightarrow$ $\angle FDE = 180° - 2A = 2\pi - 2A$

Simplarly $\angle DFE = 2\pi - 2C$ and $\angle DEF = 2\pi - 2B$

The angles of ΔDEF are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$, respectively.

Also it is given that $(z_3 - z_1)/(z_2 - z_1)$ is purely real. Hence,

$$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = 0 \text{ or } \pi$$

$\Rightarrow$ $\pi - 2A = 0$ or π

$\Rightarrow$ $A = \frac{\pi}{2}$ or 0 (not permissible)

Hence triangle ABC is right angled at A.

5.

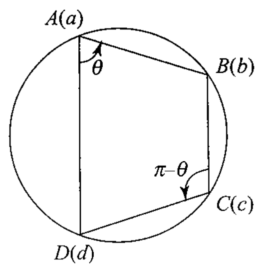


Fig. 2.50

Let complex number representing point 'D' is d and $\angle DAB = \theta$.

So, $\angle BCD = \pi - \theta$ (A, B, C, D are concyclic), Now, applying rotation formula on A and C, we get

$$\frac{b-a}{d-a} = \frac{AB}{AD}e^{i\theta} \text{ and } \frac{d-c}{b-c} = \frac{CD}{CB}e^{i(\pi-\theta)}$$

Multiplying these two, we get

$$\left(\frac{b-a}{d-a}\right)\left(\frac{d-c}{b-c}\right) = \frac{AB \times CD}{AD \times CB}e^{i\pi}$$

$$\Rightarrow \frac{d(b-a) - c(b-a)}{d(b-c) - a(b-c)} = -1 \quad \left(\because \frac{AD}{AB} = \frac{CD}{CB}\right)$$

$\Rightarrow \quad d = \dfrac{2ac - b(a+c)}{a+c-2b}$

6. $i^{2i+3} \ln\left(\dfrac{i^3x^2+2x+i}{ix^2+2x+i^3}\right) = i^{2i}i^3 \ln\left(\dfrac{2x+i(1-x^2)}{2x-i(1-x^2)}\right)$

Let $2x = r\cos\theta$ and $1 - x^2 = r\sin\theta$. Then, above expression becomes

$$(i)^{2i}(-i)\ln\left(\frac{r(\cos\theta+i\sin\theta)}{r(\cos\theta-i\sin\theta)}\right)$$

$$= (e^{i\pi/2})^{2i}(-i)\ln(e^{2i\theta})$$

$$= e^{-\pi}(-i)(2i\theta) = \frac{1}{e^{\pi}}(2\theta)$$

Now,

$$\theta = \tan^{-1}\left(\frac{1-x^2}{2x}\right) = \cot^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\left[\because \frac{1-x^2}{2x} > 0 \text{ when } x \in (0, 1)\right]$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{2} - 2\tan^{-1}x \quad [\because x \in (0, 1)]$$

So,

$$\frac{1}{e^{\pi}}(2\theta) = \frac{1}{e^{\pi}}(\pi - 4\tan^{-1}x)$$

7. Let α be the real and $i\beta$ be the imaginary roots of the given equation. Then

$\alpha + i\beta = -a \Rightarrow \alpha - i\beta = -\bar{a}$

$\Rightarrow \quad 2\alpha = -(a+\bar{a})$ and $2i\beta = -(a-\bar{a})$

$\therefore \quad 4i\alpha\beta = a^2 - \bar{a}^2 \Rightarrow 4b = a^2 - \bar{a}^2$

Alternative solution:

If one root is real and the other is imaginary, their product will be imaginary $\Rightarrow b$ is purely imaginary. Let $b = ik$, so that the equation $x^2 + ax + ik = 0$ has one purely real root. Let it be α. Then,

$\alpha^2 + a\alpha + ik = 0 \Rightarrow \alpha^2 + \bar{a}\alpha - ik = 0$

Hence,

$$\frac{\alpha^2}{-ika - i\bar{a}k} = \frac{\alpha}{ik+ik} = \frac{1}{\bar{a}-a}$$

$\Rightarrow \quad \alpha^2 = \dfrac{ik(a+\bar{a})}{a-\bar{a}}$ and $\alpha = \dfrac{-2ik}{a-\bar{a}}$

$\therefore \quad \dfrac{ik(a+\bar{a})}{a-\bar{a}} = \dfrac{-4k^2}{(a-\bar{a})^2} \Rightarrow a^2 - \bar{a}^2 = 4ik = 4b$

8. Put $z = a + ib$. Then,

$a^2 + b^2 - 2ai + 2b + 2c + 2ci = 0$

$\Rightarrow \quad (a^2 + b^2 + 2b + 2c) + (2c - 2a)i = 0$

$\Rightarrow \quad a^2 + b^2 + 2b + 2c = 0$ and $2c - 2a = 0$

$\Rightarrow \quad a = c$

Now,

$b^2 + 2b + (c^2 + 2c) = 0$

$\Rightarrow \quad b = -1 \pm \sqrt{1 - 2c - c^2}$

Since b is real,

$1 - 2c - c^2 \geq 0$

$\Rightarrow \quad c \in [-1-\sqrt{2}, -1+\sqrt{2}]$

$\Rightarrow \quad z = c + i(-1 \pm \sqrt{1-2c-c^2})$, where $c \in [-1-\sqrt{2}, -1+\sqrt{2}]$

9. $az^2 + z + 1 = 0$ (1)

Taking conjugate of both sides,

$\overline{az^2 + z + 1} = \bar{0}$

$\Rightarrow \quad \bar{a}(\bar{z})^2 + \bar{z} + \bar{1} = 0$

$\bar{a}z^2 - z + 1 = 0$ (2)

(since $\bar{z} = -z$ as z is purely imaginary)

Eliminating z from both the equations, we get

$(\bar{a} - a)^2 + 2(a + \bar{a}) = 0$

Let,

$a = \cos\theta + i\sin\theta \quad (\because |a| = 1)$

$\Rightarrow \quad (-2i\sin\theta)^2 + 2(2\cos\theta) = 0$

$\Rightarrow \quad \cos\theta = \dfrac{-1 \pm \sqrt{1+4}}{2}$

$= \dfrac{\sqrt{5}-1}{2}$

Hence, $a = \cos\theta + i\sin\theta$, where $\theta = \cos^{-1}\left(\dfrac{\sqrt{5}-1}{2}\right)$.

10. a. Let

$z = |z|(\cos\theta + i\sin\theta)$

$\Rightarrow \quad \left|\dfrac{z}{|z|} - 1\right| = |\cos\theta + i\sin\theta - 1|$

$= \sqrt{(\cos\theta-1)^2 + \sin^2\theta}$

$= \sqrt{2 - 2\cos\theta}$

$= \sqrt{4\sin^2\dfrac{\theta}{2}}$

$= 2\left|\sin\dfrac{\theta}{2}\right|$

$\leq 2\left|\dfrac{\theta}{2}\right| \quad (\because |\sin\theta| \leq |\theta|)$

$= |\theta| = |\arg z|$ (1)

b. $|z - 1| = |z - |z| + |z| - 1|$

$\leq |z - |z|| + ||z| - 1|$

$= |z|\left|\dfrac{z}{|z|} - 1\right| + ||z| - 1|$

$\leq |z| \, |\arg z| + ||z| - 1|$ [From (1)]

11. We have to prove that

$|\text{Im}(z^n)| \leq n|\text{Im}(z)||z|^{n-1}$

$\Rightarrow \quad \left|\dfrac{z^n - \bar{z}^n}{2i}\right| \leq n\left|\dfrac{z-\bar{z}}{2i}\right| |z|^{n-1}$

$\Rightarrow \quad \left|\dfrac{z^n - \bar{z}^n}{z - \bar{z}}\right| \leq n|z|^{n-1}$

Now,

$$\left|\frac{z^n - \bar{z}^n}{z - \bar{z}}\right| = \left|z^{n-1} + z^{n-2}\bar{z} + z^{n-3}\bar{z}^2 + \cdots + \bar{z}^n\right|$$

$$\leq \left|z^{n-1}\right| + \left|z^{n-2}\bar{z}\right| + \left|z^{n-3}\bar{z}^2\right| + \cdots + \left|\bar{z}^n\right|$$

$$= |z|^{n-1} + |z|^{n-2}|\bar{z}| + |z|^{n-3}|\bar{z}|^2 + \cdots + |\bar{z}|^n$$

$$= |z|^{n-1} + |z|^{n-1} + |z|^{n-1} + \cdots + |z|^{n-1} \quad [\because |\bar{z}| = |z|]$$

$$= n|z|^{n-1}$$

Hence proved.

12. Given $OA = 1$ and $|z| = 1$.

$\therefore$ $OP = |z - 0| = |z| = 1 \Rightarrow OP = OA$

$OP_0 = |z_0 - 0| = |z_0|$

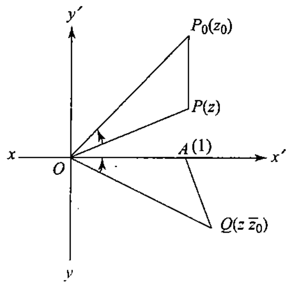


Fig. 2.51

$OQ = |zz_0 - 0| = |zz_0| = |z|\,|z| = |z_0|$

Also,

$$\angle P_0OA = \arg\left(\frac{z_0 - 0}{z - 0}\right) = \arg\left(\frac{z_0}{z}\right)$$

$$= \arg\left(\frac{\bar{z}z_0}{z\bar{z}}\right)$$

$$= \arg\left(\frac{\bar{z}z_0}{1}\right) = -\arg\left(\overline{\bar{z}z_0}\right)$$

$$= -\arg(z\,\bar{z}_0) = \arg\left(\frac{1}{z\bar{z}_0}\right)$$

$$= \arg\left(\frac{1-0}{z\bar{z}_0 - 0}\right) = \angle AOQ$$

Thus, the triangles POP_0 and AOQ are congruent. Hence,

$PP_0 = AQ \Rightarrow |z - z_0| = |z\bar{z}_0 - 1|$

13. We have

$$az^3 + bz^2 + \bar{b}z + \bar{a} = 0 \qquad (1)$$

Taking conjugate of both sides,

$$\bar{a}\bar{z}^3 + \bar{b}\bar{z}^2 + b\bar{z} + a = 0$$

Dividing this equation by $\bar{z}^3$ and writing the terms in reverse order, we get

$$\frac{a}{\bar{z}^3} + \frac{b}{\bar{z}^2} + \frac{\bar{b}}{\bar{z}} + \bar{a} = 0 \quad (\text{for } \bar{z} \neq 0) \qquad (2)$$

Since Eqs. (1) and (2) are identical,

$$z^3 = \frac{1}{\bar{z}^3}$$

$\Rightarrow |z|^6 = 1 \Rightarrow |z| = 1$

Hence, α is a root of the given equation, such that $|\alpha| = 1$.

14. $z = t^2 - 1 + \sqrt{t^4 - t^2}$

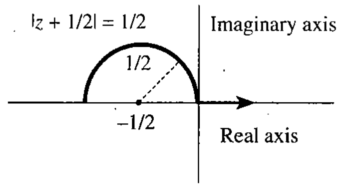


Fig. 2.52

Case I: $t^2 < 1$

$$z = t^2 - 1 + i\sqrt{t^2 - t^4}$$

$$\Rightarrow \quad z + \frac{1}{2} = \left(t^2 - \frac{1}{2}\right) + i\sqrt{\frac{1}{4} - \left(t^2 - \frac{1}{2}\right)^2}$$

$$\Rightarrow \quad \left|z + \frac{1}{2}\right|^2 = \left(t^2 - \frac{1}{2}\right)^2 + \frac{1}{4} - \left(t^2 - \frac{1}{2}\right)^2 \Rightarrow \left|z + \frac{1}{2}\right| = \frac{1}{2}$$

In this case z will lie on the upper half of circle of radius 1/2, centred at –1/2, in the Argand plane.

Case II: $t^2 \geq 1 \Rightarrow t \in (-\infty, -1] \cup [1, \infty)$

Clearly, 'z' is purely real. Also 'z' is an even function of t. Hence,

$$\frac{dz}{dt} = 2t + \frac{(4t^3 - 2t)}{2\sqrt{t^4 - t^2}}$$

$$= 2t + \frac{2t(t^2 - 1)}{2\sqrt{t^2(t^2 - 1)}}$$

$$> 0 \ \forall \ t \in (1, \infty)$$

Hence, 'z' will attain all values in the interval $[0, \infty)$ (as z is equal to zero at $t = 1$). In this case z will lie on the positive real axis in the Argand plane $\forall \ t \in (-\infty, -1) \cup (1, \infty)$.

15.

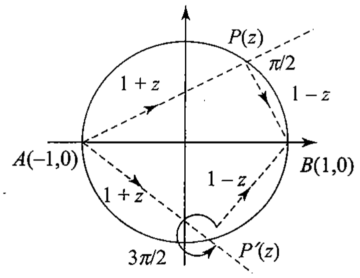


Fig. 2.53

Since $|z| = 1$, z lies on a unit circle having centre at the origin.

$$\arg\left(\frac{1+z}{1-z}\right) = +\frac{\pi}{2} \text{ or } +\frac{3\pi}{2}$$

$$\Rightarrow \quad \frac{1+z}{1-z} = ke^{i\pi/2} \text{ or } ke^{i3\pi/2}$$

where k is a real parameter and its value depends upon the position of z. Let,

$$\alpha = \sqrt{\frac{1+z}{1-z}}$$

$$= \sqrt{k}\, e^{i\pi/4} \text{ or } \sqrt{k}\, e^{i3\pi/4}$$

Therefore, α lies on one of the two perpendicular lines.

16. From figure, it is clear that $|z - i| < |z + i|$ (as z lies above the real axis). Hence,

$$|\alpha| = \frac{|z-i|}{|z+i|} < 1$$

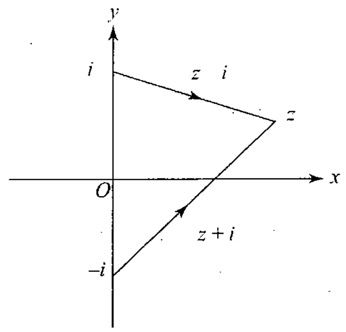


Fig. 2.54

Therefore, α lies within the unit circle which has centre at the origin. Now, if z is travelling on the real axis Im(z) = 0, Re(z) varies from $-\infty$ to $+\infty$. Let,

$$z = x + i0$$

$$\Rightarrow \quad \alpha = \frac{x-i}{x+i} \Rightarrow |\alpha| = \frac{|x-i|}{|x+i|} = 1 \text{ (as } |x-i| = |x+i| \ \forall\, x \in R$$

Hence, α moves on the unit circle which has centre at the origin

17. Suppose $x^2 + ax + b = 0$ has roots x_1 and x_2. Then,

$\therefore \quad x_1 + x_2 = -a$ (1)

and

$x_1 x_2 = b$ (2)

From (2),

$|x_1|\,|x_2| = |b|$

$\Rightarrow \quad |b| = 1$ (3)

Also,

$|-a| = |x_1 + x_2|$

$\Rightarrow \quad |a| \le |x_1| + |x_2|$

or

$|a| \le 2$ (4)

Now suppose $y^2 + |a|y + |b| = 0$ has roots y_1 and y_2. Then,

$$y_1, y_2 = \frac{-|a| \pm \sqrt{|a|^2 - 4|b|}}{2}$$

$$= \frac{-|a| \pm \left(\sqrt{4-|a|^2}\right)i}{2}$$

$$\Rightarrow \quad |y_1|, |y_2| = \frac{\sqrt{|a|^2 + 4 - |a|^2}}{2} = 1$$

Hence, $|y_1| = |y_2| = 1$.

18.

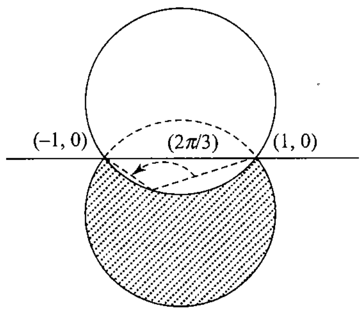


Fig. 2.55

Let us take arg $[(z+1)/(z-1)] = 2\pi/3$. Clearly, z lies on the minor arc of the circle passing through (1, 0) and (–1, 0). Similarly, arg $[(z+1)/(z-1)] = \pi/3$ means that 'a' is lying on the major arc of the circle passing through (1, 0) and (–1, 0). Now, if we take any point in the region included between the two arcs, say $P_1(z_1)$, we get

$$\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$$

Thus $\pi/3 \le \arg[(z+1)/(z-1)] \le 2\pi/3$ represents the shaded region [excluding the points (1, 0) and (– 1, 0)].

19.

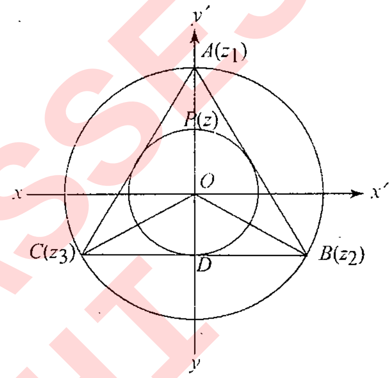


Fig. 2.56

Let,

$$z_1 = \frac{2}{\sqrt{3}} e^{i\frac{\pi}{2}},\ z_2 = \frac{2}{\sqrt{3}} e^{-i\frac{\pi}{6}},\ z_3 = \frac{2}{\sqrt{3}} e^{-i\frac{5}{6}\pi}$$

Clearly, the points lie on the circle $|z| = 2/\sqrt{3}$. And ΔABC is equilateral and its centroid coincides with circumcentre. Hence,

$$z_1 + z_2 + z_3 = 0 \text{ and } \bar{z}_1 + \bar{z}_2 + \bar{z}_3 = 0$$

Since triangle is equilateral, inradius $r = OD = 1/\sqrt{3}$. Theequation of incircle is

$$|z| = 1/\sqrt{3}$$

Let $P(z)$ be any point on the incircle. Now,

$$AP^2 = |z - z_1|^2 = |z|^2 + |z_1|^2 - (z\bar{z}_1 + \bar{z}z_1)$$

Similarly,

$$BP^2 = |z|^2 + |z_2|^2 - (z\bar{z}_2 + \bar{z}z_2)$$

$$CP^2 = |z|^2 + |z_3|^2 - (z\bar{z}_3 + \bar{z}z_3)$$

$\therefore \quad AP^2 + BP^2 + CP^2$

$$= 3|z|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 - z(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) - \bar{z}(z_1 + z_2 + z_3)$$

$$= 3 \times \frac{4}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - z(0) - \bar{z}(0)$$

$$= 5$$

20. Given equation of line is

$$a\bar{z} + \bar{a}z + b = 0,\ \forall\, b \in R$$

Let PQ be the segment intercepted between the axes. For real intercept Z_R,

$$z = \bar{z}$$

$$\Rightarrow \quad Z_R(a + \bar{a}) + b = 0$$

$$\Rightarrow \quad Z_R = \frac{-b}{(a+\bar{a})}$$

For imaginary intercept Z_I,

$$z + \bar{z} = 0$$

$\Rightarrow \quad Z_l(a-\bar{a})+b=0$

$\Rightarrow \quad Z_l = -\dfrac{b}{\bar{a}-a}$

Mid-point is

$$z = \frac{Z_R+Z_l}{2}$$

$$= \frac{-b}{2}\left[\frac{1}{\bar{a}+a}+\frac{1}{\bar{a}-a}\right]$$

$$= \frac{\bar{a}b}{(a+\bar{a})(a-\bar{a})}$$

$$= \frac{\bar{a}b}{a^2-(\bar{a})^2}$$

$$\Rightarrow \quad \frac{z[a^2-(\bar{a})^2]}{\bar{a}} = b \;= b\,(=\text{real})$$

$$\Rightarrow \quad z\frac{a^2-(\bar{a})^2}{\bar{a}} = \bar{z}\frac{(\bar{a})^2-(a)^2}{a}$$

$$\Rightarrow \quad az+\bar{a}\bar{z}=0$$

Objective Type

1. b. Verify by selecting particular values of a and b.

Let $a=-9$ and $b=4$. Then,

$$\sqrt{a}\sqrt{b} = \sqrt{-9}\sqrt{4} = (3i)(2) = 6i$$

From option (a), we have

$$-\sqrt{|a|b} = -\sqrt{|-9|\times 4} = -\sqrt{36} = -6$$

From option (b), we have

$$\sqrt{|a|b}\,i = \sqrt{|-9|\times 4}\;i = 6i$$

2. c. $$x = 9^{\frac13+\frac19+\frac1{27}+\cdots} = 9^{\frac{\frac13}{1-\frac13}} = 9^{\frac12} = 3$$

$$y = 4^{\frac13-\frac19+\frac1{27}+\cdots} = 4^{\frac{\frac13}{1+\frac13}} = 4^{\frac14} = \sqrt{2}$$

$$z = \sum_{r=1}^{\infty}(1+i)^{-r} = \frac{1}{1+i}+\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}+\cdots$$

$$= \frac{\frac{1}{1+i}}{1-\frac{1}{1+i}} = \frac{1}{i} = -i$$

Let $a = x+yz = 3-\sqrt{2}\,i$ (fourth quadrant). Then,

$$\arg a = -\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$$

3. d. $10z^2-3iz-k=0$

$$\Rightarrow \quad z = \frac{3i\pm\sqrt{-9+40k}}{20}$$

Now, $D=-9+40k$. If $k=1$, then $D=31$. So (a) is false.

If k is a negative real number, then D is a negative real number. So (d) is true.

If $k=i$, then $D=-9+40i = 16+40i-25 = (4+5i)^2$, and the roots are $(1/5)+(2/5)i$ and $-(1/5-1/10)i$. So (c) is false.

If $k=0$ (which is a complex number), then the roots are 0 and $(3/10)i$. So (b) is false.

4. d. Let $z=x+iy$, so that $\bar{z}=x-iy$.

$\therefore \quad z^2+\bar{z}=0$

$\Rightarrow \quad (x^2-y^2+x)+i(2xy-y)=0$

Equating real and imaginary parts, we get

$$x^2-y^2+x=0 \qquad (1)$$

and

$$2xy-y=0 \Rightarrow y=0 \text{ or } x=\frac12$$

If $y=0$, then (1) gives $x^2+x=0 \Rightarrow x=0$ or $x=-1$

If $x=1/2$, then from (1),

$$y^2 = \frac14+\frac12=\frac34 \Rightarrow y=\pm\frac{\sqrt3}{2}$$

Hence, there are four solutions in all.

5. c. Given that $a^2+b^2=1$. Therefore,

$$\frac{1+b+ia}{1+b-ia} = \frac{(1+b+ia)(1+b+ia)}{(1+b-ia)(1+b+ia)}$$

$$= \frac{(1+b)^2-a^2+2ia(1+b)}{1+b^2+2b+a^2}$$

$$= \frac{(1-a^2)+2b+b^2+2ia(1+b)}{2(1+b)}$$

$$= \frac{2b^2+2b+2ia(1+b)}{2(1+b)}$$

$$= b+ia$$

6. b. Let,

$$\sin\frac{\pi}{8}+i\cos\frac{\pi}{8} = z$$

$$\Rightarrow \left[\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right]^8$$

$$= \left(\frac{1+z}{1+\frac{1}{z}}\right)^8$$

$$= z^8$$

$$= \left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)^8$$

$$= \left(\cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right)\right)^8$$

$$= \left(\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}\right)^8$$

$$= \cos 3\pi = -1$$

7. a. Let $z_1 = a+ib$ and $z_2 = c-id$, where $a>0$ and $d>0$. Then,

$$|z_1| = |z_2| \Rightarrow a^2+b^2 = c^2+d^2 \qquad (1)$$

Now,

$$\frac{z_1+z_2}{z_1-z_2} = \frac{(a+ib)+(c-id)}{(a+ib)-(c-id)}$$

$$= \frac{[(a+c)+i(b-d)][(a-c)-i(b+d)]}{[(a-c)+i(b+d)][(a-c)-i(b+d)]}$$

$$= \frac{(a^2+b^2)-(c^2+d^2)-2(ad+bc)i}{a^2+c^2-2ac+b^2+d^2+2bd}$$

$$= \frac{-(ad+bc)i}{a^2+b^2-ac+bd} \quad \text{[Using (1)]}$$

Hence, $(z_1+z_2)/(z_1-z_2)$ is purely imaginary. However, if $ad + bc = 0$, then $(z_1+z_2)/(z_1-z_2)$ will be equal to zero. According to the conditions of the equation, we can have $ad + bc = 0$.

8. a. We have,

$$\arg\left(\frac{z_1}{z_2}\right) = \pi$$

$\Rightarrow$ $\arg(z_1) - \arg(z_2) = \pi$

$\Rightarrow$ $\arg(z_1) = \arg(z_2) + \pi$

Let $\arg(z_2) = \theta$. Then $\arg(z_1) = \pi + \theta$.

$\therefore$ $z_1 = |z_1|[\cos(\pi+\theta) + i\sin(\pi+\theta)]$
$= |z_1|(-\cos\theta - i\sin\theta)$

and

$z_2 = |z_2|(\cos\theta + i\sin\theta)$
$= |z_1|(\cos\theta + i\sin\theta)$ $\quad (\because |z_1| = |z_2|)$
$= -z_1$

$\Rightarrow$ $z_1 + z_2 = 0$

9. c. Let
$a = \cos\alpha + i\sin\alpha$
$b = \cos\beta + i\sin\beta$
$c = \cos\gamma + i\sin\gamma$

Then,
$a + 2b + 3c = (\cos\alpha + 2\cos\beta + 3\cos\gamma)$
$\qquad + i(\sin\alpha + 2\sin\beta + 3\sin\gamma) = 0$

$\Rightarrow$ $a^3 + 8b^3 + 27c^3 = 18\,abc$

$\Rightarrow$ $\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$

and

$\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$

10. d.

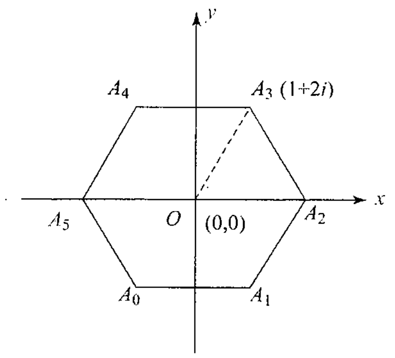


Fig. 2.57

Let the vertices be $z_0, z_1, \ldots, z_5$ w.r.t. centre O at origin and $|z_0| = \sqrt{5}$.

Now ΔOA_2A_3 is equilateral $\Rightarrow$ $OA_2 = OA_3 = A_2A_3 = \sqrt{5}$
$= |z_0|\,|\cos\theta + i\sin\theta - 1|$

Perimeter $= 6\sqrt{5}$.

11. a. $$\frac{1+iz}{1-iz} = \frac{1+i(b+ic)/(1+a)}{1-i(b+ic)/(1+a)}$$

$$= \frac{1+a-c+ib}{1+a+c-ib}$$

$$= \frac{(1+a-c+ib)(1+a+c+ib)}{(1+a+c)^2+b^2}$$

$$= \frac{1+2a+a^2-b^2-c^2+2ib+2iab}{1+a^2+c^2+b^2+2ac+2(a+c)}$$

$$= \frac{2a+2a^2+2ib+2iab}{2+2ac+2(a+c)} \quad (\because a^2+b^2+c^2=1)$$

$$= \frac{a+a^2+ib+iab}{1+ac+(a+c)}$$

$$= \frac{a(a+1)+ib(a+1)}{(a+1)(c+1)}$$

$$= \frac{a+ib}{c+1}$$

12. b. If z_1, z_2, z_3 are three complex numbers, then

$$A = \begin{vmatrix} \arg z_1 & \arg z_2 & \arg z_3 \\ \arg z_2 & \arg z_3 & \arg z_1 \\ \arg z_3 & \arg z_1 & \arg z_2 \end{vmatrix}$$

$$\Rightarrow \quad A = (\arg z_1 + \arg z_2 + \arg z_3)\begin{vmatrix} 1 & \arg z_2 & \arg z_3 \\ 1 & \arg z_3 & \arg z_1 \\ 1 & \arg z_1 & \arg z_2 \end{vmatrix}$$

(Using $C_1 \to C_1 + C_2 + C_3$)

$$\Rightarrow \quad A = \arg(z_1z_2z_3)\begin{vmatrix} 1 & \arg z_2 & \arg z_3 \\ 1 & \arg z_3 & \arg z_1 \\ 1 & \arg z_1 & \arg z_2 \end{vmatrix}$$

Hence, A is divisible by $\arg(z_1z_2z_3)$.

13. c. $\bar{z} + i\bar{w} = 0$

$\Rightarrow$ $z - iw = 0$

$\Rightarrow$ $z = iw$ $\qquad (1)$

$\arg zw = \pi$

$\Rightarrow$ $\arg z + \arg w = \pi$

$\Rightarrow$ $\arg z + \arg \dfrac{z}{i} = \pi$ $\qquad$ [Using (1)]

$\Rightarrow$ $\arg z + \arg z - \arg i = \pi$

$\Rightarrow$ $2\arg z - \dfrac{\pi}{2} = \pi$

$\Rightarrow$ $2\arg z = \dfrac{3\pi}{2}$

$\Rightarrow$ $\arg z = \dfrac{3\pi}{4}$

14. c. We have,

$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$

where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$. Given,

$\arg(z_1 - z_2) = 0$

$\Rightarrow \quad |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1|\,|z_2|$

$= (|z_1| - |z_2|)^2$

$\Rightarrow \quad |z_1 - z_2| = ||z_1| - |z_2||$

15. a. $\alpha = \dfrac{z - \bar{w}}{k^2 + z\bar{w}} \Rightarrow \bar{\alpha} = \dfrac{\bar{z} - w}{k^2 + \bar{z}w}$

But $z\bar{z} = w\bar{w} = k^2$. Hence,

$$\Rightarrow \quad \bar{\alpha} = \frac{\dfrac{k^2}{z} - \dfrac{k^2}{\bar{w}}}{k^2 + \dfrac{k^2}{z}\dfrac{k^2}{\bar{w}}} = \frac{\bar{w} - z}{z\bar{w} + k^2} = -\alpha$$

$\Rightarrow \quad \alpha + \bar{\alpha} = 0$

$\Rightarrow \quad \mathrm{Re}(\alpha) = 0$

16. a. Here $x = 4\cos\theta$, $y = 4\sin\theta$.

$\therefore \quad ||x| - |y||$

$= |4|\cos\theta| - 4|\sin\theta||$

$= 4||\cos\theta| - |\sin\theta||$

$= 4\sqrt{1 - 2|\cos\theta||\sin\theta|}$

$= 4\sqrt{1 - |\sin 2\theta|}$

Hence, the range is $[0, 4]$.

17. c. $|k + z^2| = |z^2| - k = |z^2| + |k|$

$\Rightarrow \quad k, z^2$ and $0 + i0$ are collinear

$\Rightarrow \quad \arg(z^2) = \arg(k)$

$\Rightarrow \quad 2\arg(z) = \pi$

$\Rightarrow \quad \arg(z) = \dfrac{\pi}{2}$

18. b. The given equation is

$|z|^n = (z^2 + z)\,|z|^{n-2} + 1$

$\Rightarrow \quad z^2 + z$ is real

$\Rightarrow \quad z^2 + z = \bar{z}^2 + \bar{z}$

$\Rightarrow \quad (z - \bar{z})(z + \bar{z} + 1) = 0$

$\Rightarrow \quad z = \bar{z} = x$ as $z + \bar{z} + 1 \neq 0$ $(x \neq -1/2)$

Hence, the given equation reduces to

$x^n = x^n + x|x|^{n-2} + 1$

$\Rightarrow \quad x|x|^{n-2} = -1$

$\Rightarrow \quad x = -1$

So number of solutions is 1.

19. c. Observing carefully the system of equations, we find

$$\frac{1+i}{2i} = \frac{1-i}{2} = \frac{1}{1+i}$$

Hence, there are infinite number of solutions.

20. d. Given,

$$z^3 + \frac{3(\bar{z})^2}{|z|} = 0$$

Let,

$z = re^{i\theta}$

$\Rightarrow \quad r^3 e^{i3\theta} + 3re^{-i2\theta} = 0$

Since 'r' cannot be zero, so

$re^{i5\theta} = -3$

which will hold for $r = 3$ and five distinct values of 'θ'. Thus there are five solutions.

21. c. $$\frac{(1+i)^5 \left(1 + \sqrt{3}\,i\right)^2}{-2i\left(-\sqrt{3} + i\right)} = \frac{(\sqrt{2})^5 \left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^5 2^2 \left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2} i\right)^2}{2i2\left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)}$$

$$\Rightarrow \quad \text{Argument} = \frac{5\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{2} + \frac{\pi}{6} = \frac{19\pi}{12}$$

Therefore, the principal argument is $-5\pi/12$.

22. d. Let $f(x) = x^6 + 4x^5 + 3x^4 + 2x^3 + x + 1$. Hence,

$f(\omega) = \omega^6 + 4\omega^5 + 3\omega^4 + 2\omega^3 + \omega + 1$

$= 1 + 4\omega^2 + 3\omega + 2 + \omega + 1$

$= 4(\omega^2 + \omega + 1)$

$= 0$

Hence $f(x)$ is divisible by $x - \omega$. Then $f(x)$ is also divisible by $x - \omega^2$ (as complex roots occur in conjugate pairs).

$f(-\omega) = (-\omega)^6 + 4(-\omega)^5 + 3(-\omega)^4 + 2(-\omega)^3 + (-\omega) + 1$

$= \omega^6 - 4\omega^5 + 3\omega^4 - 2\omega^3 - \omega + 1$

$= 1 - 4\omega^2 + 3\omega - 2 - \omega + 1$

$\neq 0$

23. c. We have,

$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\cdots$

$\times(\cos n\theta + i\sin n\theta) = 1$

$\Rightarrow \quad \cos(\theta + 2\theta + 3\theta + \cdots + n\theta) + i\sin(\theta + 2\theta + \cdots + n\theta) = 1$

$$\Rightarrow \quad \cos\left(\frac{n(n+1)}{2}\theta\right) + i\sin\left(\frac{n(n+1)}{2}\theta\right) = 1$$

$$\Rightarrow \quad \cos\left(\frac{n(n+1)}{2}\theta\right) = 1 \text{ and } \sin\left(\frac{n(n+1)}{2}\theta\right) = 0$$

$$\Rightarrow \quad \frac{n(n+1)}{2}\theta = 2m\pi \Rightarrow \theta = \frac{4m\pi}{n(n+1)}, \text{ where } m \in Z$$

24. c. $$z = (1 + i\sqrt{3})^{100} = 2^{100}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{100}$$

$$= 2^{100}\left(\cos\frac{100\pi}{3} + i\sin\frac{100\pi}{3}\right)$$

$$= 2^{100}\left(-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$= 2^{100}\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$$

$$\Rightarrow \quad \frac{\mathrm{Re}(z)}{\mathrm{Im}(z)} = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

25. a. $$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{\frac{i\pi}{2}}$$

$$\Rightarrow \quad i^i = \left(e^{\frac{i\pi}{2}}\right)^i = e^{-\frac{\pi}{2}}$$

$\Rightarrow \quad z = (i)^{(i)^i} = i^{e^{-\frac{\pi}{2}}}$

$\Rightarrow \quad |z| = 1$

26. d. $z = i\log(2-\sqrt{3})$

$\Rightarrow \quad e^{iz} = e^{i^2\log(2-\sqrt{3})} = e^{-\log(2-\sqrt{3})}$

$\Rightarrow \quad e^{iz} = e^{\log(2-\sqrt{3})^{-1}} = e^{\log(2+\sqrt{3})} = (2+\sqrt{3})$

$\Rightarrow \quad \cos z = \dfrac{e^{iz}+e^{-iz}}{2} = \dfrac{(2+\sqrt{3})+(2-\sqrt{3})}{2} = 2$

27. b. Let xi be the root where $x \neq 0$ and $x \in R$

$x^4 - a_1x^3i - a_2x^2 + a_3xi + a_4 = 0$

$\Rightarrow \quad x^4 - a_2x^2 + a_4 = 0$ (1)

and

$a_1x^3 - a_3x = 0$ (2)

From Eq. (2),

$a_1x^2 - a_3 = 0$

$\Rightarrow \quad x^2 = a_3/a_1$ (as $x \neq 0$)

Putting the value of x^2 in Eq. (1), we get

$\dfrac{a_3^2}{a_1^2} - \dfrac{a_2\,a_3}{a_1} + a_4 = 0$

$\Rightarrow \quad a_3^2 + a_4a_1^2 = a_1a_2a_3$

$\Rightarrow \quad \dfrac{a_3}{a_1\,a_2} + \dfrac{a_1a_4}{a_2\,a_3} = 1$ (dividing by $a_1a_2a_3$)

28. b. Let $A = x + iy$. Given,

$|A| = 1 \Rightarrow x^2 + y^2 = 1$

and

$|A+1| = 1 \Rightarrow (x+1)^2 + y^2 = 1$

$\Rightarrow \quad x = -\dfrac{1}{2}$ and $y = \pm\dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad A = \omega$ or ω^2

$\Rightarrow \quad (\omega)^n = (1+\omega)^n = (-\omega^2)^n$

Therefore, n must be even and divisible by 3.

29. c. Let $z = \cos x + i\sin x$, $x \in [0, 2\pi)$. Then,

$1 = \left|\dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z}\right|$

$= \dfrac{|z^2 + \bar{z}^2|}{|z|^2}$

$= |\cos 2x + i\sin 2x + \cos 2x - i\sin 2x|$

$= 2|\cos 2x|$

$\Rightarrow \quad \cos 2x = \pm 1/2$

Now,

$\cos 2x = 1/2$

$\Rightarrow \quad x_1 = \dfrac{\pi}{6}, x_2 = \dfrac{5\pi}{6}, x_3 = \dfrac{7\pi}{6}, x_4 = \dfrac{11\pi}{6}$

$\cos 2x = -\dfrac{1}{2}$

$\Rightarrow \quad x_5 = \dfrac{\pi}{3}, x_6 = \dfrac{2\pi}{3}, x_7 = \dfrac{4\pi}{3}, x_8 = \dfrac{5\pi}{3}$

30. a. $u^2 - 2u + 2 = 0 \Rightarrow u = 1 \pm i$

$\Rightarrow \quad \dfrac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta}$

$= \dfrac{[(\cot\theta - 1) + (1+i)]^n - [(\cot\theta - 1) + (1-i)]^n}{2i}$

($\because \cot\theta - 1 = x$)

$= \dfrac{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n}{\sin^n\theta\; 2i}$

$= \dfrac{2i\sin n\theta}{\sin^n\theta\; 2i}$

$= \dfrac{\sin n\theta}{\sin^n\theta}$

31. b. $f(z) = g(z)(z-i)(z+i) + az + b;\ a, b \in C$

Given,

$f(i) = i \Rightarrow ai + b = i$ (1)

and

$f(-i) = 1 + i$

$\Rightarrow \quad a(-i) + b = 1 + i$ (2)

From (1) and (2), we have

$a = \dfrac{i}{2}, b = \dfrac{1}{2} + i$

Hence, the required remainder is $az + b = (1/2)iz + (1/2) + i$.

32. c. We have,

$z_1(z_1^2 - 3z_2^2) = 2$ (1)

$z_2(3z_1^2 - z_2^2) = 11$ (2)

Multiplying (2) by i and adding it to (1), we get

$z_1^3 - 3z_2^2z_1 + i(3z_1^2z_2 - z_2^3) = 2 + 11i$

$\Rightarrow \quad (z_1 + iz_2)^3 = 2 + 11i$ (3)

Multiplying (2) by i and subtracting it from (1), we get

$z_1^3 - 3z_2^2z_1 - i(3z_1^2z_2 - z_2^3) = 2 - 11i$

$\Rightarrow \quad (z_1 - iz_2)^3 = 2 - 11i$ (4)

Multiplying (3) and (4), we get

$(z_1^2 + z_2^2)^3 = 2^2 - 121i^2 = 4 + 121 = 125$

$\Rightarrow \quad z_1^2 + z_2^2 = 5$

33. a. Assuming arg $z_1 = \theta$ and arg $z_2 = \theta + \alpha$,

$\dfrac{az_1}{bz_2} + \dfrac{bz_2}{az_1} = \dfrac{a|z_1|e^{i\theta}}{b|z_2|e^{i(\theta+\alpha)}} + \dfrac{b|z_2|e^{i(\theta+\alpha)}}{a|z_1|e^{i\theta}}$

$= e^{-i\alpha} + e^{i\alpha} = 2\cos\alpha$

Hence, the point lies on the line segment $[-2, 2]$ of the real axis.

34. d. $x^2 + x + 1 = 0 \Rightarrow x = \omega$ or ω^2

Let $x = \omega$. Then,

$x + \dfrac{1}{x} = \omega + \dfrac{1}{\omega} = \omega + \omega^2 = -1$

$$x^2+\frac{1}{x^2}=\omega^2+\frac{1}{\omega^2}=\omega^2+\omega=-1$$

$$x^3+\frac{1}{x^3}=\omega^3+\frac{1}{\omega^3}=2$$

$$x^4+\frac{1}{x^4}=\omega^4+\frac{1}{\omega^4}=\omega+\omega^2=-1, \text{ etc.}$$

$$\therefore \left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^3+\frac{1}{x^3}\right)^2+\cdots+\left(x^{27}+\frac{1}{x^{27}}\right)^2$$

$$=\left[\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^4+\frac{1}{x^4}\right)^2+\cdots+\left(x^{26}+\frac{1}{x^{26}}\right)^2\right]$$

$$+\left[\left(x^3+\frac{1}{x^3}\right)^2+\left(x^6+\frac{1}{x^6}\right)^2+\left(x^9+\frac{1}{x^9}\right)\right.$$

$$\left.\cdots+\left(x^{27}+\frac{1}{x^{27}}\right)^2\right]$$

$= 18 + 9(2^2) = 54$

35. c. $E=\sum_{r=1}^{n}(ar+b)\omega^{r-1}$

$=(a+b)+(2a+b)\omega+(3a+b)\omega^2+\cdots+(na+b)\omega^{n-1}$

$=b\underbrace{(1+\omega+\omega^2+\cdots+\omega^{n-1})}_{\text{zero}}+a(1+2\omega+3\omega^2+\cdots+n\omega^{n-1})$

Now,

$S=1+2\omega+3\omega^2+\cdots+n\omega^{n-1}$

$S\omega=\omega+2\omega^2+\cdots+(n-1)\omega^{n-1}+n\omega^n$

$\therefore\quad S(1-\omega)=\underbrace{1+\omega+\omega^2+\cdots+\omega^{n-1}}_{\text{zero}}-n\omega^n=-n\ (\because \omega^n=1)$

$\Rightarrow\quad S=\dfrac{n}{\omega-1}$

$\Rightarrow\quad E=\dfrac{na}{\omega-1}$

36. b. Taking cube roots of both sides, we get

$z+ab=a(1)^{1/3}=a, a\omega, a\omega^2$

where

$\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2},\ \omega^2=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$

$\therefore\quad z_1=a-ab,\ z_2=a\omega-ab,\ z_3=a\omega^2-ab$

$|z_1-z_2|=|a(1-\omega)|$

$=|a|\left|1-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right|$

$=|a|\left|\frac{3}{2}-i\frac{\sqrt{3}}{2}\right|$

$=|a|\left(\frac{9}{4}+\frac{3}{4}\right)^{1/2}=\sqrt{3}|a|$

Similarly,

$|z_2-z_3|=|z_3-z_1|=\sqrt{3}|a|$

37. a. We have,

$z^3+2z^2+2z+1=0$

$\Rightarrow\quad (z^3+1)+2z(z+1)=0$

$\Rightarrow\quad (z+1)(z^2+z+1)=0$

$\Rightarrow\quad z=-1, \omega, \omega^2$

Since $z=-1$ does not satisfy $z^{1985}+z^{100}+1=0$ while $z=\omega, \omega^2$ satisfy it, hence sum is $\omega+\omega^2=-1$.

38. b. Let $z_1=|z_1|(\cos\theta_1+i\sin\theta_1)$. Now,

$\left|\frac{z_1}{z_2}\right|=1\Rightarrow|z_1|=|z_2|$

Also,

$\arg(z_1z_2)=0\Rightarrow\arg(z_1)+\arg(z_2)=0$

$\Rightarrow\quad \arg(z_2)=-\theta_1$

$\Rightarrow\quad z_2=|z_2|(\cos(-\theta_1)+i\sin(-\theta_1))$

$=|z_1|(\cos\theta_1-i\sin\theta_1)=\bar{z}_1$

$\Rightarrow\quad \bar{z}_2=\overline{(\bar{z}_1)}=z_1$

$\Rightarrow\quad |z_2|^2=z_1z_2$

39. d. $(x-3)^3+1=0$

$\Rightarrow\quad \left(\frac{x-3}{-1}\right)^3=1$

$\Rightarrow\quad \frac{x-3}{-1}=1, \omega, \omega^2$

$\Rightarrow\quad x=2, 3-\omega, 3-\omega^2$

Hence, the um of complex roots is $6-(\omega+\omega)^2=6+1=7$.

40. b. $x=\sqrt[3]{-1}$

$\Rightarrow\quad x^3=-1$

$\Rightarrow\quad (-x)^3=1$

$\Rightarrow\quad -x=1, \omega, \omega^2$

$\Rightarrow\quad x=-1, -\omega, -\omega^2$

$=-1, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$

$=-1, \frac{-\sqrt{3}+i}{2i}, \frac{\sqrt{3}+i}{2i}$

$=-1, \frac{-\sqrt{3}+\sqrt{-1}}{\sqrt{-4}}, \frac{\sqrt{3}+\sqrt{-1}}{\sqrt{-4}}$

41. d. $|\omega z-1-\omega^2|=a$

$\Rightarrow\quad |z+1|=a\Rightarrow|z-1+2|=a$

$\Rightarrow\quad |z-1|+2\ge a\Rightarrow 0\le a\le 4$

42. b. $|z^2-3|\ge|z|^2-3$

$\Rightarrow\quad 3|z|\ge|z|^2-3$

$\Rightarrow\quad |z|^2-3|z|-3\le 0$

$\Rightarrow\quad 0<|z|\le\frac{3+\sqrt{21}}{2}$

43. b. $|2z-1|=|z-2|$

$\Rightarrow\quad |2z-1|^2=|z-2|^2$

$\Rightarrow\quad (2z-1)(2\bar{z}-1)=(z-2)(\bar{z}-2)$

$\Rightarrow\ 4z\bar{z} - 2\bar{z} - 2z + 1 = z\bar{z} - 2\bar{z} - 2z + 4$

$\Rightarrow\ 3|z|^2 = 3$

$\Rightarrow\ |z| = 1$

Again,

$$|z_1 + z_2| = |z_1 - \alpha + z_2 - \beta + \alpha + \beta|$$
$$\le |z_1 - \alpha| + |z_2 - \beta| + |\alpha + \beta|$$
$$< \alpha + \beta + |\alpha + \beta|$$
$$= 2|\alpha + \beta| \qquad [\because \alpha, \beta > 0]$$

$$\therefore \left|\frac{z_1 + z_2}{\alpha + \beta}\right| < 2$$

$$\Rightarrow \left|\frac{z_1 + z_2}{\alpha + \beta}\right| < 2\,|z|$$

44. a. $a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 3$

$\Rightarrow\ |3| = |a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n|$

$\Rightarrow\ 3 \le |a_0|\,|z|^n + |a_1|\,|z|^{n-1} + \cdots + |a_{n-1}|\,|z| + |a_n|$

$\Rightarrow\ 3 < 2(|z|^n + |z|^{n-1} + \cdots + |z| + 1)$

$\Rightarrow\ 1 + |z| + |z|^2 + \cdots + |z|^n > \frac{3}{2}$

If $|z| \ge 1$, the inequality is clearly satisfied. For $|z| < 1$, we must have,

$$\frac{1 - |z|^{n+1}}{1 - |z|} > \frac{3}{2}$$

$\Rightarrow\ 2 - 2|z|^{n+1} > 3 - 3|z|$

$\Rightarrow\ 2|z|^{n+1} < 3|z| - 1$

$\Rightarrow\ 3|z| - 1 > 0$

$\Rightarrow\ |z| > \frac{1}{3}$

45. a. $8iz^3 + 12z^2 - 18z + 27i = 0$

$\Rightarrow\ 4iz^2(2z - 3i) - 9(2z - 3i) = 0$

$\Rightarrow\ (2z - 3i)(4iz^2 - 9) = 0$

$\Rightarrow\ z = \frac{3i}{2}$ and $z^2 = \frac{9}{4i}$

$\Rightarrow\ |z| = \frac{3}{2}$ and $|z^2| = \frac{9}{4}$

$\Rightarrow\ |z| = \frac{3}{2}$

46. a. $|z^2 + 2z\cos\alpha| \le |z^2| + |2z\cos\alpha|$

$= |z|^2 + 2|z|\,|\cos\alpha|$

$\le |z|^2 + 2|z|$

$< (\sqrt{2} - 1)^2 + 2(\sqrt{2} - 1) = 1$

47. b. $\left|z + \frac{1}{z}\right| \ge \left||z| - \frac{1}{|z|}\right|$

Hence the least value occurs when $|z| = 3$.

$$\therefore \left|z + \frac{1}{z}\right|_{\text{least}} = 3 - \frac{1}{3} = \frac{8}{3}$$

48. c. $\left|\sum_{r=1}^{n} z_r\right| \le \sum_{r=1}^{n} |z_r| \le \sum_{r=1}^{n} |z_r - r| + \sum_{r=1}^{n} r \le 2\sum_{r=1}^{n} r$

49. d. Let $z = x + iy$. Then,

$|z^2 - 1| = |z|^2 + 1$

$\Rightarrow\ |(x^2 - y^2 - 1) + 2ixy| = x^2 + y^2 + 1$

$\Rightarrow\ (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$

$\Rightarrow\ x = 0$

Hence, z lies on imaginary axis.

50. a. $|z| = 1$, let $\alpha = -1 + 3z$

$\Rightarrow\ \alpha + 1 = 3z$

$\Rightarrow\ |\alpha + 1| = 3|z| = 3$

Hence, 'α' lies on a circle centred at -1 and radius equal to 3.

51. b. Let $z = x + iy$. Then,

$x = \lambda + 3$ and $y = -\sqrt{5 - \lambda^2}$

$\Rightarrow\ (x - 3)^2 = \lambda^2$ (1)

and

$y^2 = 5 - \lambda^2$ (2)

From (1) and (2),

$(x - 3)^2 = 5 - y^2 \Rightarrow (x - 3)^2 + y^2 = 5$

Obviously it is a semicircle as $y < 0$. Hence part of the circle lies below the x-axis.

52. c.

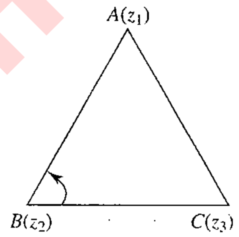


Fig. 2.58

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = e^{i\pi/4}$$

$$\angle CBA = \frac{\pi}{4}$$

Also,

$|z_1 - z_2| = |z_3 - z_2|$

Hence, ΔABC is isosceles.

53. c. Given that

$|z_1 - i| = |z_2 - i| = |z_3 - i|$

Hence, z_1, z_2, z_3 lie on the circle whose centre is i. Also given that the triangle is equilateral. Hence centroid and cirucmcentre coincides.

$$\therefore \frac{z_1 + z_2 + z_3}{3} = i$$

$\Rightarrow\ |z_1 + z_2 + z_3| = 3$

54. d.

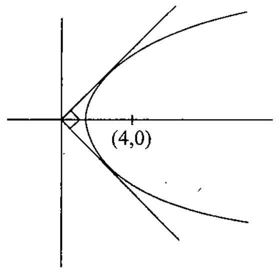


Fig. 2.59

$|z - 4| = \text{Re}(z)$

$\Rightarrow \quad \sqrt{(x-4)^2 + y^2} = x$

$\Rightarrow \quad x^2 - 8x + 16 + y^2 = x^2$

$\Rightarrow \quad y^2 = 8(x - 2)$

The given relation represents the part of the parabola with focus (4, 0) lying above x-axis and the imaginary axis as the directrix. The two tangents from directrix are at right angle. Hence greatest positive argument of z is $\pi/4$.

55. d.

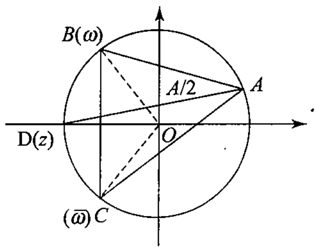


Fig. 2.60

Clearly,

$\angle DOB = \angle COD = A$

$\Rightarrow \quad z = \omega e^{iA}$ and $\bar{\omega} = ze^{iA}$ (Applying rotation about O)

$\Rightarrow \quad z^2 = \omega\bar{\omega} = 1$

$\Rightarrow \quad z = -1$ (As A and D are on opposite sides of BC)

56. b.

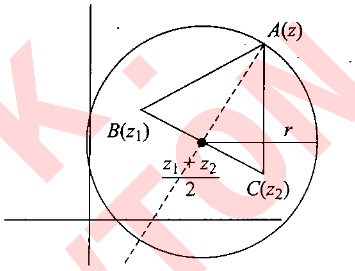


Fig. 2.61

By the given conditions, the area of the triangle ABC is given by $(1/2)|z_1 - z_2|r$.

57. d. The given equation is written as

$$\arg(z - (1 + i)) = \begin{cases} \dfrac{3\pi}{4}, & \text{when } x \le 2 \\ \dfrac{-\pi}{4}, & \text{when } x > 2 \end{cases}$$

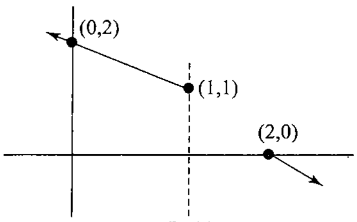


Fig. 2.62

Therefore, the locus is a set of two rays.

58. a.

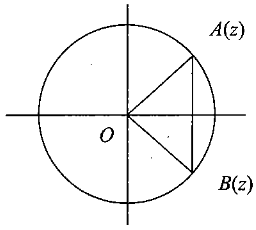


Fig. 2.63

$|z - \bar{z}|$ is the length AB while $|z|(\arg z - \arg\bar{z})$ is arc length AB.

$\therefore \quad |z - \bar{z}| \le |z|\,(\arg z - \arg\bar{z})$

59. c.

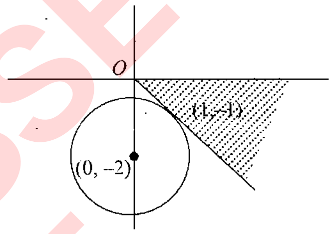


Fig. 2.64

$kz/(k + 1)$ represents any point lying on the line joining origin and z.
Given,

$$\left|\frac{kz}{k+1} + 2i\right| > \sqrt{2}$$

Hence, $kz/(k + 1)$ should lie outside the circle $|z + 2i| > \sqrt{2}$. So, z should lie in the shaded region.

$\therefore \quad -\dfrac{\pi}{4} < \arg(z) < 0$

60. b. $2\left|z - \dfrac{1}{2}\right| = |z - 1|$

$\therefore \quad \dfrac{|z-1|}{\left|z - \dfrac{1}{2}\right|} = 2$

So, locus of z is a circle.

61. d. $|z_2 + iz_1| = |z_1| + |z_2|$

$\Rightarrow \quad iz_1$, $0 + i0$ and z_2 are collinear

$\Rightarrow \quad \arg(iz_1) = \arg(z_2)$

$\Rightarrow \quad \arg(z_2) - \arg(z_1) = \dfrac{\pi}{2}$

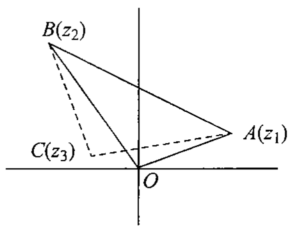


Fig. 2.65

Let,

$z_3 = \dfrac{z_2 - iz_1}{1 - i}$

$\Rightarrow \quad (1 - i)\, z_3 = z_2 - iz_1$

$\Rightarrow \quad z_2 - z_3 = i(z_1 - z_3)$

$\therefore \quad \angle ACB = \dfrac{\pi}{2}$

and

$|z_1 - z_3| = |z_2 - z_3|$

$\Rightarrow \quad AC = BC$

$\because \quad AB^2 = AC^2 + BC^2$

$\Rightarrow \quad AC = \dfrac{5}{\sqrt{2}} \qquad (\because AB = 5)$

Therefore area of ΔABC is $(1/2)AC \times BC = AC^2/2 = 25/4$ sq. units

62. a.

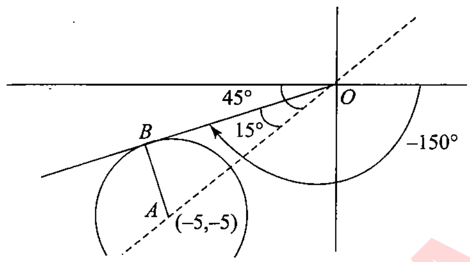


Fig. 2.66

$|2z + 10 + 10i| \le 5\sqrt{3} - 5$

$\Rightarrow \quad |z + 5 + 5i| \le \dfrac{5(\sqrt{3}-1)}{2}$

Point B has least principal argument. Now,

$AB = \dfrac{5(\sqrt{3}-1)}{2}$

$OA = 5\sqrt{2}$

$\angle AOB = \dfrac{\pi}{12}$

$\therefore \quad \arg(z) = -\dfrac{5\pi}{6}$

63. c. $z = \dfrac{at+b}{t-c} \Rightarrow t = \dfrac{b+cz}{z-a}$

Now,

$|t| = 1$

$\Rightarrow \quad \left|\dfrac{b+cz}{z-a}\right| = 1$

$\Rightarrow \quad \left|\dfrac{z+\frac{b}{c}}{z-a}\right| = \dfrac{1}{|c|} \quad (\ne 1 \text{ as } |c| \ne |t|)$

$\Rightarrow$ locus of z is a circle

64. a. Let $z = x + iy$. Then,

$|z - 3 - i| = |z - 9 - i|$

$\Rightarrow \quad \sqrt{(x-3)^2 + (y-1)^2} = \sqrt{(x-9)^2 + (y-1)^2}$

$\Rightarrow \quad x = 6$

$|z - 3 + 3i| = 3$

$\Rightarrow \quad \sqrt{(x-3)^2 + (y+3)^2} = 3$

For $x = 6$, $y = -3$.

$\therefore \quad z = 6 - 3i$

65. a. Given,

$z_k = 1 + a + a^2 + \cdots + a^{k-1} = \dfrac{1-a^k}{1-a}$

$\Rightarrow \quad z_k - \dfrac{1}{1-a} = -\dfrac{a^k}{1-a}$

$\Rightarrow \quad \left|z_k - \dfrac{1}{1-a}\right| = \dfrac{|a|^k}{|1-a|} < \dfrac{1}{|1-a|} \qquad [\because |a| < 1]$

Hence, z_k lies within the circle.

$\therefore \quad \left|z - \dfrac{1}{1-a}\right| = \dfrac{1}{|1-a|}$

66. b. $2z^2 + 2z + \lambda = 0$

Let the roots be z_1, z_2. Then,

$z_1 + z_2 = -1$ and $z_1 z_2 = \dfrac{\lambda}{2}$

0, z_1, z_2 form an equilateral triangle.

$\therefore \quad z_1^2 + z_2^2 = z_1 z_2$

$\Rightarrow \quad (z_1 + z_2)^2 = 3z_1 z_2$

$\Rightarrow \quad 1 = 3\dfrac{\lambda}{2}$

$\Rightarrow \quad \lambda = \dfrac{2}{3}$

67. a. $x + iy = 1 - t + i\sqrt{t^2 + t + 2}$

$\Rightarrow \quad x = 1 - t,\ y = \sqrt{t^2 + t + 2}$

Eliminating t,

$y^2 = t^2 + t + 2 = (1-x)^2 + 1 - x + 2 = \left(x - \dfrac{3}{2}\right)^2 + \dfrac{7}{4}$

$\Rightarrow \quad y^2 - \left(x - \dfrac{3}{2}\right)^2 = \dfrac{7}{4}$, which is a hyperbola

68. c. $z^2 + z|z| + |z|^2 = 0 \Rightarrow \left(\dfrac{z}{|z|}\right)^2 + \dfrac{z}{|z|} + 1 = 0$

$\Rightarrow \quad \dfrac{z}{|z|} = \omega, \omega^2 \Rightarrow z = \omega |z|$ or $z = \omega^2 |z|$

$\Rightarrow \quad x + iy = |z|\left(\dfrac{-1}{2} + \dfrac{i\sqrt{3}}{2}\right)$ or $x + iy = |z|\left(\dfrac{-1}{2} - \dfrac{i\sqrt{3}}{2}\right)$

$\Rightarrow \quad x = -\dfrac{1}{2}|z|,\ y = |z|\dfrac{\sqrt{3}}{2}$ or $x = -\dfrac{|z|}{2},\ y = -\dfrac{|z|\sqrt{3}}{2}$

$\Rightarrow \quad y + \sqrt{3}x = 0$ or $y - \sqrt{3}x = 0 \Rightarrow y^2 - 3x^2 = 0$

69. c. $S_1 = \Sigma z_1 = -3a$, $S_2 = \Sigma z_1 z_2 = 3b$

Since the triangle is equilateral, we have

$\Sigma z_1^2 = \Sigma z_1 z_2$

$\Rightarrow \quad (\Sigma z_1)^2 - 2\Sigma z_1 z_2 = \Sigma z_1 z_2$

$\Rightarrow \quad (\Sigma z_1)^2 = 3\Sigma z_1 z_2$

$\Rightarrow \quad (-3a)^2 = 3(3b)$

$\Rightarrow \quad 9a^2 = 9b$

$\Rightarrow \quad a^2 = b$

70. b.

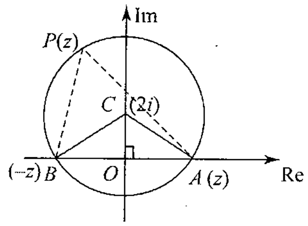


Fig. 2.67

$CA = CB = 2\sqrt{2},\ OC = 2$

$\Rightarrow \quad OA = OB = 2$

$\Rightarrow \quad A \equiv 2 + 0i,\ B = -2 + 0i$

Clearly,

$\angle BCA = \pi/2$

$\Rightarrow \quad \angle BPA = \pi/4$

$$\Rightarrow \quad \arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$$

71. a.

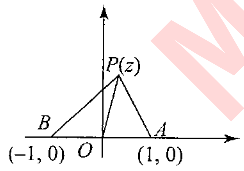


Fig. 2.68

When $|z - 1| < |z + 1|$ (or $x > 0$)

$|z| = |z - 1|$

$\Rightarrow \quad x^2 + y^2 = (x - 1)^2 + y^2$

$\Rightarrow \quad x = 1/2$

$\Rightarrow \quad z + \bar{z} = 1$

When $|z - 1| > |z + 1|$ (or $x < 0$),

$|z| = |z + 1|$

$\Rightarrow \quad x^2 + y^2 = (x + 1)^2 + y^2$

$\Rightarrow \quad x = -1/2$

$\Rightarrow \quad z + \bar{z} = -1$

72. b. Let $z = x + iy$. Then,

$$\text{Re}\left(\frac{1}{z}\right) = k$$

$$\Rightarrow \quad \text{Re}\left(\frac{1}{x+iy}\right) = k$$

$$\Rightarrow \quad \text{Re}\left(\frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}\right) = k$$

$$\Rightarrow \quad \frac{x}{x^2+y^2} = k$$

$$\Rightarrow \quad x^2 + y^2 - \frac{1}{k}x = 0$$

which is a circle.

73. b.

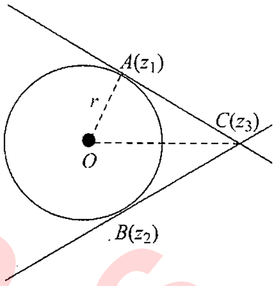


Fig. 2.69

As ΔOAC is a right-angled triangle with right angle at A, so

$|z_1|^2 + |z_3 - z_1|^2 = |z_3|^2$

$\Rightarrow \quad 2|z_1|^2 - \bar{z}_3 z_1 - \bar{z}_1 z_3 = 0$

$$\Rightarrow \quad 2\bar{z}_1 - \bar{z}_3 - \frac{\bar{z}_1}{z_1} z_3 = 0 \qquad (1)$$

Similarly,

$$2\bar{z}_2 - \bar{z}_3 - \frac{\bar{z}_2}{z_2} z_3 = 0 \qquad (2)$$

Subtracting (2) from (1), we get

$$2(\bar{z}_2 - \bar{z}_1) = z_3\left(\frac{\bar{z}_1}{z_1} - \frac{\bar{z}_2}{z_2}\right)$$

$$\Rightarrow \quad \frac{2r^2(z_1 - z_2)}{z_1 z_2} = z_3 r^2\left(\frac{z_2^2 - z_1^2}{z_1^2 z_2^2}\right) \qquad [\because |z_1|^2 = |z_2|^2 = r^2]$$

$$\Rightarrow \quad z_3 = \frac{2z_1 z_2}{z_2 + z_1}$$

74. a. $|z_1| = |z_2| = |z_3| = 1$

Hence, the circumcentre of triangle is origin. Also, centroid $(z_1 + z_2 + z_3)/3 = 0$, which coincides with the circumcentre. So the triangle is equilateral. Since radius is 1, length of side is $a = \sqrt{3}$. Therefore, the area of the triangle is $(\sqrt{3}/4)a^2 = (3\sqrt{3}/4)$.

75. a.

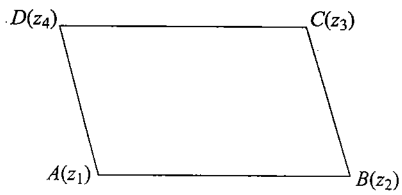


Fig. 2.70

The first condition implies that $(z_1 + z_3)/2 = (z_2 + z_4)/2$, i.e., diagonals AC and BD bisect each other. Hence, quadrilateral is a parallelogram. The second condition implies that the angle between AD and AB is $90°$. Hence the parallelogram is a rectangle.

76. b. Given that

$$\arg\left(\frac{z_1 - \dfrac{z}{|z|}}{\dfrac{z}{|z|}}\right) = \frac{\pi}{2}$$

and

$$\left|\frac{z}{|z|} - z_1\right| = 3$$

from which we can establish the following geometry.

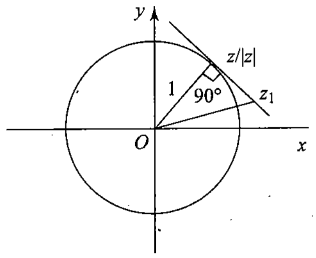


Fig. 2.71

From the diagram,

$$\left|\frac{z}{|z|}-z_1\right|=3, |z_1|=\sqrt{9+1}=\sqrt{10}$$

77. b. Note that $z_1 = 3+\sqrt{3}i$ lies on the line $y=(1/\sqrt{3})x$ and $z_2 = 2\sqrt{3}+6i$ lies on the line $y=\sqrt{3}x$.

Hence $z = 5+5i$ will only lie on the bisector of z_1 and z_2, i.e., $y = x$.

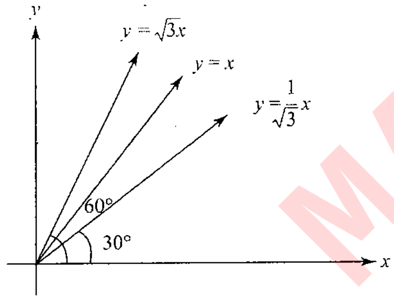


Fig. 2.72

78. a. $\log_{1/3}\left(\frac{|z-3|^2+2}{11|z-3|-2}\right)>1$

$$\Rightarrow \frac{|z-3|^2+2}{11|z-3|-2}<\frac{1}{3}$$

$\Rightarrow (3t-8)(t-1)<0$ (where $|z-3|=t$)

$\Rightarrow 1<|z-3|<8/3$

Hence, z lies between the two concentric circles.

79. c. We have,

$$|(x-2)+i(y-1)|=|z|\left|\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta\right|$$

where $\theta = \arg z$.

$$\sqrt{(x-2)^2+(y-1)^2}=\frac{1}{\sqrt{2}}|x-y|,$$

which is a parabola

80. a. We have,

$|z-2+2i|=1$

$\Rightarrow |z-(2-2i)|=1$

Hence, z lies on a circle having centre at $(2, -2)$ and radius 1. It is evident from the figure that the required complex number z is given by the point P. We find that OP makes an angle $\pi/4$ with OX and

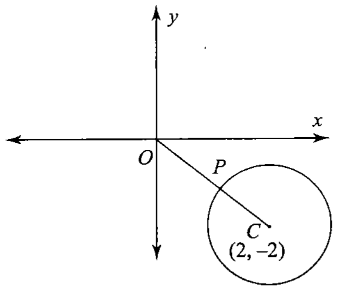


Fig. 2.73

$$OP = OC - CP = \sqrt{2^2+2^2}-1 = 2\sqrt{2}-1$$

So, coordinates of P are $[(2\sqrt{2}-1)\cos(\pi/4), -(2\sqrt{2}-1)\sin(\pi/4)]$, i.e., $((2-1/\sqrt{2}), -(2-1/\sqrt{2}))$. Hence,

$$z=\left(2-\frac{1}{\sqrt{2}}\right)+\left\{-\left(2-\frac{1}{\sqrt{2}}\right)\right\}i=\left(2-\frac{1}{\sqrt{2}}\right)(1-i)$$

81. d. Given,

$$z=\frac{3}{2+\cos\theta+i\sin\theta}$$

$$\Rightarrow \cos\theta+i\sin\theta=\frac{3}{z}-2=\frac{3-2z}{z}$$

$$\Rightarrow 1=\frac{|3-2z|}{|z|} \text{ [taking modulus]}$$

$$\Rightarrow \frac{\left|z-\frac{3}{2}\right|}{|z|}=\frac{1}{2}$$

Hence, locus of z is a circle.

82. d. We have,

$$e^{\frac{2\pi ri}{p}}=e^{\frac{2\pi m}{q}}$$

$r = 0, 1, \ldots, p-1$

$m = 0, 1, \ldots, q-1$

This is possible iff $r = m = 0$ but for $r = m = 0$ we get 1 which is not an imaginary number.

83. b. Let,

$S = 1 + 2a + 3a^2 + \cdots + na^{n-1}$

$\Rightarrow aS = a + 2a^2 + 3a^3 + \cdots + (n-1)a^{n-1} + na^n$

On subtracting, we get

$$S(1-\alpha)=1+[\alpha+\alpha^2+\alpha^{n-1}]-n\alpha^n$$

$$=1+\frac{\alpha(1-\alpha^{n-1})}{1-\alpha}-n\alpha^n$$

$$\Rightarrow S=\frac{1}{1-\alpha}+\frac{\alpha-\alpha^n}{(1-\alpha)^2}-\frac{n\alpha^n}{1-\alpha} \quad [\because \alpha^n=1]$$

$$=\frac{1}{1-\alpha}+\frac{\alpha-1}{(1-\alpha)^2}-\frac{n}{1-\alpha}=-\frac{n}{1-\alpha}$$

84. a. $\left(\frac{1+ia}{1-ia}\right)^4=z$

$$\Rightarrow \left|\frac{1+ia}{1-ia}\right|^4=|z|$$

$$\Rightarrow \left|\frac{a-i}{a+i}\right|^4=1$$

$\Rightarrow |a-i|=|a+i|$

Therefore, a lies on the perpendicular bisector of i and $-i$, which is real axis. Hence all the roots are real.

85. b. For $z \neq 1$, the given equation can be written as

$$\left(\frac{z+1}{z-1}\right)^5=1$$

$$\Rightarrow \frac{z+1}{z-1}=e^{2k\pi i/5}$$

where $k = -2, -1, 1, 2$.

If we denote this value of z by z_k, then

$$z_k = \frac{e^{2k\pi i/5}+1}{e^{2k\pi i/5}-1}$$

$$= \frac{e^{k\pi i/5}+e^{-k\pi i/5}}{e^{k\pi i/5}-e^{-k\pi i/5}}$$

$$= -i\cot\left(\frac{k\pi}{5}\right), k = -2, -1, 1, 2$$

Therefore, roots of the equation are $\pm i \cot\ (\pi/5), \pm i \cot\ (2\pi/5)$.

86. a. We have,

$\log z + \log z^2 + \log z^3 + \cdots + \log z^n = 0$

$\Rightarrow \log (z\, z^2\, z^3 \cdots z^n) = 0$

$\Rightarrow \log\left(z^{\frac{n(n+1)}{2}}\right) = 0$

$\Rightarrow z^{\frac{n(n+1)}{2}} = 1$

$\Rightarrow z = 1^{\frac{2}{n(n+1)}}$

$= (\cos 0° + i \sin 0°)^{\frac{2}{n(n+1)}}$

$= (\cos 2m\pi + i \sin 2m\pi)^{\frac{2}{n(n+1)}}$, $m = 0, 1, 2, 3, \ldots$

$= \cos\dfrac{4m\pi}{n(n+1)} + i\sin\dfrac{4m\pi}{n(n+1)}$, $m = 0, 1, 2, \ldots$

87. d. The equation $z^n = (z + 1)^n$ will have exactly $n - 1$ roots. We have,

$\left(\dfrac{z+1}{z}\right)^n = 1$

$\Rightarrow \left|\dfrac{z+1}{z}\right| = 1$

$\Rightarrow |z + 1| = |z|$

Therefore, 'z' lies on the right bisector of the segment connecting the points (0, 0) and (–1, 0). Thus Re(z) = –1/2. Hence, roots are collinear and will have their real parts equal to –1/2.

Hence sum of the real parts of roots is (–1/2)(n –1).

88. c. $\left(\dfrac{z+1}{z}\right)^4 = 16$

$\Rightarrow \dfrac{z+1}{z} = \pm 2, \pm 2i$

The roots are $1, -1/3, (-1/5-(2/5)i)$ and $(-1/5+(2/5)i)$.

Note that (–1/3, 0) and (1, 0) are equidistant from (1/3, /0) and since it lies on the perpendicular bisector of AB, it will be equidistant from A and B also.

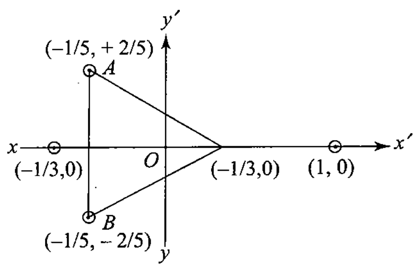


Fig. 2.74

89. d. $(z^n - 1) = (z - 1)(z - z_1)(z - z_2) \ldots (z - z_{n-1})$ (1)

Differentiating w.r.t. x, and then dividing by (1), we have

$$\frac{nz^{n-1}}{z^n-1} = \frac{1}{z-1}+\frac{1}{z-z_1}+\frac{1}{z-z_2}+\cdots+\frac{1}{z-z_{n-1}}$$

Putting $z = 3$, we get

$$\frac{n3^{n-1}}{3^n-1} = \frac{1}{2}+\frac{1}{3-z_1}+\frac{1}{3-z_2}+\cdots+\frac{1}{3-z_{n-1}}$$

$$\Rightarrow \frac{1}{3-z_1}+\frac{1}{3-z_2}+\cdots+\frac{1}{3-z_{n-1}} = \frac{n3^{n-1}}{3^n-1}-\frac{1}{2}$$

Multiple Correct Answers Type

1. a, c.

Clearly, we have to find it for real z. Let $z = x$. Then,

$|x - w| = |x - w^2| = |w - w^2|$

$$\Rightarrow \left(x+\frac{1}{2}\right)^2+\frac{3}{4} = \left|\frac{-1+\sqrt{3}i}{2}-\frac{-1-\sqrt{3}i}{2}\right|^2 = 3 \Rightarrow x+\frac{1}{2} = \pm\frac{3}{2}$$

$\Rightarrow x = 1, -2$

2. a, b, c, d.

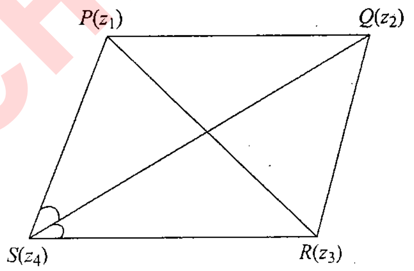


Fig. 2.75

a. $PS \parallel QR \Rightarrow \arg\left(\dfrac{z_1 - z_4}{z_2 - z_3}\right) = 0 \Rightarrow \dfrac{z_1 - z_4}{z_2 - z_3}$ is purely real

b. Since diagonal bisect the angle

$\Rightarrow \text{amp}\left(\dfrac{z_1 - z_4}{z_2 - z_4}\right) = \text{amp}\left(\dfrac{z_2 - z_4}{z_3 - z_4}\right)$

c. Diagonals of rhombus are perpendicular. Hence, $(z_1 - z_3)/(z_2 - z_4)$ is purely imaginary.

d. Diagonals of rhombus are not equal. Hence, $|z_1 - z_3| \neq |z_2 - z_4|$.

3. a, b, d.

Let $z = \alpha$ be a real root. Then,

$\alpha^3 + (3 + 2i)\alpha + (-1 + ia) = 0$

$\Rightarrow (\alpha^3 + 3\alpha - 1) + i(a + 2\alpha) = 0$

$\Rightarrow \alpha^3 + 3\alpha - 1 = 0$ and $\alpha = -a/2$

$\Rightarrow -\dfrac{a^3}{8} - \dfrac{3a}{2} - 1 = 0$

$\Rightarrow a^3 + 12a + 8 = 0$

Let $f(a) = a^3 + 12a + 8$.

$\therefore f(-1) < 0, f(0) > 0, f(-2) < 0, f(1) > 0$ and $f(3) > 0$

Hence, $a \in (-1, 0)$ or $a \in (-2, 1)$ or $a \in (-2, 3)$.

4. b, c.

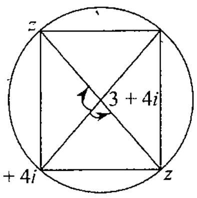


Fig. 2.76

Clearly, the inscribed rectangle is a square. Let the adjacent vertex be z. Then,

$$\frac{3+4i-(z)}{(3+4i)-(4+4i)} = e^{\pm i\pi/2} \text{ (by rotation about center)}$$

$\Rightarrow \quad 3+4i-z = \pm i(-1)$

$\Rightarrow \quad z = 3+4i \pm i = 3+5i$ or $3+3i$

5. a, d.

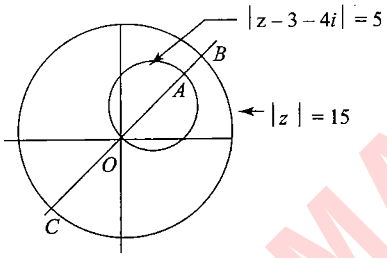


Fig. 2.77

We have,

$|z_1| = 15, |z_2 - 3 - 4i| = 5$

Minimum value of $|z_1 - z_2|$ is $AB = OB - OA = 15 - 10 = 5$. Maximum value of $|z_1 - z_2|$ is $CA = OA + OC = 10 + 15 = 25$.

6. a, c.

Triangle ABC is equilateral. Hence,

$z^2 + (-z)^2 + (1-z)^2 = z(-z) + z(1-z) + (-z)(1-z)$

$\Rightarrow \quad 3z^2 - 2z + 1 = -z^2$

$\Rightarrow \quad 4z^2 - 2z + 1 = 0$

sum of roots = 2

and product of roots is $= \dfrac{1}{4}$

7. b, c.

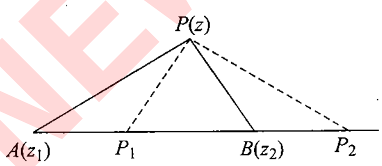


Fig. 2.78

Let internal and external bisectors of $\angle APB$ meet the line joining A and B at P_1 and P_2, respectively. Hence,

$AP_1 : P_1B \equiv PA : PB \equiv 3:1$ (internal division)

$AP_2 : P_2B \equiv PA : PB \equiv 3:1$ (external division)

Thus, P_1 and P_2 are fixed points. Also,

$\angle P_1PP_2 = \dfrac{\pi}{2}$

Thus 'P' lies on a circle having P_1P_2 as its diameter. Clearly, $B(z_2)$ lies inside this circle.

8. a, d.

Refer the figure, z lies on the point of intersection of the rays from A and B. $\angle ACB$ is a right angle and OBC is an equilateral triangle. Hence,

$$OC = a \Rightarrow z = a\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

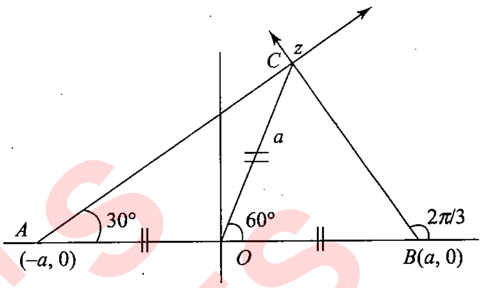


Fig. 2.79

9. a, c.

$(-i)^{1/3} = (i^3)^{1/3} = i, i\omega, i\omega^2$

where

$$\omega = \frac{-1+\sqrt{3}i}{2}$$

Hence roots are i, $(-\sqrt{3}-i)/2, (\sqrt{3}-i)/2$.

10. a, b, d.

$x^n - 1 = (x-1)(x-z_1)(x-z_2)\cdots(x-z_{n-1})$

$\Rightarrow \quad \dfrac{x^n-1}{x-1} = (x-z_1)(x-z_2)\cdots(x-z_{n-1})$

Putting $x = \omega$, we have

$$\prod_{r=1}^{n-1}(\omega - z_r) = \frac{\omega^n - 1}{\omega - 1} = \begin{cases} 0, & \text{if } n = 3k,\ k \in Z \\ 1, & \text{if } n = 3k+1,\ k \in Z \\ 1+\omega, & \text{if } n = 3k+2,\ k \in Z \end{cases}$$

11. a, d.

If α is a real root, then

$\alpha^3 + (3+i)\alpha^2 - 3\alpha - (m+i) = 0$

$\therefore \quad \alpha^3 + 3\alpha^2 - 3\alpha - m = 0$ and $\alpha^2 - 1 = 0$

$\Rightarrow \quad \alpha = 1$ or -1

$\alpha = 1 \Rightarrow m = 1$

$\alpha = -1 \Rightarrow m = 5$

12. c, d.

We have,

$x^2 + x + 1 = (x-\omega)(x-\omega^2)$

Since $f(x)$ is divisible by x^2+x+1, $f(\omega) = 0, f(\omega^2) = 0$, so

$P(\omega^3) + \omega Q(\omega^3) = 0 \Rightarrow P(1) + \omega Q(1) = 0 \quad (1)$

$P(\omega^6) + \omega^2 Q(\omega^6) = 0 \Rightarrow P(1) + \omega^2 Q(1) = 0 \quad (2)$

Solving (1) and (2), we obtain

$P(1) = 0$ and $Q(1) = 0$

Therefore, both $P(x)$ and $Q(x)$ are divisible by $x - 1$. Hence, $P(x^3)$ and $Q(x^3)$ are divisible by $x^3 - 1$ and so by $x - 1$ Since $f(x) = P(x^3) + xQ(x^3)$, we get $f(x)$ is divisible by $x - 1$.

13. b, c.

$\text{amp}(z_1z_2) = 0 \Rightarrow \text{amp } z_1 + \text{amp } z_2 = 0$

$\therefore \quad \text{amp } z_1 = -\text{amp } z_2 = \text{amp } \bar{z}_2$

Since $|z_1| = |z_2|$, we get $|z_1| = |\bar{z}_2|$. So, $z_1 = \bar{z}_2$. Also, $z_1z_2 = \bar{z}_2z_2 = |z_2|^2 = 1$ because $|z_2| = 1$.

14. a, b.

$$\left|z-\frac{1}{z}\right|=1$$

$\Rightarrow \quad 1\ge\left||z|-\frac{1}{|z|}\right|$

$\Rightarrow \quad -1\le|z|-\frac{1}{|z|}\le 1$

$\Rightarrow \quad -|z|\le|z|^2-1\le|z|$

From $|z|^2-1\ge-|z|$, we get

$|z|^2+|z|-1\ge 0$

$\Rightarrow \quad |z|\ge\frac{-1+\sqrt{5}}{2}$ (1)

From $|z|^2-1\le|z|$, we get

$|z|^2-|z|-1\le 0$

$\Rightarrow \quad \frac{1-\sqrt{5}}{2}\le|z|\le\frac{1+\sqrt{5}}{2}$ (2)

From (1) and (2), we get

$\Rightarrow \quad \frac{-1+\sqrt{5}}{2}\le|z|\le\frac{1+\sqrt{5}}{2}$

$\Rightarrow \quad |z|_{min}=\frac{\sqrt{5}-1}{2},\ |z|_{max}=\frac{1+\sqrt{5}}{2}$

15. a, b, c.

$z^n\cos\theta_0+z^{n-1}\cos\theta_1+\cdots+z\cos\theta_{n-1}+\cos\theta_n=2$

$\Rightarrow \quad 2=|z_0^n\cos\theta_0+z_0^{n-1}\cos\theta_1+\cdots+z_0\cos\theta_{n-1}+\cos\theta_n|$

$\Rightarrow \quad 2\le|z_0|^n|\cos\theta_0|+|z|^{n-1}|\cos\theta_1|+\cdots+|z_0||\cos\theta_{n-1}|+|\cos\theta_n|$

$\Rightarrow \quad 2\le|z_0|^n+|z_0|^{n-1}+|z_0|^{n-2}+\cdots+|z_0|+1$

which is clearly satisfied for $|z_0|\ge1$. If $|z_0|<1$, then

$2<1+|z_0|+|z_0|^2+\cdots+|z|^n+\cdots\infty$

$\Rightarrow \quad 2<\frac{1}{1-|z_0|}$

$\Rightarrow \quad |z_0|>\frac{1}{2}$

16. a, b, c, d.

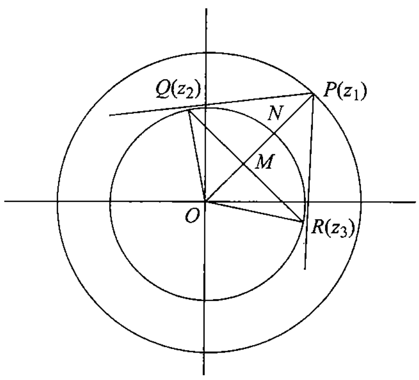


Fig. 2.80

Since $OQ=1$ and $OP=2$, so $\sin(\angle OPQ)=1/2$ and hence $\angle QPR=\pi/3$. Then $\angle PQR$ is equilateral. Also, $OM\perp QR$. Then from $\angle OMQ$, $OM=1/2$. Hence $MN=1/2$. Then centroid of $\angle PQR$ lies on $|z|=1$.

As PQR is an equilateral triangle, so orthocentre, circumcentre and centroid will coincide. Now,

$$\left|\frac{z_1+z_2+z_3}{3}\right|=1$$

$\Rightarrow \quad |z_1+z_2+z_3|^2=9$

$\Rightarrow \quad (z_1+z_2+z_3)(\bar{z}_1+\bar{z}_2+\bar{z}_3)=9$

$\Rightarrow \quad \left(\frac{4}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\frac{1}{\bar{z}_3}\right)\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)=9$

and

$\angle QOR=120°$

17. a, d. $|z_1^2-z_2^2|=|\bar{z}_1^2+\bar{z}_1^2-2\bar{z}_1\bar{z}_2|$

$\Rightarrow \quad |z_1-z_2||z_1+z_2|=|\bar{z}_1-\bar{z}_2|^2$

$\Rightarrow \quad |z_1+z_2|=|\bar{z}_1-\bar{z}_2|$

$|z_1+z_2|=|z_1-z_2|$

$\Rightarrow \quad \left|\frac{z_1}{z_2}+1\right|=\left|\frac{z_1}{z_2}-1\right|$

$\Rightarrow \quad \frac{z_1}{z_2}$ lies on $\perp$ bisector of 1 and -1

$\Rightarrow \quad \frac{z_1}{z_2}$ lies on imaginary axis

$\Rightarrow \quad \frac{z_1}{z_2}$ is purely imaginary

$\Rightarrow \quad \arg\left(\frac{z_1}{z_2}\right)=\pm\frac{\pi}{2}$

$|\arg(z_1)-\arg(z_2)|=\frac{\pi}{2}$

18. a, c, d.

$z'=ze^{i\alpha}$ (1)

$z''=ze^{-i\alpha}$ (2)

$\therefore \quad z'z''=z^2$

$\Rightarrow \quad z', z, z''$ are in G.P.

Also,

$$\left(\frac{z'}{z}\right)^2+\left(\frac{z''}{z}\right)^2=2\cos 2\alpha$$

$\Rightarrow \quad z'^2+z''^2=2z^2\cos2\alpha$

$\Rightarrow \quad z'^2+z''^2=2z^2(2\cos^2\alpha-1)$

$\Rightarrow \quad z'^2+z''^2+2z^2=4z^2\cos^2\alpha$

$\Rightarrow \quad z'^2+z''^2+2z'z''=4z^2\cos^2\alpha$

$\Rightarrow \quad (z'+z'')^2=4z^2\cos^2\alpha$

$\Rightarrow \quad z'+z''=2z\cos\alpha$

19. a, c. If $p = q$, then equation becomes $z^p = \bar{z}^q$ and it has infinite number of solutions because any $z \in R$ will satisfy it. If $p \neq q$, let $p > q$, then $z^p = \bar{z}^q$.

$\therefore \quad |z|^p = |z|^q$

$\Rightarrow \quad |z|^p (|z|^{p-q} - 1) = 0$

$\Rightarrow \quad |z| = 0$ or $|z| = 1$

$|z| = 0 \Rightarrow z = 0 + i0$

$|z| = 1 \Rightarrow z = e^{i\theta}$

$\Rightarrow \quad e^{(p+q)\theta i} = 1$

$\Rightarrow \quad z = 1^{1/(p+q)}$

Therefore, the number of solutions is $p + q + 1$.

20. a, b, d.

$z_1 = 5 + 12i, |z_2| = 4$

$|z_1 + iz_2| \le |z_1| + |z_2| = 13 + 4 = 17$

$\therefore \quad |z_1 + (1+i) z_2| \ge ||z_1| - |1+i|\,|z_2||$

$= 13 - 4\sqrt{2}$

$\therefore \quad \min(|z_1 + (1+i)z_2|) = 13 - 4\sqrt{2}$

$$\left|z_2 + \frac{4}{z_2}\right| \le |z_2| + \frac{4}{|z_2|} = 4 + 1 = 5$$

$$\left|z_2 + \frac{4}{z_2}\right| \ge |z_2| - \frac{4}{|z_2|} = 4 - 1 = 3$$

$$\therefore \quad \max\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{3} \text{ and } \min\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{5}$$

21. a, c.

$$\begin{aligned} p + q + r &= a + b\omega + c\omega^2 \\ &+ b + c\omega + a\omega^2 \\ &+ c + a\omega + b\omega^2 \end{aligned}$$

$$\therefore \quad p + q + r = (a + b + c)(1 + \omega + \omega^2) = 0 \qquad (1)$$

p, q, r lie on the circle $|z| = 2$, whose circumcentre is origin. Also, $(p + q + r)/3 = 0$. Hence the cenroid coincides with circumcentre. So, the triangle is equilateral. Now,

$(p + q + r)^2 = 0.$

$$\Rightarrow \quad p^2 + q^2 + r^2 = -2pqr\left[\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right]$$

$$= -2pqr\left[\frac{1}{a+b\omega+c\omega^2} + \frac{1}{b+c\omega+a\omega^2} + \frac{1}{c+a\omega+b\omega^2}\right]$$

$$= -2pqr\left[\frac{1}{\omega^2(a\omega+b\omega^2+c)} + \frac{1}{\omega(b\omega^2+c+a\omega)} + \frac{1}{c+a\omega+b\omega^2}\right]$$

$$= \frac{-2pqr}{a\omega + b\omega^2 + c}\left[\frac{1}{\omega^2} + \frac{1}{\omega} + \frac{1}{1}\right] = 0 \qquad (2)$$

Hence,

$$p^2 + q^2 + r^2 = 2(pq + qr + rp)$$

22. a, b, c, d.

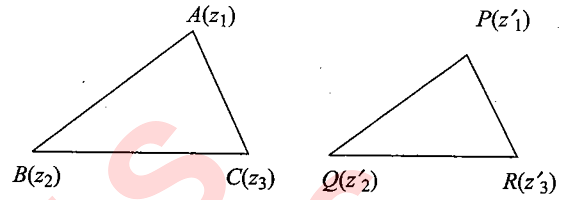


Fig. 2.81

$$z_3 = (1-\lambda)z_1 + z_2 = \frac{(1-\lambda)z_1 + \lambda z_2}{1 - \lambda + \lambda}$$

Hence, z_3 divides the line joining $A(z_1)$ and $B(z_2)$ in the ratio $\lambda:(1-\lambda)$. That means the given points are collinear. Also, the ratio $\lambda/(1-\lambda) > 0$ (or $0 < \lambda < 1$) if z_3 divides the line joining z_1 and z_2 internally and $\mu/(1-\mu) < 0$ (or $\mu < 0$ or $\mu > 1$) if z_3' divides the line joining z_1', z_2' externally.

When λ, μ are complex numbers, where $\lambda = \mu$, we have $z_3 = (1-\lambda)z_1 + \lambda z_2$ and $z_3' = (1-\lambda)z_1' + \lambda z_2'$. Comparing the value of λ, we have

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{z_3' - z_1'}{z_2' - z_1'}$$

$$\Rightarrow \quad \left|\frac{z_3 - z_1}{z_2 - z_1}\right| = \left|\frac{z_3' - z_1'}{z_2' - z_1'}\right| \text{ and } \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \arg\left(\frac{z_3' - z_1'}{z_2' - z_1'}\right)$$

$$\Rightarrow \quad \frac{AC}{AB} = \frac{PR}{PQ} \text{ and } \angle BAC = \angle QPR$$

Hence, triangles ABC and PQR are similar.

23. a, d.

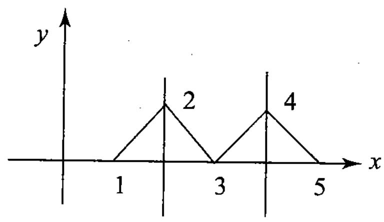


Fig. 2.82

24. a, c.

Given,

$z^n = (z+1)^n \Rightarrow |z^n| = |(z+1)^n|$

$\therefore \quad |z|^n = |z+1|^n \Rightarrow |z| = |z+1|$

$\Rightarrow \quad |z|^2 = |z+1|^2$

$\Rightarrow \quad x^2 + y^2 = (x+1)^2 + y^2$, where $z = x + iy$

$$\Rightarrow \quad x = -\frac{1}{2}$$

Hence, z lies on the line $x = -1/2$. Hence sum of real parts of the roots is $-(n-1)/2$ (since equation has $n - 1$ roots).

25. a, b, d.

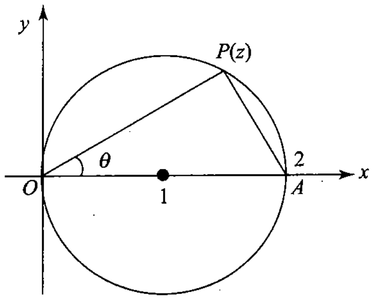


Fig. 2.83

Since arg $((z - 1 - i)/z)$ is the angle subtended by the chord joining the points O and $1 + i$ at the circumference of the circle $|z - 1| = 1$, so it is equal to $-\pi/4$. The line joining the points $z = 0$ and $z = 2 + 0i$ is the diameter.

$$\arg\frac{z-2}{z} = \pm\frac{\pi}{2}$$

$\Rightarrow$ $\dfrac{z-2}{z-0}$ is purely imaginary

We have,

$$\angle OPA = \frac{\pi}{2}$$

$$\Rightarrow \arg\left(\frac{2-z}{0-z}\right) = \frac{\pi}{2} \Rightarrow \frac{z-2}{z} = \frac{AP}{OP}i$$

Now in ΔOAP,

$$\tan\theta = \frac{AP}{OP}$$

Thus.

$$\frac{z-2}{z} = i\tan\theta$$

26. a, b, d.

$$\left|\frac{2z-i}{z+1}\right| = m \Rightarrow \left|z - \frac{i}{2}\right| = \frac{m}{2}|z+1|$$

This shows that the given equation will represent a circle, if $m/2 \neq 1$, i.e., $m \neq 2$.

27. a, d.

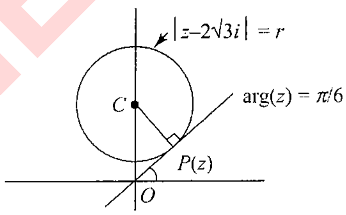


Fig. 2.84

$$CP = r,\ OC = 2\sqrt{3},\ \angle COP = \pi/3$$

$$\Rightarrow CP = OC\sin\frac{\pi}{3} = 2\sqrt{3}\ \frac{\sqrt{3}}{2} = 3$$

Thus, when $r = 3$, the circle touches the line. Hence, for two distinct points of intersection $3 < r < 2\sqrt{3}$.

28. b, c.

We have,

$$\left|\frac{1}{z_2} + \frac{1}{z_1}\right| = \left|\frac{1}{z_2} - \frac{1}{z_1}\right| \Rightarrow |z_1 + z_2| = |z_1 - z_2|$$

Squaring both sides, we have

$$|z_1|^2 + |z_2|^2 + 2\left(z_1\bar{z}_2 + \bar{z}_1 z_2\right) = |z_1|^2 + |z_2|^2 - 2\left(z_1\bar{z}_2 + \bar{z}_1 z_2\right)$$

$$\Rightarrow 4\left(z_1\bar{z}_2 + \bar{z}_1 z_2\right) = 0$$

$$\Rightarrow \frac{z_1}{z_2} + \frac{\bar{z}_1}{\bar{z}_2} = 0$$

$$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} = \arg\left(\frac{z_1 - 0}{z_2 - 0}\right)$$

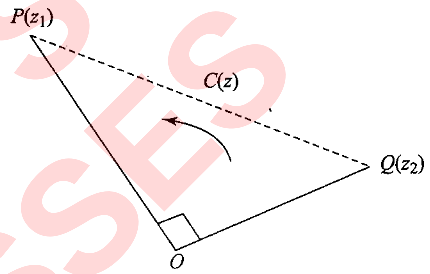


Fig. 2.85

That is angle between z_2, O and z_1 is a right angle, taken in order, as shown in the above diagram. Now, the circumcentre of the above diagram will lie on the line PQ as diameter and is represented by C which is the centre of PQ, such that $z = (z_1 + z_2)/2$, where z is the affix of circumcentre.

29. a, c, d.

Choice (a) on simplification gives

$$z = \frac{1+x}{1+x^2} + i\frac{1+x}{1+x^2}$$

For $x = 0.5$, $f(0.5) > 1$ which is out of range, Hence, (a) is not correct. From choice (b),

$$z = \frac{1-x}{1+x^2} + i\frac{1-x}{1+x^2}$$

$f(x)$ and $g(x) \in (0, 1)$ if $x \in (0, 1)$. Hence, (b) is correct. From choice (c),

$$z = \frac{1+x}{1+x^2} + \frac{1-x}{1+x^2}i$$

Hence, (c) is not correct. From choice (d),

$$z = \frac{1-x}{1+x^2} + \frac{1+x}{1+x^2}i$$

Hence, (d) is not correct.

30. a, b, c.

Let $z = x + iy$ where x, y satisfy the given equation. Hence,

$$(x^2 + y^2)(x^2 - y^2) = 175$$

$\Rightarrow$ $x^2 + y^2 = 25$ and $x^2 - y^2 = 7$ (as all other possibilities will give non-integral solutions)

Hence, possible values of z will be $4 + 3i$, $4 - 3i$, $-4 + 3i$ and $-4 - 3i$. Clearly, it will form a rectangle having length of the diagonal 10.

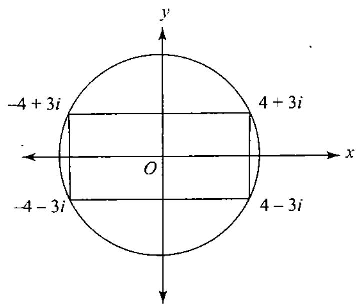


Fig. 2.86

From the diagram, options (a), (b), (c) are correct.

31. a, b.

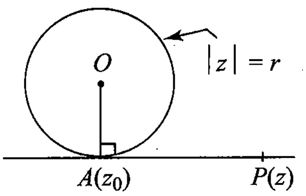


Fig. 2.87

$\angle OAP = \dfrac{\pi}{2}$

$\Rightarrow \quad \dfrac{z - z_0}{z_0}$ is purely imaginary

$\Rightarrow \quad \dfrac{z - z_0}{z_0} + \dfrac{\bar{z} - \bar{z}_0}{\bar{z}_0} = 0$

$\Rightarrow \quad \dfrac{z}{z_0} + \dfrac{\bar{z}}{\bar{z}_0} = 2$ **(1)**

$\Rightarrow \quad \text{Re}\left(\dfrac{z}{z_0}\right) = 1$

From (1),

$z\bar{z}_0 + z_0\bar{z} = 2|z_0|^2 = 2r^2$

32. b., c. z_1 and z_2 are the roots of the equation $z^2 - az + b = 0$. Hence,

$z_1 + z_2 = a, z_1z_2 = b$

Now,

$|z_1 + z_2| \le |z_1| + |z_2|$

$\Rightarrow \quad |z_1 + z_2| = |a| \le 1 + 1 = 2 \qquad (\because |z_1| = |z_2| = 1)$

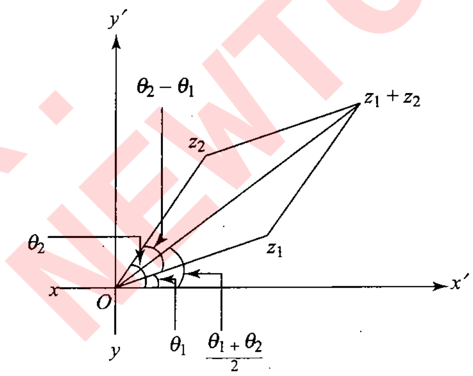


Fig. 2.88

$\Rightarrow \quad \arg(a) = \dfrac{1}{2}[\arg(z_2) + \arg(z_1)]$

Also,

$\arg(b) = \arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

$\Rightarrow \quad 2\arg(a) = \arg(b)$

33. a, b, d. Given,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$\Rightarrow \quad a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow \quad (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow \quad \dfrac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$

$\Rightarrow \quad (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Rightarrow \quad a = b = c \qquad [\because a + b + c \ne 0, \because z_1 \ne 0, \therefore |z_1| = a \ne 0 \text{ etc.}]$

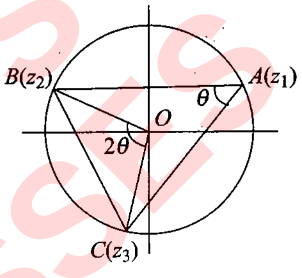


Fig. 2.89

Hence, $OA = OB = OC$, where O is the origin and A, B, C are the points representing z_1, z_2 and z_3, respectively. Therefore, O is circumcentre of $\angle ABC$. Now,

$\arg\left(\dfrac{z_3}{z_2}\right) = \angle BOC \qquad$ (i)

$= 2\angle BAC = 2\arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) \qquad$ (ii)

$= \arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)^2 \qquad [\because \angle BOC = 2\angle BAC]$

Hence,

$\arg\left(\dfrac{z_3}{z_2}\right) = \arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)^2$

Also, centroid is $(z_1 + z_2 + z_3)/3$. Since $HG:GO \equiv 2:1$ (where H is orthocentre and G is centroid), then orthocentre is $z_1 + z_2 + z_3$ (by section formula). When triangle is equilateral centroid coincides with circumcentre; hence $z_1 + z_2 + z_3 = 0$.

Also, the area for equilateral triangle is $(\sqrt{3}/4)L^2$, where L is length of side. Since radius is $|z_1|$, $L = \sqrt{3}|z_1|$, hence area is $(3\sqrt{3}/4)|z_1|^2$.

34. a, c, d.

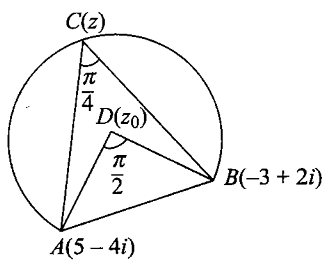


Fig. 2.90

$$\frac{z_0-(-3+2i)}{z_0-(5-4i)}=\frac{BD}{AD}e^{i\pi/2}=i$$

$\Rightarrow$ $z_0+3-2i=iz_0-5i-4$

$\Rightarrow$ $z_0=-2-5i$

$\Rightarrow$ Radius $AD=|5-4i-(-2-5i)|$

$=|7+i|$

$=\sqrt{50}=5\sqrt{2}$

Length of arc $=\frac{3}{4}$(perimeter of circle)

$=\frac{3}{4}(2\pi\times 5\sqrt{2})$

$=\frac{15\pi}{\sqrt{2}}$

35. a, c, d.

Let $z=c$ be a real root. Then,

$ac^2+c+\bar{\alpha}=0$ (1)

Putting $\alpha=p+iq$, we have

$(p+iq)c^2+c+p-iq=0$

$\Rightarrow$ $pc^2+c+p=0$ and $qc^2-q=0\Rightarrow c=\pm 1$ $(\because q\neq 0)$

$\therefore$ (1) $\Rightarrow \alpha\pm 1+\bar{\alpha}=0$

Also,

$|c|=1$

36. a, b, c, d.

$\sqrt{5-12i}=\sqrt{(3-2i)^2}=\pm(3-2i)$

$\sqrt{-5-12i}=\sqrt{(2-3i)^2}=\pm(2-3i)$

$\Rightarrow$ $z=\sqrt{5-12i}+\sqrt{-5-12i}$

$=-1-i,-5+5i,5-5i,1+i$

Therefore, principal values of arg z are $-3\pi/4$, $3\pi/4$, $-\pi/4$, $\pi/4$.

Reasoning Type

1. a. $\arg(z_1z_2)=2\pi\Rightarrow\arg(z_1)+\arg(z_2)=2\pi\Rightarrow\arg(z_1)=\arg(z_2)=\pi$, as principal arguments are from $-\pi$ to π.

Hence both the complex numbers are purely real. Hence both the statements are true and statement 2 is correct explanation of statement 1.

2. c. $x^3+x^2+x=x(x^2+x+1)=x(x-\omega)(x-\omega^2)$

Now $f(x)=(x+1)^n-x^n-1$ is divisible by x^3+x^2+x. Then $f(0)=0, f(\omega)=0\, f(\omega^2)=0$. Now,

$f(0)=(0+1)^n-0^n-1=0$

$f(\omega)=(\omega+1)^n-\omega^n-1=(-\omega^2)^n-\omega^n-1=-(\omega^{2n}+\omega^n+1)$
$=0$ (as n is not a multiple of 3)

Similarly, we have $f(\omega^2)=0$.

Hence statement 1 is correct but statement 2 is false.

3. a. First, let the two complex numbers be conjugate of each other. Let complex numbers be $z_1=x+iy$ and $z_2=x-iy$. Then, $z_1+z_2=(x+iy)+(x-ix)=2x$, which is real and $z_1z_2=(x+iy)(x-iy)=x^2-i^2y^2=x^2+y^2$, which is real.

Conversely, let z_1 and z_2 be two complex numbers such that their sum z_1+z_2 and product z_1z_2 both are real. Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$. Then,

$z_1+z_2=(x_1+x_2)+i(y_1+y_2)$ and $z_1z_2=(x_1xs_2-y_1y_2)+i(x_1y_2+x_2y_1)$

Now, z_1+z_2 and z_1z_2 are real. Hence,

$b_1+b_2=0$ and $a_1b_2+a_2b_1=0$ $[\because z$ is real $\Rightarrow \text{Im}(z)=0]$

$\Rightarrow$ $b_2=-b_1$ and $a_1b_2+a_2b_1=0$

$=-b_1$ and $-a_1b_1+a_2b_1=0$

$=-b_1$ and $(a_2-a_1)b_1=0$

$=-b_1$ and $a_2-a_1=0$

$=-b_1$ and $a_2=a_1$

$\Rightarrow$ $z_2=a_2+ib_2=a_1-ib_1$

$=\bar{z}_1$

Hence, z_1 and z_2 are conjugate of each other. Hence, statement 2 is true.

Also in statement 1, $a=\bar{a}$ and $b=\bar{b}$, then a and b are real. Thus, z_1+z_2 and z_1z_2 are real. So,

$z_2=\bar{z}_2$

$\Rightarrow$ $\arg(z_1z_2)=\arg(z_1\bar{z}_1)=\arg(|z_1|^2)=0$

Hence, statement 1 is correct and statement 2 is correct explanation of statement 1.

4 d. $x+\frac{1}{x}=1$

$\Rightarrow$ $x^2-x+1=0$

$\therefore$ $x=-\omega,-\omega^2$

Now for $x=-\omega$,

$p=\omega^{4000}+\frac{1}{\omega^{4000}}=\omega+\frac{1}{\omega}=-1$

Similarly for $x=-\omega^2$, $P=-1$. For $n>1$,

$2^n=4k$

$\therefore$ $2^{2^n}=2^{4k}=(16)^k=$ a number with last digit 6

$\Rightarrow$ $q=6+1=7$

Hence, $p+q=-1+7=6$.

5. d.

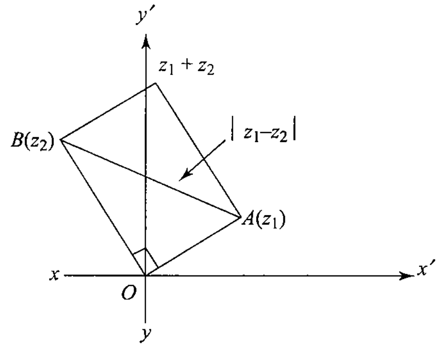


Fig. 2.91

From the diagram when $|z_1-z_2|=|z_1+z_2|$, OAB is right-angled triangle. Hence orthocentre is 0.

6. d. Statement 2 is true as it is the definition of an ellipse. Statement 1 is false as distance between 1 and 8 is 7 but $|z-1|+|z-8| = 5 < 7$. Hence no such z exists.

7. d. $|z_1 + z_2 + z_3| = |z_1 - a + z_2 - b + z_3 - c + (a+b+c)|$
$$\le |z_1 - a| + |z_2 - b| + |z_3 - c| + |a+b+c|$$
$$\le 2|a+b+c|$$

Hence, $|z_1 + z_2 + z_3|$ is less than $2|a+b+c|$.

8. b. Fourth roots of unity are $-1, 1, -i$ and i.

$\therefore \quad z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ and $z_1 + z_2 + z_3 + z_4 = 0$

9. a. If roots of $ax^2 + bx + c = 0$, $0 < a < b < c$, are non-real, then they will be the conjugate of each other. Hence,

$$z_2 = \bar{z}_1 \Rightarrow |z_1| = |z_2|$$

Now,

$$z_1 z_2 = \frac{c}{a} > 1 \Rightarrow |z_1|^2 > 1$$

$\Rightarrow \quad |z_1| > 1$

$\Rightarrow \quad |z_2| > 1$

10. a. We have,

$$az^2 + bz + c = 0 \qquad (1)$$

and z_1, z_2 [roots of (1)] are such that Im $(z_1 z_2) \neq 0$. So, z_1 and z_2 are not conjugate of each other. That is complex roots of (1) are not conjugate of each other, which implies that coefficients a, b, c cannot all be real. Hence, at least one of a, b, c is imaginary.

11. b. We have,

Let $x = (\cos\theta + i\sin\theta)^{3/5}$

$\Rightarrow \quad x^5 = (\cos\theta + i\sin\theta)^3$

$\Rightarrow \quad x^5 - (\cos 3\theta + i\sin 3\theta) = 0$

$\Rightarrow \quad$ Product of roots $= \cos 3\theta + i\sin 3\theta$

Also product of roots of the equation $x^5 - 1 = 0$ is 1. Hence statement 2 is true. But it is not correct explanation of statement 1.

12. b. Since $|z_1| = |z_2| = |z_3|$, circumcentre of Δ is origin

Also $\dfrac{z_1 + z_2 + z_3}{3} = 0$

Centroid concide with circumcentre

$\Rightarrow \quad \Delta$ is equilateral

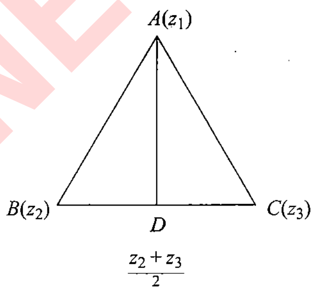


Fig. 2.92

$$\arg\left(\frac{z_2 + z_3 - 2z_1}{z_3 - z_2}\right) = \arg\left(2\left(\frac{\frac{z_2+z_3}{2} - z_1}{z_3 - z_2}\right)\right)$$

$$= \arg\left(\frac{\frac{z_2+z_3}{2} - z_1}{z_3 - z_2}\right)$$

$(z_2 + z_3)/2$ is the mid-point of side BC. Clearly, line joining A and mid-point of BC will be perpendicular to side BC. Thus,

$$\arg\left(\frac{\frac{z_2+z_3}{2} - z_1}{z_3 - z_2}\right) = \frac{\pi}{2}$$

Hence, statement 2 is also true. However, it does not explain statement 1.

13. a. Suppose there exists a complex number z which satisfies the given equation and is such that $|z| < 1$. Then,

$$z^4 + z + 2 = 0 \Rightarrow -2 = z^4 + z \Rightarrow |-2| = |z^4 + z|$$

$\Rightarrow \quad 2 \le |z^4| + |z| \Rightarrow 2 < 2$, because $|z| < 1$

But $2 < 2$ is not possible. Hence given equation cannot have a root z such that $|z| < 1$.

14. c. $|z_1 + z_2| = \left|\dfrac{z_1 + z_2}{z_1 z_2}\right|$

$$\Rightarrow \quad |z_1 + z_2|\left(1 - \frac{1}{|z_1 z_2|}\right) = 0$$

$\Rightarrow \quad |z_1 z_2| = 1$

Hence, statement 1 is true. However, it is not necessary that $|z_1| = |z_2| = 1$. Hence, statement 2 is false.

Linked Comprehension Type

For Problems 1–4

1. b, 2. b, 3. c, 4. c.

Sol. Given that

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 = |z_1|^2 + |z_2|^2$

$\Rightarrow \quad z_1\bar{z}_2 + \bar{z}_1 z_2 = 0 \qquad (1)$

$\Rightarrow \quad \dfrac{z_1}{z_2} + \dfrac{\bar{z}_1}{\bar{z}_2} = 0 \qquad$ (dividing by $z_2\,\bar{z}_2$)

$\Rightarrow \quad \dfrac{z_1}{z_2} + \overline{\left(\dfrac{z_1}{z_2}\right)} = 0 \qquad (2)$

From (1), $z_2\,\bar{z}_2$ is purely imaginary. From (2), z_1/z_2 is purely imaginary. Hence,

$$\arg\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2} \Rightarrow \arg(z_1) - \arg(z_2) = \pm\frac{\pi}{2}$$

Also, $i(z_1/z_2)$ is purely real. Hence its possible arguments are 0 and π.

For Problems 5–8

5. a, 6. d, 7. c, 8. d.

Sol.

$$z = \frac{1 - i\sin\theta}{1 + i\cos\theta} = \frac{(1 - i\sin\theta)(1 - i\cos\theta)}{(1 + i\cos\theta)(1 - i\cos\theta)}$$

$$= \frac{(1 - \sin\theta\cos\theta) - i(\cos\theta + \sin\theta)}{(1 + \cos^2\theta)}$$

If z is purely real, then

$$\cos\theta + \sin\theta = 0$$

or

$$\tan\theta = -1$$

$$\Rightarrow \quad \theta = n\pi - \frac{\pi}{4}, n \in I$$

If z is purely imaginary, $1 - \sin\theta\cos\theta = 0$ or $\sin\theta\cos\theta = 1$, which is not possible.

$$|z| = \left|\frac{1 - i\sin\theta}{1 + i\cos\theta}\right| = \frac{\sqrt{1+\sin^2\theta}}{\sqrt{1+\cos^2\theta}}$$

If $|z| = 1$, then

$$\cos^2\theta = \sin^2\theta \Rightarrow \tan^2\theta = 1 \Rightarrow \theta = n\pi \pm \frac{\pi}{4}, n \in I$$

We have,

$$\arg(z) = \tan^{-1}\left(\frac{-(\cos\theta + \sin\theta)}{(1 - \sin\theta\cos\theta)}\right)$$

Now,

$$\arg(z) = \pi/4$$

$$\Rightarrow \quad \frac{-(\cos\theta + \sin\theta)}{(1 - \sin\theta\cos\theta)} = 1$$

$$\Rightarrow \quad \cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta$$

$$= 1 + \sin^2\theta\cos^2\theta - 2\sin\theta\cos\theta$$

$$\Rightarrow \quad 1 + 4\sin\theta\cos\theta = 1 + \sin^2\theta\cos^2\theta$$

$$\Rightarrow \quad \sin^2\theta\cos^2\theta - 4\sin\theta\cos\theta = 0$$

$$\Rightarrow \quad \sin\theta\cos\theta(\sin\theta\cos\theta - 4) = 0$$

$$\Rightarrow \quad \sin\theta\cos\theta = 0 \qquad (\because \sin\theta\cos\theta = 4 \text{ is not possible})$$

$$\Rightarrow \quad \theta = (2n+1)\pi \text{ or } \theta = (4n-1)\pi/2, n \in I$$

$$(\because -\cos\theta - \sin\theta > 0)$$

For Problems 9–11

9. a, 10. d, 11. d.

Sol.

9. Let z_1 (purely imaginary) be a root of the given equation. Then,

$$z_1 = -\bar{z}_1$$

and

$$az_1^2 + bz_1 + c = 0 \qquad (1)$$

$$\Rightarrow \quad \overline{az_1^2 + bz_1 + c} = 0$$

$$\Rightarrow \quad \bar{a}\bar{z}_1^2 + \bar{b}\bar{z}_1 + \bar{c} = 0$$

$$\Rightarrow \quad \bar{a}z_1^2 - \bar{b}z_1 + \bar{c} = 0 \qquad (\text{as } \bar{z}_1 = -z_1) \qquad (2)$$

Now Eqs. (1) and (2) must have one common root.

$$\therefore \quad (c\bar{a} - a\bar{c})^2 = (b\bar{c} + c\bar{b})(-a\bar{b} - \bar{a}b)$$

Let z_1 and z_2 be two purely imaginary roots. Then,

$$\bar{z}_1 = -z_1, \bar{z}_2 = -z_2$$

Now,

$$az^2 + bz + c = 0 \qquad (1)$$

$$\Rightarrow \quad \overline{az^2 + bz + c} = \bar{0}$$

$$\Rightarrow \quad \bar{a}\bar{z}^2 + \bar{b}\bar{z} + \bar{c} = 0$$

$$\Rightarrow \quad \bar{a}z^2 - \bar{b}z + \bar{c} = 0 \qquad (2)$$

Equations (1) and (2) must be identical as their roots are same.

$$\therefore \quad \frac{a}{\bar{a}} = -\frac{b}{\bar{b}} = \frac{c}{\bar{c}}$$

$$\Rightarrow \quad a\bar{c} = \bar{a}c, a\bar{b} + \bar{a}b = 0 \text{ and } b\bar{c} + \bar{b}c = 0$$

Hence, $a\bar{c}$ is purely real and $a\bar{b}$ and $b\bar{c}$ are purely imaginary.

Let z_1 (purely real) be a root of the given equation. Then,

$$z_1 = \bar{z}_1$$

and

$$az_1^2 + bz_1 + c = 0 \qquad (1)$$

$$\Rightarrow \quad \overline{az_1^2 + bz_1 + c} = \bar{0}$$

$$\Rightarrow \quad \bar{a}\bar{z}_1^2 + \bar{b}\bar{z}_1 + \bar{c} = 0$$

$$\Rightarrow \quad \bar{a}z_1^2 + \bar{b}z_1 + \bar{c} = 0 \qquad (2)$$

Now (1) and (2) must have one common root. Hence,

$$(c\bar{a} - a\bar{c})^2 = (b\bar{c} - c\bar{b})(a\bar{b} - \bar{a}b)$$

For Problems 12–14

12. c, 13. b, 14. d.

Sol. $az + b\bar{z} + c = 0 \qquad (1)$

$$\Rightarrow \quad \bar{a}\bar{z} + \bar{b}z + \bar{c} = 0 \qquad (2)$$

Eliminating $\bar{z}$ from (1) and (2), we get

$$z = \frac{c\bar{a} - b\bar{c}}{|b|^2 - |a|^2}$$

If $|a| \neq |b|$, then z represents one point on the Argand plane. If $|a| = |b|$ and $\bar{a}c \neq b\bar{c}$, then no such z exists. Adding (1) and (2),

$$(\bar{a} + b)\bar{z} + (a + \bar{b})z + (c + \bar{c}) = 0$$

This is of the form $A\bar{z} + \bar{A}z + B = 0$, where $B = c + \bar{c}$ is real.

Hence locus of z is a straight line.

For Problems 15–17

15. a, 16. b, 17. c.

$$z = -\lambda \pm \sqrt{\lambda^2 - 1}$$

Case I:

When $-1 < \lambda < 1$, we have

$$\lambda^2 < 1 \Rightarrow \lambda^2 - 1 < 0$$

$$z = -\lambda \pm i\sqrt{1-\lambda^2} \text{ or } x = -\lambda, y = \pm\sqrt{1-\lambda^2}$$

$$\Rightarrow \quad y^2 = 1 - x^2 \text{ or } x^2 + y^2 = 1$$

Case II:

$$\lambda > 1 \Rightarrow \lambda^2 - 1 > 0$$

$$z = -\lambda \pm \sqrt{\lambda^2 - 1} \text{ or } x = -\lambda \pm \sqrt{\lambda^2 - 1}, y = 0$$

Roots are $(-\lambda + \sqrt{\lambda^2-1}, 0)$, $(-\lambda - \sqrt{\lambda^2-1}, 0)$. One root lies inside the unit circle and the other root lies outside the unit circle.

Case III:

When λ is very large, then

$$z = -\lambda - \sqrt{\lambda^2 - 1} \approx -2\lambda$$

$$z = -\lambda + \sqrt{\lambda^2 - 1} = \frac{(-\lambda + \sqrt{\lambda^2-1})(-\lambda - \sqrt{\lambda^2-1})}{(-\lambda - \sqrt{\lambda^2-1})}$$

$$= \frac{1}{-\lambda - \sqrt{\lambda^2 - 1}} = -\frac{1}{2\lambda}$$

For Problems 18–20

18. d, 19. c, 20. c.

Sol. We have,

$$az^2 + z + 1 = 0 \quad (1)$$

$\Rightarrow \quad \overline{az^2 + z + 1} = 0$ (taking conjugate of both sides)

$$\Rightarrow \quad \bar{a}z^2 - z + 1 = 0 \quad (2)$$

[since z is purely imaginary $\bar{z} = -z$]

Eliminating z from (1) and (2) by cross-multiplication rule,

$$(\bar{a} - a)^2 + 2(a + \bar{a}) = 0 \Rightarrow \left(\frac{\bar{a} - a}{2}\right)^2 + \frac{a + \bar{a}}{2} = 0$$

$$\Rightarrow \quad -\left(\frac{a - \bar{a}}{2i}\right)^2 + \left(\frac{a + \bar{a}}{2}\right) = 0 \Rightarrow -\sin^2\theta + \cos\theta = 0$$

$$\Rightarrow \quad \cos\theta = \sin^2\theta \quad (3)$$

Now,

$$f(x) = x^3 - 3x^2 + 3(1 + \cos\theta)x + 5$$

$$f'(x) = 3x^2 - 6x + 3(1 + \cos\theta)$$

Its discriminant is

$$36 - 36(1 + \cos\theta) = -36\cos\theta = -36\sin^2\theta < 0$$

$$\Rightarrow \quad f'(x) > 0 \ \forall\, x \in R$$

Hence, $f(x)$ is increasing $\forall\, x \in R$. Also, $f(0) = 5$, then $f(x) = 0$ has one negative root. Now,

$$\cos 2\theta = \cos\theta \Rightarrow 1 - 2\sin^2\theta = \cos\theta$$

$$\Rightarrow \quad 1 - 2\cos\theta = \cos\theta$$

$$\Rightarrow \quad \cos\theta = 1/3$$

which has four roots for $\theta \in [0, 4\pi]$.

For Problems 21–23

21. a, 22. b, 23. b.

Sol.

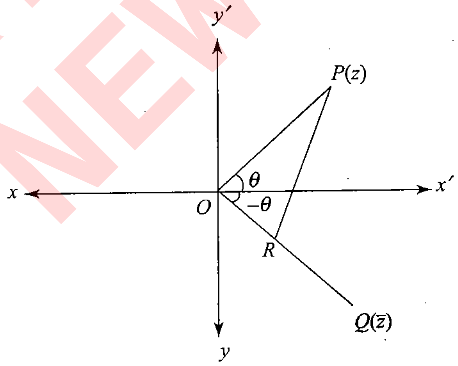


Fig. 2.93

We have,

$$\left|\,|z| - \left|\frac{4}{z}\right|\,\right| \le \left|z - \frac{4}{z}\right| = 2$$

$$\Rightarrow \quad -2 \le |z| - \frac{4}{|z|} \le 2$$

$$\Rightarrow \quad |z|^2 + 2|z| - 4 \ge 0 \text{ and } |z|^2 - 2|z| - 4 \le 0$$

$$\Rightarrow \quad (|z| + 1)^2 - 5 \ge 0 \text{ and } (|z| - 1)^2 \le 5$$

$$\Rightarrow \quad (|z| + 1 + \sqrt{5})(|z| + 1 - \sqrt{5}) \ge 0$$

$$\text{and } (|z| - 1 + \sqrt{5}) \times (|z| - 1 - \sqrt{5}) \le 0$$

$$\Rightarrow \quad |z| \le -\sqrt{5} - 1 \text{ or } |z| \ge \sqrt{5} - 1 \text{ and } \sqrt{5} - 1 \le |z| \le \sqrt{5} + 1$$

$$\Rightarrow \quad \sqrt{5} - 1 \le |z| \le \sqrt{5} + 1$$

Hence, the least modulus is $\sqrt{5} - 1$ and the greatest modulus is $\sqrt{5} + 1$. Also,

$$|z| = \sqrt{5} + 1 \Rightarrow \frac{4}{|z|} = \sqrt{5} - 1$$

Now,

$$\frac{4}{z} = \frac{4\bar{z}}{|z|^2}$$

Hence, $4/z$ lies in the direction of $\bar{z}$.

$$\left|z - \frac{4}{z}\right| = PR = 2 \text{ (given)}$$

We have,

$$OP = \sqrt{5} + 1 \text{ and } OR = \sqrt{5} - 1$$

$$\Rightarrow \quad \cos 2\theta = \frac{OP^2 + OR^2 - PR^2}{2\,OP \cdot OR}$$

$$= \frac{(\sqrt{5} + 1)^2 + (\sqrt{5} - 1)^2 - 4}{2(5 - 1)} = 1$$

$$\Rightarrow \quad 2\theta = 0, 2\pi$$

$$\Rightarrow \quad \theta = 0, \pi$$

$\Rightarrow \quad z$ is purely real

$$\Rightarrow \quad z = \pm(\sqrt{5} + 1)$$

Similarly for $|z| = \sqrt{5} - 1$, we have $z = \pm(\sqrt{5} - 1)$.

For Problems 24–26

24. a, 25. c, 26. b.

Sol. $BM \equiv y - 0 = -1(x - 1)$

$$x + y = 1$$

$$\therefore \quad \sqrt{u - 1} = t + i(1 - t)$$

$$u = 2t + 2it(1 - t)$$

$$x = 2t \text{ and } y = 2t(1 - t)$$

$$y = x(1 - x/2)$$

$$2y = 2x - x^2$$

$$\Rightarrow \quad (x - 1)^2 = -2\left(y - \frac{1}{2}\right)$$

which is a parabola. Its axis is $x = 1$, i.e., $z + \bar{z} = 2$ and directrix is $y = 1$, i.e., $z - \bar{z} = 2i$.

For Problems 27–29

27. a, 28. b, 29. c.

Sol. 27.

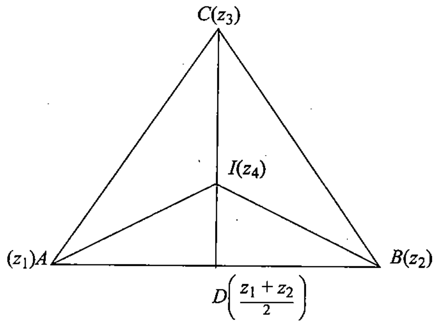


Fig. 2.94

$$\frac{AB\times AC}{(IA)^2}=\frac{AB}{IA}\times\frac{AC}{IA}$$

$$\angle IAB=\frac{\theta}{2},\ \angle IAC=\frac{\theta}{2}$$

$$\frac{z_2-z_1}{z_4-z_1}=\frac{|z_2-z_1|}{|z_4-z_1|}e^{-\frac{i\theta}{2}}$$

and

$$\frac{z_3-z_1}{z_4-z_1}=\frac{|z_3-z_1|}{|z_4-z_1|}e^{\frac{i\theta}{2}}$$

Multiplying,

$$\frac{z_2-z_1}{z_4-z_1}\,\frac{z_3-z_1}{z_4-z_1}=\frac{|z_2-z_1|}{|z_4-z_1|}\,\frac{|z_3-z_1|}{|z_4-z_1|}$$

$$\Rightarrow\quad \frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)^2}=\frac{AB\times AC}{IA^2} \qquad (1)$$

28. From (1),

$$\frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)^2}=2\left(\frac{AD}{IA}\right)^2\left(\frac{AC}{AD}\right) \qquad (\because AB=2\,AD)$$

$$\Rightarrow\quad (z_2-z_1)(z_3-z_1)=(z_4-z_1)^2\,2\cos^2\frac{\theta}{2}\sec\theta$$

$$=(z_4-z_1)^2(\cos\theta+1)\sec\theta$$

29. Keeping in mind that $\tan\theta = CD/AD$ and $\tan\theta/2 = ID/BD$, We have,

$$\frac{z_3-\frac{z_1+z_2}{2}}{z_1-\frac{z_1+z_2}{2}}=\frac{\left|z_3-\frac{z_1+z_2}{2}\right|}{\left|z_1-\frac{z_1+z_2}{2}\right|}e^{-i\frac{\pi}{2}}$$

$$\Rightarrow\quad \frac{2z_3-z_1-z_2}{z_1-z_2}=\frac{CD}{AD}e^{-i\frac{\pi}{2}} \qquad (1)$$

and

$$\frac{z_4-\frac{z_1+z_2}{2}}{z_2-\frac{z_1+z_2}{2}}=\frac{\left|z_4-\frac{z_1+z_2}{2}\right|}{\left|z_2-\frac{z_1+z_2}{2}\right|}e^{i\frac{\pi}{2}}$$

$$\Rightarrow\quad \frac{2z_4-z_1-z_2}{z_2-z_1}=\frac{ID}{BD}e^{i\frac{\pi}{2}} \qquad (2)$$

Multiplying (1) and (2), we have

$$\frac{2z_3-z_1-z_2}{z_1-z_2}\,\frac{2z_4-z_1-z_2}{z_2-z_1}=\frac{CD}{AD}\,\frac{ID}{BD}=\tan\theta\tan\frac{\theta}{2}$$

$$\Rightarrow\quad (z_2-z_1)^2\tan\theta\tan\frac{\theta}{2}=-(z_1+z_2-2z_3)(z_1+z_2-2z_4)$$

For Problems 30–32

30. d, 31. c, 32. c.

Sol.

30. $\angle BOD = 2\angle BAD = A$

$\angle COD = 2\angle CAD = A$

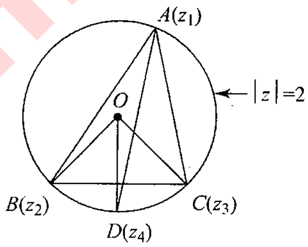


Fig. 2.95

$$\frac{z_4}{z_2}=e^{iA},\ \frac{z_3}{z_4}=e^{iA} \qquad \text{(From rotation about the point 'O')}$$

$$\Rightarrow\quad z_4^2=z_2z_3$$

31. Clearly, OD bisects $\angle BAC$ of isosceles triangle BOC.

Thus angle between segments OD and BC is $\pi/2$.

$$\therefore\quad \arg\left(\frac{z_4}{z_2-z_3}\right)=\frac{\pi}{2}$$

32. See theory.

Matrix-Match Type

1.a → r, s; b → p, q; r → t; d → u, t.

a. If $ab > 0$, then either a, b both are positive or both a, b are negative. Hence $z = a + ib$ lies in either first or third quadrant, then argument of z is $\tan^{-1} b/a$ or $-\pi + \tan^{-1} b/a$.

b. If $ab < 0$, then a and b have opposite signs, then z lies in either second or fourth quadrant, then argument of z is $-\tan^{-1} b/a$ or $\pi - \tan^{-1} |b/a|$ or $\tan^{-1} b/a$.

c. If $a^2 + b^2 = 0$, then $a = b = 0$, so $z = 0 + i0$ whose argument is not defined.

d. If $ab = 0$, then either $a = 0$ or $b = 0$ or both are 0, then argument is 0 or $\pi/2$ or not defined.

2. a → s; b → r; c → p; d → q.

a. $z^4 - 1 = 0 \Rightarrow z^4 = 1 = \cos 0 + i \sin 0 \Rightarrow z = (\cos 0 + i \sin 0)^{1/4}$
$= \cos 0 + i \sin 0$

b. $z^4 + 1 = 0 \Rightarrow z^4 = -1 = \cos \pi + i \sin \pi \Rightarrow z = (\cos\pi + i \sin\pi)^{1/4}$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

c. $iz^4 + 1 = 0 \Rightarrow z^4 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \Rightarrow z = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/4}$

$$= \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$$

d. $iz^4 - 1 = 0 \Rightarrow z^4 = -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

$$\Rightarrow \quad z = \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^{1/4} = \cos\frac{\pi}{8} - i\sin\frac{\pi}{8}$$

3. a → q; b → s; c → p; d → r.

a. $|z - 1| = |z - i|$

Hence it lies on the perpendicular bisector of the line joining (1, 0) and (0, 1) which is a straight line passing through the origin.

b. $|z + \bar{z}| + |z - \bar{z}| = 2$

$\Rightarrow$ $|x| + |y| = 1$

Hence, z lies on a square.

c. Let $z = x + iy$. Then,

$|z + \bar{z}| = |z - \bar{z}|$

$\Rightarrow$ $|2x| = |2iy|$

$\Rightarrow$ $|x| = |y|$

$\Rightarrow$ $x = \pm y$

Hence, the locus of z is a pair of straight lines.

d. Let $Z = 2/z$. Then,

$$|Z| = \left|\frac{2}{z}\right| = \frac{2}{|z|} = \frac{2}{1} = 2$$

This shows that Z lies on a circle with centre at the origin and radius 2 units.

4. a → p, r; b → p, q, r, t; c → p, r, s; d → p, q, r, s, t.

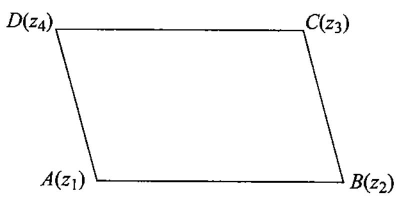


Fig. 2.96

In parallelogram, the mid-points of diagonals coincide

$$\frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$$

$\Rightarrow$ $z_1 - z_4 = z_2 - z_3$

Also in parallelogram, $AB \| CD$. Hence,

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = 0$$

$\Rightarrow$ $\frac{z_1 - z_2}{z_3 - z_4}$ is purely real

In rectangle, adjacent sides are perpendicular. Hence,

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \frac{\pi}{2}$$

$\Rightarrow$ $\frac{z_1 - z_2}{z_3 - z_2}$ is purely imaginary

Also in rectangle,

$AC = BD \Rightarrow |z_1 - z_3| = |z_2 - z_4|$

In rhombus,

$AC \perp BD \Rightarrow \frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary

5. a → p, q; b → p, q, r, s, t; c → r, s; d → p, q.

a. $|z - 2i| + |z - 7i| = k$ is ellipse if $k > |7i - 2i|$ or $k > 5$

b. $\left|\frac{2z-3}{3z-2}\right| = k \Rightarrow \left|\frac{z - \frac{3}{2}}{z - \frac{2}{3}}\right| = \frac{3k}{2} \Rightarrow 3k/2 > 1 \Rightarrow k > 2/3$

c. $|z - 3| - |z - 4i| = k$ is hyperbola, if $k < |3 - 4i| \Rightarrow 0 < k < 5$

d. $|z - (3 + 4i)| = \frac{k}{50}|a\bar{z} + \bar{a}z + b|$

$\Rightarrow$ $|z - (3 + 4i)| = \frac{k}{5}\frac{|a\bar{z} + \bar{a}z + b|}{2|3 + 4i|}$

This is hyperbola if $k/5 > 1 \Rightarrow k > 5$.

6. a → s; b → q; c → p; d → r.

a. $x^2 - x + 1 = 0$

$$\Rightarrow \quad x = \frac{1 \pm i\sqrt{3}}{2}$$

$$= -\omega, -\omega^2$$

$$\Rightarrow \quad \left(x^n + \frac{1}{x^n}\right)^2 = (-1)^{2n}\left(\omega^n + \frac{1}{\omega^n}\right)^2$$

$$= (\omega^n + \omega^{2n})^2 \qquad (1)$$

$$\because \quad \frac{1}{\omega^n} = \frac{\omega^{2n}}{\omega^{3n}} = \omega^{2n}$$

Now,

$$1 + \omega^n + \omega^{2n} = \frac{1 - \omega^{3n}}{1 - \omega^n} = 0 \text{ for } n \neq 3p$$

$\therefore$ $\omega^n + \omega^{2n} = -1$ for $n \neq 3p$

$= 2$ for $n = 3p$

$$\therefore \quad \sum_{n=1}^{5}\left(x^n + \frac{1}{x^n}\right)^2 = 8$$

b. In the expression,

$$\left[\frac{1+\cos\theta+i\sin\theta}{\sin\theta+i(1+\cos\theta)}\right]^4$$

numerator is

$$1+\cos\theta+i\sin\theta=2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]$$

$$=2\cos\frac{\theta}{2}e^{i\theta/2}$$

and denominator is

$$-i^2\sin\theta+i(1+\cos\theta)=i\text{ [conjugate of numerator]}$$

$$=i\,2\cos\frac{\theta}{2}\;e^{-\theta/2}$$

$$\therefore\quad E=\left(\frac{N^r}{D^r}\right)=\left[\frac{1}{i}\frac{e^{i\theta/2}}{e^{-i\theta/2}}\right]^4=\frac{1}{i^4}e^{4i\theta}$$

$$=\cos 4\theta+i\sin 4\theta$$

$\therefore\quad n=4$

c. We know that if $z=re^{i\theta}$, then $\bar{z}=re^{-i\theta}$.

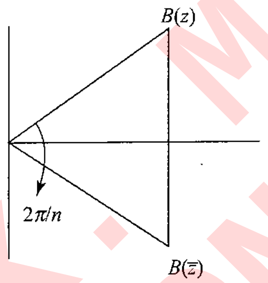


Fig. 2.97

$$\therefore\quad \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}=\frac{r\sin\theta}{r\cos\theta}=\tan\theta=\tan\frac{\pi}{n}=\sqrt{2}-1$$

$$\Rightarrow\quad \tan\frac{\pi}{n}=\tan\frac{\pi}{8}\Rightarrow n=8$$

d. $$\sum_{r=1}^{10}(r-\omega)(r-\omega^2)=\sum_{r=1}^{10}(r^2+r+1)$$

$$=\Sigma r^2+\Sigma r+10$$

$$=\frac{10\times11\times21}{6}+\frac{10\times11}{2}+10$$

$$=450$$

$$\Rightarrow\quad \frac{1}{50}\left\{\sum_{r=1}^{10}(r-\omega)(r-\omega^2)\right\}=9$$

Integer Type

1.(3) $$x=\frac{x^3}{x^2}=\frac{2+11i}{3+4i}\times\frac{3-4i}{3-4i}=\frac{50+25i}{25}=2+i$$

2.(7) We have

$x^3-y^3=98i$

$\Rightarrow (x-y)^3+3xy(x-y)=98i$

$\Rightarrow -343i+3(a+ib)(7i)=98i$

$\Rightarrow -343+3(a+bi)7=98$

$\Rightarrow a+ib=21$

$\Rightarrow a=21$ and $b=0$

$\Rightarrow a+b=21$

3.(1) We have $x=\omega-\omega^2-2$ or $x+2=\omega-\omega^2$

Squaring, $x^2+4x+4=\omega^2+\omega^4-2\omega^3=\omega^2+\omega-2=-3$

$\Rightarrow x^2+4x+7=0.$

Dividing $x^4+3x^3+2x^2-11x-6$ by x^2+4x+7, we get

$x^4+3x^3+2x^2-11x-6=(x^2+4x+7)(x^2-x-1)+1$

$=(0)(x^2-x-1)+1=0+1=1$

4.(5) $z^2=81-b^2+18bi$

$z^3=729+243bi-27b^2-b^3i$

$z^2=z^3\Rightarrow 243b-b^3=18b$ and $243-b^2=18\Rightarrow b=15$

5.(2) $\bar{z}+z=0$

$\Rightarrow\ \bar{z}=-z$ (1)

Now $|z|^2-4zi=z^2$

$\Rightarrow -z^2-4zi=z^2$ (from (1))

$\Rightarrow 2z=-4i$

$\Rightarrow z=-2i$

$\Rightarrow |z|=2$

6.(3) $(1+ri)^3=s(1+i)$

$\Rightarrow\ 1+3ri+3r^2i^2+r^3i^3=s(1+i)$

$\Rightarrow\ 1-3r^2+i(3r-r^3)=s+si$

$\Rightarrow\ 1-3r^2=s=3r-r^3$

Hence, $1-3r^2=3r-r^3$

$\Rightarrow\ r^3-3r^2-3r+1=0$

$\Rightarrow$ sum of three roots is 3.

7.(4) We have $|z|^2+\dfrac{16}{|z|^3}=z^2-4z=\bar{z}^2-4\bar{z}$

$\Rightarrow (z-\bar{z})(z+\bar{z}-4)=0$

$\Rightarrow z=\bar{z}=x\ (x\neq 2)$

So, $x^2=4x+x^2+\dfrac{16}{|x|^3}$

$\Rightarrow x=\dfrac{-4}{|x|^3}\quad\Rightarrow\quad x=-\sqrt{2}$

$\therefore\ z=-\sqrt{2}$

$\therefore\ |z|^4=4$

8.(9) Let $z=a+bi$.

$\Rightarrow |z|^2=a^2+b^2.$

Now $z+|z|=2+8i$

$\Rightarrow a+bi+\sqrt{a^2+b^2}=2+8i$

$\Rightarrow a+\sqrt{a^2+b^2}=2,\ b=8$

$\Rightarrow a+\sqrt{a^2+64}=2$

$\Rightarrow a^2+64=(2-a)^2=a^2-4a+4,$

$\Rightarrow 4a=-60,\ a=-15.$

Thus, $a^2+b^2=225+64=289$

$\therefore\ |z|=\sqrt{a^2+b^2}=\sqrt{289}=17$

9.(1) Let $z=a+ib$

Given $|z|=2\Rightarrow a^2+b^2=4\Rightarrow a,b\in[-2,2]$

Now $w = \dfrac{(a+1)+ib}{(a-1)+ib}$;

$\Rightarrow |w| = \sqrt{\dfrac{(a+1)^2+b^2}{(a-1)^2+b^2}}$

$= \sqrt{\dfrac{a^2+b^2+2a+1}{a^2+b^2-2a+1}} = \sqrt{\dfrac{5+2a}{5-2a}}$

$|w|_{max} = \sqrt{\dfrac{5+4}{1}} = 3$ (when $a = 2$)

$|w|_{min} = \sqrt{\dfrac{5-4}{9}} = \dfrac{1}{3}$ (when $a = -2$)

Hence, required product is 1.

10.(4) $= \left[\dfrac{1+\cos\theta + i\sin\theta}{\sin\theta + i(1+\cos\theta)}\right]^4$

$= i^4\left[\dfrac{1+\cos\theta+i\sin\theta}{i\sin\theta + i^2(1+\cos\theta)}\right]^4$

$= \left[\dfrac{2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]^4$

$= \left[\dfrac{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}}\right]^4$

$= \left[\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^2\right]^4$

$= \cos 8\frac{\theta}{2} + i\sin 8\frac{\theta}{2} = \cos 4\theta + i\sin 4\theta \Rightarrow n = 4$

11.(1) $z^4 + z^3 + z^2 + z^2 + z + 1 = 0$

$\Rightarrow (z^2(z^2+z+1) + (z^2+z+1) = 0$

$\Rightarrow (z^2+z+1)(z^2+1) = 0$

$\therefore\ z = i, -i, \omega, \omega^2$. For each, $|z| = 1$

12.(5) Roots are 2ω, $(2+3\omega)$, $(2+3\omega^2)$, $(2-\omega-\omega^2)$

$2+3\omega$ and $2+3\omega^2$ are conjugate to each other.

2ω is complex root, then other root must be $2\omega^2$ (as complex roots occur in conjugate pair)

$2-\omega-\omega^2 = 2-(-1) = 3$ which is real.

Hence least degree of the polynomial is 5.

13.(6) We have $|a\omega + b| = 1$

$\Rightarrow |a\omega + b|^2 = 1$

$\Rightarrow (a\bar{\omega} + b) = 1$

$\Rightarrow a^2 + ab\,(\omega + \bar{\omega}) + b^2 = 1$

$\Rightarrow a^2 - ab + b^2 = 1$

$\Rightarrow (a-b)^2 + ab = 1$ (1)

when $(a-b)^2 = 0$ and $ab = 1$ then $(1, 1)$; $(-1, -1)$

when $(a-b)^2 = 1$ and $ab = 0$ then $(0, 1)$; $(1, 0)$; $(0, -1)$; $(-1, 0)$

Hence there are 6 ordered pairs

14.(3) $|z - 2 - 2i| \leq 1$

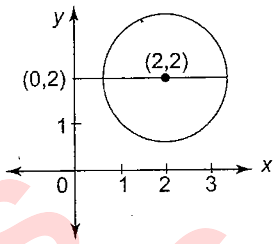


Fig. 2.98

denotes the region inside a circle with centre (2, 2) and radius is 1

$|2iz + 4| = |2i\,(z - 2i)|$

$= |2i|\,|z - 2i|$

$= 2|z - 2i|$

$|z - 2i|$ = distance of z from $P(0, 2)$

Hence, maximum value is 3.

15.(5) $|3z + 9 - 7i| = |(3z + 6 - 3i) + (3 - 4i)|$

$\leq |3z + 6 - 3i| + |3 - 4i|$

$= 3|z + 2 - i| + 5$

$= 20$

16.(5) $Z_1 = (8\sin\theta + 7\cos\theta) + i\,(\sin\theta + 4\cos\theta)$

$Z_2 = (\sin\theta + 4\cos\theta) + i\,(8\sin\theta + 4\cos\theta)$

Hence, $Z_1 = x + iy$ and $Z_2 = y + ix$

where $x = (8\sin\theta + 7\cos\theta)$ and $y = (\sin\theta + 4\cos\theta)$

$Z_1 \cdot Z_2 = (xy - xy) + i(x^2 + y^2) = i(x^2 + y^2) = a + ib$

$\Rightarrow\ a = 0;\ b = x^2 + y^2$

Now, $x^2 + y^2 = (8\sin\theta + 7\cos\theta)^2 + (\sin\theta + 4\cos\theta)^2$

$= 65\sin^2\theta + 65\cos^2\theta + 120\sin\theta\,.\cos\theta$

$= 65 + 60\sin 2\theta$

$\Rightarrow\ Z_1 \cdot Z_2|_{max} = 125$

17.(9) $A \equiv (1+2i)x^3 - 2\,(3+i)\,x^2 + (5-4i)x + 2a^2 = 0$

Let the real root of equation be α

Then $(1+2i)\,\alpha^3 - 2(3+i)\,\alpha^2 + (5-4i)\,\alpha + 2a^2 = 0$

equating imaginary part zero, we get

$2\alpha^3 - 2\alpha^2 - 4\alpha = 0$

$\Rightarrow$ or $\alpha\,(\alpha^2 - \alpha - 2) = 0$

$\Rightarrow\ \alpha = 0$ or $\alpha = -1, 2$

Now equating real part zero

$\alpha^3 - 6\alpha^2 + 5\alpha + 2a^2 = 0$

$\alpha = 0 \Rightarrow a = 0$

$\alpha = -1 \Rightarrow a = \pm\sqrt{6}$

$\alpha = 2 \Rightarrow a = \pm\sqrt{3}$

$\Rightarrow\ \sum a^2 = (0)^2 + (+\sqrt{6})^2 + (-\sqrt{6})^2 + (+\sqrt{3})^2 + (-\sqrt{3})^2 = 18$

18.(6) Let $z = x + iy$

$\therefore\ E = z\bar{z} + (z-3)(\bar{z}-3) + (z-6i)(\bar{z}+6i)$

$= 3z\bar{z} - 3(z+\bar{z}) + 9 + 6(z-\bar{z})i + 36$

$= 3(x^2 + y^2) - 6x - 12y + 45$

$= 3[x^2 + y^2 - 2x - 4y + 15]$

$= 3[(x-1)^2 + (y-2)^2 + 10]$

$\therefore\ E_{min} = 30$ when $x = 1$ and $y = 2$

Archives

Subjective Type

1. $\dfrac{1}{1-\cos\theta+2i\sin\theta}=\dfrac{1}{2\sin^2\theta/2+4i\sin\theta/2\ \cos\theta/2}$

$$=\frac{1}{2\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{(\sin\theta/2+2i\cos\theta/2)(\sin\theta/2-2i\cos\theta/2)}\right]$$

$$=\frac{1}{2\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{\sin^2\theta/2+4\cos^2\theta/2}\right]$$

$$=\frac{1}{2\sin\theta/2}\left[\frac{2\sin\theta/2-4i\cos\theta/2}{1-\cos\theta+4+4\cos\theta}\right]$$

$$=\frac{1}{\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{5+3\cos\theta}\right]$$

$$=\left(\frac{1}{5+3\cos\theta}\right)+\left(\frac{-2\cot\theta/2}{5+3\cos\theta}\right)i$$

2. As β and γ are the complex cube roots of unity, therefore let $\beta=\omega$ and $\gamma=\omega^2$ so that $\omega+\omega^2+1=0$ and $\omega^3=1$. Then,

$$xyz=(a+b)(a\omega+b\omega^2)(a\omega^2+b\omega)$$
$$=(a+b)(a^2\omega^3+ab\omega^4+ab\omega^2+b^2\omega^3)$$
$$=(a+b)(a^2+ab\omega+ab\omega^2+b^2)\qquad(\text{Using }\omega^3=1)$$
$$=(a+b)(a^2+ab(\omega+\omega^2)+b^2)$$
$$=(a+b)(a^2-ab+b^2)\qquad(\text{Using }\omega+\omega^2=-1)$$
$$=a^3+b^3$$

3. Given,

$$x+iy=\sqrt{\frac{a+ib}{c+id}}$$

$$\Rightarrow\ (x+iy)^2=\frac{a+ib}{c+id}\qquad(1)$$

$$\Rightarrow\ \left|(x+iy)^2\right|=\left|\frac{a+ib}{c+id}\right|$$

$$\Rightarrow\ |x+iy|^4=\left|\frac{a+ib}{c+id}\right|^2$$

$$\Rightarrow\ (x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$$

4. Given that n is an odd integer >3 and n is not a multiple of 3.

Let,

$$p(x)=(x+1)^n-x^n-1$$

and

$$q(x)=x^3+x^2+x$$
$$=x(x^2+x+1)$$
$$=x(x-\omega)(x-\omega^2)$$

where ω and ω^2 are cube roots of unity. Clearly, 0, ω, ω^2 are zeros of the polynomial $q(x)$. Now,

$$p(0)=1^n-0-1=0$$

Hence, 0 is a zero of $p(x)$.

$$p(\omega)=(\omega+1)^n-\omega^n-1$$
$$=(-\omega^2)^n-\omega^n-1$$
$$=-(\omega^{2n}+\omega^n+1)\qquad[\because n\text{ is odd}]$$
$$=0\qquad[\because \omega^n+\omega^{2n}+1=0\text{ if }n\neq 3m]$$

Therefore, ω is a zero of $p(x)$. Also,

$$p(\omega^2)=(\omega^2+1)^n-(\omega^2)^n-1$$
$$=(-\omega)^n-\omega^{2n}-1$$
$$=-\omega^n-\omega^{2n}-1$$
$$=-(1+\omega^n+\omega^{2n})$$
$$=0\ [\text{for }n\neq 3m]$$

Hence, ω^2 is a zero of $p(x)$.

Since 0, ω, ω^2 are zeros of $p(x)$, hence x, $x-\omega$, $x-\omega^2$ are factors of $p(x)$. Hence, $x(x-\omega)(x-\omega^2)$ is a factor of $p(x)$, i.e, x^3+x^2+x is a factor of $p(x)$.

5. $\dfrac{(1+i)x-2i}{3+i}+\dfrac{(2-3i)y+i}{3-i}=i$

$$\Rightarrow\ (4+2i)x-6i-2+(9-7i)y+3i-1=10i$$
$$\Rightarrow\ (4x+9y-3)+(2x-7y-3)i=10i$$
$$\Rightarrow\ 4x+9y-3=0\text{ and }2x-7y-3=10$$

On solving, we get $x=3$, $y=-1$.

6. $A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of an equilateral triangle. Hence,

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$$

Now,

$$(z_1+z_2+z_3)^2=z_1^2+z_2^2+z_3^2+2(z_1z_2+z_2z_3+z_3z_1)$$
$$=3(z_1^2+z_2^2+z_3^2)$$

We also have,

$$z_0=\frac{z_1+z_2+z_3}{3}\ (\text{as centroid will coincide with circumcentre})$$

$$\Rightarrow\ z_1^2+z_2^2+z_3^2=3z_0^2$$

7. We know that if z_1, z_2, z_3 form an equilateral triangle, then

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$$

Putting $z_3=0$, we get

$$z_1^2+z_2^2=z_1z_2$$

$$\Rightarrow\ z_1^2+z_2^2-z_1z_2=0$$

8.

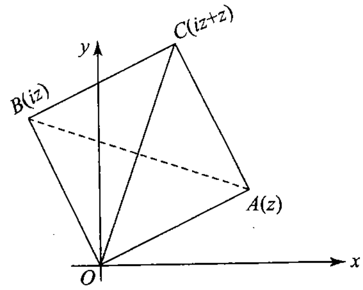


Fig. 2.99

Let the vertices of the triangle be $A(z)$, $B(iz)$, $C(z + iz)$. We know that iz is obtained by rotating OA through an angle 90°. Also point $z + iz$ can be obtained by completing the parallelogram two of whose adjacent sides are OA and OB. From Argand diagram, it is clear that

Area of ΔABC = Area of ΔOAB

$$= \frac{1}{2} \times OA \times OB \quad [\because \text{ it is right angled at point } O]$$

$$= \frac{1}{2}|z| \times |iz|$$

$$= \frac{1}{2}|z|^2$$

9.

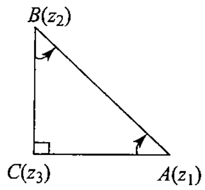


Fig. 2.100

Applying rotation about point C,

$$\frac{z_2 - z_3}{z_1 - z_3} = e^{i\pi/2} \quad (1)$$

Applying rotation about point B,

$$\frac{z_1 - z_2}{z_3 - z_2} = \sqrt{2}e^{i\pi/4} \quad (2)$$

Applying rotation about point A,

$$\frac{z_2 - z_1}{z_3 - z_1} = \sqrt{2}\,e^{-i\pi/4} \quad (3)$$

Multiplying (2) and (3), we get

$$\frac{(z_1 - z_2)(z_2 - z_1)}{(z_3 - z_2)(z_3 - z_1)} = 2$$

$$(z_1 - z_2)^2 = -2(z_3 - z_2)(z_3 - z_1)$$
$$= 2(z_1 - z_3)(z_3 - z_2)$$

10.

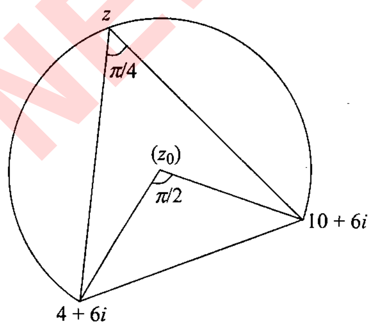


Fig. 2.101

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$$

Locus of z is major arc whose centre is at z_0. Applying rotation at z_0, we have

$$\frac{z_0 - (10 + 6i)}{z_0 - (4 + 6i)} = \frac{|z_0 - (10 + 6i)|}{|z_0 - (4 + 6i)|}e^{i\frac{\pi}{2}}$$

$$\Rightarrow \frac{z_0 - (10 + 6i)}{z_0 - (4 + 6i)} = i$$

$$\Rightarrow z_0 - 10 - 6i = iz_0 - 4i + 6$$

$$\Rightarrow z_0 = 7 + 9i$$

Thus centre is at $7 + 9i$ and z is any point on the arc.

Hence, $|z - (7 + 9i)| = |10 + 6i - (7 + 9i)| = 3\sqrt{2}$.

11. Dividing throughout by i, we get

$$z^3 - iz^2 + iz + 1 = 0$$

$$\Rightarrow z^2(z - i) + i(z - i) = 0 \text{ as } 1 = -i^2$$

$$\Rightarrow (z - i)(z^2 + i) = 0$$

$$\Rightarrow z = i \text{ or } z^2 = -i$$

$$\Rightarrow |z| = |i| = 1 \text{ or } |z^2| = |z|^2 = |-i| = 1$$

$$\Rightarrow |z| = 1$$

Hence, in either case $|z| = 1$.

12. Let $z = |z|e^{i\alpha}$ and $w = |w|e^{i\beta}$. Now,

$$|z - w|^2 = |z|^2 + |w|^2 - z\bar{w} - \bar{z}w$$

$$= (|z| - |w|)^2 + 2|z||w| - |z||w|e^{i(\alpha-\beta)} - |z||w|e^{-i(\alpha-\beta)}$$

$$= (|z| - |w|)^2 + |z||w|(2 - 2\cos(\alpha - \beta))$$

$$\leq (|z| - |w|)^2 + 4\sin^2\left(\frac{\alpha - \beta}{2}\right)^2 \quad (\because |z| \leq 1, |w| \leq 1)$$

$$\leq (|z| - |w|)^2 + 4\left(\frac{\alpha - \beta}{2}\right)^2 \quad [\because \sin\theta < \theta \text{ for } \theta \in (0, \pi/2)]$$

$$= (|z| - |w|)^2 + 4(\alpha - \beta)^2$$

$$= (|z| - |w|)^2 + (\arg z - \arg w)^2$$

13. Let $z = x + iy$. Then,

$$\bar{z} = iz^2$$

$$\Rightarrow x - iy = i(x^2 - y^2 + 2ixy)$$

$$\Rightarrow x - iy = i(x^2 - y^2) - 2xy$$

$$\Rightarrow x(1 + 2y) = 0 \quad (1)$$

and

$$x^2 - y^2 + y = 0 \quad (2)$$

From (1), $x = 0$ or $y = -1/2$. From (2), when $x = 0$, $y = 0$, 1 and when $y = -1/2$, $x = \pm(\sqrt{3}/2)$. For non-zero complex number z,

$$z = i, \frac{\sqrt{3}}{2} - \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2}$$

14. Given that z_1 is the reflection of z_2 through the line

$$b\bar{z} + \bar{b}z = c \quad (1)$$

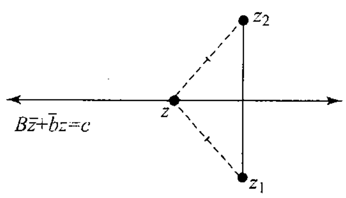


Fig. 2.102

Therefore, for any arbitrary point z on the line, we must have

$$|z - z_1| = |z - z_2|$$

$$\Rightarrow |z - z_1|^2 = |z - z_2|^2$$

$\Rightarrow \quad |z|^2 + |z_1|^2 - z\bar{z}_1 - \bar{z}z_1 = |z|^2 + |z_2|^2 - z\bar{z}_2 - \bar{z}z_2$

$\Rightarrow \quad (\bar{z}_2 - \bar{z}_1)z + (z_2 - z_1)\bar{z} = |z_2|^2 - |z_1|^2 \qquad (1)$

Comparing (1) with (2), we have

$b = z_2 - z_1$ and $c = |z_2|^2 - |z_1|^2$

$\Rightarrow \quad \bar{z}_1 b + z_2\bar{b} = \bar{z}_1(z_2 - z_1) + z_2(\bar{z}_2 - \bar{z}_1) = |z_2|^2 - |z_1|^2 = c$

15.

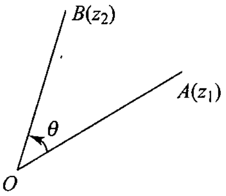


Fig. 2.103

Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$. Then,

$z_1 + z_2 = -p,\ z_1z_2 = q$

Also,

$$\frac{z_2}{z_1} = e^{i\theta} \Rightarrow z_2 = z_1e^{i\theta}$$

$\Rightarrow \quad z_1(1 + e^{i\theta}) = -p,\ z_1^2e^{i\theta} = q$

$$z_1^2 = q\,e^{-i\theta} = \frac{p^2}{(1+e^{i\theta})^2}$$

$\Rightarrow \quad p^2 = qe^{-i\theta}(1 + e^{2i\theta} + 2e^{i\theta})$

$= q(e^{-i\theta} + e^{i\theta} + 2)$

$= q(2\cos\theta + 2)$

$= 4q\cos^2\dfrac{\theta}{2}$

16. Given that z and ω are two complex numbers. To prove

$|z|^2 w - |w|^2 z = z - w \Leftrightarrow z = w$ or $z\bar{w} = 1$

First let us consider

$|z|^2 w - |w|^2 z = z - w \qquad (1)$

$\Rightarrow \quad z(1 + |w|^2) = w(1 + |z|^2)$

$$\Rightarrow \quad \frac{z}{w} = \frac{1+|z|^2}{1+|w|^2} = \text{a real number}$$

$$\Rightarrow \quad \overline{\left(\frac{z}{w}\right)} = \frac{z}{w} \Rightarrow \frac{\bar{z}}{\bar{w}} = \frac{z}{w}$$

$\Rightarrow \bar{z}w = z\bar{w} \qquad (2)$

Again from Eq. (1),

$z\bar{z}\,w - w\bar{w}z = z - w$

$z(\bar{z}w - 1) - w(\bar{w}z - 1) = 0$

$z(z\bar{w} - 1) - w(z\bar{w} - 1) = 0 \qquad$ [Using Eq. (2)]

$\Rightarrow \quad (z\bar{w} - 1)(z - w) = 0$

$\Rightarrow \quad z\bar{w} = 1$ or $z = w$

Conversely if $z = w$, then L.H.S. of (1) is $|w|^2 w - |w|^2 w = 0$ and R.H.S. of (1) is $w - w = 0$. Therefore, Eq. (1) holds. Also, if $\bar{w}z = 1$, then $w\bar{z} = 1$. L.H.S. of (1) is $z\bar{z}w - w\bar{w}z = zz\bar{w} - ww\bar{z}$ = R.H.S. Hence proved.

17. $z^{p+q} - z^p - z^q + 1 = 0$

$\Rightarrow \quad (z^p - 1)(z^q - 1) = 0$

$\Rightarrow \quad z = (1)^{1/p}$ or $(1)^{1/q} \qquad (1)$

where p and q are distinct prime numbers. Hence both the equations will have distinct roots and as $z \neq 01$, both will be simultaneously zero for any value of z given by Eq. (1). Also,

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1} = \frac{1-\alpha^p}{1-\alpha} \qquad (\alpha \neq 1)$$

or

$$1 + \alpha + \alpha^2 + \cdots + \alpha^q = \frac{1-\alpha^q}{1-\alpha} \qquad (\alpha \neq 1)$$

Because of (1), either $\alpha^p = 1$ or $\alpha^q = 1$ but not both simultaneously as p and q are distinct primes.

18. Given that $|z_1| < 1 < |z_2|$. Now,

$$\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$$

$\Rightarrow \quad |1 - z_1\bar{z}_2| < |z_1 - z_2|$

$\Rightarrow \quad |1 - z_1\bar{z}_2|^2 < |z_1 - z_2|^2$

$\Rightarrow \quad (1 - z_1\bar{z}_2)\left(\overline{1 - z_1\bar{z}_2}\right) < (z_1 - z_2)\left(\overline{z_1 - z_2}\right)$

$\Rightarrow \quad (1 - z_1\bar{z}_2)(1 - \bar{z}_1z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$\Rightarrow \quad 1 - z_1\bar{z}_2 - \bar{z}_1z_2 + z_1\bar{z}_1z_2\bar{z}_2 < z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1z_2 + z_2\bar{z}_2$

$\Rightarrow \quad 1 + |z_1|^2|z_2|^2 < |z_1|^2 + |z_2|^2$

$\Rightarrow \quad (1 - |z_1|^2)(1 - |z_2|^2) < 0$

which is obviously true as

$|z_1| < 1 < |z_2|$

$\Rightarrow \quad |z_1|^2 < 1 < |z_2|^2$

$\Rightarrow \quad (1 - |z_1|^2) > 0$ and $(1 - |z_2|^2) < 0$

19. $\displaystyle\sum_{r=1}^{n} a_r\, z^r = 1$ (where $|a_r| < 2$)

$\Rightarrow \quad a_1z + a_2z^2 + a_3z^3 + \cdots + a_nz^n = 1$

$\Rightarrow \quad |a_1z + a_2z^2 + a_3z^3 + \cdots + a_nz^n| = 1 \qquad (1)$

$\Rightarrow \quad 1 = |a_1z + a_2z^2 + a_3z^3 + \cdots + a_nz^n|$

$\leq |a_1z| + |a_2z^2| + \cdots + |a_nz^n|$

$= |a_1||z| + |a_2||z^2| + |a_3||z^3| + \cdots + |a_n||z^n|$

$< 2\,[|z| + |z|^2 + |z|^3 + \cdots + |z|^n] \qquad (\because |a_r| < 2,\ \forall\, r$ and $|z^n| = |z|^n)$

$$= 2\left[\frac{|z|(1 - |z|^n)}{1 - |z|}\right]$$

$$= 2\left[\frac{|z| - |z|^{n+1}}{1 - |z|}\right]$$

$\Rightarrow \quad 2[|z| - |z|^{n+1}] > 1 - |z| \qquad (\because 1 - |z| > 0$ as $|z| < 1/3)$

$\Rightarrow \frac{3}{2}|z| > \frac{1}{2} + |z|^{n+1}$

$\Rightarrow |z| > \frac{1}{3} + \frac{2}{3}|z|^{n+1}$

$\Rightarrow |z| > \frac{1}{3}$

which is a contradiction. Hence, there exists no such complex number.

20.

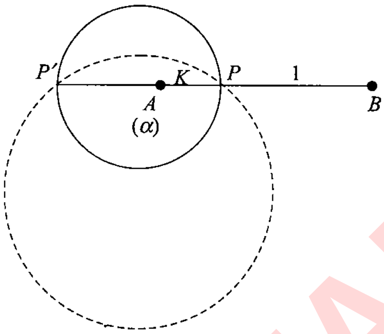


Fig. 2.104

$\left|\frac{z-\alpha}{z-\beta}\right| = k$

$\Rightarrow |z-\alpha| = k|z-\beta|$

Let points A, B and P represent complex numbers α, β and z, respectively. Then,

$|z-\alpha| = k|z-\beta|$

Therefore, z is the complex number whose distance from A is k times its distance from B, i.e.,

$PA = k\,PB$

Hence, P divides AB in the ratio k:1 internally or externally (at P'). Then,

$P \equiv \left(\frac{k\beta+\alpha}{k+1}\right)$ and $P' \equiv \left(\frac{k\beta-\alpha}{k-1}\right)$

Now through PP' there can pass a number of circles, but with given data we can find radius and centre of that circle for which PP' is diameter. Hence the centre is the mid-point of PP', and is given by

$$\frac{\left(\frac{k\beta+\alpha}{k+1} + \frac{k\beta-\alpha}{k-1}\right)}{2}$$

$$= \frac{k^2\beta + k\alpha - k\beta - \alpha + k^2\beta - k\alpha + k\beta - \alpha}{2(k^2-1)}$$

$$= \frac{k^2\beta - \alpha}{k^2-1}$$

$$= \frac{\alpha - k^2\beta}{1-k^2}$$

Radius $= \frac{1}{2}|PP'|$

$$= \frac{1}{2}\left|\frac{k\beta+\alpha}{k+1} - \frac{k\beta-\alpha}{k-1}\right|$$

$$= \frac{1}{2}\left|\frac{k^2\beta + k\alpha - k\beta - \alpha - k^2\beta + k\alpha - k\beta + \alpha}{k^2-1}\right|$$

$$= \frac{k|\alpha-\beta|}{|1-k^2|}$$

21. The given circle is $|z-1| = \sqrt{2}$ where $z_0 = 1$ is the centre and $\sqrt{2}$ is radius of the circle. z_1 is one of the vertices of the square inscribed in the given circle.

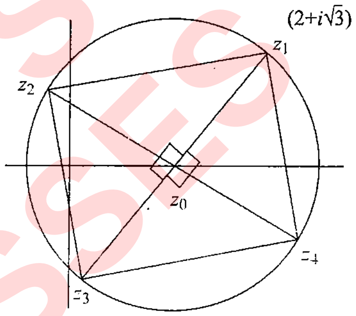


Fig. 2.105

Clearly, z_2 can be obtained by rotating z_1 by an angle of 90° in anticlockwise sense about centre z_0. Thus,

$z_2 - z_0 = (z_1 - z_0)\,e^{i\pi/2}$

$\Rightarrow z_2 - 1 = (2 + i\sqrt{3} - 1)i$

$\Rightarrow z_2 = i - \sqrt{3} + 1$

$\Rightarrow z_2 = (1-\sqrt{3}) + i$

Now z_0 is mid-point of z_1 and z_3 and z_2 and z_4

$\therefore \frac{z_1+z_3}{2} = z_0 \Rightarrow \frac{2+i\sqrt{3}+z_3}{2} = 1$

$\Rightarrow z_3 = -i\sqrt{3}$

and

$\frac{z_2+z_4}{2} = z_0 \Rightarrow \frac{(1-\sqrt{3})+i+z_4}{2} = 1$

$\Rightarrow z_4 = (\sqrt{3}+1) - i$

22. Let $u = \frac{1}{1-z} \Rightarrow z = 1 - \frac{1}{u}$

$|z| = 1 \Rightarrow = \left|1-\frac{1}{u}\right| = 1$

$\Rightarrow |u-1| = |u|$

$\therefore$ locus of u is perpendicular bisector of line segment joining 0 and 1

$\Rightarrow$ maximum arg u approaches $\frac{\pi}{2}$ but will not attain.

23. $z = \frac{2i(x+iy)}{1-(x+iy)^2} = \frac{2i(x+iy)}{1-(x^2-y^2+2ixy)}$

Using $1-x^2 = y^2$

$z = \frac{2ix-2y}{2y^2-2ixy} = -\frac{1}{y}$.

$\because -1 \le y \le 1 \Rightarrow -\frac{1}{y} \le -1$ or $-\frac{1}{y} \ge 1$.

Objective Type

Fill in the blanks

1. Let,

$$z=\frac{\sin\frac{x}{2}+\cos\frac{x}{2}+i\tan x}{1+2i\sin\frac{x}{2}}$$

$$=\frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}+i\tan x\right)\left(1-2i\sin\frac{x}{2}\right)}{\left(1+2i\sin\frac{x}{2}\right)\left(1-2i\sin\frac{x}{2}\right)}$$

$$=\frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}+2\sin\frac{x}{2}\tan x\right)+i\left(\tan x-2\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{1+4\sin^2\frac{x}{2}}$$

Now,

Im(z) = 0 (as z is real)

$$\Rightarrow \tan x-2\sin\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)=0$$

$$\Rightarrow \frac{\sin x}{\cos x}-2\sin^2(x/2)-2\sin(x/2)\cos(x/2)=0$$

$$\Rightarrow \frac{\sin x}{\cos x}-(1-\cos x)-\sin x=0$$

$$\Rightarrow \sin x\left[\frac{1}{\cos x}-1\right]-[1-\cos x]=0$$

$$\Rightarrow (1-\cos x)\left[\frac{\sin x}{\cos x}-1\right]=0$$

$\Rightarrow$ $\cos x = 1 \Rightarrow x = 2n\pi$ or $\tan x = 1 \Rightarrow x = n\pi + \pi/4$, $n \in Z$

2. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$= a^2|z_1|^2 + b^2|z_2|^2 - 2ab\,\text{Re}(z_1\bar{z}_2) + b^2|z_1|^2 + a^2|z_2|^2 + 2ab \times \text{Re}(z_1\bar{z}_2)$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

3. As $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ from an equilateral triangle, therefore

$|z_1 - z_3| = |z_2 - z_3| = |z_1 - z_2|$

$\Rightarrow$ $|a + i| = |1 + bi| = |(a - 1) + i(1 - b)|$

$\Rightarrow$ $a^2 + 1 = 1 + b^2 = (a - 1)^2 + (1 - b)^2$

$\Rightarrow$ $a^2 = b^2 = a^2 + b^2 - 2a - 2b + 1$

$\Rightarrow$ $a = b$ $\quad (\because a, b > 0 \therefore a \neq -b)$ $\quad$ (1)

and

$b^2 - 2a - 2b + 1 = 0$ $\quad$ (2)

Solving $a^2 - 2a - 2a + 1 = 0$, we get

$a^2 - 4a + 1 = 0$

$$\Rightarrow a=\frac{4\pm2\sqrt{3}}{2}=2\pm\sqrt{3}$$

But $0 < a, b < 1$.

$\therefore$ $a = 2 - \sqrt{3}$ and $b = 2 - \sqrt{3}$

4.

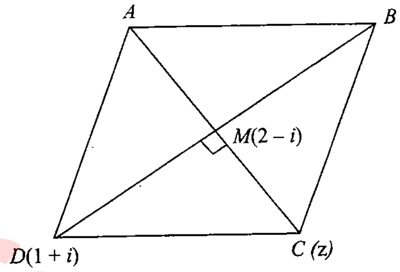


Fig. 2.106

Rotating DM about M by an angle 90°, we have

$$\frac{z-(2-i)}{(1+i)-(2-i)}=\frac{|z-(2-i)|}{|(1+i)-(2-1)|}e^{\pm i\frac{\pi}{2}}$$

$$\Rightarrow \frac{z-(2-i)}{-1+2i}=\pm\frac{i}{2}$$

$\Rightarrow$ $2z = (-i - 2) + (4 - 2i)$ or $(i + 2) + (4 - 2i)$

$$\Rightarrow z=1-\frac{3}{2}i \text{ or } 3-\frac{i}{2}$$

5. Let z_1, z_2, z_3 be the vertices A, B and C, respectively, of equilateral ΔABC, inscribed in a circle $|z| = 2$ with centre (0, 0) and radius = 2. Given $z_1 = 1 + i\sqrt{3}$.

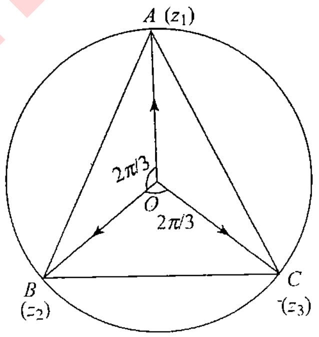


Fig. 2.107

Rotating OA about O by an angle $2\pi/3$, we have

$$\frac{z-0}{1+i\sqrt{3}-0}=\frac{|z-0|}{|1+i\sqrt{3}-0|}e^{\pm i\frac{2\pi}{3}}$$

$$\Rightarrow z=(1+i\sqrt{3})\left(\cos\frac{2\pi}{3}\pm i\sin\frac{2\pi}{3}\right)$$

$$\Rightarrow z=(1+i\sqrt{3})\left(-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow z=-\frac{(1+i\sqrt{3})^2}{2} \text{ or } -\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{2}$$

$$\Rightarrow z=1-i\sqrt{3} \text{ or } -2$$

6. $S = 1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) + \cdots + (n-1)(n-\omega)(n-\omega^2)$

Here,

$T_n = (n-1)(n-\omega)(n-\omega^2)$

$= n^3 - 1$

$$S=\sum_{n=2}^{n}(n^3-1)$$

$$= \sum_{n=1}^{n}(n^3-1)$$

$$= \left[\left(\frac{n(n+1)}{2}\right)^2 - n\right]$$

$$= \frac{n^2(n^2+2n+1)-4n}{4}$$

$$= \frac{1}{4}n(n^3+2n^2+n-4)$$

$$= \frac{1}{2}n[n-1]\left[n^2+3n+4\right]$$

True or false

1. Let $z = x + iy$. Then

$1 \cap z \Rightarrow 1 \;\; \le x$ and $0 \le y$ (by definition)

$$\frac{1-z}{1+z} = \frac{1-(x+iy)}{1+(x+iy)}$$

$$= \frac{(1-x)-iy}{(1+x)+iy} \times \frac{(1+x)-iy}{(1+x)-iy}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2} - \frac{iy(1-x+1+x)}{(1+x)^2+y^2}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2} - \frac{2iy}{(1+x)^2+y^2}$$

Now,

$$\frac{1-z}{1+z} \le 0 \Rightarrow \frac{1-x^2-y^2}{(1+x)^2+y^2} \le 0 \text{ and } \frac{-2y}{(1+x)^2+y^2} \le 0$$

$\Rightarrow \; 1 - x^2 - y^2 \le 0$ and $-2y \le 0$

$\Rightarrow \; x^2 + y^2 \ge 1$ and $y \ge 0$

which is true as $x > 1$ and $y > 0$. Therefore, the given statement is true, $\forall\, z \in C$.

2. As $|z_1| = |z_2| = |z_3|$, therefore, z_1, z_2, z_3 are equidistant from origin.

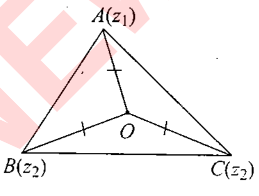


Fig. 2.108

Hence O is circumcentre of ΔABC. But according to the question, ΔABC is equilateral and we know that in an equilateral triangle circumcnetre and centroid coincide. Hence, centroid of ΔABC is 0. Hence,

$$\frac{z_1+z_2+z_3}{3} = 0$$

$\Rightarrow \; z_1 + z_2 + z_3 = 0$

Therefore, the statement is true.

3. If z_1, z_2, z_3 are in A.P., then $(z_1 + z_3)/2 = z_2$. So, z_2 is mid-point of line joining z_1 and z_3. Hence, z_1, z_2, z_3 lie on a straight line. Hence, given statement is false.

4. See the theory of cube roots of unity. The given statement is true.

Multiple choice questions with one correct answer

1. b. $\left(\dfrac{x-1}{-2}\right)^3 = 1$

$\Rightarrow \; \dfrac{x-1}{-2} = 1, \omega, \omega^2$

$\Rightarrow \; x = -1,\ 1-2\omega,\ 1-2\omega^2$

2. d. $\dfrac{1+i}{1-i} = \dfrac{(1+i)^2}{(1-i)(1+i)} = \dfrac{1-1+2i}{2} = i$

Now $i^n = 1$. Hence, the smallest positive integral value of n should be 4.

3. a. We know that $|z - z_1| = |z - z_2|$. Then locus of z is the line, which is a perpendicular bisector of line segment joining z_1 and z_2. Hence,

$z = x + iy$

$\Rightarrow \; |z - 5i| = |z + 5i|$

Therefore, z remains equidistant from $z_1 = 5i$ and $z_2 = -5i$. Hence, z lies on perpendicular bisector of line segment joining z_1 and z_2, which is clearly the real axis or $y = 0$.

Alternative solution:

$$\left|\frac{z-5i}{z+5i}\right| = 1$$

$\Rightarrow \; |x + iy - 5i| = |x + iy + 5i|$

$\Rightarrow \; |x + (y-5)i| = |x + (y+5)i|$

$\Rightarrow \; x^2 + (y-5)^2 = x^2 + (y+5)^2$

$\Rightarrow \; x^2 + y^2 - 10y + 25 = x^2 + y^2 + 10y + 25$

$\Rightarrow \; 20y = 0$

$\Rightarrow \; y = 0$

4. b. $z = \left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^5 + \left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)^5$

$$= \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5 + \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^5$$

$$= \left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) + \left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)$$

$$= 2\cos\frac{5\pi}{6}$$

$$= -\sqrt{3}$$

$\Rightarrow \; \text{Re}(z) < 0$ and $\text{Im}(z) = 0$

Alternative solution:

$z = \bar{z}_1 + \bar{z}_1$

where

$$\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5$$

$\Rightarrow \; z$ is real

$\Rightarrow \; \text{Im}(z) = 0$

5. d. $|z-4| < |z-2|$

$\Rightarrow |(x-4)+iy| < |(x-2)+iy|$

$\Rightarrow (x-4)^2+y^2 < (x-2)^2+y^2$

$\Rightarrow -8x+16 < -4x+4$

$\Rightarrow 4x-12 > 0$

$\Rightarrow x > 3$

$\Rightarrow \text{Re}(z) > 3$

6. b. $|\omega| = 1$

$$\Rightarrow \left|\frac{1-iz}{z-i}\right| = 1$$

$\Rightarrow |1-iz| = |z-i|$

$\Rightarrow |-i|\,|z+i| = |z-i|$

$\Rightarrow |z+i| = |z-i|$

Hence, z is equidistant from $(0, -1)$ and $(0, 1)$. So, z lies on perpendicular bisector of $(0, -1)$ and $(0, 1)$. i.e., x-axis, and $y = 0$. Therefore, z lies on real axis.

7. b. If vertices of a parallelogram are z_1, z_2, z_3, z_4, then as diagonals bisect each other as given,

$$\frac{z_1+z_3}{2} = \frac{z_2+z_4}{2}$$

$\Rightarrow z_1+z_3 = z_2+z_4$

8. d. Let $z_1 = \sin x + i\cos 2x$; $z_2 = \cos x - i\sin 2x$. Then

$\bar{z}_1 = z_2$

$\Rightarrow \sin x - i\cos 2x = \cos x - i\sin 2x$

$\sin x = \cos x$ and $\cos 2x = \sin 2x$

$\Rightarrow \tan x = 1$ and $\tan 2x = 1$

$$\Rightarrow x = \frac{\pi}{4} \text{ and } x = \frac{\pi}{8}$$

which is not possible. Hence, there is no value of x.

9. b. $(1+\omega)^7 = A + B\omega$

$\Rightarrow (-\omega^2)^7 = A + B\omega \quad (\because 1+\omega+\omega^2 = 0)$

$\Rightarrow -\omega^{14} = A + B\omega$

$\Rightarrow -\omega^2 = A + B\omega \quad (\because \omega^3 = 1)$

$\Rightarrow 1 + \omega = A + B\omega$

$\Rightarrow A = 1, B = 1$

10. d. We have,

$|z| = |\omega|$ and $\arg z = \pi - \arg \omega$

Let $\omega = re^{i\theta}$. Then

$z = re^{i(\pi-\theta)}$

$\Rightarrow z = re^{i\pi}e^{-i\theta} = (re^{-i\theta})(\cos\pi + i\sin\pi)$

$= \bar{\omega}(-1) = -\bar{\omega}$

11. c. We have,

$2 = |z - i\omega| \le |z| + |\omega| \quad (\because |z_1+z_2| \le |z_1| + |z_2|)$

$\therefore |z| + |\omega| \ge 2$ (i)

But given that $|z| \le 1$ and $|\omega| \le 1$. Hence,

$\Rightarrow |z| + |\omega| \le 2$ (ii)

From (i) and (ii),

$|z| = |\omega| = 1$

Also,

$|z + i\omega| = |z - i\bar{\omega}|$

$\Rightarrow |z - (-i\omega)| = |z - i\bar{\omega}|$

Hence, z lies on perpendicular bisector of the line segment joining $(-i\omega)$ and $(i\bar{\omega})$, which is a real axis, as $(-i\omega)$ and $(i\bar{\omega})$ are conjugate to each other. For z, $\text{Im}(z) = 0$. If $z = x$, then

$|z| \le 1 \Rightarrow x^2 \le 1$

$\Rightarrow -1 \le x \le 1$

12. d. $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$

$= \left[(1+i)^{n_1} + (1-i)^{n_1}\right] + \left[(1+i)^{n_2} + (1-i)^{n_2}\right]$

$= \left[(1+i)^{n_1} + \overline{(1+i)^{n_1}}\right] + \left[(1+i)^{n_2} + \overline{(1+i)^{n_2}}\right]$

$=$ [purely real number] + [purely real number]

Hence, n_1 and n_2 are any integers.

13. c. $E = 4 + 5(\omega)^{334} + 3(\omega)^{365}$

$= 4 + 5\omega + 3\omega^2$

$= 1 + 2\omega + 3(1 + \omega + \omega^2)$

$= 1 + (-1 + i\sqrt{3})$

$= i\sqrt{3}$

14. a. $\arg(-z) - \arg(z) = \arg\left(\frac{-z}{z}\right) = \arg(-1) = \pi$

15. a. $|z_1| = |z_2| = |z_3|$ (given)

Now,

$|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\bar{z}_1 = 1$

Similarly,

$z_2\bar{z}_2 = 1, z_3\bar{z}_3 = 1$

Now,

$$\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$$

$\Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1$

$\Rightarrow \left|\overline{z_1 + z_2 + z_3}\right| = 1$

$\Rightarrow |z_1 + z_2 + z_3| = 1$

16. d. Let,

$z = (1)^{1/n} = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$

$$= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, k = 0, 1, 2, \ldots, n-1$$

Let

$$z_1 = \cos\left(\frac{2k_1\pi}{n}\right) + i\sin\left(\frac{2k_1\pi}{n}\right)$$

and

$$z_2 = \cos\left(\frac{2k_2\pi}{n}\right) + i\sin\left(\frac{2k_2\pi}{n}\right)$$

be the two values of z such that they subtend angle of 90° at origin. Then,

$$\Rightarrow \frac{2k_1\pi}{n} - \frac{2k_2\pi}{n} = \pm\frac{\pi}{2} \Rightarrow 4(k_1 - k_2) = \pm n$$

As k_1 and k_2 are integers and $k_1 \ne k_2$, therefore $n = 4m$, $m \in Z$.

17. c. $$\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$$

$$\Rightarrow \arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \arg\left(\frac{1 - i\sqrt{3}}{2}\right)$$

Hence, the angle between $z_1 - z_3$ and $z_2 - z_3$ is 60°. Also,

$$\left|\frac{z_1 - z_3}{z_2 - z_3}\right| = \left|\frac{1 - i\sqrt{3}}{2}\right|$$

$$\Rightarrow \left|\frac{z_1 - z_3}{z_2 - z_3}\right| = 1$$

$\Rightarrow$ $|z_1 - z_3| = |z_2 - z_3|$

Hence, the triangle with vertices z_1, z_2 and z_3 is isosceles with vertical angle 60°. Hence rest of the two angles should also be 60° each. Therefore, the required triangle is an equilateral triangle.

18. b. $|z_1| = 12$. Therefore, z_1 lies on a circle with centre (0, 0) and radius 12 units. As $|z_2 - 3 - 4i| = 5$, so z_2 lies on a circle with centre (3, 4) and radius 5 units.

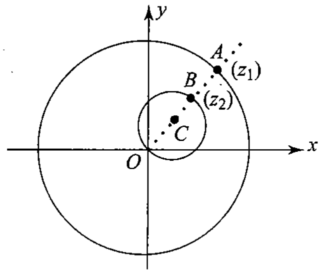


Fig. 2.109

From the above figure it is clear that $|z_1 - z_2|$, i.e., distance between z_1 and z_2 will be minimum when they lie at A and B, respectively, i.e., on diagram as shown. Then $|z_1 - z_2| = AB = OA - OB = 12 - 2(5) = 2$. As it is the minimum value, we must have $|z_1 - z_2| \geq 2$.

19. a. $\omega = \dfrac{z-1}{z+1}$

$$\Rightarrow z = \frac{1+\omega}{1-\omega}$$

Now,

$|z| = 1$

$$\Rightarrow \left|\frac{1+\omega}{1-\omega}\right| = 1$$

$\Rightarrow$ $|\omega + 1| = |\omega - 1|$

Therefore, ω is equidistant from (1, 0) and (–1, 0) and hence must lie on perpendicular bisector of line segment joining (1, 0) and (–1, 0), i.e., imaginary axis. Hence, ω is purely imaginary, i.e., $\text{Re}(\omega) = 0$.

20. b. $(1 + \omega^2)^n = (1 + \omega^4)^n$

$\Rightarrow$ $(-\omega)^n = (1 + \omega)^n = (-\omega^2)^n$

$\Rightarrow$ $\omega^n = 1$

Hence, the least positive value of n is 3.

21. a. Here we observe that

$PA = AQ = AR = 2$

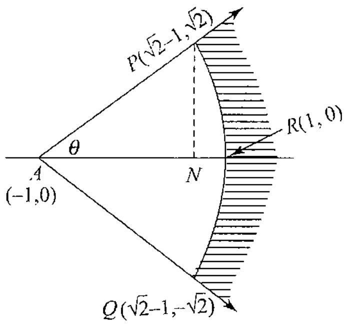


Fig. 2.110

Therefore, PRQ is an arc of a circle with centre at A and radius 2. Shaded region is outer (exterior) part of the sector $APRQA$.

Hence, for any point x on arc PRQ, we should have

$|z - (-1)| = 2$

and for shaded region,

$|z + 1| > 2$ (1)

Also,

$$\tan\theta = \frac{PN}{AN} = \frac{\sqrt{2}}{(\sqrt{2}-1)-(-1)} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

$\Rightarrow$ $\theta = \pi/4$

and by symmetry, $\arg(z + 1)$ varies from $-\pi/4$ to $\pi/4$ as it moves from Q to P on arc QRP. Hence, for shaded region, we also have

$-\pi/4 < \arg(z + 1) < \pi/4$

or

$|\arg(z + 1)| < \pi/4$ (2)

Combining (i) and (ii), we find that (a) is the correct option.

22. b. Given that a, b, c are integers not all equal, ω is cube root of unity $\neq 1$. Then

$$|a + b\omega + c\omega^2| = \left|a + b\left(\frac{-1+i\sqrt{3}}{2}\right) + c\left(\frac{-1-i\sqrt{3}}{2}\right)\right|$$

$$= \left|\left(\frac{2a-b-c}{2}\right) + i\left(\frac{b\sqrt{3}-c\sqrt{3}}{2}\right)\right|$$

$$= \frac{1}{2}\sqrt{(2a-b-c)^2 + 3(b-c)^2}$$

$$= \frac{1}{2}\sqrt{4a^2+b^2+c^2-4ab+2bc-4ac+3b^2+3c^2-6bc}$$

$$= \sqrt{a^2+b^2+c^2-ab-bc-ca}$$

$$= \sqrt{\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]}$$

R.H.S. will be minmum when $a = b = c$, but we cannot take $a = b = c$ as per the question. Hence, the minimum value is obtained when any two are zero and third is a minimum magnitude integer, i.e., 1. Thus $b = c = 0$, $a = 1$ gives us the minimum value of 1.

23. b. If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that they are similar, then

$$\begin{vmatrix} a & u & 1 \\ b & v & 1 \\ c & w & 1 \end{vmatrix} = 0$$

or

$$\frac{a-c}{a-b} = \frac{u-w}{u-v} = r$$

24. c. Let $z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$. Also,

$|z_1 + z_2| = |z_1| + |z_2|$

$\Rightarrow$ $|z_1 + z_2|^2 = (|z_1| + |z_2|)^2$

$\Rightarrow$ $|z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2) = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow$ $\text{Re}(z_1\bar{z}_2) = 2|z_1||z_2|$

$\Rightarrow$ $2|z_1||z_2|\cos(\theta_1 - \theta_2) = 2|z_1||z_2|$

$\Rightarrow$ $\cos(\theta_1 - \theta_2) = 1$

$\Rightarrow$ $\theta_1 - \theta_2 = 0$

$\Rightarrow$ $\arg z_1 - \arg z_2 = 0$

25. d. Let $z = \cos(2\pi/7) + i \sin(2\pi/7)$. Then by De Moivre's theorem, we have

$$z^k = \cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7}$$

Now,

$$\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right) = \sum_{k=1}^{6}(-i)\left(\cos\frac{2\pi k}{7} + i\sin\frac{2k\pi}{7}\right)$$

$$= (-i)\sum_{k=1}^{6} z^k$$

$$= -i\frac{z(1-z^6)}{1-z}$$

$$= -i\left(\frac{z-z^7}{1-z}\right)$$

$$= (-i)\left(\frac{z-1}{1-z}\right)$$

[Using $z^7 = \cos 2\pi + i\sin 2\pi = 1$]

$$= (-i)\left(\frac{1-z}{1-z}\right)$$

$$= i$$

26. d. We have,

$$(1+\omega-\omega^2)^7 = (-\omega^2-\omega^2)^7$$
$$= (-2)^7(\omega^2)^7$$
$$= -128\omega^{14}$$
$$= -128\omega^2$$

27. b. $\displaystyle\sum_{i=1}^{13}(i^n + i^{n+1}) = \sum_{i=1}^{13} i^n(1+i)$

$$= (1+i)\sum_{i=1}^{13} i^n$$

$$= i(1+i)\frac{(1-i^{13})}{1-i}$$

$$= i - 1 \text{ as } i^{13} = i$$

28. d. Taking $-3i$ common from C_2, we get

$$-3i\begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} = 0 \qquad (\because C_2 \equiv C_3)$$

$\Rightarrow$ $x = 0, y = 0$

29. b. Operating $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 3 & 0 & 0 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} = 3[-\omega^4 - \omega^6 - \omega^4]$$

$$= 3(-1-2\omega)$$
$$= 3(\omega^2-\omega)$$
$$= 3\omega(\omega-1)$$

30. d. Since $(w - \bar{w}z)/(1-z)$ is purely real, therefore

$$\overline{\left(\frac{w-\bar{w}z}{1-z}\right)} = \left(\frac{w-\bar{w}z}{1-z}\right)$$

$\Rightarrow$ $\dfrac{\bar{w}-w\bar{z}}{1-\bar{z}} = \dfrac{w-\bar{w}z}{1-z}$

$\Rightarrow$ $\bar{w} - \bar{w}z - w\bar{z} + wz\bar{z} = w - w\bar{z} - \bar{w}z + \bar{w}z\bar{z}$

$\Rightarrow$ $w - \bar{w} = (w-\bar{w})|z|^2$

$\Rightarrow$ $|z|^2 = 1 \qquad (\because w = \alpha + i\beta \text{ and } \beta \neq 0)$

$\Rightarrow$ $|z| = 1$

Also given $z \neq 1$. Therefore, the required set is $\{z : |z| = 1, z \neq 1\}$.

31. d. $\overline{OP} = \overline{OA} + \overline{AP}$

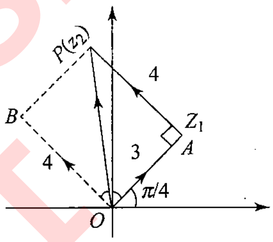


Fig. 2.111

Rotating OA by an angle 45° in anticlockwise direction to get OP, we have

$$\frac{z_2-0}{z_1-0} = \frac{|z_2|}{|z_1|}e^{i\theta} \text{ (where } \tan\theta = 4/3)$$

$\Rightarrow$ $\dfrac{z_2-0}{3e^{i\pi/4}} = \dfrac{5}{3}(\cos\theta + i\sin\theta)$

$\Rightarrow$ $\dfrac{z_2-0}{e^{i\pi/4}} = 5\left(\dfrac{3}{5} + i\dfrac{4}{5}\right)$

$\Rightarrow$ $z_2 = (3+4i)e^{i\pi/4}$

32. d. Given $|z| = 1$ and $z \neq \pm 1$. To find locus of $\omega = z/(1-z^2)$. We have,

$$\omega = \frac{z}{1-z^2} = \frac{z}{z\bar{z}-z^2} \qquad (\because |z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow z\bar{z} = 1)$$

$$= \frac{1}{\bar{z}-z}$$

which is a purely imaginary number. Therfore, ω must lie on y-axis.

33. d.

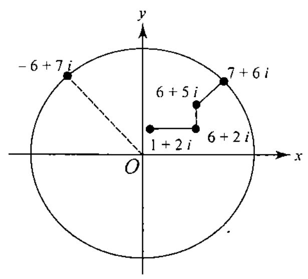


Fig. 2.112

$z_0 \equiv (1+2i)$

$z_1 \equiv (6+5i)$

$z_2 \equiv (-6+7i)$

34. a. $z\bar{z}(\bar{z}^2 + z^2) = 350$

Puting $z = x + iy$, we have

$(x^2+y^2)(x^2-y^2) = 175$

$(x^2+y^2)(x^2-y^2) = 5 \times 5 \times 7$

$x^2 + y^2 = 25$

and

$x^2 - y^2 = 7$

(as other combinations give non-integral values of x and y)

$\therefore \quad x = \pm 4, y = \pm 3 \ (x, y \in I)$

Hence, area is $8 \times 6 = 48$ sq. units.

Multiple choice questions with one or more than one correct answer

1. a, b, c.

We have,

$|z_1| = |z_2| = 1 \Rightarrow a^2 + b^2 = c^2 + d^2 = 1 \quad (1)$

and

$\text{Re}(z_1\bar{z}_2) = 0 \Rightarrow \text{Re}\{(a+ib)(c-id)\} = 0 \Rightarrow ac + bd = 0 \quad (2)$

Now from (1) and (2),

$$a^2 + b^2 = 1 \Rightarrow a^2 + \frac{a^2c^2}{d^2} = 1 \Rightarrow a^2 = d^2 \quad (3)$$

Also,

$$c^2 + d^2 = 1 \Rightarrow c^2 + \frac{a^2c^2}{b^2} = 1 \Rightarrow b^2 = c^2 \quad (4)$$

$$|\omega_1| = \sqrt{a^2+c^2} = \sqrt{a^2+b^2} = 1 \quad \text{[From (1) and (4)]}$$

and

$$|\omega_2| = \sqrt{b^2+d^2} = \sqrt{a^2+b^2} = 1 \quad \text{[From (1) and (4)]}$$

Further,

$$\text{Re}(\omega_1\bar{\omega}_2) = \text{Re}\{(a+ic)(b-id)\}$$

$$= ab + cd$$

$$= ab + \left(-\frac{ac^2}{b}\right) \quad \text{[From (2)]}$$

$$= \frac{ab^2 - ac^2}{b} = 0 \quad \text{[From (4)]}$$

Also,

$$\text{Im}(\omega_1\bar{\omega}_2)\ bc - ad = bc - a\left(-\frac{ac}{b}\right) = \frac{(a^2+b^2)c}{b} = \frac{c}{b} = \pm 1 \neq 0$$

$\therefore \quad |\omega_1| = 1, |\omega_2| = 1$ and $\text{Re}(\omega_1\bar{\omega}_2) = 0$

2. a, d. Let $z_1 = a + ib$, $a > 0$ and $b \in R$; $z_2 = c + id$, $d < 0$, $c \in R$.

Given,

$|z_1| = |z_2|$

$\Rightarrow \quad a^2 + b^2 = c^2 + d^2$

$\Rightarrow \quad a^2 - c^2 = d^2 - b^2 \quad (1)$

Now,

$$\frac{z_1+z_2}{z_1-z_2} = \frac{(a+c)+i(b+d)}{(a-c)+i(b-d)}$$

$$= \frac{[(a^2-c^2)+(b^2-d^2)]+i[(a-c)(b+d)-(a+c)(b-d)]}{(a-c)^2+(b-d)^2}$$

which is a purely imaginary number or zero in case $a + c = b + d = 0$.

3. a, c, d.

Given $z = (1-t)z_1 + tz_2$

$$\Rightarrow \quad z = \frac{(1-t)z_1 + tz_2}{(1-t)+t}$$

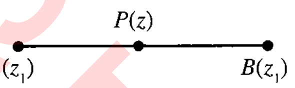
$\Rightarrow$ z divides the line segment joining z_1 and z_2 in ration $(1-t) : t$ internally as $0 < t < 1$

$\Rightarrow$ z, z_1, and z_2 are collinear.

$\Rightarrow$ $\arg(z - z_1) = \arg(z_2 - z_1)$

$$\Rightarrow \quad \frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$$

$$\Rightarrow \quad \begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix} = 0$$



AP + PB = AB

$\Rightarrow \quad |z - z_1| + |z - z_2| = |z_1 - z_2|$

Comprehension

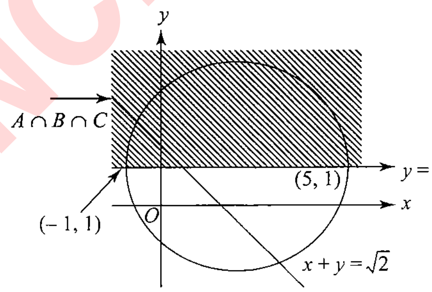


Fig. 2.113

1. b. A is the set of points on and above the line $y = 1$ in the Argand plane. B is the set of points on the circle $(x-2)^2 + (y-1)^2 = 9$ and

$C = \text{Re}(1-i)z = \text{Re}((1-i)(x+iy))$

$\Rightarrow \quad x + y = \sqrt{2}$

Hence $A \cap B \cap C$ has only one point of intersection.

2. c. The points $(-1, 1)$ and $(5, 1)$ are the extremities of a diameter of the given circle. Hence,

$|z + 1 - i|^2 + |z - 5 - i|^2 = 36$

3. d. $||z| - |w|| < |z - w|$ and $|z - w|$ is the distance between z and w. Here, z is fixed. Hence distance between z and w would be maximum for diametrically opposite points. Therefore,

$|z - w| < 6$

$\Rightarrow \quad -6 < |z| - |w| < 6$

$\Rightarrow \quad -3 < |z| - |w| + 3 < 9$

Matrix-match type

a→q

$$\left|\frac{z}{|z|} - i\right| = \left|\frac{z}{|z|} + i\right|, z \neq 0$$

$\frac{z}{|z|}$ is unimodular complex number

and lies on perpendicular bisector of i and $-i$

$\Rightarrow \frac{z}{|z|} = \pm 1 \Rightarrow z = \pm 1 |z|$

$\Rightarrow z$ is real number $\Rightarrow \text{Im}(z) = 0$.

b→p

$|z + 4| + |z - 4| = 10$

z lies on an ellipse whose focus are (4,0) and (–4,0) and length of major axis is 10

$\Rightarrow 2ae = 8$ and $2a = 10 \Rightarrow e = 4/5$

$|\text{Re}(z)| \le 5$.

c→p, t

$|\omega| = 2 \Rightarrow w = 2(\cos\theta + i\sin\theta)$

$\Rightarrow z = x + iy = 2(\cos\theta + i\sin\theta) - \frac{1}{2}(\cos\theta - i\sin\theta)$

$= \frac{3}{2}\cos\theta + i\frac{5}{2}\sin\theta \Rightarrow \frac{x^2}{(3/2)^2} + \frac{y^2}{(5/2)^2} = 1$

$\Rightarrow e^2 = 1 - \frac{9/4}{25/4} = 1 - \frac{9}{25} = \frac{16}{25} \Rightarrow e = \frac{4}{5}$

d→q, t

$|\omega| = 1 \Rightarrow x + iy = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$x + iy = 2\cos\theta$

$|\text{Re}(z)| \le 1$, $\text{Im}(z) = 0$.

Integer type

1. (1)

$\omega = e^{i2\pi/3}$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

Applying ($C_1 \to C_1 + C_2 + C_3$)

$$\Rightarrow z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$\Rightarrow z^3 = 0$

$z = 0$ is only solution.

2.(5) $|z - 3 - 2i| \le 2$

$\Rightarrow z$ lies on or inside the circle radius 2 and centre (3, 2)

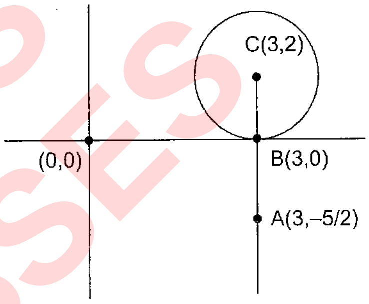


Fig. 2.114

$|2z - 6 + 5i|_{\text{min.}}$

$= 2|z - 3 + (5/2)i|_{\text{min.}}$

= 2(minimum distance of any point on the circle to the point (3, – 5/2)

= 2(5/2) = 5

3.(0) The expression may not attain integral value for all a, b, c

If we consider $a = b = c$, then

$y = a(1 + \omega + \omega^2) = a(1 + i\sqrt{3})$

$z = a(1 + \omega^2 + \omega) = a(1 + i\sqrt{3})$

$\therefore |x|^2 + |y|^2 + |z|^2 = 9|a|^2 + 4|a|^2 + 4|a|^2 = 17|a|^2$

$\therefore \frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = \frac{17}{3}$

***Note:** However if $\omega = e^{i(2\pi/3)}$, then the value of the expression = 3.*

CHAPTER 3

Progression and Series

- Introduction
- Arithmetic Progression (A.P.)
- Geometric Progression (G.P.)
- Harmonic Progression (H.P.)
- Miscellaneous Series

INTRODUCTION

In mathematics, a **sequence** is an ordered list of objects (or events). Like a set, it contains members (also called *elements* or *terms*), and the number of terms (possibly infinite) is called the *length* of the sequence. Unlike a set, order matters, and exactly the same elements can appear multiple times at different positions in the sequence. A sequence is a discrete function.

The '*n*th' term is a formula with 'n' in it which enables you to find **any** term of a sequence without having to go up from one term to the next.

'***n***' stands for the ***term number*** so to find the 50th term we would just substitute 50 in the formula in place of '***n***'.

Thus *n*th term of A.P. is linear in n. Infact $I_n = a + bn$, where $n \in N$,

Real Sequence

A sequence whose range is a subset of R is called a real sequence.

Finite and Infinite Sequences

On the basis of the number of terms, there are two types of sequences.

(i) Finite sequences: A sequence is said to be finite if it has finite number of terms.

(ii) Infinite sequences: A sequence is said to be infinite if it has infinite number of terms i.e. sequence of all even natural numbers (2, 4, 6, 8, ...).

Example 3.1 **Write down the sequence whose n^{th} term is a. $2^n/n$ and b. $[3 + (-1)^n]/3^n$.**

Sol.

a. Let $t_n = 2^n/n$ and put $n = 1, 2, 3, 4, \ldots$. We get

$t_1 = 2, t_2 = 2, t_3 = 8/3, t_4 = 4$

So, the sequence is 2, 2, 8/3, 4, ...

b. Let $t_n = [3 + (-1)^n]/3^n$ and put $n = 1, 2, 3, 4, \ldots$

So, the sequence is 2/3, 4/9, 2/27, 4/81, ...

Example 3.2 **Find the sequence of the numbers defined by**

$$a_n = \begin{cases} \dfrac{1}{n}, & \text{when } n \text{ is odd} \\ -\dfrac{1}{n}, & \text{when } n \text{ is even} \end{cases}$$

Sol. We have $a_1 = 1, a_3 = \frac{1}{3}, a_5 = \frac{1}{5}, \ldots$

and $a_2 = -\frac{1}{2}, a_4 = -\frac{1}{4}, a_6 = -\frac{1}{6}, \ldots$

Hence the sequence is $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots\ldots\ldots$

Example 3.3 **Write the first three terms of the sequence defined by $a_1 = 2, a_{n+1} = \dfrac{2a_n + 3}{a_n + 2}$.**

Sol. Put $n = 1$ in $a_{n+1} = \dfrac{2a_n + 3}{a_n + 2}$, we have

$$a_{1+1} = a_2 = \frac{2a_1 + 3}{a_1 + 2} = \frac{2(2) + 3}{(2) + 2} = \frac{7}{4}$$

Put $n = 2$, then we have $a_{2+1} = a_3 = \dfrac{2a_2 + 3}{a_2 + 2}$

$$= \frac{2\left(\frac{7}{4}\right) + 3}{\left(\frac{7}{4}\right) + 2} = \frac{26}{15}$$

Example 3.4 **Consider the sequence defined by $a_n = an^2 + bn + c$. If $a_1 = 1$, $a_2 = 5$ and $a_3 = 11$ then find the value of a_{10}.**

Sol. $a_1 = 1 \Rightarrow a + b + c = 1$ (i)

$a_2 = 5 \Rightarrow 4a + 2b + c = 5$ (ii)

$a_3 = 11, \Rightarrow 9a + 3b + c = 11$ (iii)

Now from (ii) – (i), we have $3a + b = 4$ (iv)

From (iii) – (ii), we have $5a + b = 6$ (v)

From (v) – (iv), we have $2a = 2$ or $a = 1$

$\Rightarrow b = 1$ (from (iv)), and $c = -1$ (from (i))

Hence $a_n = n^2 + n - 1$

Hence $a_{10} = 100 + 10 - 1 = 109$

Series

By adding or subtracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, \ldots, a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + \cdots + a_n$ is a series. For example,

(i) $1 + 2 + 3 + 4 + \cdots + n$

(ii) $2 + 4 + 8 + 16 + \cdots$

Progression

It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the *n*th term. Those sequences whose terms follow certain patterns are called progressions.

ARITHMETIC PROGRESSION (A.P.)

A.P. is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference.

If a is the first term and d is the common difference, then A.P. can be written as

$$a, a + d, a + 2d, \ldots, a + (n - 1)\, d, \ldots$$

n^{th} term: $T_n = a + (n - 1)\, d = l$ (last term), where $d = T_n - T_{n-1}$.

n^{th} term from end: $T_n' = l - (n - 1)d$.

The n^{th} term of A.P. is linear in n.

Example 3.5 **Show that the sequence 9, 12, 15, 18, ... is an A.P. Find its 16^{th} term and the general term.**

Sol. Since $(12-9)=(15-12)=(18-15)=3$, therefore the given sequence is an A.P. with common difference 3. First term is 9. Therefore, the 16th term is

$$a_{16}=a+(16-1)d \quad [\because a_n=a+(n-1)d]$$
$$=a+15d$$
$$\Rightarrow a_{16}=9+15\times 3=54$$

The general term (n^{th} term) is given by

$$a_n=a+(n-1)d$$
$$=9+(n-1)\times 3=3n+6$$

Example 3.6 **Show that the sequence log a, log (ab), log (ab^2), log(ab^3), ... is an A.P. Find its n^{th} term.**

Sol. We have,

$$\log(ab)-\log a=\log\left(\frac{ab}{a}\right)=\log b$$

$$\log(ab^2)-\log(ab)=\log\left(\frac{ab^2}{ab}\right)=\log b$$

$$\log(ab^3)-\log(ab^2)=\log\left(\frac{ab^3}{ab^2}\right)=\log b$$

It follows from the above results that the difference of a term and the preceding term is always same. So, the given sequence is an A.P. with common difference log b. Now,

$$a_n=a+(n-1)d$$
$$=\log a+(n-1)\log b$$
$$=\log a+\log b^{n-1}$$
$$=\log(ab^{n-1})$$

Example 3.7 **Find the sum to n terms of the sequence $\langle a_n\rangle$, where $a_n=5-6n$, $n\in N$.**

Sol. We have,

$$a_n=5-6n \Rightarrow a_{n+1}=5-6(n+1)=-1-6n$$
$$\therefore \quad a_{n+1}-a_n=(-1-6n)-(5-6n)=-6, \text{ for all } n\in N$$

Since $a_{n+1}-a_n$ is constant for all $n\in N$. So, the given sequence is an A.P. with common difference -6. Putting $n=1$ in $a_n=5-6n$, we get $a_1=-1$. So, the sum S_n to n terms is given by

$$S_n=(n/2)(a_1+a_n)=(n/2)(-1+5-6n)=n(2-3n)$$

Example 3.8 **How many terms are there in the A.P. 3, 7, 11, ..., 407?**

Sol. We know that last term $a_n=a+(n-1)d$

Where d = common difference = 4

and a = first term = 3

$$407=3+(n-1)4$$
$$\Rightarrow n=102$$

Hence there are 102 terms in A.P.

Example 3.9 **If a, b, c, d, e are in A.P., then find the the value of $a-4b+6c-4d+e$.**

Sol. $E=(a+e)-4(b+d)+6c$.

Now b, c, d in A.P. $\Rightarrow b+d=2c$

Again a, c, e are also in A.P.

$$\therefore \quad a+e=2c$$
$$\therefore \quad E=2c-4(2c)+6c=0$$

Example 3.10 **In a certain A.P., 5 times the 5th term is equal to 8 times the 8th term, then prove that its 13th term is 0.**

Sol. $5T_5=8T_8$

$$\Rightarrow 5(a+4d)=8(a+7d)$$
$$\Rightarrow 3a+36d=0$$
$$\Rightarrow a+12d=0$$
$$\Rightarrow T_{13}=0$$

Example 3.11 **Find the term of the series 25, $22\frac{3}{4}, 20\frac{1}{2}, 18\frac{1}{4}$, ... which is numerically the smallest.**

Sol. The given series is an A.P. $a=25$, $d=-9/4$.

$$T_n=a+(n-1)d=\left(25+\frac{9}{4}\right)-\frac{9}{4}n$$

or $$T_n=\frac{109}{4}-\frac{9}{4}n$$

Now T_n will be –ive if $\frac{109}{4}-\frac{9}{4}n<0$ or $n>12\frac{1}{9}$

Above shows that T_{13} will be first –ive terms and hence T_{12} will be smallest +ive terms. $T_{13}=-2$, $T_{12}=\frac{1}{4}$ is numerically smallest

Example 3.12 **Given two A.P.'s 2, 5, 8, 11, ..., T_{60} and 3, 5, 7, 9, ... T_{50}. Then find the number of terms which are identical.**

Sol. 2, 5, 8, 11, ..., $T_{60} \Rightarrow T_{60}=2+(60-1)3=179$

3, 5, 7, 9, ... $T_{50} \Rightarrow T_{50}=3+(50-1)2=101$

Hence, last common term ≤ 101.

Now common difference of first A.P. is 3 and common difference of second A.P. is 2.

Hence common difference of A.P. formed by common terms is L.C.M. of 3 and 2 which 6. Also common terms are are 5, 11, ...

For last term let $101=5+(n-1)6$

$$\Rightarrow n=17$$

Hence 101 is the actual last common term.

Example 3.13 **Consider two A.P.s:**

S_1: 2, 7, 12, 17, ... 500 terms

and S_2: 1, 8, 15, 22, ... 300 terms

Find the number of common terms. Also find the last common term.

Sol. S_1 :2, 7, 12, 17,... 500 terms

$$\Rightarrow T_{500}=2+(500-1)5=2497$$

S_2 :1,8, 15, 22, ... 300 terms

$$\Rightarrow T_{300}=1+(300-1)7=2094$$

Common differences of S_1 and S_2 are 5 and 7 respectively.

Hence common difference of common term series is 35

A.P. of common terms is 22, 57, 92, ...

Let last term is 2094 $\Rightarrow 22 + (n-1)35 = 2094 \Rightarrow n = 60.2$
But n is natural number $\Rightarrow n = 60$
Then actual last common term $= 22 + (60-1)35 = 2062$

Example 3.14 **If p^{th}, q^{th} and r^{th} terms of an A.P. are a, b, c, respectively, then show that**

a. $a(q-r) + b(r-p) + c(p-q) = 0$

b. $(a-b)r + (b-c)p + (c-a)q = 0$

Sol. Let A be the first term and D be the common difference of the given A.P. Then,

$a = p^{th}$ term $\Rightarrow a = A + (p-1)D$ (i)

$b = q^{th}$ term $\Rightarrow b = A + (q-1)D$ (ii)

$c = r^{th}$ term $\Rightarrow c = A + (r-1)D$ (iii)

a. $a(q-r) + b(r-p) + c(p-q) = \{A + (p-1)D\}(q-r) + \{A + (q-1)D\}(r-p) + \{A + (r-1)D\}(p-q)$

[Using (i), (ii) and (iii)]

$= A\{(q-r) + (r-p) + (p-q)\} + D\{(p-1)(q-r) + (q-1)(r-p) + (r-1)(p-q)\}$

$= A \times 0 + D\{p(q-r) + q(r-p) + r(p-q) - (q-r) - (r-p) - (p-q)\}$

$= A \times 0 + D \times 0 = 0$

b. On subtracting (ii) from (i), (iii) from (ii) and (i) from (iii), we get

$a - b = (p-q)D$ (iv)

$b - c = (q-r)D$ (v)

$c - a = (r-p)D$ (vi)

Now,

$(a-b)r + (b-c)p + (c-a)q$
$= (p-q)Dr + (q-r)Dp + (r-p)Dq$
$= D[(p-q)r + (q-r)p + (r-p)q]$
$= D \cdot 0 = 0$

Some Important Facts about A.P.

1. If a fixed number is added or subtracted to each term of a given A.P., then the resulting series is also an A.P., and its common difference remains the same.
2. If each term of an A.P. is multiplied by a fixed constant or divided by a fixed non-zero constant, then the resulting series in also an A.P.
3. If $x_1 + x_2 + x_3 + \cdots$ and $y_1 + y_2 + y_3 + \cdots$ are two A.P.'s, then $x_1 \pm y_1, x_2 \pm y_2, x_3 \pm y_3, \ldots$ are also A.P.'s.
4. Three terms in an A.P. should preferably be taken as $a-d, a, a+d$ and four terms as $a-3d, a-d, a+d, a+3d$.
5. In A.P., $a_n = \dfrac{a_{n-k} + a_{n+k}}{2}$, for $k \le n$.
6. It $a, a_2, a_3, \ldots a_n$ are in A.P. Then $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots = a_r + a_{n-r+1}$.

Example 3.15 **If $(b+c-a)/a$, $(c+a-b)/b$, $(a+b-c)/c$ are in A.P., then prove that $1/a$, $1/b$, $1/c$ are also in A.P.**

Sol. $\dfrac{b+c-a}{a}, \dfrac{c+a-b}{b}, \dfrac{a+b-c}{c}$ are in A.P.

$\Rightarrow \left\{\dfrac{b+c-a}{a}+2\right\}, \left\{\dfrac{c+a-b}{b}+2\right\}, \left\{\dfrac{a+b-c}{c}+2\right\}$ are in A.P.

[Adding 2 to each term]

$\Rightarrow \dfrac{a+b+c}{a}, \dfrac{a+b+c}{b}, \dfrac{a+b+c}{c}$ are in A.P.

$\Rightarrow \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P. [Dividing each term by $a+b+c$]

Example 3.16 **If $a, b, c \in R+$ form an A.P., then prove that $a + 1/(bc)$, $b + 1/(ac)$, $c + 1/(ab)$ are also in A.P.**

Sol. a, b, c are in A.P.

$\Rightarrow \dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ are also in A.P. [Dividing by abc]

$\Rightarrow a + \dfrac{1}{bc}, b + \dfrac{1}{ca}, c + \dfrac{1}{ab}$ will also be in A.P.

[$\because$ Sum of two A.P.'s is also an A.P.]

Example 3.17 **If a, b, c are in A.P., then prove that the following are also in A.P.**

a. $a^2(b+c), b^2(c+a), c^2(a+b)$

b. $\dfrac{1}{\sqrt{b}+\sqrt{c}}, \dfrac{1}{\sqrt{c}+\sqrt{a}}, \dfrac{1}{\sqrt{a}+\sqrt{b}}$

c. $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right), b\left(\dfrac{1}{c}+\dfrac{1}{a}\right), c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$

Sol.

a. Let $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

$\Rightarrow b^2(c+a) - a^2(b+c) = c^2(a+b) - b^2(c+a)$

$\Rightarrow c(b^2 - a^2) + ab(b-a) = a(c^2 - b^2) + bc(c-b)$

$\Rightarrow (b-a)(ab+bc+ca) = (c-b)(ab+bc+ca)$

$\Rightarrow b - a = c - b$

$\Rightarrow 2b = a + c$

$\Rightarrow a, b, c$ are in A.P.

$\Rightarrow a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

b. Let $\dfrac{1}{\sqrt{b}+\sqrt{c}}, \dfrac{1}{\sqrt{c}+\sqrt{a}}, \dfrac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P.

$\Rightarrow \dfrac{1}{\sqrt{c}+\sqrt{a}} - \dfrac{1}{\sqrt{b}+\sqrt{c}} = \dfrac{1}{\sqrt{a}+\sqrt{b}} - \dfrac{1}{\sqrt{c}+\sqrt{a}}$

$\Rightarrow \dfrac{\sqrt{b}-\sqrt{a}}{(\sqrt{c}+\sqrt{a})(\sqrt{b}+\sqrt{c})} = \dfrac{(\sqrt{c}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$

$\Rightarrow \dfrac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{c}} = \dfrac{\sqrt{c}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

$\Rightarrow b - a = c - b$

$\Rightarrow 2b = a + c$

$\Rightarrow a, b, c$ are in A.P.

$\Rightarrow \dfrac{1}{\sqrt{b}+\sqrt{c}}, \dfrac{1}{\sqrt{c}+\sqrt{a}}, \dfrac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P.

c. a, b, c are in A.P.

$\Rightarrow \frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are in A.P.

[On dividing each term by abc]

$\Rightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

$\Rightarrow \frac{ab+bc+ca}{bc}, \frac{ab+bc+ca}{ca}, \frac{ab+bc+ca}{ab}$ are in A.P.

[On multiplying each term by $ab + bc + ca$]

$\Rightarrow \frac{ab+bc+ca}{bc}-1, \frac{ab+bc+ca}{ca}-1, \frac{ab+bc+ca}{ab}-1$ are in A.P. [On adding –1 to each term]

$\Rightarrow \frac{ab+ac}{bc}, \frac{ab+bc}{ca}, \frac{bc+ca}{ab}$ are in A.P.

$\Rightarrow a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

Example 3.18 **The sum of three numbers in A.P. is –3 and their product is 8. Find the numbers.**

Sol. Let the numbers be $(a - d), a, (a + d)$. Therefore,

$(a - d) + a + (a + d) = -3$

$\Rightarrow 3a = -3$

$\Rightarrow a = -1$

And

$(a - d)(a)(a + d) = 8$

$\Rightarrow a(a^2 - d^2) = 8$

$\Rightarrow (-1)(1 - d^2) = 8 \quad [\because a = -1]$

$\Rightarrow d^2 = 9$

$\Rightarrow d = \pm 3$

If $d = 3$, the numbers are $-4, -1, 2$. If $d = -3$, the numbers are $2, -1, -4$. So, the numbers are $-4, -1, 2$ or $2, -1, -4$.

Example 3.19 **Divide 32 into four parts which are in A.P. such that the ratio of the product of extremes to the product of means is 7:15.**

Sol. Let the four parts be $(a - 3d), (a - d), (a + d)$ and $(a + 3d)$. Then,

$(a - 3d) + (a - d) + (a + d) + (a + 3d) = 32$

$\Rightarrow 4a = 32$

$\Rightarrow a = 8$

Also,

$$\frac{(a-3d)(a+3d)}{(a-d)(a+d)} = \frac{7}{15}$$

$$\Rightarrow \frac{a^2-9d^2}{a^2-d^2} = \frac{7}{15}$$

$$\Rightarrow \frac{64-9d^2}{64-d^2} = \frac{7}{15}$$

$\Rightarrow 128d^2 = 512$

$\Rightarrow d^2 = 4$

$\Rightarrow d = \pm 2$

Thus, the four parts are 2, 6, 10, 14.

Example 3.20 **The digits of a positive integer, having three digits, are in A.P. and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.**

Sol. Let the digits at ones, tens and hundreds place be $(a - d)$, a and $(a + d)$, respectively. Then the number is

$(a + d) \times 100 + a \times 10 + (a - d) = 111a + 99d$

The number obtained by reversing the digits is,

$(a - d) \times 100 + a \times 10 + (a + d) = 111a - 99d$

It is given that

$(a - d) + a + (a + d) = 15 \quad$ (i)

and

$111a - 99d = 111a + 99d - 594 \quad$ (ii)

$\therefore \ 3a = 15$ and $198d = 594$

$\Rightarrow a = 5$ and $d = 3$

So, the number is $111 \times 5 + 99 \times 3 = 852$.

Sum of *n* terms of an A.P.

The sum S_n of n terms of an A.P. with the first term 'a' and the common difference 'd' is

$$S_n = \frac{n}{2}[2a + (n - 1)d] \text{ or } S_n = \frac{n}{2}[a + l]$$

where l = last term = $a + (n - 1)d$.

Proof: $S_n = a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n \quad$ (i)

$= a_n + a_{n-1} + a_{n-2} + \cdots + a_3 + a_2 + a_1 \quad$ (ii)

Adding corresponding terms in (i) and (ii), we get

$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \cdots + (a_{n-1} + a_2) + (a_n + a_1)$

$= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n)$

$= n(a_1 + a_n)$

$[\because a_1 + a_n = a_k + a_{n-k+1}$ for $k = 2, 3, \ldots, n]$

$\Rightarrow S_n = \frac{n}{2}(a_1 + a_n)$

$= \frac{n}{2}\{a_1 + a_1 + (n - 1)d\} \quad [\because a_n = a_1 + (n - 1)d]$

$= \frac{n}{2}[2a_1 + (n - 1)d]$

Example 3.21 **If the sum of the series 2, 5, 8, 11, ... is 60100, then find the value of *n*.**

Sol. Here first term is $a = 2$ and common difference $d = 3$

Hence sum of n terms of A.P. is, $(n/2)(4 + 3(n - 1)) = 60100$

$\Rightarrow 3n^2 + n - 120200 = 0$

$\Rightarrow (n - 200)(3n + 601) = 0$

$\Rightarrow n = 200$

Example 3.22 **In an A.P. if $S_1 = T_1 + T_2 + T_3 + \cdots + T_n$ (n odd), $S_2 = T_2 + T_4 + T_6 + \cdots + T_{n-1}$, then find the value of S_1/S_2 in terms of n.**

Sol. S_1 is an A.P. of n terms, but S_2 is an A.P. of $\frac{n-1}{2}$ terms with common difference $2d$

$$S_1 = \frac{n}{2}[T_1 + T_n] \quad (1)$$

$$S_1 = \frac{1}{2}\left(\frac{n-1}{2}\right)[T_2 + T_{n-1}] = \frac{1}{2}\left(\frac{n-1}{2}\right)[T_1 + T_n]$$

$$\therefore \frac{S_1}{S_2} = \frac{2n}{n-1}$$

Example 3.23 **Prove that sum of n number of terms of two different A.P.s can be same for only one value of n.**

Sol. According to the given condition

$$\frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[2A+(n-1)D]$$

$\Rightarrow$ $[2a+(n-1)d] = [2A+(n-1)D]$, from this we get one integral value of n or no value of n.

Example 3.24 **In an A.P. of 99 terms, the sum of all the odd numbered terms is 2550. Then find the sum of all the 99 terms of the A.P.**

Sol. Given $\frac{50}{2}[a_1 + a_{99}] = 2550$

$\Rightarrow$ $a_1 + a_{99} = 102$

Now sum of all the terms is

$$\frac{99}{2}[a_1 + a_{99}] = \frac{99}{2} \times 102 = 5049$$

Example 3.25 **Find the degree of the expression $(1 + x)(1 + x^6)(1 + x^{11}) \ldots (1 + x^{101})$.**

Sol. The degree of the expression is $1 + 6 + 11 + \cdots + 101$ which is an A.P.

Now $101 = 1 + 5(n-1) \Rightarrow n = 21$

$\Rightarrow$ $1 + 6 + 11 + \ldots + 101$

$$= \frac{21}{2}[1+101] = 21 \times 51 = 1071$$

Example 3.26 **Find the number of terms in the series 20, $19\frac{1}{3}$, $18\frac{2}{3}$, ... the sum of which is 300. Explain the answer.**

Sol. The given sequence is an A.P. with first term $a = 20$ and the common difference $d = -2/3$. Let the sum of n terms be 300. Then,

$$S_n = 300 \Rightarrow \frac{n}{2}[2a+(n-1)d] = 300$$

$\Rightarrow$ $\frac{n}{2}[2 \times 20 + (n-1)(-2/3)] = 300$

$\Rightarrow$ $n^2 - 61n + 900 = 0$

$\Rightarrow$ $(n-25)(n-36) = 0$

$\Rightarrow$ $n = 25$ or 36

So, sum of 25 terms is equal to sum of 36 terms, which is equal to 300.

Here the common difference is negative, therefore terms go on diminishing and the 31st term becomes zero. All terms after the 31st term are negative. These negative terms when added to positive terms from 26th term to 30th term, they cancel out each other and the sum remains same. Hence, the sum of 25 terms as well as that of 36 terms is 300.

Example 3.27 **Find the sum of all three-digit natural numbers, which are divisible by 7.**

Sol. The smallest and the largest numbers of three digits, which are divisible by 7 are 105 and 994, respectively. So, the sequence of three digit numbers which are divisible by 7 is 105, 112, 119, ..., 994. Clearly, it is an A.P. with first term $a = 105$ and common difference $d = 7$. Let there be n terms in this sequence. Then,

$a_n = 994$

$\Rightarrow$ $a + (n-1)d = 994$

$\Rightarrow$ $105 + (n-1) \times 7 = 994$

$\Rightarrow$ $n = 128$

Now, required sum is

$$\frac{n}{2}[2a+(n-1)d]$$

$$= \frac{128}{2}[2 \times 105 + (128-1) \times 7]$$

$$= 70336$$

Example 3.28 **Prove that a sequence is an A.P. if the sum of its n terms is of the form $An^2 + Bn$, where A, B are constants.**

Sol. Let S_n be the sum of n terms of an A.P. with first term a and common difference d. Then,

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$= an + \frac{n^2}{2}d - \frac{n}{2}d$$

$$= \left(\frac{d}{2}\right)n^2 + \left(a - \frac{d}{2}\right)n$$

$$= An^2 + Bn$$

where $A = d/2$ and $B = a - d/2$.

Thus, the sum of n terms of an A.P. is of the form $An^2 + Bn$. Conversely, let the sum S_n of n terms of a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ be of the form $An^2 + Bn$.

Then, we have to show that the sequence is an A.P. We have,

$S_n = An^2 + Bn$

$\Rightarrow$ $S_{n-1} = A(n-1)^2 + B(n-1)$ [On replacing n by $(n-1)$]

Now,

$a_n = S_n - S_{n-1}$

$\Rightarrow$ $a_n = \{An^2 + Bn\} - \{A(n-1)^2 + B(n-1)\}$

$= 2An + (B - A)$

$\Rightarrow \quad a_{n+1} = 2A(n+1) + (B - A)$ [On replacing n by $(n+1)$]

$\Rightarrow \quad a_{n+1} - a_n = \{2A(n+1) + B - A\} - \{2An + (B - A)\} = 2A$

Since $a_{n+1} - a_n = 2A$ for all $n \in N$, so the sequence is an A.P. with common difference $2A$.

Example 3.29 **If the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ forms an A.P., then prove that**

$$a_1^2 - a_2^2 + a_3^2 - a_4^2 + \cdots + a_{2n-1}^2 - a_{2n}^2 = \frac{n}{2n-1}(a_1^2 - a_{2n}^2)$$

Sol. Let d be the common difference of the A.P. Then,

$$d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = a_{2n} - a_{2n-1}$$

Now,

$$a_1^2 - a_2^2 + a_3^2 - a_4^2 + \cdots + a_{2n-1}^2 - a_{2n}^2$$
$$= (a_1 + a_2)(a_1 - a_2) + (a_3 + a_4)(a_3 - a_4) + \cdots + (a_{2n-1} + a_{2n}) \times (a_{2n-1} - a_{2n})$$
$$= -d(a_1 + a_2 + a_3 + \cdots + a_{2n})$$
$$= -d\,\frac{2n}{2}(a_1 + a_{2n})$$
$$= -dn\,\frac{(a_1^2 - a_{2n}^2)}{a_1 - a_{2n}}$$
$$= \frac{dn(a_1^2 - a_{2n}^2)}{a_{2n} - a_1}$$
$$= \frac{n}{2n-1}(a_1^2 - a_{2n}^2) \quad [\text{Using } a_{2n} = a_1 + (2n-1)d]$$

Example 3.30 **Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots$, if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.**

Sol. We know that in an A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term, i.e., $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \cdots$.

So, if an A.P. consists of 24 terms, then

$$a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15}$$

$$a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$$

$\Rightarrow \quad (a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225$

$\Rightarrow \quad 3(a_1 + a_{24}) = 225$

$\Rightarrow \quad a_1 + a_{24} = \dfrac{225}{3} = 75 \qquad$ (i)

$\therefore \quad S_{24} = \dfrac{24}{2}(a_1 + a_{24}) \quad \left[\text{Using } S_n = \dfrac{n}{2}(a_1 + a_n)\right]$

$= 12\,(75)$

$= 900$ [Using (i)]

Example 3.31 **If the arithmetic progression whose common difference is non-zero, the sum of first $3n$ terms is equal to the sum of next n terms. Then, find the ratio of the sum of the first $2n$ terms to the sum of next $2n$ terms.**

Sol.

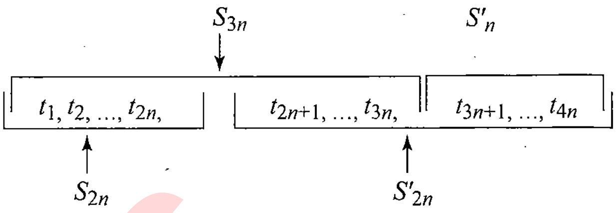


Fig. 3.1

Given,

$$S_{3n} = S_n' = S_{4n} - S_{3n}$$

$\Rightarrow \quad 2S_{3n} = S_{4n}$

$\Rightarrow \quad 2\,\dfrac{3n}{2}(2a + (3n-1)d) = \dfrac{4n}{2}(2a + (4n-1)d)$

$\Rightarrow \quad 12a + (18n - 6)d = 8a + (16n - 4)d$

$\Rightarrow \quad 4a = (-2n + 2)d$

$\Rightarrow \quad 2a = (1 - n)d \qquad$ (1)

Now we have to find $\dfrac{S_{2n}}{S'_{2n}}$.

$$\frac{S_{2n}}{S'_{2n}} = \frac{S_{2n}}{S_{4n} - S_{2n}}$$
$$= \frac{\frac{2n}{2}(2a + (2n-1)d)}{\frac{4n}{2}[2a + (4n-1)d] - \frac{2n}{2}[2a + (2n-1)d]}$$
$$= \frac{2[(1-n)d + (2n-1)d]}{4[(1-n)d + (4n-1)d] - 2[(1-n)d + (2n-1)d]}$$
$$= \frac{2nd}{10nd} = \frac{1}{5}$$

Arithmetic Means

If between a and b, two given quantities, we have to insert n quantities $A_1, A_2, \ldots, A_n$ such that $a, A_1, A_2, \ldots, A_n, b$ forms an A.P., then we say that $A_1, A_2, \ldots, A_n$ are arithmetic means between a and b.

For example, 15, 11, 7, 3, –1, –5 are in A.P. It follows that 11, 7, 3, –1 are four arithmetic means between 15 and –5.

If a, A, b are in A.P., we say that A is the arithmetic mean between a and b.

Insertion of n Arithmetic Means Between a and b

Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between two quantities a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ is an A.P. Let d be the common difference of this A.P. Clearly, it contains $n + 2$ terms.

$\therefore \quad b = (n+2)^{\text{th}}$ term

$\Rightarrow \quad b = a + (n+1)d$

$\Rightarrow \quad d = \dfrac{b-a}{n+1}$

Now,

$$A_1 = a + d \Rightarrow A_1 = \left(a + \frac{b-a}{n+1}\right)$$

$$A_2 = a + 2d \Rightarrow A_2 = \left(a + \frac{2(b-a)}{n+1}\right)$$

$$A_n = a + nd \Rightarrow A_n = \left(a + n\frac{(b-a)}{n+1}\right)$$

These are the required arithmetic means between a and b.

An Important Property of A.M.'s

The sum of n arithmetic means between two numbers is n times the single A.M. between them.

Proof:

Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ is an A.P. with common difference $d = \frac{b-a}{n+1}$.

Now,

$$A_1 + A_2 + \cdots + A_n$$

$$= \frac{n}{2}[A_1 + A_n]$$

$$= \frac{n}{2}[a+b]$$

$[\because a, A_1, A_2, \ldots, A_n, b$ is an A.P., $\therefore a + b = A_1 + A_n]$

$$= n\left(\frac{a+b}{2}\right)$$

$= n \times$ (A.M. between a and b)

Example 3.32 **Insert three arithmetic means between 3 and 19.**

Sol. Let A_1, A_2, and A_3 be three A.M.'s between 3 and 19. Then $3, A_1, A_2, A_3$, 19 are in A.P. whose common difference is

$$d = \frac{19-3}{3+1} = 4$$

$\therefore \quad A_1 = 3 + d = 7$

$\Rightarrow \quad A_2 = 3 + 2d = 11$

$A_3 = 3 + 3d = 15$

Hence, the required A.M.'s are 7, 11, 15.

Example 3.33 **If eleven A.M.'s are inserted between 28 and 10, then find the number of integral A.M.'s.**

Sol. Assume $A_1, A_2, A_3, \ldots, A_{11}$ be the eleven A.M.'s between 28 and 10, so $28, A_1, A_2, \ldots, A_{11}$, 10 are in A.P. Let d be the common difference of the A.P. The number of terms is 13. Now,

$$10 = T_{13} = T_1 + 12d = 28 + 12d$$

$$\Rightarrow \quad d = \frac{10-28}{12} = -\frac{18}{12} = -\frac{3}{2}$$

Here integral A.M.'s are

$$28 - 2\left(\frac{3}{2}\right), 28 - 4\left(\frac{3}{2}\right), 28 - 6\left(\frac{3}{2}\right),$$

$$28 - 8\left(\frac{3}{2}\right), 28 - 10\left(\frac{3}{2}\right).$$

Thus, the number of integral A.M.'s is 5.

Example 3.34 **Between 1 and 31 are inserted m arithmetic means so that the ratio of the 7[th] and $(m-1)$[th] means is 5:9. Find the value of m.**

Sol. Let $A_1, A_2, \ldots, A_m$ be m arithmetic means between 1 and 31. Then, $1, A_1, A_2, \ldots, A_m$, 31 is an A.P. with common difference

$$d = \frac{31-1}{m+1} = \frac{30}{m+1} \quad \left[\text{Using } d = \frac{b-a}{n+1}\right]$$

Now,

$$A_7 = 1 + 7d$$

$$\Rightarrow \quad A_7 = 1 + \frac{7 \times 30}{m+1} = \frac{m+211}{m+1}$$

$$A_{m-1} = 1 + (m-1)d$$

$$= 1 + \frac{30}{m+1}(m-1)$$

$$= \frac{31m - 29}{m+1}$$

It is given that

$$\frac{A_7}{A_{m-1}} = \frac{5}{9}$$

$$\Rightarrow \quad \frac{m+211}{31m-29} = \frac{5}{9}$$

$\Rightarrow \quad 9m + 1899 = 155m - 145$

$\Rightarrow \quad 146m = 2044$

$\Rightarrow \quad m = 14$

Example 3.35 **For what value of n, $(a^{n+1} + b^{n+1})/(a^n + b^n)$ is the arithmetic mean of a and b?**

Sol. Since A.M. of a and b is $(a + b)/2$, we have

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \frac{a+b}{2}$$

$\Rightarrow \quad 2(a^{n+1} + b^{n+1}) = (a^n + b^n)(a + b)$

$\Rightarrow \quad 2a^{n+1} + 2b^{n+1} = a^{n+1} + a^n b + b^n a + b^{n+1}$

$\Rightarrow \quad a^{n+1} + b^{n+1} = a^n b + b^n a$

$\Rightarrow \quad a^n(a - b) = b^n(a - b)$

$\Rightarrow \quad a^n = b^n$

$$\Rightarrow \quad \frac{a^n}{b^n} = 1$$

$$\Rightarrow \quad \left(\frac{a}{b}\right)^n = 1$$

$$\Rightarrow \quad \left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0$$

$\Rightarrow \quad n = 0$

Concept Application Exercise 3.1

1. If the p^{th} term of an A.P. is q and the q^{th} term is p, then find its r^{th} term.
2. If x is a positive real number different from 1, then prove that the numbers $\frac{1}{1+\sqrt{x}}, \frac{1}{1-x}, \frac{1}{1-\sqrt{x}}, \ldots$ are in A.P. Also find their common difference.
3. In a certain A.P., 5 times the 5^{th} term is equal to 8 times the 8^{th} term, then find its 13^{th} term.
4. If $S_n = nP + \frac{n(n-1)}{2}Q$, where S_n denotes the sum of the first n terms of an A.P., then find the common difference.
5. Find the first negative term of the sequence $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$
6. Solve the equation $(x+1)+(x+4)+(x+7)+\cdots+(x+28)=155$.
7. The p^{th} term of an A.P. is a and q^{th} term is b. Then find the sum of its $(p+q)$ terms.
8. The sum of n, $2n$, $3n$ terms of an A.P. are S_1, S_2, S_3, respectively. Prove that $S_3 = 3(S_2 - S_1)$.
9. The ratio of the sums of m and n terms of an A.P. is $m^2:n^2$. Show that the ratio of the m^{th} and n^{th} terms is $(2m-1):(2n-1)$.
10. Find the number of common terms to the two sequences 17, 21, 25, ..., 417 and 16, 21, 26, ..., 466.
11. If a, b, c, d are distinct integers in an A.P. such that $d = a^2 + b^2 + c^2$, then find the value of $a+b+c+d$.
12. Let S_n denote the sum of first n terms of an A.P. If $S_{2n} = 3S_n$, then find the ratio S_{3n}/S_n.
13. Find four numbers in an A.P. whose sum is 20 and sum of their squares is 120.
14. Divide 28 into four parts in an A.P. so that the ratio of the product of first and third with the product of second and fourth is 8:15.
15. If $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P. then prove that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are also in A.P.
16. If n arithmetic means are inserted between 2 and 38, then the sum of the resulting series is obtained as 200. Then find the value of n.
17. If a, b, c, d, e, f are A.M.'s between 2 and 12, then find the sum: $a+b+c+d+e+f$.
18. n arithmetic means are inserted between x and $2y$ and then between $2x$ and y. If the r^{th} means in each case be equal, then find the ratio x/y.

GEOMETRIC PROGRESSION (G.P.)

G.P. is a sequence of numbers whose first term is non-zero and each of the succeeding terms is equal to the proceeding terms multiplied by a constant. Thus in a G.P., the ratio of successive terms is constant. This constant factor is called the common ratio of the series and is obtained by dividing any term by that which immediately proceeds it.

If a is the first term and r is the common ratio, then G.P. can be written as $a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}$.

n^{th} term: $T_n = ar^{n-1} = l$ (last term), where $r = \frac{T_n}{T_{n-1}}$.

n^{th} term from end: $T_n' = \frac{l}{r^{n-1}}$.

Increasing and Decreasing G.P.

For a G.P. to be increasing or decreasing, $r > 0$. Since if $r < 0$, terms of G.P. are alternately positive and negative and so neither increasing nor decreasing.

If $a > 0$, then G.P. is increasing if $r > 1$ and decreasing if $0 < r < 1$. If $a < 0$, then G.P. is decreasing if $r > 1$ and increasing if $0 < r < 1$. The above discussion can be exhibited as follows:

a	$a > 0$	$a > 0$	$a < 0$	$a < 0$
r	$r > 1$	$0 < r < 1$	$r > 1$	$0 < r < 1$
Result	Increasing	Decreasing	Decreasing	Increasing

Example 3.36 **Which term of the G.P. 2, 1, 1/2, 1/4, ... is 1/128?**

Sol. Clearly, the given progression is a G.P. with first term $a = 2$ and common ratio $r = 1/2$. Let the n^{th} term be 1/128. Then,

$$a_n = \frac{1}{128}$$

$$\Rightarrow ar^{n-1} = \frac{1}{128}$$

$$\Rightarrow 2\left(\frac{1}{2}\right)^{n-1} = \frac{1}{128}$$

$$\Rightarrow \left(\frac{1}{2}\right)^{n-2} = \left(\frac{1}{2}\right)^7$$

$$\Rightarrow n-2=7$$

$$\Rightarrow n=9$$

Thus, 9^{th} term of the given G.P. is 1/128.

Example 3.37 **The first term of a G.P. is 1. The sum of the third and fifth terms is 90. Find the common ratio of the G.P.**

Sol. Let r be the common ratio of the G.P. It is given that the first term is $a = 1$. Now,

$$a_3 + a_5 = 90$$

$$\Rightarrow ar^2 + ar^4 = 90$$

$$\Rightarrow r^2 + r^4 = 90$$

$$\Rightarrow r^4 + r^2 - 90 = 0$$

$\Rightarrow\ r^4 + 10r^2 - 9r^2 - 90 = 0$

$\Rightarrow\ (r^2 + 10)(r^2 - 9) = 0$

$\Rightarrow\ r^2 - 9 = 0$

$\Rightarrow\ r = \pm 3$

Example 3.38 **Fifth term of a G.P. is 2. Find the product of its first nine terms.**

Sol. $t_5 = a r^4 = 2$

Product of its first 9 terms is

$$a\,(ar)\,(ar^2)\cdots(ar^8) = a^9 r^{1+2+\cdots+8}$$
$$= a^9 r^{\frac{8}{2}(1+8)}$$
$$= a^9 r^{36}$$
$$= (ar^4)^9 = 2^9 = 512$$

Example 3.39 **The fourth, seventh and the last term of a G.P. are 10, 80 and 2560, respectively. Find the first term and the number of terms in the G.P.**

Sol. Let a be the first term and r be the common ratio of the given G.P. Then,

$$a_4 = 10,\ a_7 = 80 \Rightarrow ar^3 = 10 \text{ and } ar^6 = 80$$

$$\Rightarrow\ \frac{ar^6}{ar^3} = \frac{80}{10} \Rightarrow r^3 = 8 \Rightarrow r = 2$$

Putting $r = 2$ in $ar^3 = 10$, we get $a = 10/8$.

Let there be n terms in the given G.P. Then,

$$a_n = 2560 \Rightarrow ar^{n-1} = 2560$$

$$\Rightarrow\ \frac{10}{8}(2^{n-1}) = 2560$$

$$\Rightarrow\ 2^{n-4} = 256 \Rightarrow 2^{n-4} = 2^8$$

$$\Rightarrow\ n - 4 = 8 \Rightarrow n = 12$$

Example 3.40 **Three numbers are in G.P. If we double the middle term, we get an A.P. Then find the common ratio of the G.P.**

Sol. Let the three numbers in G.P. be a, ar, and ar^2. By the given condition, a, $2ar$, and ar^2 are in A.P. Hence,

$$4ar = a + ar^2$$

$$\Rightarrow\ 4r = 1 + r^2$$

$$\Rightarrow\ r^2 - 4r + 1 = 0$$

$$\Rightarrow\ r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Example 3.41 **If p, q, and r are in A.P., show that the p^{th}, q^{th} and r^{th} terms of any G.P. are in G.P.**

Sol. Let A be the first term and R the common ratio of a G.P. Then, $a_p = AR^{p-1}$, $a_q = AR^{q-1}$ and $a_r = AR^{r-1}$. We have to prove that a_p, a_q, and a_r are in G.P. For this, it is sufficient to show that $(a_q)^2 = a_p a_r$. We have,

$$(a_q)^2 = (AR^{q-1})^2$$
$$= A^2 R^{2q-2}$$
$$= A^2 R^{p+r-2} \quad [\because p, q, r \text{ are in A.P. } \therefore 2q = p + r]$$
$$= (AR^{p-1})(AR^{r-1}) = a_p a_r$$

Hence, a_p, a_q and a_r are in G.P.

Example 3.42 **If a, b, c, and d are in G.P., show that $(ab + bc + cd)^2 = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$.**

Sol. Let r be the common ratio of the G.P., a, b, c, d. Then $b = ar$, $c = ar^2$ and $d = ar^3$.

L.H.S. $= (ab + bc + cd)^2$
$= (aar + arar^2 + ar^2ar^3)^2$
$= a^4r^2(1 + r^2 + r^4)^2$

R.H.S. $= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$
$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$
$= a^2(1 + r^2 + r^4)a^2r^2(1 + r^2 + r^4)$
$= a^4r^2(1 + r^2 + r^4)^2$

$\therefore$ L.H.S. = R.H.S.

Example 3.43 **Three non-zero numbers a, b and c are in A.P. Increasing a by 1 or increasing c by 2, the numbers are a in G.P. Then find b.**

Sol. a, b, and c are in A.P. Hence,

$$2b = a + c \quad (1)$$

Again by the given condition, $a + 1$, b, and c are in G.P. and a, b, c, and $+ 2$ are in G.P. Hence,

$$b^2 = (a + 1)\,c \quad (2)$$

and

$$b^2 = a(c + 2) \quad (3)$$

By (2) and (3),

$$(a + 1)c = a(c + 2)$$

$$\Rightarrow\ ac + c = ac + 2a$$

$$\Rightarrow\ c = 2a$$

Equation (2) gives $b^2 = (a + 1)\,2a$

Also, Eq. (1) gives

$$2b = a + 2a = 3a$$

$$\Rightarrow\ b = \frac{3a}{2}$$

$$\Rightarrow\ \frac{9a^2}{4} = (a + 1)\,2a$$

$$\Rightarrow\ \frac{9a}{8} = a + 1$$

$$\Rightarrow\ a = 8$$

$$\Rightarrow\ c = 2(8) = 16$$

$$\Rightarrow\ 2b = 8 + 16 = 24$$

$$\Rightarrow\ b = 12$$

Example 3.44 **Three numbers are in G.P. whose sum is 70. If the extremes be each multiplied by 4 and the means by 5, they will be in A.P. Find the numbers.**

Sol. Let the numbers be a, ar, and ar^2. Then,

$$a(1 + r + r^2) = 70 \quad (1)$$

It is given that $4a$, $5ar$, and $4ar^2$ are in A.P. Therefore,

$$2(5ar) = 4a + 4ar^2$$
$$\Rightarrow 5r = 2 + 2r^2$$
$$\Rightarrow 2r^2 - 5r + 2 = 0$$
$$\Rightarrow (2r - 1)(r - 2) = 0$$
$$\Rightarrow r = 2 \text{ or } r = 1/2$$

Putting $r = 2$ in (1), we obtain $a = 10$. Putting $r = 1/2$ in (i), we get $a = 40$. Hence, the numbers are 10, 20, 40 or 40, 20, 10.

Some Important Facts about G.P.

1. If each term of a G.P. is multiplied or divided by some fixed non-zero number, then the resulting sequence is also a G.P.
2. If $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ are two G.P.'s, then $x_1y_1, x_2y_2, x_3y_3, \ldots$ and $\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3}, \cdots$ are also G.P.'s.
3. If $x_1, x_2, x_3, \ldots$ is a G.P. of positive terms, then $\log x_1, \log x_2, \log x_3, \ldots$ is an A.P. and vice versa.
4. Three terms of a G.P. can be taken as a/r, a, ar and four terms in G.P. as a/r^3 a/r, ar, ar^3. This presentation is useful if the product of terms is involved in the problem. In other problems, terms should be taken as a, ar, ar^2, ...

Example 3.45 **If the continued product of three numbers in a G.P. is 216 and the sum of their products in pairs is 156, find the numbers.**

Sol. Let the three numbers be a/r, a, and ar. Then, product $= 216$. Hence, $a/r \times a \times ar = 216 \Rightarrow a^3 = 216 \Rightarrow a = 6$ Sum of the products in pairs is 156. Hence,

$$\frac{a}{r}a + aar + \frac{a}{r}ar = 156$$
$$\Rightarrow a^2\left(\frac{1}{r} + r + 1\right) = 156$$
$$\Rightarrow 36\left(\frac{1 + r^2 + r}{r}\right) = 156$$
$$\Rightarrow 3(r^2 + r + 1) = 13r$$
$$\Rightarrow 3r^2 - 10r + 3 = 0$$
$$\Rightarrow (3r - 1)(r - 3) = 0$$
$$\Rightarrow r = \frac{1}{3} \text{ or } r = 3$$

Hence, putting the values of a and r, the required numbers are 18, 6, 2 or 2, 6, 18.

Sum of n Terms of a G.P.

The sum of n terms of a G.P. with first term 'a' and common ratio 'r' is given by

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right) \text{ or } S_n = a\left(\frac{1 - r^n}{1 - r}\right), r \neq 1$$

Proof:

Let S_n denote the sum of n terms of the G.P. with first term 'a' and common ration 'r'. Then,

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \quad (1)$$

Multiplying both sides by r, we get

$$rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n \quad (2)$$

On subtracting (2) from (1), we get

$$S_n - rS_n = a - ar^n$$
$$\Rightarrow S_n(1 - r) = a(1 - r^n)$$
$$\Rightarrow S_n = a\left(\frac{1 - r^n}{1 - r}\right)$$
$$\Rightarrow S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

The above formulas do not hold for $r = 1$. For $r = 1$, the sum of n terms of the G.P. is $S_n = na$.

Example 3.46 **Determine the number of terms in a G.P., if $a_1 = 3$, $a_n = 96$ and $S_n = 189$.**

Sol. Let r be the common ratio of the given G.P. Then,

$$a_n = 96$$
$$\Rightarrow a_1 r^{n-1} = 96$$
$$\Rightarrow 3r^{n-1} = 96$$
$$\Rightarrow r^{n-1} = 32 \quad (1)$$

Now,

$$S_n = 189$$
$$\Rightarrow a_1\left(\frac{r^n - 1}{r - 1}\right) = 189$$
$$\Rightarrow 3\left(\frac{(r^{n-1})r - 1}{r - 1}\right) = 189$$
$$\Rightarrow 3\left(\frac{32r - 1}{r - 1}\right) = 189 \quad \text{[Using (1)]}$$
$$\Rightarrow 32r - 1 = 63r - 63$$
$$\Rightarrow 31r = 62$$
$$\Rightarrow r = 2$$

Putting $r = 2$ in (1), we get

$$2^{n-1} = 32$$
$$\Rightarrow 2^{n-1} = 2^5$$
$$\Rightarrow n - 1 = 5$$
$$\Rightarrow n = 6$$

Example 3.47 **Find the sum to n terms of the sequence $(x + 1/x)^2, (x^2 + 1/x^2)^2, (x^3 + 1/x^3)^2, \ldots$.**

Sol. Let S_n denote the sum to n terms of the given sequence. Then,

$$S_n = \left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + \cdots + \left(x^n + \frac{1}{x^n}\right)^2$$
$$= \left(x^2 + \frac{1}{x^2} + 2\right) + \left(x^4 + \frac{1}{x^4} + 2\right)$$
$$+ \left(x^6 + \frac{1}{x^6} + 2\right) + \cdots + \left(x^{2n} + \frac{1}{x^{2n}} + 2\right)$$

$$= (x^2 + x^4 + x^6 + \cdots + x^{2n}) + \left(\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+\cdots+\frac{1}{x^{2n}}\right)+\underbrace{(2+2+\cdots)}_{n\text{ times}}$$

$$= x^2\left(\frac{(x^2)^n-1}{x^2-1}\right)+\frac{1}{x^2}\left(\frac{(1/x^2)^n-1}{(1/x^2)-1}\right)+2n$$

$$= x^2\left(\frac{x^{2n}-1}{x^2-1}\right)+\frac{1}{x^{2n}}\left(\frac{1-x^{2n}}{1-x^2}\right)+2n$$

$$= x^2\left(\frac{x^{2n}-1}{x^2-1}\right)+\frac{1}{x^{2n}}\left(\frac{x^{2n}-1}{x^2-1}\right)+2n$$

$$= \left(\frac{x^{2n}-1}{x^2-1}\right)\left(x^2+\frac{1}{x^{2n}}\right)+2n$$

Example 3.48 **Prove that the sum to n terms of the series $11 + 103 + 1005 + \cdots$ is $(10/9)(10^n - 1) + n^2$.**

Sol. Let S_n denote the sum to n terms of the given series. Then,

$$S_n = 11 + 103 + 1005 + \cdots \text{ to } n \text{ terms}$$

$$= (10 + 1) + (10^2 + 3) + (10^3 + 5) + \cdots + \{10^n + (2n - 1)\}$$

$$= (10 + 10^2 + \cdots + 10^n) + \{1 + 3 + 5 + \cdots + (2n - 1)\}$$

$$= \frac{10(10^n-1)}{(10-1)} + \frac{n}{2}(1+2n-1)$$

$$= \frac{10}{9}(10^n-1)+n^2$$

Example 3.49 **Find the sum of the following series: $5 + 55 + 555 + \cdots$ to n terms.**

Sol. We have,

$$5 + 55 + 555 + \cdots \text{ to } n \text{ terms}$$
$$= 5[1 + 11 + 111 + \cdots \text{ to } n \text{ terms}]$$

$$= \frac{5}{9}[9 + 99 + 999 + \cdots \text{ to } n \text{ terms}]$$

$$= \frac{5}{9}[(10-1)+(10^2-1)+(10^3-1)+\cdots+(10^n-1)]$$

$$= \frac{5}{9}[(10+10^2+10^3+\cdots+10^n)-(1+1+1+\cdots n\text{ times})]$$

$$= \frac{5}{9}\left[10\times\frac{(10^n-1)}{10-1}-n\right]$$

$$= \frac{5}{9}\left[\frac{10}{9}(10^n-1)-n\right]$$

$$= \frac{5}{81}[10^{n+1}-10-9n]$$

Important Result

- $a^n - b^n$ is divisible by $a - b$ for any $n \in N$.

$$\frac{a^n-b^n}{a-b} = \frac{a^n\left(1-\left(\frac{b}{a}\right)^n\right)}{a\left(1-\frac{b}{a}\right)}$$

$$= a^{n-1}\left(\frac{1-\left(\frac{b}{a}\right)^n}{1-\frac{b}{a}}\right)$$

$$= a^{n-1}\left(1+\frac{b}{a}+\left(\frac{b}{a}\right)^2+\left(\frac{b}{a}\right)^3+\cdots+\left(\frac{b}{a}\right)^{n-1}\right)$$

$$= a^{n-1} + ba^{n-2} + b^2a^{n-3} + \cdots + b^{n-1}$$

- $a^n + b^n$ is divisible by $a + b$ for odd positive natural numbers.

$$\frac{a^n+b^n}{a+b} = \frac{a^n\left(1-\left(-\frac{b}{a}\right)^n\right)}{a\left(1-\left(-\frac{b}{a}\right)\right)}$$

$$= a^{n-1}\left(\frac{1-\left(-\frac{b}{a}\right)^n}{1-\left(-\frac{b}{a}\right)}\right)$$

$$= a^{n-1}\left(1-\frac{b}{a}+\left(\frac{b}{a}\right)^2-\left(\frac{b}{a}\right)^3+\cdots+(-1)^{n-1}\left(\frac{b}{a}\right)^{n-1}\right)$$

$$= a^{n-1} - ba^{n-2} + b^2a^{n-3} - \cdots + (-1)^{n-1}b^{n-1}$$

Example 3.50 **If $p(x) = (1 + x^2 + x^4 + \cdots + x^{2n-2})(1 + x + x^2 + \cdots + x^{n-1})$ is a polynomial in x, then find possible values of n.**

Sol. $$p(x)=\left(\frac{1-x^{2n}}{1-x^2}\right)\left(\frac{1-x}{1-x^n}\right)=\frac{1+x^n}{1+x}$$

As $p(x)$ is a polynomial, $x = -1$ must be a zero of $1 + x^n = 0$, i.e., $1 + (-1)^n = 0$. Hence, n is odd.

Sum of an Infinite G.P.

The sum of an infinite G.P. with first term a and common ratio $r(-1 < r < 1, r \neq 0$ or $0 < |r| < 1)$ is $S = a/(1 - r)$

Proof:

Consider an infinite G.P. with first term a and common ratio r, where $0 < |r| < 1$. The sum of n terms of the G.P. is given by

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r} - \frac{ar^n}{1-r} \quad (1)$$

Since $0 < |r| < 1$, therefore r^n decreases as n increases and r^n tends to zero as n tends to infinity, i.e., $r^n \to 0$ as $n \to \infty$.

$$\therefore \quad \frac{ar^n}{1-r} \to 0 \text{ as } n \to \infty$$

Hence, from (1), the sum of an infinite G.P. is given by

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right) = \frac{a}{1-r} \text{ if } 0 < |r| < 1$$

Note: *If $r \geq 1$, then the sum of an infinite G.P. tends to infinity.*

Example 3.51 **Find the sum of the following series:**

a. $(\sqrt{2}+1) + 1 + (\sqrt{2}-1) + \cdots \infty$

b. $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \cdots \infty$

Sol. a. The given series is a geometric series with first term

$a = \sqrt{2}+1$ and the common ratio

$$r = \frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} = \sqrt{2}-1$$

Hence, the sum to infinity is given by

$$S = \frac{a}{1-r}$$
$$= \frac{\sqrt{2}+1}{1-(\sqrt{2}-1)}$$
$$= \frac{\sqrt{2}+1}{2-\sqrt{2}}$$
$$= \frac{\sqrt{2}+1}{\sqrt{2}(\sqrt{2}-1)}$$
$$= \frac{(\sqrt{2}+1)^2}{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}+1)}$$
$$= \frac{3+2\sqrt{2}}{\sqrt{2}}$$
$$= \frac{4+3\sqrt{2}}{2}$$

b. We have,

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \cdots \text{ to } \infty$$
$$= \left(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots\right)$$
$$= \left(\frac{(1/2)}{1-(1/2^2)}\right) + \left(\frac{(1/3^2)}{1-(1/3^2)}\right)$$
$$= \frac{2}{3} + \frac{1}{8}$$
$$= \frac{19}{24}$$

Example 3.52 **If each term of an infinite G.P. is twice the sum of the terms following it, then find the common ratio of the G.P.**

Sol. Let a be the first term and r the common ratio of the G.P. Then,

$$a_n = 2[a_{n+1} + a_{n+2} + a_{n+3} + \cdots \infty], \text{ for all } n \in N$$

[Given]

$$\Rightarrow \quad ar^{n-1} = 2[ar^n + ar^{n+1} + \cdots \infty]$$
$$\Rightarrow \quad ar^{n-1} = \frac{2ar^n}{1-r}$$
$$\Rightarrow \quad 1 = \frac{2r}{1-r}$$
$$\Rightarrow \quad r = \frac{1}{3}$$

Example 3.53 **If $x = a + a/r + a/r^2 + \cdots \infty$, $y = b - b/r + b/r^2 - \cdots \infty$ and $z = c + c/r^2 + c/r^4 + \cdots \infty$, then prove that $xy/z = ab/c$.**

Sol. We have,

$$x = \frac{a}{1-\frac{1}{r}} = \frac{ar}{r-1}$$
$$y = \frac{b}{1-\left(-\frac{1}{r}\right)} = \frac{br}{1+r}$$
$$z = \frac{c}{1-\frac{1}{r^2}} = \frac{cr^2}{r^2-1}$$
$$\therefore \quad xy = \left(\frac{ar}{r-1}\right)\left(\frac{br}{r+1}\right) = \frac{abr^2}{r^2-1}$$
$$\therefore \quad \frac{xy}{z} = \left[\frac{\frac{abr^2}{r^2-1}}{\frac{cr^2}{r^2-1}}\right] = \frac{ab}{c}$$

Example 3.54 **Prove that $6^{1/2} \times 6^{1/4} \times 6^{1/8} \cdots \infty = 6$.**

Sol. We have,

$$6^{1/2} \times 6^{1/4} \times 6^{1/8} \cdots \infty$$
$$= 6^{[1/2 + 1/4 + 1/8 + \cdots \text{ to } \infty]}$$
$$= 6^{[(1/2)/(1-1/2)]} = 6^1 = 6$$
$$\left[\because \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \text{ to } \infty = \frac{1/2}{1-1/2} = 1\right]$$

Example 3.55 **Sum of infinite number of terms in G.P. is 20 and sum of their squares is 100. Then find the common ratio of G.P.**

Sol. $a + ar + ar^2 + \cdots \text{ to } \infty = 20$

$$\Rightarrow \quad \frac{a}{1-r} = 20 \qquad (1)$$

$$a^2 + a^2r^2 + a^2r^4 + \cdots \text{ to } \infty = 100$$

$$\Rightarrow \frac{a^2}{1-r^2} = 100 \qquad (2)$$

Squaring (1), we have

$$\frac{a^2}{(1-r)^2} = 400 \qquad (3)$$

Dividing (3) by (2), we get

$$\frac{\frac{a^2}{(1-r)^2}}{\frac{a^2}{1-r^2}} = \frac{400}{100}$$

$$\Rightarrow \frac{1-r^2}{(1-r)^2} = 4$$

$$\Rightarrow \frac{1+r}{1-r} = 4$$

$$\Rightarrow 1 + r = 4 - 4r$$

$$\Rightarrow 5r = 3$$

$$\Rightarrow r = \frac{3}{5}$$

Geometric Means (G.M.'s)

Let a and b be two given numbers. If n numbers $G_1, G_2, \ldots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \ldots, G_n, b$ is a G.P., then the numbers $G_1, G_2, \ldots, G_n$ are known as n G.M.'s between a and b.

If a single G.M. G is inserted between two given numbers a and b, then G is known as the G.M. between a and b. Thus, G is the G.M. between a and b.

$\Leftrightarrow$ a, G, b are in G.P.

$$\Leftrightarrow G^2 = ab \Leftrightarrow G = \sqrt{ab}$$

For example, the G.M. between 4 and 9 is given by

$$G = \sqrt{4 \times 9} = 6$$

The G.M. between –9 and –4 is given by

$$G = \sqrt{-9 \times -4} = -6$$

Note: *If a and b are two numbers of opposite signs, then geometric mean between them does not exist.*

Insertion of n G.M.'s Between Two Given Numbers a and b

Let $G_1, G_2, \ldots, G_n$ be n G.M. between two given numbers a and b. Then, $a, G_1, G_2, \ldots, G_n, b$ is a G.P. consisting of $n + 2$ terms. Let r be the common ratio of this G.P. Then,

$$b = (n+2)^{\text{th}} \text{ term} = ar^{n+1}$$

$$\Rightarrow r^{n+1} = \frac{b}{a}$$

$$\Rightarrow r = (b/a)^{\frac{1}{n+1}}$$

$$\Rightarrow G_1 = ar = a\left(\frac{b}{a}\right)^{1/(n+1)}$$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/(n+1)}$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{n/(n+1)}$$

An Important Property of G.M.'s

If n G.M.'s are inserted between two quantities, then the product of n G.M.'s is the n^{th} power of the single G.M. between the two quantities.

Proof:

Let $G_1, G_2, G_3, \ldots, G_n$ be n G.M.'s between two quantities a and b. Then, $a, G_1, G_2, \ldots, G_n, b$ is a G.P. Let r be the common ratio of this G.P. Then, $r = (b/a)^{1/(n+1)}$ and $G_1 = ar$, $G_2 = ar^2$, $G_3 = ar^3, \ldots, G_n = ar^n$. Now,

$$G_1 G_2 G_3 \cdots G_n = (ar)(ar^2)(ar^3)\cdots(ar^n)$$

$$= a^n r^{\frac{n(n+1)}{2}}$$

$$= a^n \left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^{\frac{n(n+1)}{2}}$$

$$= a^n \left(\frac{b}{a}\right)^{n/2} = a^{n/2} b^{n/2}$$

$$= (\sqrt{ab})^n$$

$$= G^n$$

where $G = \sqrt{ab}$ is the single G.M. between a and b.

Example 3.56 **If G be the geometric mean of x and y, then prove that $1/(G^2 - x^2) + 1/(G^2 - y^2) = 1/G^2$.**

Sol. Given, $G = \sqrt{xy}$

$$\therefore \frac{1}{G^2 - x^2} + \frac{1}{G^2 - y^2}$$

$$= \frac{1}{xy - x^2} + \frac{1}{xy - y^2}$$

$$= \frac{1}{x-y}\left\{-\frac{1}{x} + \frac{1}{y}\right\}$$

$$= \frac{1}{xy} = \frac{1}{G^2}$$

Example 3.57 **Insert four G.M.'s between 2 and 486.**

Sol. Common ratio of the series is given by

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$= \left(\frac{486}{2}\right)^{\frac{1}{4+1}}$$

$$= (243)^{1/5}$$

$$r = 3$$

Hence, the four G.M.'s are 6, 18, 54, 162.

Example 3.58 **Find the product of three geometric means between 4 and 1/4.**

Sol. Product of n G.M.'s is $(\sqrt{ab})^n = \left(\sqrt{4\frac{1}{4}}\right)^3 = 1$

Example 3.59 **Find two numbers whose arithmetic mean is 34 and the geometric mean is 16.**

Sol. Let the two numbers be a and b. Then,

$$\text{A.M.} = 34 \Rightarrow \frac{a+b}{2} = 34 \Rightarrow a + b = 68,$$

$$\text{G.M.} = 16$$

$$\Rightarrow \sqrt{ab} = 16 \Rightarrow ab = 256 \quad (1)$$

$$\therefore (a-b)^2 = (a+b)^2 - 4ab$$

$$\Rightarrow (a-b)^2 = (68)^2 - 4 \times 256 = 3600$$

$$\Rightarrow a - b = 60 \quad (2)$$

On solving (1) and (2), we get $a = 64$ and $b = 4$. Hence, the required numbers are 64 and 4.

Example 3.60 **If the A.M. and G.M. between two numbers is in the ratio $m:n$, then prove that the numbers are in the ratio $(m+\sqrt{m^2-n^2}):(m-\sqrt{m^2-n^2})$.**

Sol. Let the two numbers be a and b. Let A and G, be, respectively, the arithmetic and geometric means between a and b. Then,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

$$\therefore a + b = 2A \text{ and } G^2 = ab \quad (1)$$

The equation having a, and b as its roots is

$$x^2 - (a+b)x + ab = 0$$

or

$$x^2 - 2Ax + G^2 = 0 \quad \text{[Using (1)]}$$

$$\Rightarrow x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2}$$

$$\Rightarrow x = A \pm \sqrt{A^2 - G^2}$$

So, the two numbers are $a = A + \sqrt{A^2 - G^2}$ and $b = A - \sqrt{A^2 - G^2}$.

It is given that $A:G = m:n$. Therefore, $A = \lambda m$ and $G = \lambda n$ for some λ Substituting the values of A and G in $a = A + \sqrt{A^2 - G^2}$ and $b = A - \sqrt{A^2 - G^2}$, we get

$$\frac{a}{b} = \frac{\lambda m + \sqrt{\lambda^2 m^2 - \lambda^2 n^2}}{\lambda m - \sqrt{\lambda^2 m^2 - \lambda^2 n^2}}$$

$$\Rightarrow \frac{a}{b} = \frac{m + \sqrt{m^2 - n^2}}{m - \sqrt{m^2 - n^2}}$$

$$\Rightarrow a:b = (m + \sqrt{m^2 - n^2}) : (m - \sqrt{m^2 - n^2})$$

Example 3.61 **If a is the A.M. of b and c and the two geometric means are G_1 and G_2, then prove that $G_1^3 + G_2^3 = 2abc$.**

Sol. It is given that a is the A.M. of b and c. So,

$$a = \frac{b+c}{2} \Rightarrow b + c = 2a \quad (1)$$

Since G_1 and G_2 are two geometric means between b and c, so b, G_1, G_2, c is a G.P. with common ratio $r = (c/b)^{1/3}$.

$$\therefore G_1^3 = b^2c \text{ and } G_2^3 = bc^2$$

$$\Rightarrow G_1^3 + G_2^3 = b^2c + bc^2$$

$$\Rightarrow G_1^3 + G_2^3 = bc(b+c)$$

$$\Rightarrow G_1^3 + G_2^3 = 2abc \quad \text{[Using (1)]}$$

Concept Application Exercise 3.2

1. The first and second terms of a G.P. are x^{-4} and x^n, respectively. If x^{52} is the 8th term, then find the value of n.
2. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of the terms occupying odd places, then find the common ratio.
3. If the sum of n terms of a G.P. is $3 - \frac{3^{n+1}}{4^{2n}}$, then find the common ratio.
4. Prove that $(\underbrace{666\cdots6}_{n \text{ digits}})^2 + (\underbrace{888\cdots8}_{n \text{ digits}}) = \underbrace{4444\cdots4}_{2n \text{ digits}}$.
5. Find the sum $(x+y) + (x^2+xy+y^2) + (x^3+x^2y+xy^2+y^3) + \cdots n$ terms.
6. Find the sum of n terms of the series $4/3 + 10/9 + 28/27 + \cdots$.
7. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then find the common ratio.
8. If a, b, c, d are in G.P., then prove that $(a^3+b^3)^{-1}, (b^3+c^3)^{-1}, (c^3+d^3)^{-1}$ are also in G.P.
9. If the sum of the series $\sum_{n=0}^{\infty} r^n$, $|r| < 1$, is s, then find the sum of the series $\sum_{n=0}^{\infty} r^{2n}$.
10. Let T_r denote the r^{th} term of a G.P. for $r = 1, 2, 3, \ldots$. If for some postivie integers m and n, we have $T_m = 1/n^2$ and $T_n = 1/m^2$, then find the value of $T_{(m+n)/2}$.
11. If a, b, and c are in G.P., then prove that $\frac{1}{a^2-b^2} + \frac{1}{b^2} = \frac{1}{b^2-c^2}$.
12. Find the value of $(32)(32)^{1/6}(32)^{1/36}\cdots\infty$.
13. If a, b, and c are, respectively, the p^{th}, q^{th} and r^{th} terms of a G.P., show that $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$.
14. If a, b, c, d and p are distinct real numbers such that $(a^2+b^2+c^2)p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \le 0$, then prove that a, b, c, d are in G.P.
15. The product of three numbers in G.P. is 125 and sum of their products taken in pairs is 87/2. Find them.
16. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the geometric mean of a and b?

If a, b, c, are in H.P., then $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$ or $b = \frac{2ac}{a+c}$, here b is called harmonic mean of a and c.

HARMONIC PROGRESSION (H.P.)

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ of non-zero numbers is called a harmonic progression or a harmonic sequence, if the sequence $1/a_1$, $1/a_2$, $1/a_3$, $\ldots$, $1/a_n$, $\ldots$ is an arithmetic progression.

If a, b, c are in H.P., then

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow b = \frac{2ac}{a+c}$$

Hence b is called harmonic mean of a and c.

n^{th} Term of a H.P.

The n^{th} term of a H.P. is the reciprocal of the nth term of the corresponding A.P. Thus, if a_1, a_2, a_3, $\ldots$, a_n is a H.P. and the common difference of the corresponding A.P. is d, i.e., $d = 1/a_{n+1} - 1/a_n$, then the n^{th} term of the H.P. is given by

$$a_n = \frac{1}{\frac{1}{a_1} + (n-1)d}$$

In other words, the n^{th} term of a H.P. is the reciprocal of the n^{th} term of the corresponding A.P.

Example 3.62 **The 8^{th} and 14^{th} term of a H.P. are 1/2 and 1/3, respectively. Find its 20^{th} term. Also, find its general term.**

Sol. Let the H.P. be $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots, \frac{1}{a+(n-1)d}, \ldots$.

Then,

$$a_8 = \frac{1}{2} \text{ and } a_{14} = \frac{1}{3}.$$

$$\Rightarrow \frac{1}{a+7d} = \frac{1}{2} \text{ and } \frac{1}{a+13d} = \frac{1}{3} \quad \left[\because a_n = \frac{1}{a+(n-1)d}\right]$$

$$\Rightarrow a + 7d = 2 \text{ and } a + 13d = 3$$

$$\Rightarrow a = \frac{5}{6},\ d = \frac{1}{6}$$

Now,

$$a_{20} = \frac{1}{a+19d} = \frac{1}{\frac{5}{6} + \frac{19}{6}} = \frac{1}{4}$$

and

$$a_n = \frac{1}{a+(n-1)d}$$

$$= \frac{1}{\frac{5}{6} + (n-1) \times \frac{1}{6}}$$

$$= \frac{6}{n+4}$$

Example 3.63 **If the 20^{th} term of a H.P. is 1 and the 30^{th} term is –1/17, then find its largest term.**

Sol. Let the H.P. be $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \cdots$. We have,

$$a_{20} = 1 \text{ and } a_{30} = -\frac{1}{17}$$

$$\Rightarrow \frac{1}{a+19d} = 1 \text{ and } \frac{1}{a+29d} = \frac{-1}{17}.$$

$$\Rightarrow a + 19d = 1 \text{ and } a + 29d = -17$$

$$\Rightarrow a = \frac{176}{5},\ d = -\frac{9}{5}$$

Let n^{th} term be the largest term. We have,

$$a_n = \frac{1}{a+(n-1)d}$$

$$= \frac{1}{\frac{176}{5} + (n-1)\left(\frac{-9}{5}\right)}$$

$$= \frac{5}{176 - 9(n-1)}$$

$$= \frac{5}{185 - 9n}$$

Now, a_n is the largest term if $185 - 9n$ is the least positive integer.

Clearly, $185 - 9n$ attains the least positive integral value, if $n = 20$. Thus, 20^{th} term of the given H.P. is the largest term which is equal to 1.

Example 3.64 **If a, b, c, and d are in H.P., then prove that $(b+c+d)/a$, $(c+d+a)/b$, $(d+a+b)/c$, and $(a+b+c)/d$, are in A.P.**

Sol. a, b, c, and d are in H.P.

$$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d} \text{ are in A.P.}$$

$$\Rightarrow \frac{a+b+c+d}{a}, \frac{a+b+c+d}{b}, \frac{a+b+c+d}{c}, \frac{a+b+c+d}{d} \text{ are in A.P.}$$

$$\Rightarrow 1 + \frac{b+c+d}{a}, \frac{a+c+d}{b} + 1, \frac{a+b+d}{c} + 1, \frac{a+b+c}{d} + 1 \text{ are in A.P.}$$

$$\Rightarrow \frac{b+c+d}{a}, \frac{a+c+d}{b}, \frac{a+b+d}{c}, \frac{a+b+c}{d} \text{ are in A.P.}$$

Example 3.65 **The m^{th} term of a H.P. is n and the n^{th} term is m. Prove that its r^{th} term is mn/r.**

Sol. Let the H.P. be $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots$

Then,

$$a_m = n \text{ and } a_n = m$$

$$\Rightarrow \frac{1}{a+(m-1)d} = n \text{ and } \frac{1}{a+(n-1)d} = m$$

$$\Rightarrow a+(m-1)d = \frac{1}{n} \text{ and } a+(n-1)d = \frac{1}{m}$$

$$\Rightarrow \{a+(m-1)d\} - \{a+(n-1)d\} = \frac{1}{n} - \frac{1}{m}$$

[On subtracting]

$$\Rightarrow (m-n)d = \frac{m-n}{mn}$$

$$\Rightarrow d = \frac{1}{mn}$$

Putting $d = \frac{1}{mn}$ in $a+(m-1)d = \frac{1}{n}$, we get

$$a + \frac{(m-1)}{mn} = \frac{1}{n}$$

$$\Rightarrow a + \frac{1}{n} - \frac{1}{mn} = \frac{1}{n}$$

$$\Rightarrow a = \frac{1}{mn}$$

Now,

$$a_r = \frac{1}{a+(r-1)d}$$

$$= \frac{1}{\frac{1}{mn} + \frac{(r-1)}{mn}}$$

$$= \frac{mn}{1+r-1}$$

$$= \frac{mn}{r}$$

Example 3.66 **If $a > 1$, $b > 1$, and $c > 1$ are in G.P., then show that $\frac{1}{1+\log_e a}$, $\frac{1}{1+\log_e b}$, and $\frac{1}{1+\log_e c}$ are in H.P.**

Sol. It is given that a, b, c are in G.P. Hence,

$$b^2 = ac$$

$$\Rightarrow 2\log_e b = \log_e a + \log_e c$$

$\Rightarrow$ $\log_e a$, $\log_e b$, and $\log_e c$ are in A.P.

$\Rightarrow$ $1+\log_e a$, $1+\log_e b$, and $1+\log_e c$ are in A.P.

$\Rightarrow$ $\frac{1}{1+\log_e a}, \frac{1}{1+\log_e b}$, and $\frac{1}{1+\log_e c}$ are in H.P.

Example 3.67 **If a, b, and c be in G.P. and $a+x, b+x$, and $c+x$ in H.P. then find the value of x (a, b, and c are distinct numbers).**

Sol. $a+x$, $b+x$, and $c+x$ are in H.P.

$$\Rightarrow b+x = \frac{2(a+x)(c+x)}{(a+x)+(c+x)}$$

$$\Rightarrow (b+x)(a+c+2x) = 2(a+x)(c+x)$$

$$\Rightarrow (a+c+2b)x + 2x^2 + ab + bc = 2ac + 2x(a+c) + 2x^2$$

$$\Rightarrow x(c+a-2b) = bc+ab-2ac$$

$$\Rightarrow x(c+a-2b) = bc+ab-2b^2 \quad (\because a, b, c \text{ are in G.P.})$$

$$\Rightarrow x(c+a-2b) = b(c+a-2b)$$

$\Rightarrow$ $x = b$, (as a, b, c, are G.P. and distinct hence a, b, c, cannot be in A.P.)

Example 3.68 **If first three terms of the sequence 1/16, a, b, 1/6 are in geometric series and last three terms are in harmonic series, then find the values of a and b.**

Sol. 1/16, a, b are in G.P. Hence,

$$a^2 = \frac{b}{16} \text{ or } 16a^2 = b \qquad (1)$$

$a, b, \frac{1}{6}$ are in H.P. Hence,

$$b = \frac{2a\frac{1}{6}}{a+\frac{1}{6}} = \frac{2a}{6a+1} \qquad (2)$$

From (1) and (2),

$$16a^2 = \frac{2a}{6a+1}$$

$$\Rightarrow 2a\left(8a - \frac{1}{6a+1}\right) = 0$$

$$\Rightarrow 8a(6a+1) - 1 = 0$$

$$\Rightarrow 48a^2 + 8a - 1 = 0 \quad (\because a \neq 0)$$

$$\Rightarrow (4a+1)(12a-1) = 0$$

$$\therefore a = -\frac{1}{4}, \frac{1}{12}$$

When $a = -1/4$, then from (1),

$$b = 16\left(-\frac{1}{4}\right)^2 = 1$$

When $a = 1/12$, then from (1),

$$b = 16\left(\frac{1}{12}\right)^2 = \frac{1}{9}.$$

Therefore, $a = -1/4$, $b = 1$ or $a = 1/12$, $b = 1/9$.

Harmonic Means

Let a and b be two given numbers. If n numbers $H_1, H_2, \ldots, H_n$ are inserted between a and b such that the sequence a, $H_1, H_2, H_3, \ldots, H_n, b$ is a H.P., then $H_1, H_2, \ldots, H_n$ are called n harmonic means between a and b. Now, $a, H_1, H_2, \ldots, H_n, b$ are in H.P. Hence,

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \ldots, \frac{1}{H_n}, \frac{1}{b} \text{ are in A.P.}$$

Let D be the common difference of this A.P. Then,

$$\frac{1}{b} = (n+2)^{\text{th}} \text{ term}$$

$$\Rightarrow \quad \frac{1}{b} = \frac{1}{a} + (n+1)D$$

$$\Rightarrow \quad D = \frac{a-b}{(n+1)ab}$$

Thus, if n harmonic means are inserted between two given numbers a and b, then the common diference of the corresponding A.P. is given by

$$D = \frac{a-b}{(n+1)ab}$$

Also,

$$\frac{1}{H_1} = \frac{1}{a} + D,\ \frac{1}{H_2} = \frac{1}{a} + 2D,\ \ldots,\ \frac{1}{H_n} = \frac{1}{a} + nD$$

On putting the value of D, we can obtain the values of H_1, H_2, …, H_n.

Harmonic Means of Two Given Numbers

If a and b are two non-zero numbers, then the harmonic mean of a and b is a number H such that the sequence a, H, and b is a H.P. Now, a, H, and b is a H.P. Hence,

$$\frac{1}{a}, \frac{1}{H}, \frac{1}{b} \text{ is an A.P.}$$

$$\Rightarrow \quad \frac{2}{H} = \frac{1}{a} + \frac{1}{b}$$

$$\Rightarrow \quad H = \frac{2ab}{a+b}$$

Thus, the harmonic mean H between two numbers a and b is given by $H = (2ab)/(a+b)$.

Example 3.69 **Insert four H.M.'s between 2/3 and 2/13.**

Sol. Let d be the common difference of corresponding A.P. So,

$$d = \frac{\frac{13}{2} - \frac{3}{2}}{5} = 1$$

$$\therefore \quad \frac{1}{H_1} = \frac{3}{2} + 1 = \frac{5}{2} \quad \text{or} \quad H_1 = \frac{2}{5}$$

$$\frac{1}{H_2} = \frac{3}{2} + 2 = \frac{7}{2} \quad \text{or} \quad H_2 = \frac{2}{7}$$

$$\frac{1}{H_3} = \frac{3}{2} + 3 = \frac{9}{2} \quad \text{or} \quad H_3 = \frac{2}{9}$$

$$\frac{1}{H_4} = \frac{3}{2} + 4 = \frac{11}{2} \quad \text{or} \quad H_4 = \frac{2}{11}$$

Example 3.70 **If H is the harmonic mean between P and Q, then find the value of $H/P + H/Q$.**

Sol. $$\frac{H}{P} + \frac{H}{Q} = H\left(\frac{1}{P} + \frac{1}{Q}\right)$$

$$= \frac{2PQ}{P+Q}\,\frac{P+Q}{PQ} = 2$$

Example 3.71 **If nine arithmetic means and 9 harmonic means are inserted between 2 and 3 alternatively, then prove that $A + 6/H = 5$ (where A is any of the A.M.'s and H the corresponding H.M.).**

Sol. Let $H_1, H_2, \ldots, H_9$ be nine harmonic means between 2 and 3. Then 2, $H_1, H_2, \ldots, H_9$, 3 are in H.P. Therefore, 1/2, $1/H_1$, $1/H_2$, ⋯, $1/H_9$, 1/3 are in A.P. with common difference

$$D = \frac{2-3}{(9+1)\times 2\times 3} = -\frac{1}{60} \quad \left[\because D = \frac{a-b}{(n+1)ab}\right]$$

$$\therefore \quad \frac{1}{H_i} = \frac{1}{2} + iD;\ i = 1, 2, 3, \ldots, 9$$

$$\Rightarrow \quad \frac{1}{H_i} = \frac{1}{2} - \frac{i}{60}$$

$$\Rightarrow \quad \frac{6}{H_i} = 3 - \frac{i}{10} \qquad (1)$$

Let $A_1, A_2, \ldots, A_9$ be 9 A.M.'s between 2 and 3. Then, 2, $A_1, A_2, \ldots, A_9$, 3 are in A.P. with common difference

$$d = \frac{3-2}{9+1} = \frac{1}{10} \quad \left[\because d = \frac{b-a}{n+1}\right]$$

$$\therefore \quad A_i = 2 + id;\ i = 1, 2, \ldots, 9$$

$$\Rightarrow \quad A_i = 2 + \frac{i}{10} \qquad (2)$$

From (1) and (2), we have

$$A_i + \frac{6}{H_i} = 3 - \frac{i}{10} + 2 + \frac{i}{10} = 5 \text{ for } i = 1, 2, \ldots, 9$$

Concept Application Exercise 3.3

1. If the first two terms of a H.P. are 2/5 and 12/13, respectively. Then find the largest term.
2. If a, b, c are in G.P. and $a-b$, $c-a$ and $b-c$ are in H.P., then prove that $a + 4b + c$ is equal to 0.
3. If x, y, and z are in A.P., ax, by, and cz in G.P. and a, b, c in H.P., then prove that $\frac{x}{z} + \frac{z}{x} = \frac{a}{c} + \frac{c}{a}$.
4. If a, b, c, and d are in H.P., then find the value of $\dfrac{a^{-2} - d^{-2}}{b^{-2} - c^{-2}}$.
5. If $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz}$ and p, q, and r be in A.P., then prove that x, y, z are in H.P.
6. If $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$, where a, b, and c are in A.P. and $|a| < 1$, $|b| < 1$, and $|c| < 1$, then prove that x, y, and z are in H.P.
7. If x, 1, and z are in A.P. and x, 2, and z are in G.P., then prove that x, and 4, z are in H.P.

8. If $a, a_1, a_2, a_3, \ldots, a_{2n}, b$ are in A.P. and $a, g_1, g_2, g_3, \ldots, g_{2n}, b$ are in G.P. and h is the H.M. of a and b, then prove that
$$\frac{a_1 + a_{2n}}{g_1 g_{2n}} + \frac{a_2 + a_{2n-1}}{g_2 g_{2n-1}} + \cdots + \frac{a_n + a_{n+1}}{g_n g_{n+1}} = \frac{2n}{h}$$

9. If the $(m+1)^{\text{th}}$, $(n+1)^{\text{th}}$ and $(r+1)^{\text{th}}$ terms of an A.P. are in G.P. and m, n, r are in H.P., then find the value of the ratio of the common difference to the first term of the A.P.

10. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then prove that $\frac{a}{c}, \frac{b}{a}$ and $\frac{c}{b}$ are in H.P.

Properties of A.M., G.M. and H.M. of Two Positive Real Numbers

Let A, G and H be arithmetic, geometric and harmonic means of two positive numbers a and b. Then,
$$A = \frac{a+b}{2},\ G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b}$$

These three means possess the following properties.

1. $A \ge G \ge H$

Proof: We have,
$$A = \frac{a+b}{2},\ G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b}$$
$$\Rightarrow\ A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a} - \sqrt{b})^2}{2} \ge 0$$
$$\Rightarrow\ A \ge G \qquad \text{(i)}$$
$$G - H = \sqrt{ab} - \frac{2ab}{a+b}$$
$$= \sqrt{ab}\left\{\frac{a+b-2\sqrt{ab}}{a+b}\right\}$$
$$= \frac{\sqrt{ab}}{a+b}(\sqrt{a} - \sqrt{b})^2 \ge 0$$
$$\Rightarrow\ G \ge H \qquad \text{(ii)}$$
From (i) and (ii), we get $A \ge G \ge H$.

Note: *The equality holds in the above result only when $a = b$.*

A.M., G.M. and H.M. of n positive quantities,

$a_1, a_2, a_3, \ldots, a_n$ is given by
$$A = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$$
$$G = (a_1 a_2 a_3 \cdots a_n)^{1/n}$$
and
$$H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}}$$

$A \ge G \ge H$ also holds here.

2. A, G, and H form a G.P., i.e., $G^2 = AH$.

Proof: We have,
$$AH = \frac{a+b}{2} \times \frac{2ab}{a+b}$$
$$= ab = (\sqrt{ab})^2 = G^2$$
Hence, $G^2 = AH$.

3. The equation having a and b as its roots is $x^2 - 2Ax + G^2 = 0$.

Proof: The equation having a and b as its roots is
$$x^2 - (a+b)x + ab = 0$$
$$\Rightarrow\ x^2 - 2Ax + G^2 = 0 \qquad \left[\because A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}\right]$$

4. If A and G be the A.M. and G.M. between two positive numbers, then the numbers are $A \pm \sqrt{A^2 - G^2}$.

Proof: The equation having its roots as the given numbers is
$$x^2 - 2Ax + G^2 = 0$$
$$\Rightarrow\ x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2}$$
$$= A \pm \sqrt{A^2 + G^2}$$

5. If A, G, and H are arithmetic, geometric and harmonic means between three given numbers a, b and c, then the equation having a, b, and c as its roots is
$$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

Proof: We have,
$$A = \frac{a+b+c}{3},\ G = (abc)^{1/3},$$
$$\frac{1}{H} = \frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3}$$
$$\therefore\ a + b + c = 3A,\ abc = G^3$$
and
$$\frac{3G^3}{H} = ab + bc + ca$$
The equation having a, b, and c as its roots is
$$x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc = 0$$
$$\Rightarrow\ x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

Example 3.72 **The A.M. and H.M. between two numbers are 27 and 12, respectively, then find their G.M.**

Sol. Let A, G and H denote, respectively, the A.M., G.M. and H.M. between the two numbers. Then, $A = 27$ and $H = 12$. Since A, G, and H are in G.P. Therefore,
$$G^2 = AH$$

$= 27 \times 12$

$\Rightarrow \quad G = 18$

Example 3.73 **If the A.M. between two numbers exceeds their G.M. by 2 and the G.M. exceeds their H.M. by 8/5, find the numbers.**

Sol. $A - G = 2$ (1)

$G - H = 8/5$ (2)

$G^2 = AH$

$= (G + 2)(G - 8/5)$

$\Rightarrow \quad G = 8$

$\Rightarrow \quad ab = 64$ (3)

From (1),

$A = 10$

$\Rightarrow \quad a + b = 20$ (4)

Solving (3) and (4), we get $a = 4$ and $b = 16$ or $a = 16$ and $b = 4$.

Concept Application Exercise 3.4

1. If the arithmetic mean of two positive numbers a and b $(a > b)$ is twice their geometric mean, then find the ratio $a:b$.
2. The A.M. of two given positive numbers is 2. If the larger number is increased by 1, the G.M. of the numbers becomes equal to the A.M. of the given numbers. Then find the H.M.
3. The harmonic mean between two numbers is 21/5, their A.M. 'A' and G.M. 'G' satisfy the relation $3A + G^2 = 36$. Then find the sum of square of numbers.

MISCELLANEOUS SERIES

Arithmetico-Geometric Sequence

Let a, $(a + d)r$, $(a + 2d)r^2$, $(a + 3d)r^3$, … be an arithmetico-geometric sequence. Then, $a + (a + d)r + (a + 2d)r^2 + (a + 3d)r^3 + \cdots$ is an arithmetico-geometric series.

Sum of n Terms of an Arithmetico-geometric Sequence

Sum of n terms of an arithmetico-geometric sequence a, $(a + d)r$, $(a + 2d)r^2$, $(a + 3d)r^3$, … is given by

$$S_n = \begin{cases} \dfrac{a}{1-r} + dr\dfrac{(1-r^{n-1})}{(1-r)^2} - \dfrac{\{a+(n-1)d\}r^n}{1-r}, & \text{when } r \neq 1 \\ \dfrac{n}{2}[2a+(n-1)d], & \text{when } r = 1 \end{cases}$$

Proof:

Let S_n denote the sum of n terms of the given sequence. Then,

$$S_n = a + (a + d)r + (a + 2d)r^2 + \cdots + \{a + (n - 1)d\}r^{n-1} \quad (1)$$

$$\Rightarrow \quad rS_n = ar + (a + d)r^2 + \cdots + \{a + (n - 2)d\}r^{n-1} + \{a + (n - 1)d\}r^n \quad (2)$$

Subtracting (2) from (1), we get

$$S_n - rS_n = a + [dr + dr^2 + \cdots + dr^{n-1}] - \{a + (n - 1)d\}\, r^n$$

$$\Rightarrow \quad S_n(1 - r) = a + dr\left(\frac{1-r^{n-1}}{1-r}\right) - [a+(n-1)d]r^n$$

$$\Rightarrow \quad S_n = \frac{a}{1-r} + dr\frac{1-r^{n-1}}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r} \quad (3)$$

Note: *Generally we dont use this formula to find sum of n terms infact we use the mechanism by which we derived this formula.*

Example 3.74 **Find the sum of the series $1 + 3x + 5x^2 + 7x^3 + \cdots$ to n terms.**

Sol. The given series is an arithmetico-geometric series whose corresponding A.P. and G.P. are 1, 3, 5, 7, … and 1, x, x^2, …, respectively. The n^{th} term of A.P. is $[1 + (n - 1) \times 2] = (2n - 1)$. The n^{th} term of G.P. is $[1 \times x^{n-1}] = x^{n-1}$. So, the n^{th} term of the given series is $(2n - 1)x^{n-1}$. Let S_n denote the sum of n terms of the given series. Then,

$$S_n = 1 + 3x + 5x^2 + 7x^3 + \cdots + (2n - 3)\, x^{n-2} + (2n - 1)x^{n-1} \quad (1)$$

$$\therefore \quad xS_n = x + 3x^2 + 5x^3 + \cdots + (2n-3)\, x^{n-1} + (2n-1)\, x^n \quad (2)$$

Subtracting (2) from (1), we have

$$S_n - xS_n = 1 + [2x + 2x^2 + 2x^3 + \cdots + 2x^{n-1}] - (2n - 1)x^n$$

$$\Rightarrow \quad S_n(1 - x) = 1 + 2x\left(\frac{1-x^{n-1}}{1-x}\right) - (2n-1)x^n$$

$$\Rightarrow \quad S_n = \frac{1}{1-x} + 2x\frac{(1-x^{n-1})}{(1-x)^2} - \frac{(2n-1)x^n}{1-x}$$

Example 3.75 **Find the sum of n terms of the series $1 + 4/5 + 7/5^2 + 10/5^3 + \cdots$.**

Sol. Clearly, the given series is an arithmetico-geometric series whose corresponding A.P. and G.P. are, respectively, 1, 4, 7, 10, … and 1, 1/5, $1/5^2$, $1/5^3$, ….

The n^{th} term of A.P. is $[1 + (n - 1) \times 3] = 3n - 2$.

The n^{th} term of G.P. is $[1 \times (1/5)^{n-1}] = (1/5)^{n-1}$.

So, the n^{th} term of the given series is $(3n - 2) \times (1/5^{\,n-1}) = (3n - 2)/5^{\,n-1}$. Let,

$$S_n = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots + \frac{3n-5}{5^{n-2}} + \frac{3n-2}{5^{n-1}} \quad (1)$$

$$\frac{1}{5}S_n = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \cdots + \frac{(3n-5)}{5^{n-1}} + \frac{3n-2}{5^n} \quad (2)$$

Subtracting (2) from (1), we get

$$S_n - \frac{1}{5}S_n = 1 + \left[\frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^2} + \cdots + \frac{3}{5^{n-1}}\right] - \frac{(3n-2)}{5^n}$$

$$\Rightarrow \quad \frac{4}{5}S_n = 1+\frac{3}{5}\frac{\left(1-\left(\frac{1}{5}\right)^{n-1}\right)}{\left(1-\frac{1}{5}\right)} - \frac{(3n-2)}{5^n}$$

$$= 1+\frac{3}{5}\frac{\left[1-\frac{1}{5^{n-1}}\right]}{\left(\frac{4}{5}\right)} - \frac{(3n-2)}{5^n}$$

$$= 1+\frac{3}{4}\left(1-\frac{1}{5^{n-1}}\right) - \frac{(3n-2)}{5^n}$$

$$\Rightarrow \quad S_n = \frac{5}{4}+\frac{15}{16}\left(1-\frac{1}{5^{n-1}}\right) - \frac{(3n-2)}{4(5^{n-1})}$$

Sum of an Infinite Arithmetico-geometric Sequence

If $|r| < 1$, then $r^n, r^{n-1} \to 0$ as $n \to \infty$ and it can also be shown that $nr^n \to 0$ as $n \to \infty$. So we have

$$S_n \to \frac{a}{1-r}+\frac{dr}{(1-r)^2}, \text{ as } n \to \infty$$

In other words, when $|r| < 1$, the sum to infinity of an arithmetico-geometric series is

$$S_\infty = \frac{a}{1-r}+\frac{dr}{(1-r)^2}$$

Example 3.76 **Find the sum to infinity of the series $1 - 3x + 5x^2 + 7x^3 + \cdots \infty$ when $|x| < 1$.**

Sol. Let S_∞ denote the sum of the given infinite series. Now,

$$S_\infty = 1 - 3x + 5x^2 - 7x^3 + \cdots \infty$$
$$= 1 + 3(-x) + 5(-x)^2 + 7(-x)^3 + \cdots \infty$$

Here, $a = 1$, $r = -x$ and $d = 2$. Hence,

$$S_\infty = \frac{1}{1-(-x)}+\frac{2(-x)}{[1-(-x)]^2}$$

$$= \frac{1}{1+x}-\frac{2}{(1+x)^2}$$

$$= \frac{1-x}{(1+x)^2}$$

Sum to Infinity of the Series Reducible to Arithmetico-geometric Series

Example 3.77 **Find the sum to infinity of the series $1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots \infty$.**

Sol. The given series is not an arithmetico-geometric series, because $1^2, 2^2, 3^2, 4^2, \ldots$ are not in A.P. However, their successive differences $(2^2 - 1^2), (3^2 - 2^2), (4^2 - 3^2), \ldots$, i.e., $3, 5, 7, \ldots$ form an A.P. So, the process of finding the sum to infinity of an arithmetico-geometric series will be repeated twice as given below. Let,

$$S_\infty = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots \infty \quad (1)$$

$$\Rightarrow \quad xS_\infty = 1^2x + 2^2x^2 + 3^2x^3 + \cdots \infty \quad (2)$$

Subtracting (2) from (1), we get

$$(1-x)S_\infty = 1^2 + (2^2 - 1^2)x + (3^2 - 2^2)x^2 + (4^2 - 3^2)x^3 + \cdots$$

$$\Rightarrow \quad (1-x)S_\infty = 1 + 3x + 5x^2 + 7x^3 + \cdots \quad (3)$$

This is an arithmetico-geometric series in which $a = 1$, $d = 2$, $r = x$.

$$\therefore \quad (1-x)S_\infty = \frac{1}{1-x}+\frac{2x}{(1-x)^2} = \frac{1+x}{(1-x)^2}$$

$$\Rightarrow \quad S_\infty = \frac{1+x}{(1-x)^3}$$

Summation by Sigma (Σ) Operator

Properties of Sigma Operator

1. $\sum_{r=1}^{n} T_r = T_1 + T_2 + T_3 + \cdots + T_n$, where T_r is the general term of the series.
2. $\sum_{r=1}^{n} (T_r \pm T_r') = \sum_{r=1}^{n} T_r \pm \sum_{r=1}^{n} T_r'$ (sigma operator is distributive over addition and subtraction)
3. $\sum_{r=1}^{n} T_rT_r' \neq \left(\sum_{r=1}^{n} T_r\right)\left(\sum_{r=1}^{n} T_r'\right)$ (sigma operator is not distributive over multiplication)
4. $\sum_{r=1}^{n} \frac{T_r}{T_r'} \neq \frac{\sum_{r=1}^{n} T_r}{\sum_{r=1}^{n} T_r'}$ (sigma operator is not distributive over division)
5. $\sum_{r=1}^{n} 1 = 1 + 1 + 1 + \cdots n \text{ times} = n$
6. $\sum_{r=1}^{n} aT_r = a\sum_{r=1}^{n} T_r$ (where a is constant)
7. $\sum_{j=1}^{n}\sum_{i=1}^{n} T_iT_j = \left(\sum_{i=1}^{n} T_i\right)\left(\sum_{j=1}^{n} T_j\right)$ (here i and j are independent)

Sum of the Squares of the First n Natural Numbers

Let the sum be denoted by S; then $S = 1^2 + 2^2 + 3^2 + \cdots + n^2$.

We have, $n^3 - (n-1)^3 = 3n^2 - 3n + 1$; and by changing n to $n - 1$, we get

$$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$$
$$(n-2)^3 - (n-3)^3 = 3(n-2)^2 - 3(n-2) + 1$$
$$\vdots$$
$$3^3 - 2^3 = 3 \times 3^2 - 3 \times 3 + 1$$
$$2^3 - 1^2 = 3 \times 2^2 - 3 \times 2 + 1$$
$$1^2 - 0^2 = 3 \times 1^2 - 3 \times 1 + 1$$

Hence, by addition,

$$n^3 = 3(1^2 + 2^2 + 3^2 + \cdots + n^2) - 3(1 + 2 + 3 + \cdots + n) + n$$

$$= 3S - \frac{3n(n+1)}{2} + n$$

$$\Rightarrow \quad 3S = n^3 - n + \frac{3n(n+1)}{2}$$

$$= n(n+1)\left(n - 1 + \frac{3}{2}\right)$$

$$\Rightarrow \quad S = \frac{n(n+1)(2n+1)}{6}$$

Sum of the Cubes of the First n Natural Numbers

Let the sum be denoted by S; then $S = 1^3 + 2^3 + 3^3 + \cdots + n^3$.

We have,

$$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$$
$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1$$
$$(n-2)^4 - (n-3)^4 = 4(n-2)^3 - 6(n-2)^2 + 4(n-2) - 1$$
$$\vdots$$
$$3^4 - 2^4 = 4 \times 3^3 - 6 \times 3^2 + 4 \times 3 - 1$$
$$2^4 - 1^4 = 4 \times 2^3 - 6 \times 2^2 + 4 \times 2 - 1$$
$$1^4 - 0^4 = 4 \times 1^3 - 6 \times 1^2 + 4 \times 1 - 1$$

Hence, by addition,

$$n^4 = 4S - 6(1^2 + 2^2 + \cdots + n^2) + 4(1 + 2 + \cdots + n) - n$$

$$\therefore \quad 4S = n^4 + n + 6(1^2 + 2^2 + \cdots + n^2) - 4(1 + 2 + \cdots + n)$$

$$= n^4 + n + n(n+1)(2n+1) - 2n(n+1)$$

$$= n(n+1)(n^2 - n + 1 + 2n + 1 - 2)$$

$$= n(n+1)(n^2 + n)$$

$$\therefore \quad S = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2$$

Thus, the sum of the cubes of the first n natural numbers is equal to the square of the sum of these numbers.

Example 3.78 **Find the sum to n terms of the series $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \cdots$.**

Sol. $T_n = n(n+1)(n+2)$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(k+1)(k+2)$$

$$= \sum_{k=1}^{n} (k^3 + 3k^2 + 2k)$$

$$= \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right)$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\}$$

$$= \frac{n(n+1)}{4}\{n^2 + n + 4n + 2 + 4\}$$

$$= \frac{n(n+1)}{4}(n^2 + 5n + 6)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

Example 3.79 **Find the sum of the series $1 \times n + 2(n-1) + 3 \times (n-2) + \cdots + (n-1) \times 2 + n \times 1$.**

Sol. Let T_r be the r^{th} term of the given series. Then,

$$T_r = r \times \{n - (r-1)\}$$
$$= r(n - r + 1)$$
$$= r\{(n+1) - r\}$$
$$= (n+1)r - r^2$$

$$\therefore \quad \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left[(n+1)r - r^2\right]$$

$$= (n+1)\left(\sum_{r=1}^{n} r\right) - \left(\sum_{r=1}^{n} r^2\right)$$

$$= (n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)(n+2)}{6}$$

Example 3.80 **Find the sum of the series:**

$$\frac{1^2}{1} + \frac{1^2 + 2^2}{1+2} + \frac{1^2 + 2^2 + 3^2}{1+2+3} + \cdots n \text{ terms.}$$

Sol. $$T_r = \frac{1^2 + 2^2 + \cdots + r^2}{1 + 2 + \cdots + r}$$

$$= \frac{r(r+1)(2r+1)2}{6r(r+1)}$$

$$= \frac{1}{3}(2r+1)$$

$$\Rightarrow \quad \sum_{r=1}^{n} T_r = \frac{2}{3}\left(\sum_{r=1}^{n} r\right) + \frac{n}{3}$$

$$= \frac{1}{3}n(n+1) + \frac{n}{3}$$

$$= \frac{n(n+2)}{3}$$

Example 3.81 **Find the sum of the series $31^3 + 32^3 + \cdots + 50^3$.**

Sol. $S = (1^3 + 2^3 + \cdots + 50^3) - (1^3 + 2^3 + \cdots + 30^3)$

$$= \left(\frac{50 \times 51}{2}\right)^2 - \left(\frac{30 \times 31}{2}\right)^2$$

$$\left[\text{Using } \Sigma n^3 = \left(\frac{n(n+1)}{2}\right)^2\right]$$

$$= \frac{1}{4}(50 \times 51 - 30 \times 31)(50 \times 51 + 30 \times 31)$$

$$= 1409400$$

Example 3.82 Find the sum of first n terms of the series $1^3 + 3 \times 2^2 + 3^3 + 3 \times 4^2 + 5^3 + 3 \times 6^2 + \cdots$ when

a. n is even

b. n is odd

Sol. a. n is even. Let $n = 2m$.

$$S_n = S_{2m} = \sum_{r=1}^{m}(2r-1)^3 + 3\sum_{r=1}^{m}(2r)^2$$

$$= \sum_{r=1}^{m}[8r^3 - 3(2r)^2 + 3(2r) - 1)] + 12\sum_{r=1}^{m} r^2$$

$$= 8\sum_{r=1}^{m} r^3 + 6\sum_{r=1}^{m} r - \sum_{r=1}^{m} 1$$

$$= 2m^2(m+1)^2 + 3m(m+1) - m$$

$$= m[2m^3 + 4m^2 + 5m + 2]$$

Put $2m = n$ or $m = n/2$.

$$\therefore \quad S_n = \frac{n}{8}[n^3 + 4n^2 + 10n + 8] \qquad \text{(i)}$$

b. If n is odd, then $n + 1$ is even. Now,

$$S_n = S_{n+1} - T_{n+1} \qquad \text{(ii)}$$

S_{n+1} is obtained from (1) by replacing n by $n + 1$ and T_{n+1} = $(n + 1)^{th}$ even term = $3(n + 1)^2$. Hence from (ii),

$$S_n(n \text{ odd}) = \frac{n+1}{8}[(n+1)^3 + 4(n+1)^2 + 10(n+1) + 8] - 3(n+1)^2$$

$$S_n = \frac{n+1}{8}[n^3 + 7n^2 - 3n - 1] \qquad \text{(iii)}$$

Equations (i) and (iii) give the required results.

Example 3.83 Find the sum to n terms of the series $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \cdots$.

Sol. Clearly, n^{th} term of the given series is negative or positive accordingly as n is even or odd, respectively.

a. n is even:

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \cdots + (n-1)^2 - n^2$$

$$= (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + \cdots + ((n-1)^2 - n^2)$$

$$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \cdots + ((n-1) - (n))(n-1+n)$$

$$= -(1 + 2 + 3 + 4 + \cdots + (n-1) + n)$$

$$= -\frac{n(n+1)}{2}$$

b. n is odd:

$$(1^2 - 2^2) + (3^2 - 4^2) + \cdots + \{(n-2)^2 - (n-1)^2\} + n^2$$

$$= (1-2)(1+2) + (3-4)(3+4) + \cdots + [(n-2) - (n-1)][(n-2) + (n-1)] + n^2$$

$$= -(1 + 2 + 3 + 4 + \cdots + (n-2) + (n-1)) + n^2$$

$$= -\frac{(n-1)(n-1+1)}{2} + n^2$$

$$= \frac{n(n+1)}{2}$$

Getting n^{th} Term T_n from Sum of n Terms

Example 3.84 If $\sum_{r=1}^{n} T_r = n(2n^2 + 9n + 13)$, then find the sum $\sum_{r=1}^{n} \sqrt{T_r}$.

Sol. $S_n = \sum_{r=1}^{n} T_r = n(2n^2 + 9n + 13)$

$$\Rightarrow \quad T_r = S_r - S_{r-1}$$

$$= r(2r^2 + 9r + 13) - (r-1)(2(r-1)^2 + 9(r-1) + 13)$$

$$= 6r^2 + 12r + 6 = 6(r+1)^2$$

$$\Rightarrow \quad \sqrt{T_r} = \sqrt{6}(r+1)$$

$$\Rightarrow \quad \sum_{r=1}^{n}\sqrt{T_r} = \sqrt{6}\sum_{r=1}^{n}(r+1)$$

$$= \sqrt{6}\left(\frac{n^2 + 3n}{2}\right)$$

$$= \sqrt{\frac{3}{2}}(n^2 + 3n)$$

Example 3.85 If $\sum_{r=1}^{n} T_r = (3^n - 1)$, then find the sum of $\sum_{r=1}^{n} \frac{1}{T_r}$.

Sol. $\sum_{r=1}^{n} T_r = 3^n - 1$

$$\Rightarrow \quad T_r = (3^r - 1) - (3^{r-1} - 1) = 3^{r-1}(3-1) = 2(3^{r-1})$$

$$\Rightarrow \quad \frac{1}{T_r} = \frac{1}{2} \times \left(\frac{1}{3}\right)^{r-1}$$

$$\Rightarrow \quad \sum_{r=1}^{n}\frac{1}{T_r} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{3}\right)^{r-1}$$

$$= \frac{1}{2}\frac{\left(1 - \left(\frac{1}{3}\right)^n\right)}{1 - \frac{1}{3}}$$

$$= \frac{3}{4}\left(1 - \left(\frac{1}{3}\right)^n\right)$$

Sum of Series by Method of Difference

Sometimes, the n^{th} term of a series cannot be determined by the methods discussed so far. If a series is such that the difference between successive terms are either in A.P. or in G.P., then we determine its n^{th} term by the method of difference and then find the sum of the series by using the formulas for Σn, Σn^2 and Σn^3. The method of difference is illustrated in the following examples.

Example 3.86 **Find the sum to n terms of the series $3 + 15 + 35 + 63 + \cdots$.**

Sol. The differences between the successive terms are $15 - 3 = 12$, $35 - 15 = 20$, $63 - 35 = 28$, ... Clearly, these differences are in A.P. Let T_n be the n^{th} term and S_n denote the sum to n terms of the given series. Then,

$$S_n = 3 + 15 + 35 + 63 + \cdots + T_{n-1} + T_n \quad (1)$$
$$S_n = \quad 3 + 15 + 35 + 63 \cdots + T_{n-1} + T_n \quad (2)$$

$$0 = 3 + [12 + 20 + 28 + \cdots + (n-1) \text{ terms}] - T_n$$

[Subtracting (2) from (1)]

$$\Rightarrow T_n = 3 + \frac{(n-1)}{2}[2\times12+(n-1-1)\times8]$$
$$= 3 + (n-1)(12 + 4n - 8)$$
$$= 3 + (n-1)(4n+4)$$
$$= 4n^2 - 1$$

$$\Rightarrow S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n}(4k^2-1)$$
$$= 4\sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}1$$
$$= 4\left\{\frac{n(n+1)(2n+1)}{6}\right\} - n$$
$$= \frac{n}{3}(4n^2+6n-1)$$

Example 3.87 **Find the sum of the following series to n terms $5 + 7 + 13 + 31 + 85 + \cdots$.**

Sol. The sequence of differences between successive terms is 2, 6, 18, 54, ... Clearly, it is a G.P. Let T_n be the n^{th} term of the given series and S_n be the sum of its n terms. Then,

$$S_n = 5 + 7 + 13 + 31 + \cdots + T_{n-1} + T_n \quad (1)$$
$$S_n = \quad 5 + 7 + 13 + 31 + \cdots + T_{n-1} + T_n \quad (2)$$

$$0 = 5 + [2 + 6 + 18 + \cdots + (n-1) \text{ terms}] - T_n$$

[Subtracting (2) from (1)]

$$\Rightarrow 0 = 5 + 2\,\frac{(3^{n-1}-1)}{(3-1)} - T_n$$

$$\Rightarrow T_n = 5 + (3^{n-1} - 1) = 4 + 3^{n-1}$$

$$\therefore S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n}(4+3^{k-1})$$
$$= \sum_{k=1}^{n}4 + \sum_{k=1}^{n}3^{k-1}$$
$$= 4n + (1 + 3 + 3^2 + \cdots + 3^{n-1})$$
$$= 4n + 1 \times \left(\frac{3^n-1}{3-1}\right)$$
$$= 4n + \left(\frac{3^n-1}{2}\right)$$
$$= \frac{1}{2}[3^n + 8n - 1]$$

Sum of Some Special Series

Consider the series

$$\frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \frac{1}{(a+2d)(a+3d)} + \cdots + \frac{1}{(a+(n-2)d)(a+(n-1)d)}$$

In order to find the sum of a finite number of terms of such series, we write its each term as the difference of two terms as given below:

$$\frac{1}{a(a+d)} = \frac{1}{d}\left(\frac{1}{a} - \frac{1}{a+d}\right)$$
$$\frac{1}{(a+d)(a+2d)} = \frac{1}{d}\left(\frac{1}{a+d} - \frac{1}{a+2d}\right)$$
$$\frac{1}{(a+2d)(a+3d)} = \frac{1}{d}\left(\frac{1}{a+2d} - \frac{1}{a+3d}\right)$$

and so on. Therefore,

$$\frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \cdots + \frac{1}{(a+(n-2)d)(a+(n-1)d)}$$
$$= \frac{1}{d}\left[\left(\frac{1}{a} - \frac{1}{a+d}\right) + \left(\frac{1}{a+d} - \frac{1}{a+2d}\right) + \cdots + \left(\frac{1}{a+(n-2)d} - \frac{1}{a+(n-1)d}\right)\right]$$
$$= \frac{1}{d}\left[\frac{1}{a} - \frac{1}{a+(n-1)d}\right]$$
$$= \frac{n-1}{a(a+(n-1)d)}$$

Example 3.88 **Find the sum to n terms of the series $1/(1 \times 2) + 1/(2 \times 3) + 1/(3 \times 4) + \cdots + 1/n(n + 1)$.**

Sol. Let T_r be the r^{th} term of the given series. Then,

$$T_r = \frac{1}{r(r+1)}, \quad r = 1, 2, \ldots, n$$

$$= \frac{1}{r} - \frac{1}{r+1}$$

Hence, the required sum is

$$\sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

Example 3.89 **Find the sum to n terms of the series $1/(1 \times 3) + 1/(3 \times 5) + 1/(5 \times 7) + \cdots$.**

Sol. Let T_r be the r^{th} term of the given series. Then,

$$T_r = \frac{1}{(2r-1)(2r+1)}, \quad r = 1, 2, 3, \ldots, n$$

$$= \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

Hence, the required sum is

$$\sum_{r=1}^{n} T_r = \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2n+1}\right]$$

$$= \frac{n}{2n+1}$$

Example 3.90 **Find the sum $\sum_{r=1}^{n} \frac{1}{(ar+b)(ar+a+b)}$.**

Sol. $$\sum_{r=1}^{n} \frac{1}{(ar+b)(ar+a+b)} = \sum_{r=1}^{n} \frac{1}{a}\left(\frac{1}{ar+b} - \frac{1}{ar+a+b}\right)$$

$$= \frac{1}{a}\sum_{r=1}^{n}\left(\frac{1}{ar+b} - \frac{1}{ar+a+b}\right)$$

$$= \frac{1}{b}\left[\left(\frac{1}{a+b}-\frac{1}{2a+b}\right)+\left(\frac{1}{2a+b}-\frac{1}{3a+b}\right)+\cdots+\left(\frac{1}{na+b}-\frac{1}{(n+1)a+b}\right)\right]$$

$$= \frac{1}{a}\left(\frac{1}{a+b} - \frac{1}{(n+1)a+b}\right)$$

$$= \frac{n}{(a+b)((n+1)a+b)}$$

Example 3.91 **Find the sum to n terms of the series $3/(1^2 \times 2^2) + 5/(2^2 \times 3^2) + 7/(3^2 \times 4^2) + \cdots$.**

Sol. Let T_r be the r^{th} term of the given series. Then,

$$T_r = \frac{(2r+1)}{r^2(r+1)^2}, \quad r = 1, 2, 3, \ldots$$

$$= \left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right], \quad r = 1, 2, 3, \ldots$$

Hence, the required sum is

$$\sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right]$$

$$= \left(\frac{1}{1^2}-\frac{1}{2^2}\right)+\left(\frac{1}{2^2}-\frac{1}{3^2}\right)+\cdots+\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)$$

$$= 1 - \frac{1}{(n+1)^2}.$$

$$= \frac{2n+n^2}{(n+1)^2}$$

Example 3.92 **Find the sum to n terms of the series $\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \cdots$.**

Sol. Let T_r be the r^{th} term of the given series. Then,

$$T_r = \frac{r}{1+r^2+r^4}, \quad r = 1, 2, 3, \ldots, n$$

$$= \frac{r}{(r^2+r+1)(r^2-r+1)}$$

$$= \frac{1}{2}\left[\frac{1}{r^2-r+1} - \frac{1}{r^2+r+1}\right]$$

Therefore sum of the series is

$$\sum_{r=1}^{n} T_r = \frac{1}{2}\left[\sum_{r=1}^{n}\left(\frac{1}{r^2-r+1} - \frac{1}{r^2+r+1}\right)\right]$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{13}\right)+\cdots+\left(\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{n^2+n+1}\right]$$

$$= \frac{n^2+n}{2(n^2+n+1)}$$

Example 3.93 **Find the sum**

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+\cdots+n}$$

Sol. $T_r = \dfrac{1}{1+2+3+\cdots+r} = \dfrac{2}{r(r+1)} = 2\left[\dfrac{1}{r}-\dfrac{1}{r+1}\right]$

$$\Rightarrow \sum_{r=1}^{n} T_r = \frac{2n}{n+1}$$

Example 3.94 **Find the sum**

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)}$$

Sol. $T_r = \dfrac{1}{r(r+1)(r+2)(r+3)}$

Here factors in determinator are in A.P. In such cases we multiply T_r by difference of last factor and first factor

$$\Rightarrow T_r = \frac{r+3-r}{3[r(r+1)(r+2)(r+3)]}$$

$$= -\frac{1}{3}\left[\frac{1}{(r+1)(r+2)(r+3)} - \frac{1}{r(r+1)(r+2)}\right]$$

$$= -\frac{1}{3}[V(r)-V(r-1)]$$

$$\Rightarrow \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)}$$

$$= -\frac{1}{3}[V(n)-V(0)]$$

$$= \frac{1}{3}\left[\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right]$$

Example 3.95 **Find the sum** $\sum\limits_{r=1}^{n} \dfrac{r}{(r+1)!}$ **where** $n!$ **= 1.2.3 ... n.**

Sol. $T_r = \dfrac{r}{(r+1)!}$

$$= \frac{r+1-1}{(r+1)!}$$

$$= \frac{1}{r!} - \frac{1}{(r+1)!}$$

$$\Rightarrow \sum_{r=1}^{n} \left(\frac{1}{r!} - \frac{1}{(r+1)!}\right)$$

$$= \left(\frac{1}{1!}-\frac{1}{2!}\right) + \left(\frac{1}{2!}-\frac{1}{3!}\right) + \cdots + \left(\frac{1}{ni} - \frac{1}{(n+1)!}\right)$$

$$= 1 - \frac{1}{(n+1)!}$$

Example 3.96 **Find the sum** $\sum\limits_{r=1}^{n} r(r+1)(r+2)(r+3)$.

Sol. $T_r = r(r+1)(r+2)(r+3)$

Here factors are in A.P., we multiply T_r by difference of factor after $r+3$ and factor before r.

$$\Rightarrow T_r = \frac{1}{5} r(r+1)(r+2)(r+3)[r+4-(r-1)]$$

$$= \frac{1}{5}[r(r+1)(r+2)(r+3)(r+4)$$

$$-(r-1)r(r+1)(r+2)(r+3)]$$

$$= \frac{1}{5}[V(r)-V(r-1)]$$

$$\Rightarrow \sum_{r=1}^{n} r(r+1)(r+2)(r+3)$$

$$= \frac{1}{5}[V(n)-V(0)]$$

$$= \frac{1}{5} n(n+1)(n+2)(n+3)(n+4)$$

Note: *Also we have*

$$\sum_{r=1}^{n} r(r+1)(r+2)(r+3)(r+4)$$

$$= \frac{1}{6} n(n+1)(n+2)(n+3)(n+4)(n+5)$$

$$\sum_{r=1}^{n} r(r+1)(r+2)$$

$$= \frac{1}{4} n(n+1)(n+2)(n+3) \text{ etc.}$$

Example 3.97 **Find the sum**

$$\frac{1^4}{1\times 3} + \frac{2^4}{3\times 5} + \frac{3^4}{5\times 7} + \cdots + \frac{n^4}{(2n-1)(2n+1)}$$

Sol. $T_n = \dfrac{1}{16}\,\dfrac{16n^4-1+1}{(4n^2-1)}$

$$= \frac{1}{16}\left[4n^2+1+\frac{1}{(2n-1)(2n+1)}\right]$$

$$\therefore T_n = \frac{1}{16}[4n^2+1] + \frac{1}{32}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

Now putting $n = 1, 2, 3, \ldots, n$ and adding, we have

$$S_n = \frac{1}{16}[4\Sigma n^2 + n] + \frac{1}{32}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{16}\times 4\frac{n(n+1)(2n+1)}{6} + \frac{1}{16}n + \frac{1}{32}\left(1-\frac{1}{2n+1}\right)$$

$$= \frac{1}{16}\left[\frac{2}{3}\times n(n+1)(2n+1) + n + \frac{n}{2n+1}\right]$$

$$= \frac{1}{16}\left[\frac{2}{3}n(n+1)(2n+1) + \frac{n\times 2(n+1)}{2n+1}\right]$$

$$= \frac{1}{16}2n(n+1)\left[\frac{(2n+1)^2+3}{3(2n+1)}\right]$$

$$= \frac{n(n+1)}{8}\times\frac{4n^2+4n+4}{3(2n+1)}$$

$$= \frac{n(n+1)(n^2+n+1)}{6(2n+1)}$$

Example 3.98 **Find the sum of the series**

$$\sum_{k=1}^{360}\left(\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}\right)$$

Sol. $T_k = \dfrac{1}{\sqrt{k}\sqrt{k+1}\,[\sqrt{k}+\sqrt{k+1}]}$

$$= \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\,\sqrt{k+1}}$$

$$= \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$$

$$\therefore\quad S = \sum_{k=1}^{360}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)$$

$$= 1 - \frac{1}{\sqrt{361}}$$

$$= 1 - \frac{1}{19} = \frac{18}{19}$$

Example 3.99 **Find the sum of the series**

$$\frac{1}{3^2+1}+\frac{1}{4^2+2}+\frac{1}{5^2+3}+\frac{1}{6^2+4}+\cdots\infty$$

Sol. $T_n = \dfrac{1}{n^2+(n-2)} = \dfrac{1}{(n+2)(n-1)}$, where $n = 3, 4, 5\ldots$

$$= \frac{1}{3}\left[\frac{1}{n-1}-\frac{1}{n+2}\right]$$

$$\therefore\quad S = \sum_{n=3}^{\infty}T_n = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{3}-\frac{1}{6}\right) + \frac{1}{3}\left(\frac{1}{4}-\frac{1}{7}\right) + \frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right)$$

$$\vdots\ \vdots$$

$$S = \frac{1}{3}\left[\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right] = \frac{1}{3}\left[\frac{6+4+3}{12}\right] = \frac{13}{36}$$

Example 3.100 **Find the sum of firs 100 terms of the series whose general term is given by $a_k = (k^2+1)\,k\,!$**

Sol. $a_k = (k^2+1)\,k\,!$

$= (k(k+1)-(k-1))\,k\,!$

$= k\,(k+1)\,!-(k-1)\,k\,!$

so $\quad k\,(k+1)\,!-(k-1)\,k\,!$

$a_1 = 1\,.2\,!-0$

$a_2 = 2\,.3\,!-1\,.2\,!$

$a_3 = 3\,.4\,!-2\,.3\,!$

....

....

$a_{100} = 100\,.101\,!-99.100\,!$

$a_1 + a_2 + \cdots + a_{100} = 100.\,101\,!$

Sum of the Series when *i* and *j* are Dependent

Consider sum of the series $\sum\sum_{0\le i<j\le n} ij$. In the given summation, i and j are not independent. In the sum of series $\sum_{i=1}^{n}\sum_{j=1}^{n} ij = \sum_{i=1}^{n}\left(i\left(\sum_{j=1}^{n} j\right)\right)$. Here i and j are independent. In this summation, there are three types of terms, those when $i<j$ (upper triangle), $i>j$ (lower triangle) and $i=j$ (diagonal) as shown in the diagram below.

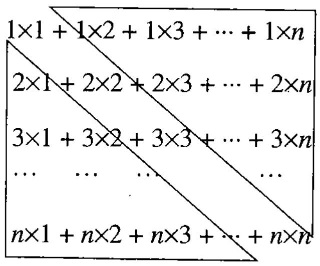


Fig. 3.2

Also, the sum of terms when $i<j$ is equal to the sum of the terms when $i>j$ as terms in both the triangles are symmetrical. So, in that case

$\sum_{i=1}^{n}\sum_{j=1}^{n} ij$ = Sum of terms in upper triangle + sum of terms in lower triangle + sum of terms in diagonal

$$= 2\sum\sum_{0\le i<j\le n} ij + \sum\sum_{i=j} ij \quad (\because \text{ sums of terms in upper and lower triangles are same})$$

$$\Rightarrow \sum\sum_{0\le i<j\le n} ij = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} ij - \sum\sum_{i=j} ij}{2}$$

$$= \frac{\sum_{i=1}^{n} i \sum_{j=1}^{n} j - \sum_{i=1}^{n} i^2}{2}$$

$$= \frac{\frac{n(n+1)}{2}\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}}{2}$$

When $f(i)$ and $f(j)$ are not symmetrical, we find the sum by listing all the terms.

Consider the sum $\sum\sum_{0\le i\le j\le n} ij$.

In this sum, we have to find the sum of the upper triangle and the diagonal of the above square. Hence,

$$\sum\sum_{0\le i\le j\le n} ij = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} ij - \sum\sum_{i=j} ij}{2} + \sum\sum_{i=j} ij$$

$$= \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} ij + \sum\sum_{i=j} ij}{2}$$

Alternative method:

$$(1+2+3+4+\cdots+n)^2 = (1^2+2^2+3^2+\cdots+n^2) + 2\sum\sum_{0\le i<j\le n} ij$$

$$\Rightarrow \left(\frac{n(n+1)}{2}\right)^2 = \frac{n(n+1)(2n+1)}{6} + 2\sum\sum_{0\le i<j\le n} ij$$

$$\Rightarrow \sum\sum_{0\le i<j\le n} ij = \frac{\left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6}}{2}$$

Example 3.101 **Find the sum of the products of the ten numbers ±1, ±2, ±3, ±4, and ±5 taking two at a time.**

Sol. We have,

$$(1-1+2-2+\cdots+5-5)^2 = 1^2+1^2+2^2+2^2+\cdots+5^2+5^2+2S,$$

where S is the required sum. Hence,

$$0 = 2(1^2+2^2+3^2+4^2+5^2)+2S$$

$$\Rightarrow S = -55$$

Example 3.102 **Find the sum $\sum\sum_{0\le i<j\le n} 1$.**

Sol.
$$\sum\sum_{0\le i<j\le n} 1 = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} 1 - \sum\sum_{i=j} 1}{2}$$

$$= \frac{\left(\sum_{j=1}^{n} 1\right)\left(\sum_{j=1}^{n} 1\right) - \sum_{j=1}^{n} 1}{2}$$

$$= \frac{n^2-n}{2}$$

$$= \frac{n(n-1)}{2}$$

Concept Application Exercise 3.5

1. Find the sum $\frac{3}{2}-\frac{5}{6}+\frac{7}{18}-\frac{9}{54}+\cdots\infty$.
2. Find the sum $\frac{1^2}{2}+\frac{3^2}{2^2}+\frac{5^2}{2^3}+\frac{7^2}{2^4}+\cdots\infty$.
3. If the sum to infinity of the series $3+(3+d)\frac{1}{4}+(3+2d)\frac{1}{4^2}+\cdots\infty$ is $\frac{44}{9}$, then find d.
4. Find the sum of the series $1^2+3^2+5^2+\cdots$ to n terms.
5. Find the sum up to 20 terms.
$$1+\frac{1}{2}(1+2)+\frac{1}{3}(1+2+3)+\frac{1}{4}(1+2+3+4)+\cdots$$
6. If the sum of the squares of first n natural numbers exceeds their sum by 330, then find n.
7. Find the value of $11^2+12^2+13^2+\cdots+20^2$.
8. Find the sum $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\cdots$ up to 22nd term.
9. Find the sum $2+5+10+17+26+\cdots$.
10. Find the sum $1+4+13+40+121+\cdots$.
11. If the set of natural numbers is partitioned into subsets $S_1=\{1\}$, $S_2=\{2,3\}$, $S_3=\{4,5,6\}$ and so on, then find the sum of the terms in S_{50}.
12. If $T_r = r(r^2-1)$, then find $\sum_{r=2}^{\infty}\frac{1}{T_r}$.
13. If $S=\frac{1}{1\times3\times5}+\frac{1}{3\times5\times7}+\frac{1}{5\times7\times9}+\cdots$ to infinity, then find the value of $[36S]$, where $[\cdot]$ represents the greatest integer function.
14. If $\sum_{r=1}^{n} t_r = \frac{n}{8}(n+1)(n+2)(n+3)$, then find $\sum_{r=1}^{n}\frac{1}{t_r}$.
15. Find the sum of the series
$1+2(1-x)+3(1-x)(1-2x)+\cdots$
$+n(1-x)(1-2x)(1-3x)\cdots[1-(n-1)x]$

EXERCISES

Subjective Type

Solutions on page 3.43

1. For $a, x > 0$ prove that at the most one term of the G.P. $\sqrt{a-x}, \sqrt{x}, \sqrt{a+x}$ can be rational.
2. If the terms of the A.P. $\sqrt{a-x}, \sqrt{x}, \sqrt{a+x}, \ldots$ are all integers, where $a, x > 0$, then find the least composite value of a.
3. Find a three-digit number such that its digits are in increasing G.P. (from left to right) and the digits of the number obtained from it by subtracting 100 form an A.P.
4. Along a road lies an odd number of stones placed at intervals of 10 m. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried the job with one of the end stones by carrying them in succession. In carrying all the stones, he covered a distance of 3 km. Find the number of stones.
5. If the first and the n^{th} terms of a G.P. are a and b, respectively, and if P is the product of the first n terms, prove that $P^2 = (ab)^n$.
6. Let $x = 1 + 3a + 6a^2 + 10a^3 + \cdots, |a| < 1$.
   $y = 1 + 4b + 10b^2 + 20b^3 + \cdots, |b| < 1$.
   Find $S = 1 + 3(ab) + 5(ab)^2 + \cdots$ in terms of x and y.
7. If the sum of n terms of the series $\frac{2n+1}{2n-1} + 3\left(\frac{2n+1}{2n-1}\right)^2 + 5\left(\frac{2n+1}{2n-1}\right)^3 + \cdots$ is 36, then find the value of n.
8. Find the sum $\frac{3}{1\times 2}\times\frac{1}{2} + \frac{4}{2\times 3}\times\left(\frac{1}{2}\right)^2 + \frac{5}{3\times 4}\times\left(\frac{1}{2}\right)^3 + \cdots$ to n terms.
9. If $S_1, S_2, S_3, \ldots, S_m$ are the sums of n terms of m A.P.'s whose first terms are $1, 2, 3, \ldots, m$ and common differences are $1, 3, 5, \ldots, (2m-1)$, respectively, show that
   $$S_1 + S_2 + \cdots + S_m = \frac{mn}{2}(mn+1)$$
10. If S_1, S_2 and S_3 be, respectively, the sum of n, $2n$ and $3n$ terms of a G.P., prove that $S_1(S_3 - S_2) = (S_2 - S_1)^2$.
11. Find four numbers in a G.P. whose sum is 85 and product is 4096.
12. There are $(4n + 1)$ terms in a certain sequence of which the first $(2n + 1)$ terms form an A.P. of common difference 2 and the last $(2n + 1)$ terms are in G.P. of common ratio 1/2. If the middle term of both A.P. and G.P. be the same, then find the mid-term of this sequence.
13. Let there be $a_1, a_2, a_3, \ldots, a_n$ terms in G.P. whose common ratio is r. Let S_k denote the sum of first k terms of this G.P. Prove that $S_{m-1}\, S_m = \frac{r+1}{r}\sum_{i<j}^{m} a_i a_j$.
14. Find the sum of n terms of the series whose n^{th} term is
   $$T(n) = \tan\frac{x}{2^n} \times \sec\frac{x}{2^{n-1}}.$$
15. Find the value of $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\substack{k=0\\(i\neq j\neq k)}}^{\infty}\frac{1}{3^i 3^j 3^k}$.
16. Let $a_1, a_2, \ldots, a_n$ be real numbers such that
   $$\sqrt{a_1} + \sqrt{a_2 - 1} + \sqrt{a_3 - 2} + \cdots + \sqrt{a_n - (n-1)} = \frac{1}{2}(a_1 + a_2 + \ldots + a_n) - \frac{n(n-3)}{4}$$
   Compute the value of $\sum_{i=1}^{100} a_i$.

Objective Type

Solutions on page 3.46

Each question has four choices a, b, c and d, out of which *only one* is correct.

1. If $\log_2(5\times 2^x + 1)$, $\log_4(2^{1-x} + 1)$ and 1 are in A.P., then x equals
   a. $\log_2 5$ **b.** $1 - \log_5 2$
   c. $\log_5 2$ **d.** none of these
2. If three positive real numbers a, b, c are in A.P. such that $abc = 4$, then the minimum value of b is
   a. $2^{1/3}$ **b.** $2^{2/3}$
   c. $2^{1/2}$ **d.** $2^{3/2}$
3. The maximum sum of the series $20 + 19\frac{1}{3} + 18\frac{2}{3} + \cdots$ is
   a. 310 **b.** 300
   c. 320 **d.** none of these
4. The largest term common to the sequences 1, 11, 21, 31, … to 100 terms and 31, 36, 41, 46, … to 100 terms is
   a. 381 **b.** 471
   c. 281 **d.** none of these
5. If the sum of m terms of an A.P. is the same as the sum of its n terms, then the sum of its $(m + n)$ terms is
   a. mn **b.** $-mn$
   c. $1/mn$ **d.** 0
6. If the sides of a right angled triangle are in A.P. then the sines of the acute angles are
   a. $\frac{3}{5}, \frac{4}{5}$ **b.** $\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}$
   c. $\frac{1}{2}, \frac{\sqrt{3}}{2}$ **d.** none of these
7. If the ratio of the sum to n terms of two A.P.'s is $(5n + 3)$:$(3n + 4)$, then the ratio of their 17^{th} terms is
   a. 172:99 **b.** 168:103
   c. 175:99 **d.** 171:103
8. 150 workers were engaged to finish a piece of work in a certain number of days. Four workers stopped working on the second day, four more workers stopped their work on the third day and so on. It took 8 more days to finish the work. Then the number of days in which the work was completed is
   a. 29 days **b.** 24 days
   c. 25 days **d.** none of these
9. In an A.P. of which a is the first term, if the sum of the first p terms is zero, then the sum of the next q terms is
   a. $-\frac{a(p+q)p}{q+1}$ **b.** $\frac{a(p+q)p}{p+1}$

c. $-\frac{a(p+q)q}{p-1}$ d. none of these

10. If S_n denotes the sum of first 'n' terms of an A.P. and $\frac{S_{3n}-S_{n-1}}{S_{2n}-S_{2n-1}}=31$, then the value of n is
a. 21 b. 15
c. 16 d. 19

11. If a, b, and c are in A.P. then $a^3+c^3-8b^3$ is equal to
a. $2\,abc$ b. $6\,abc$
c. $4\,abc$ d. none of these

12. The number of terms of an A.P. is even; the sum of the odd terms is 24, and of the even terms is 30, and the last term exceeds the first by 10/2, then the number of terms in the series is
a. 8 b. 4
c. 6 d. 10

13. If $a, \frac{1}{b}, c$ and $\frac{1}{p}, q, \frac{1}{r}$ form two arithmetic progressions of the same common difference, then a, q, c are in A.P. if
a. p, b, r are in A.P. b. $\frac{1}{p}, \frac{1}{b}, \frac{1}{r}$ are in A.P.
c. p, b, r are in G.P. d. none of these

14. Suppose that $F(n+1)=\frac{2F(n)+1}{2}$ for $n = 1, 2, 3, \ldots$ and $F(1)=2$. Then, $F(101)$ equals
a. 50 b. 52
c. 54 d. none of these

15. If the sum of n terms of an A.P. is $cn\,(n-1)$, where $c \neq 0$, then sum of the squares of these terms is
a. $c^2 n\,(n+1)^2$ b. $\frac{2}{3}c^2 n\,(n-1)\,(2n-1)$
c. $\frac{2c^2}{3} n\,(n+1)\,(2n+1)$ d. none of these

16. Consider an A.P. $a_1, a_2, a_3, \ldots$ such that $a_3+a_5+a_8=11$ and $a_4+a_2=-2$, then the value of $a_1+a_6+a_7$ is
a. -8 b. 5
c. 7 d. 9

17. If $a_1, a_2, a_3, \ldots$ are in A.P., then a_p, a_q, a_r are in A.P. if p, q, r are in
a. A.P. b. G.P.
c. H.P. d. none of these

18. Let $a_1, a_2, a_3, \ldots$ be terms of an A.P. If $\frac{a_1+a_2+\cdots+a_p}{a_1+a_2+\cdots+a_q}=\frac{p^2}{q^2}$, $p \neq q$, then $\frac{a_6}{a_{21}}$ equals
a. 41/11 b. 7/2
c. 2/7 d. 11/41

19. If S_n denotes the sum of first n terms of an A.P. whose first term is a and $\frac{S_{nx}}{S_x}$ is independent of x, then $S_p=$
a. p^3 b. p^2a
c. pa^2 d. a^3

20. Three numbers form an increasing G.P. If the middle number is doubled, then the new numbers are in A.P. The common ratio of the G.P. is
a. $2-\sqrt{3}$ b. $2+\sqrt{3}$
c. $\sqrt{3}-2$ d. $3+\sqrt{2}$

21. If a_1, a_2, a_3 $(a_1>0)$ are three successive terms of a G.P. with common ratio r, the value of r for which $a_3>4a_2-3a_1$ holds is given by
a. $1<r<3$ b. $-3<r<-1$
c. $r>3$ or $r<1$ d. none of these

22. Let $S \subset (0, \pi)$ denote the set of values of x satisfying the equation $8^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\cdots \text{ to } \infty}=4^3$. Then, $S=$
a. $\{\pi/3\}$ b. $\{\pi/3, -2\pi/3\}$
c. $\{-\pi/3, 2\pi/3\}$ d. $\{\pi/3, 2\pi/3\}$

23. If $|a|<1$ and $|b|<1$, then the sum of the series $1+(1+a)b+(1+a+a^2)b^2+(1+a+a^2+a^3)b^3+\cdots$ is
a. $\frac{1}{(1-a)(1-b)}$ b. $\frac{1}{(1-a)(1-ab)}$
c. $\frac{1}{(1-b)(1-ab)}$ d. $\frac{1}{(1-a)(1-b)(1-ab)}$

24. If $(p+q)^{\text{th}}$ term of a G.P. is 'a' and its $(p-q)^{\text{th}}$ term is 'b' where $a, b \in R^+$, then its p^{th} term is
a. $\sqrt{\frac{a^3}{b}}$ b. $\sqrt{\frac{b^3}{a}}$
c. $\sqrt{ab}$ d. none of these

25. If the sides of a triangle are in G.P., and its largest angle is twice the smallest, then the common ratio r satisfies the inequality
a. $0<r<\sqrt{2}$ b. $1<r<\sqrt{2}$
c. $1<r<2$ d. none of these

26. The value of $0.2^{\log_{\sqrt{5}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)}$ is
a. 4 b. log 4
c. log 2 d. none of these

27. If $(1+x)(1+x^2)(1+x^4)\cdots(1+x^{128})=\sum_{r=0}^{n}x^r$, then n is equal to
a. 256 b. 255
c. 254 d. none of these

28. If x, y, z are in G.P. and $a^x=b^y=c^z$, then
a. $\log_b a=\log_a c$ b. $\log_c b=\log_a c$
c. $\log_b a=\log_c b$ d. none of these

29. The geometric mean between -9 and -16 is
a. 12 b. -12
c. -13 d. none of these

30. If S denotes the sum to infinity and S_n the sum of n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$, such that $S-S_n<\frac{1}{1000}$, then the least value of n is

a. 8
b. 9
c. 10
d. 11

31. The first term of an infinite geometric series is 21. The second term and the sum of the series are both positive integers. Then which of the following is not the possible value of the second term

a. 12
b. 14
c. 18
d. None of these

32. The number of terms common between the series $1 + 2 + 4 + 8 + \cdots$ to 100 terms and $1 + 4 + 7 + 10 + \cdots$ to 100 terms is

a. 6
b. 4
c. 5
d. none of these

33. After striking the floor, a certain ball rebounds $(4/5)^{th}$ of height from which it has fallen. Then the total distance that it travels before coming to rest, if it is gently dropped from a height of 120 m is

a. 1260 m
b. 600 m
c. 1080 m
d. none of these

34. The sum of an infinite G.P. is 57 and the sum of their cubes is 9747, then common ratio of the G.P. is

a. 1/3
b. 2/3
c. 1/6
d. none of these

35. If $a^2 + b^2$, $ab + bc$ and $b^2 + c^2$ are in G.P., then a, b, c are in

a. A.P.
b. G.P.
c. H.P.
d. none of these

36. Consider the ten numbers $ar, ar^2, ar^3, \ldots, ar^{10}$. If their sum is 18 and the sum of their reciprocals is 6 then the product of these ten numbers, is

a. 81
b. 243
c. 343
d. 324

37. Let $a = 1\,1\,1 \ldots 1$ (55 digits), $b = 1 + 10 + 10^2 + \cdots + 10^4$, $c = 1 + 10^5 + 10^{10} + 10^{15} + \cdots + 10^{50}$, then

a. $a = b + c$
b. $a = bc$
c. $b = ac$
d. $c = ab$

38. Let a_n be the n^{th} term of a G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, such that $\alpha \neq \beta$, then the common ratio is

a. α/β
b. β/α
c. $\sqrt{\alpha/\beta}$
d. $\sqrt{\beta/\alpha}$

39. The sum of 20 terms of a series of which every even term is 2 times the term before it, and every odd term is 3 times the term before it, the first term being unity is

a. $\left(\frac{2}{7}\right)(6^{10} - 1)$
b. $\left(\frac{3}{7}\right)(6^{10} - 1)$
c. $\left(\frac{3}{5}\right)(6^{10} - 1)$
d. none of these

40. In a G.P. the first, third and fifth terms may be considered as the first, fourth and sixteenth terms of an A.P. Then the fourth term of the A.P., knowing that its first term is 5 is

a. 10
b. 12
c. 16
d. 20

41. If the p^{th}, q^{th}, and r^{th} terms of an A.P. are in G.P., then common ratio of the G.P. is

a. $\frac{pr}{q^2}$
b. $\frac{r}{p}$
c. $\frac{q+r}{p+q}$
d. $\frac{q-r}{p-q}$

42. If the p^{th}, q^{th}, r^{th} and s^{th} terms of an A.P. are in G.P., then $p - q$, $q - r$, $r - s$ are in

a. A.P.
b. G.P.
c. H.P.
d. none of these

43. If a, b, c, d are in G.P., then $(b-c)^2 + (c-a)^2 + (d-b)^2$ in equal to

a. $(a-d)^2$
b. $(ad)^2$
c. $(a+d)^2$
d. $(a/d)^2$

44. If a, b, c are digits, then the rational number represented by $0.cababab \ldots$ is

a. $cab/990$
b. $(99c + ba)/990$
c. $(99c + 10a + b)/99$
d. $(99c + 10a + b)/990$

45. The sum of an infinite geometric series is 162 and the sum of its first n terms is 160. If the inverse of its common ratio is an integer, then which of the following is not a possible first term?

a. 108
b. 144
c. 160
d. none of these

46. Let $f(x) = 2x + 1$. Then the number of real number of real values of x for which the three unequal numbers $f(x)$, $f(2x)$, $f(4x)$ are in G.P. is

a. 1
b. 2
c. 0
d. none of these

47. Concentric circles of radii 1, 2, 3, ..., 100 cm are drawn. The interior of the smallest circle is coloured red and the angular regions are coloured alternately green and red, so that no two adjacent regions are of the same colour. Then, the total area of the green regions in sq. cm is equal to

a. 1000π
b. 5050π
c. 4950π
d. 5151π

48. Let $\{t_n\}$ be a sequence of integers in G.P. in which $t_4 : t_6 = 1:4$ and $t_2 + t_5 = 216$. Then t_1 is

a. 12
b. 14
c. 16
d. none of these

49. If x, $2y$, $3z$ are in A.P., where the distinct numbers x, y, z are in G.P., then the common ratio of the G.P. is

a. 3
b. $\frac{1}{3}$
c. 2
d. $\frac{1}{2}$

50. If S_p denotes the sum of the series $1 + r^p + r^{2p} + \cdots$ to ∞ and s_p the sum of the series $1 - r^p + r^{2p} - r^{3p} + \cdots$ to ∞, $|r| < 1$, then $S_p + s_p$ in terms of S_{2p} is

a. $2S_{2p}$
b. 0
c. $\frac{1}{2}S_{2p}$
d. $-\frac{1}{2}S_{2p}$

51. If x, y, z are real and $4x^2 + 9y^2 + 16z^2 - 6xy - 12yz - 8zx = 0$, then x, y, z are in

a. A.P. **b.** G.P.
c. H.P. **d.** none of these

52. If $a_1, a_2, \ldots, a_n$ are in H.P., then

$$\frac{a_1}{a_2+a_3+\cdots+a_n}, \frac{a_2}{a_1+a_3+\cdots+a_n}, \ldots, \frac{a_n}{a_1+a_2+\cdots+a_{n-1}}$$ are in

a. A.P. **b.** G.P.
c. H.P. **d.** none of these

53. If $H_1, H_2, \ldots, H_{20}$ be 20 harmonic means between 2 and 3, then $\frac{H_1+2}{H_1-2}+\frac{H_{20}+3}{H_{20}-3} =$

a. 20 **b.** 21
c. 40 **d.** 38

54. Let a_1, a_2, a_3, a_4 and a_5 be such that a_1, a_2 and a_3 are in A.P., a_2, a_3 and a_4 are in G.P., and a_3, a_4 and a_5 are in H.P. Then $\log_e a_1$, $\log_e a_3$ and $\log_e a_5$ are in

a. G.P. **b.** A.P.
c. H.P. **d.** none of these

55. If a, b, c are in A.P., then $\frac{a}{bc}, \frac{1}{c}, \frac{2}{b}$ will be in

a. A.P. **b.** G.P.
c. H.P. **d.** none of these

56. If $x, 2x + 2$, and $3x + 3$ are first three terms of a G.P., then the fourth term is

a. 27 **b.** -27
c. 13.5 **d.** -13.5

57. Sum of three numbers in G.P. be 14. If one is added to first and second and 1 is subtracted from the third, the new numbers are in A.P. The smallest of them is

a. 2 **b.** 4
c. 6 **d.** 10

58. If a, b, and c are in A.P., p, q, and r are in H.P. and ap, bq, and cr are in G.P., then $\frac{p}{r}+\frac{r}{p}$ is equal to

a. $\frac{a}{c}-\frac{c}{a}$ **b.** $\frac{a}{c}+\frac{c}{a}$
c. $\frac{b}{q}+\frac{q}{b}$ **d.** $\frac{b}{q}-\frac{q}{b}$

59. If a, b, and c are in A.P. and $b - a, c - b$ and a are in G.P., then $a:b:c$ is

a. 1:2:3 **b.** 1:3:5
c. 2:3:4 **d.** 1:2:4

60. Let $a \in (0, 1]$ satisfies the equation $a^{2008} - 2a + 1 = 0$ and $S = 1 + a + a^2 + \ldots + a^{2007}$. Sum of all possible value(s) of S, is

a. 2010 **b.** 2009
c. 2008 **d.** 2

61. Let $\alpha, \beta \in R$. If α, β^2 be the roots of quadratic equation $x^2 - px + 1 = 0$ and α^2, β be the roots of quadratic equation $x^2 - qx + 8 = 0$, then the value of 'r' if $\frac{r}{8}$ be arithmetic mean of p and q, is

a. $\frac{83}{2}$ **b.** 83
c. $\frac{83}{8}$ **d.** $\frac{83}{4}$

62. $a, b, c, d \in R^+$ such that a, b, and c are in A.P. and b, c and, d are in H.P., then

a. $ab = cd$ **b.** $ac = bd$
c. $bc = ad$ **d.** none of these

63. If in a progression $a_1, a_2, a_3, \ldots$, etc., $(a_r - a_{r+1})$ bears a constant ratio with $a_r \times a_{r+1}$, then the terms of the progression are in

a. A.P. **b.** G.P.
c. H.P. **d.** none of these

64. If a, b, and c are in G.P., then $a + b$, $2b$, and $b + c$ are in

a. A.P. **b.** G.P.
c. H.P. **d.** none of these

65. If a, x, and b are in A.P., a, y, and b are in G.P. and a, z, b are in H.P. such that $x = 9z$ and $a > 0, b > 0$, then

a. $|y| = 3z$ **b.** $x = 3|y|$
c. $2y = x + z$ **d.** none of these

66. Let $n \in N, n > 25$. Let A, G, H denote the arithmetic mean, geometric mean and harmonic mean of 25 and n. The least value of n for which $A, G, H \in \{25, 26, \ldots, n\}$ is

a. 49 **b.** 81
c. 169 **d.** 225

67. The 15th term of the series $2\frac{1}{2}+1\frac{7}{13}+1\frac{1}{9}+\frac{20}{23}+\cdots$ is

a. $\frac{10}{39}$ **b.** $\frac{10}{21}$
c. $\frac{10}{23}$ **d.** none of these

68. If a, b, and c are in G.P. and x, y, respectively, be arithmetic means between a, b and b, c, then the value of $\frac{a}{x}+\frac{c}{y}$ is

a. 1 **b.** 2
c. 1/2 **d.** none of these

69. If a, b, and c are in A.P. and p, p' are, respectively, A.M. and G.M. between a and b while q, q' are, respectively, the A.M. and G.M. between b and c, then

a. $p^2 + q^2 = p'^2 + q'^2$ **b.** $pq = p'q'$
c. $p^2 - q^2 = p'^2 - q'^2$ **d.** none of these

70. If $a_1, a_2, \ldots, a_n$ are in A.P. with common difference $d \neq 0$, then sum of the series $\sin d[\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \cdots + \sec a_{n-1} \sec a_n]$ is

a. $\operatorname{cosec} a_n - \operatorname{cosec} a$ **b.** $\cot a_n - \cot a$
c. $\sec a_n - \sec a_1$ **d.** $\tan a_n - \tan a_1$

71. The sum of the series $a - (a + d) + (a + 2d) - (a + 3d) + \cdots$ up to $(2n + 1)$ terms is

a. $-nd$ **b.** $a + 2nd$
c. $a + nd$ **d.** $2nd$

72. The sum to 50 terms of the series $1+2\left(1+\frac{1}{50}\right)+3\left(1+\frac{1}{50}\right)^2+\cdots$ is given by

a. 2500 **b.** 2550
c. 2450 **d.** none of these

73. The sum to 50 terms of the series $\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\cdots$ is

a. $\frac{100}{17}$ b. $\frac{150}{17}$

c. $\frac{200}{51}$ d. $\frac{50}{17}$

74. If $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$ to $\infty=\frac{\pi^2}{6}$, then $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ equals

a. $\pi^2/8$ b. $\pi^2/12$

c. $\pi^2/3$ d. $\pi^2/2$

75. Coefficient of x^{18} in $(1+x+2x^2+3x^3+\cdots+18x^{18})^2$ is equal to

a. 995 b. 1005

c. 1235 d. none of these

76. $\lim\limits_{n\to\infty}\sum_{r=1}^{n}\frac{r}{1\times3\times5\times7\times9\times\cdots\times(2r+1)}$ is equal to

a. $\frac{1}{3}$ b. $\frac{3}{2}$

c. $\frac{1}{2}$ d. none of these

77. Greatest integer by which $1+\sum_{r=1}^{30} r\times r!$ is divisible is

a. composite number b. odd number

c. divisible by 3 d. none of these

78. If $\sum_{r=1}^{n} r^4 = I(n)$, then $\sum_{r=1}^{n}(2r-1)^4$ is equal to

a. $I(2n)-I(n)$ b. $I(2n)-16I(n)$

c. $I(2n)-8I(n)$ d. $I(2n)-4I(n)$

79. Value of $\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\cdots\infty$ is equal to

a. 3 b. $\frac{6}{5}$

c. $\frac{3}{2}$ d. none of these

80. If $x_1, x_2, \ldots, x_{20}$ are in H.P. and $x_1, 2, x_{20}$ are in G.P., then $\sum_{r=1}^{19} x_r x_{r+1} =$

a. 76 b. 80

c. 84 d. none of these

81. The value of $\sum_{r=0}^{n}(a+r+ar)(-a)^r$ is equal to

a. $(-1)^n[(n+1)a^{n+1}-a]$ b. $(-1)^n(n+1)a^{n+1}$

c. $(-1)^n\frac{(n+2)a^{n+1}}{2}$ d. $(-1)^n\frac{na^n}{2}$

82. If $b_i = 1-a_i$, $na=\sum_{i=1}^{n} a_i$, $nb=\sum_{i=1}^{n} b_i$, then $\sum_{i=1}^{n} a_i b_i+\sum_{i=1}^{n}(a_i-a)^2=$

a. ab b. $-nab$ c. $(n+1)ab$ d. nab

83. The sum of the series $\frac{x}{1-x^2}+\frac{x^2}{1-x^4}+\frac{x^4}{1-x^8}+\cdots$ to infinite terms, if $|x|<1$, is

a. $\frac{x}{1-x}$ b. $\frac{1}{1-x}$

c. $\frac{1+x}{1-x}$ d. 1

84. If $a_1, a_2, a_3, \ldots, a_{2n+1}$ are in A.P., then $\frac{a_{2n+1}-a_1}{a_{2n+1}+a_1}+\frac{a_{2n}-a_2}{a_{2n}+a_2}+\cdots+\frac{a_{n+2}-a_n}{a_{n+2}+a_n}$ is equal to

a. $\frac{n(n+1)}{2}\times\frac{a_2-a_1}{a_{n+1}}$ b. $\frac{n(n+1)}{2}$

c. $(n+1)(a_2-a_1)$ d. none of these

85. The sum of $i-2-3i+4\cdots$ up to 100 terms, where $i=\sqrt{-1}$ is

a. $50(1-i)$ b. $25i$

c. $25(1+i)$ d. $100(1-i)$

86. Let $S=\frac{4}{19}+\frac{44}{19^2}+\frac{444}{19^3}+\cdots$ up to ∞. Then S is equal to

a. 40/9 b. 38/81

c. 36/171 d. none of these

87. If $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$, then value of $S_n = 1+\frac{3}{2}+\frac{5}{3}+\cdots+\frac{99}{50}$ is

a. $H_{50}+50$ b. $100-H_{50}$

c. $49+H_{50}$ d. $H_{50}+100$

88. If the sum to infinity of the series $1+2r+3r^2+4r^3+\cdots$ is 9/4, then value of r is

a. 1/2 b. 1/3

c. 1/4 d. none of these

89. The sum of series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\cdots\infty$ is

a. 7/16 b. 5/16

d. 105/64 d. 35/16

90. The sum $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots$ to 16 terms is

a. 246 b. 646

c. 446 d. 746

91. The sum $1+3+7+15+31+\cdots$ to 100 terms is

a. $2^{100}-102$ b. $2^{99}-101$

c. $2^{101}-102$ d. none of these

92. In a sequence of $(4n+1)$ terms the first $(2n+1)$ terms are in AP whose common difference is 2, and the last $(2n+1)$ terms are in GP whose common ratio is 0.5 if the middle terms of the AP and GP are equal then the middle term of the sequence is

a. $\frac{n\cdot2^{n+1}}{2^n-1}$ b. $\frac{n\cdot2^{n+1}}{2^{2n}-1}$

c. $n\cdot2^n$ d. none of these

93. The coefficient of x^{49} in the product $(x-1)(x-3)\cdots(x-99)$ is

a. -99^2 b. 1

c. -2500 d. none of these

94. The sum of 20 terms of the series whose r^{th} term is given by $T(n) = (-1)^n \frac{n^2+n+1}{n!}$ is

a. $\frac{20}{19!} - 2$ **b.** $\frac{21}{20!} - 1$

c. $\frac{21}{20!}$ **d.** none of these

95. Consider the sequence 1, 2, 2, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, ... Then 1025th term will be

a. 2^9 **b.** 2^{11}

c. 2^{10} **d.** 2^{12}

96. If $t_n = \frac{1}{4}(n+2)(n+3)$ for $n = 1, 2, 3, \ldots$, then

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \cdots + \frac{1}{t_{2003}} =$$

a. $\frac{4006}{3006}$ **b.** $\frac{4003}{3007}$

c. $\frac{4006}{3008}$ **d.** $\frac{4006}{3009}$

97. The sum of $0.2 + 0.004 + 0.00006 + 0.0000008 + \cdots$ to ∞ is

a. $\frac{200}{891}$ **b.** $\frac{2000}{9801}$

c. $\frac{1000}{9801}$ **d.** none of these

98. The value of $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = 220$, then the value of n equals

a. 11 **b.** 12

c. 10 **d.** 9

99. If $1^2 + 2^2 + 3^2 + \cdots + 2003^2 = (2003)(4007)(334)$ and $(1)(2003) + (2)(2002) + (3)(2001) + \cdots + (2003)(1) = (2003)(334)(x)$, then x equals

a. 2005 **b.** 2004

c. 2003 **d.** 2001

100. If t_n denotes the n^{th} term of the series $2 + 3 + 6 + 11 + 18 + \cdots$ then t_{50} is

a. $49^2 - 1$ **b.** 49^2

c. $50^2 + 1$ **d.** $49^2 + 2$

101. The positive integer n for which $2 \times 2^2 + 3 \times 2^3 + 4 \times 2^4 + \cdots + n \times 2^n = 2^{n+10}$ is

a. 510 **b.** 511

c. 512 **d.** 513

102. If $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}$, then value of $\frac{1}{1\times 3} + \frac{1}{5\times 7} + \frac{1}{9\times 11} + \cdots$ is

a. $\pi/8$ **b.** $\pi/6$

c. $\pi/4$ **d.** $\pi/36$

103. The coefficient of x^{19} in the polynomial $(x-1)(x-2)(x-2^2)\cdots(x-2^{19})$ is

a. $2^{20} - 2^{19}$ **b.** $1 - 2^{20}$

c. 2^{20} **d.** none of these

104. If $b_{n+1} = \frac{1}{1-b_n}$ for $n \ge 1$ and $b_1 = b_3$, then $\sum_{r=1}^{2001} b_r^{2001}$ is equal to

a. 2001 **b.** –2001

c. 0 **d.** none of these

105. If $(1^2 - t_1) + (2^2 - t_2) + \cdots + (n^2 - t_n) = \frac{n(n^2-1)}{3}$, then t_n is equal to

a. n^2 **b.** $2n$

c. $n^2 - 2n$ **d.** none of these

106. If $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$, then a, b, c, d are in

a. A.P. **b.** G.P.

c. H.P. **d.** none of these

107. If $(1 + 3 + 5 + \cdots + p) + (1 + 3 + 5 + \cdots + q) = (1 + 3 + 5 + \cdots + r)$ where each set of parentheses contains the sum of consecutive odd integers as shown, the smallest possible value of $p + q + r$ (where $p > 6$) is

a. 12 **b.** 21

c. 45 **d.** 54

108. In a geometric series, the first term is a and common ratio is r. If S_n denotes the sum of the n terms and $U_n = \sum_{n=1}^{n} S_n$ then $rS_n + (1-r)U_n$ equals

a. 0 **b.** n

c. na **d.** nar

109. The line $x + y = 1$ meets x-axis at A and y-axis at B, P is the mid-point of AB;
P_1 is the foot of the perpendicular from P to OA;
M_1 is that of P_1 from OP;
P_2 is that of M_1 from OA;
M_2 is that of P_2 from OP;
P_3 is that of M_2 from OA; and so on.
If P_n denotes the n^{th} foot of the perpendicular on OA;
then OP_n is

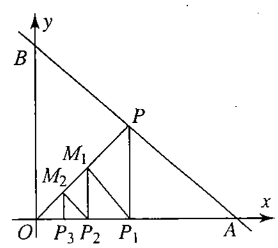


Fig. 3.3

a. $\left(\frac{1}{2}\right)^{n-1}$ **b.** $\left(\frac{1}{2}\right)^n$

c. $\left(\frac{1}{2}\right)^{n+1}$ **d.** none of these

110. If $(1-p)(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5) = 1 - p^6$, $p \ne 1$, then the value of $\frac{p}{x}$ is

a. $\frac{1}{3}$ **b.** 3

c. $\frac{1}{2}$ **d.** 2

111. ABC is a right-angled triangle in which $\angle B = 90°$ and $BC = a$. If n points $L_1, L_2, \ldots, L_n$ on AB is divided in $n + 1$ equal parts and $L_1M_1, L_2M_2, \ldots, L_nM_n$ are line segments parallel to BC and $M_1, M_2, \ldots, M_n$ are on AC, then the sum of the lengths of $L_1M_1, L_2M_2, \ldots, L_nM_n$ is

a. $\frac{a(n+1)}{2}$ **b.** $\frac{a(n-1)}{2}$

c. $\frac{an}{2}$ d. none of these

112. Let T_r and S_r be the r^{th} term and sum up to r^{th} term of a series respectively. If for an odd number n, $S_n = n$ and $T_n = \frac{T_{n-1}}{n^2}$, then T_m (m being even) is

a. $\frac{2}{1+m^2}$ b. $\frac{2m^2}{1+m^2}$

c. $\frac{(m+1)^2}{2+(m+1)^2}$ d. $\frac{2(m+1)^2}{1+(m+1)^2}$

113. $ABCD$ is a square of length a, $a \in N$, $a > 1$. Let $L_1, L_2, L_3, \ldots$ be points on BC such that $BL_1 = L_1L_2 = L_2L_3 = \cdots = 1$ and $M_1, M_2, M_3, \ldots$ be points on CD such that $CM_1 = M_1M_2 = M_2M_3 = \cdots = 1$. Then $\sum_{n=1}^{a-1}(AL_n^2 + L_nM_n^2)$ is equal to

a. $\frac{1}{2}a(a-1)^2$ b. $\frac{1}{2}(a-1)(2a-1)(4a-1)$

c. $\frac{1}{2}a(a-1)(4a-1)$ d. none of these

114. If x, y, and z are in G.P. and $x+3$, $y+3$, and $z+3$ are in H.P., then

a. $y = 2$ b. $y = 3$

c. $y = 1$ d. $y = 0$

115. If x, y, and z are distinct prime numbers, then

a. x, y, and z may be in A.P. but not in G.P.

b. x, y, and z may be in G.P. but not in A.P.

c. x, y, and z can neither be in A.P. nor in G.P.

d. none of these

Multiple Correct Answers Type

Solutions on page 3.59

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. For the series, $S = 1 + \frac{1}{(1+3)}(1+2)^2 + \frac{1}{(1+3+5)}(1+2+3)^2 + \frac{1}{(1+3+5+7)}(1+2+3+4)^2 + \cdots$

a. 7^{th} term is 16

b. 7^{th} term is 18

c. sum of first 10 terms is $\frac{505}{4}$

d. sum of first 10 terms is $\frac{405}{4}$

2. If sum of an infinite G.P. $p, 1, 1/p, 1/p^2, \ldots$ is 9/2, then value of p is

a. 2 b. 3/2

c. 3 d. 9/2

3. If $\sum_{r=1}^{n} r(r+1)(2r+3) = an^4 + bn^3 + cn^2 + dn + e$, then

a. $a - b = d - c$

b. $e = 0$

c. a, $b - 2/3$, $c - 1$ are in A.P.

d. $(b+d)/a$ is an integer

4. The terms of an infinitely decreasing G.P. in which all the terms are positive, the first term is 4, and the difference between the third and fifth term is 32/81, then

a. $r = 1/3$ b. $r = 2\sqrt{2}/3$

c. $S_\infty = 6$ d. none of these

5. The consecutive digits of a three digit number are in G.P. If the middle digit be increased by 2, then they form an A.P. If 792 is subtracted from this, then we get the number constituting of same three digits but in reverse order. Then number is divisible by

a. 7 b. 49

c. 19 d. none of these

6. If $S_n = 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \cdots$, then

a. $S_{40} = -820$ b. $S_{2n} > S_{2n+2}$

c. $S_{51} = 1326$ d. $S_{2n+1} > S_{2n-1}$

7. Given that $x + y + z = 15$ when a, x, y, z, b are in A.P. and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{3}$ when a, x, y, z, b are in H.P. Then

a. G.M. of a and b is 3

b. one possible value of $a + 2b$ is 11

c. A.M. of a and b is 6

d. greatest value of $a - b$ is 8

8. Let $a_1, a_2, a_3, \ldots, a_n$ be in G.P. such that $3a_1 + 7a_2 + 3a_3 - 4a_5 = 0$. Then common ratio of G.P. can be

a. 2 b. $\frac{3}{2}$

c. $\frac{5}{2}$ d. $-\frac{1}{2}$

9. $\frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \cdots n$ terms, is equal to

a. $\frac{\sqrt{3n+2}-\sqrt{2}}{3}$ b. $\frac{n}{\sqrt{2+3n}+\sqrt{2}}$

c. less than n d. less than $\sqrt{\frac{n}{3}}$

10. If a, b, and c are in H.P. then the value of $\frac{(ac+ab-bc)(ab+bc-ac)}{(abc)^2}$ is

a. $\frac{(a+c)(3a-c)}{4a^2c^2}$ b. $\frac{2}{bc} - \frac{1}{b^2}$

c. $\frac{2}{bc} - \frac{1}{a^2}$ d. $\frac{(a-c)(3a+c)}{4a^2c^2}$

11. If $p(x) = \frac{1+x^2+x^4+\cdots+x^{2n-2}}{1+x+x^2+\cdots+x^{n-1}}$ is a polynomial in x, then n can be

a. 5 b. 10

c. 20 d. 17

12. For an increasing A.P. $a_1, a_2, \ldots, a_n$ if $a_1 + a_3 + a_5 = -12$ and $a_1a_3a_5 = 80$, then which of the following is/are true?

a. $a_1 = -10$ **b.** $a_2 = -1$
c. $a_3 = -4$ **d.** $a_5 = +2$

13. If $n > 1$, the values of the positive integer m for which $n^m + 1$ divides $a = 1 + n + n^2 + \cdots + n^{63}$ is/are

a. 8 **b.** 16
c. 32 **d.** 64

14. If p, q, and r are in A.P. then which of the following is/are true?

a. p^{th}, q^{th} and r^{th} terms of A.P, are in A.P.
b. p^{th}, q^{th} and r^{th} terms of G.P. are in G.P.
c. p^{th}, q^{th} and r^{th} terms of H.P, are in H.P.
d. none of these

15. If $1 + 2x + 3x^2 + 4x^3 + \cdots \infty \geq 4$, then

a. least value of x is 1/2
b. greatest value of x is 4/3
c. least value of is x 2/3
d. greatest value of x does not exists

16. Let $E = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$. Then,

a. $E < 3$ **b.** $E > 3/2$ **c.** $E > 2$ **d.** $E < 2$

17. If the sum of n terms of an A.P. is given by $S_n = a + bn + cn^2$, where a, b, c are independent of n, then

a. $a = 0$
b. common difference of A.P. must be $2b$
c. common difference of A.P. must be $2c$
d. first term of A.P. is $b + c$

18. If $x^2 + 9y^2 + 25z^2 = xyz\left(\frac{15}{x} + \frac{5}{y} + \frac{3}{z}\right)$, then

a. x, y, and z are in H.P. **b.** $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.
c. x, y, z are in G.P. **d.** $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in G.P.

19. If a, b, c, and d are four unequal positive numbers which are in A.P., then

a. $\frac{1}{a} + \frac{1}{d} > \frac{1}{b} + \frac{1}{c}$ **b.** $\frac{1}{a} + \frac{1}{d} < \frac{1}{b} + \frac{1}{c}$
c. $\frac{1}{b} + \frac{1}{c} > \frac{4}{a+d}$ **d.** $\frac{1}{a} + \frac{1}{d} = \frac{1}{b} + \frac{1}{c}$

20. The next term of the G.P. x, $x^2 + 2$, and $x^3 + 10$ is

a. $\frac{729}{16}$ **b.** 6
c. 0 **d.** 54

21. In the 20^{th} row of the triangle

1
2 3
4 5 6
7 8 9 10
⋰ ⋰ ⋱ ⋱

a. last term = 210 **b.** first term = 191
c. sum = 4010 **d.** sum = 4200

22. If A_1, A_2; G_1, G_2; and H_1, H_2 are two arithmetic, geometric and harmonic means respectively, between two quantities a and b, then ab is equal to

a. A_1H_2 **b.** A_2H_1
c. $G_1 G_2$ **d.** none of these

23. Let $S_1, S_2, \ldots$ be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 sq. cm?

a. 7 **b.** 8
c. 9 **d.** 10

24. If $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$, then

a. a, b, and c are in H.P.
b. a, b, and c are in A.P.
c. $b = a + c$
d. $3a = b + c$

25. If a, b, and c are in G.P. and x and y, respectively, be arithmetic means between a, b and b, c, then

a. $\frac{a}{x} + \frac{c}{y} = 2$ **b.** $\frac{a}{x} + \frac{c}{y} = \frac{c}{a}$
c. $\frac{1}{x} + \frac{1}{y} = \frac{2}{b}$ **d.** $\frac{1}{x} + \frac{1}{y} = \frac{2}{ac}$

26. Consider a sequence $\{a_n\}$ with $a_1 = 2$ and $a_n = \frac{a_{n-1}^2}{a_{n-2}}$ for all $n \geq 3$, terms of the sequence being distinct. Given that a_2 and a_5 are positive integers and $a_5 \leq 162$ then the possible value(s) of a_5 can be

a. 162 **b.** 64
c. 32 **d.** 2

27. Which of the following can be terms (not necessarily consecutive) of any A.P.

a. 1, 6, 19 **b.** $\sqrt{2}, \sqrt{50}, \sqrt{98}$
c. $\log 2, \log 16, \log 128$ **d.** $\sqrt{2}, \sqrt{3}, \sqrt{7}$

28. The numbers 1, 4, 16 can be three terms (not necessarily consecutive) of

a. no AP **b.** only one GP
c. infinite number of APs **d.** infinite number of GPs

Reasoning Type

Solutions on page 3.63

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.
b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.
c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** The numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ cannot be the terms of a single A.P. with non-zero common difference.
   Statement 2: If p, q, r ($p \neq q$) are terms (not necessarily consecutive) of an A.P., then there exists a rational number k such that $(r - q)/(q - p) = k$.
2. **Statement 1:** In a G.P. if the $(m + n)^{\text{th}}$ term be p and $(m - n)^{\text{th}}$ term be q, then its m^{th} term is $\sqrt{pq}$.
   Statement 2: T_{m+n}, T_m, T_{m-n} are in G.P.
3. **Statement 1:** There are infinite geometric progressions for which 27, 8 and 12 are three of its terms (not necessarily consecutive).
   Statement 2: Given terms are integers.
4. **Statement 1:** If $x^2 + 9y^2 + 25z^2 = xyz\left(\frac{15}{x} + \frac{5}{y} + \frac{3}{z}\right)$, then x, y, z are in H.P.
   Statement 2: If $a_1^2 + a_2^2 + \cdots + a_n^2 = 0$, then $a_1 = a_2 = a_3 = \cdots = a_n = 0$.
5. **Statement 1:** Coefficient of x^{14} in $(1 + 2x + 3x^2 + \cdots + 16x^{15})^2$ is 560.
   Statement 2: $\sum_{r=1}^{n} r(n-r) = \frac{n(n^2-1)}{6}$.
6. **Statement 1:** $x = 1111\cdots91$ times is composite number.
   Statement 2: 91 is composite number.
7. Let $a, r \in R - \{0, 1, -1\}$ and n be an even number.
   Statement 1: $a \times ar \times ar^2 \cdots ar^{n-1} = (a^2 r^{n-1})^{n/2}$.
   Statement 2: Product of i^{th} term from the beginning and from the end in a G.P. is independent of i.
8. **Statement 1:** Sum of the series $1^3 - 2^3 + 3^3 - 4^3 + \cdots + 11^3 = 378$.
   Statement 2: For any odd integer $n \geq 1$, $n^3 - (n-1)^3 + \cdots + (-1)^{n-1}\,1^3 = \frac{1}{4}(2n-1)(n+1)^2$.
9. **Statement 1:** If an infinite G.P. has 2^{nd} term x and its sum is 4, then x belongs to $(-8, 1)$.
   Statement 2: Sum of an infinite G.P. is finite if for its common ratio r, $0 < |r| < 1$.
10. **Statement 1:** Let $p_1, p_2, \ldots, p_n$ and x be distinct real number such that $\left(\sum_{r=1}^{n-1} p_r^2\right)x^2 + 2\left(\sum_{r=1}^{n-1} p_r p_{r+1}\right)x + \sum_{r=2}^{n} p_r^2 \leq 0$, then $p_1, p_2, \ldots, p_n$ are in G.P. and when $a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2 = 0$, $a_1 = a_2 = a_3 = \cdots = a_n = 0$
    Statement 2: If $\frac{p_2}{p_1} = \frac{p_3}{p_2} = \cdots = \frac{p_n}{p_{n-1}}$, then $p_1, p_2, \ldots, p_n$ are in G.P.
11. **Statement 1:** $1^{99} + 2^{99} + \cdots + 100^{99}$ is divisible by 10100.
    Statement 2: $a^n + b^n$ is divisible by $a + b$ if n is odd.
12. **Statement 1:** If the arithmetic mean of two numbers is 5/2, geometric mean of the numbers is 2, then the harmonic mean will be 8/5.
    Statement 2: For a group of positive numbers $(\text{G.M.})^2 = (\text{A.M.}) \times (\text{H.M.})$.

Linked Comprehension Type

Solutions on page 3.64

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

Sum of certain consecutive odd positive integers is $57^2 - 13^2$.

1. Number of integers are
   a. 40 **b.** 37
   c. 44 **d.** 51
2. The least value of an integer is
   a. 22 **b.** 27
   c. 31 **d.** 43
3. The greatest integer is
   a. divisible by 7 **b.** divisible by 11
   c. divisible by 9 **d.** none of these

For Problems 4–6

Consider three distinct real numbers a, b, c in a G.P. with $a^2 + b^2 + c^2 = t^2$ and $a + b + c = \alpha t$. Sum of the common ratio and its reciprocal is denoted by S.

4. Complete set of α^2 is
   a. $\left(\frac{1}{3}, 3\right)$ **b.** $\left[\frac{1}{3}, 3\right]$
   c. $\left(\frac{1}{3}, 3\right) - \{1\}$ **d.** $\left(-\infty, \frac{1}{3}\right) \cup (3, \infty)$
5. Complete set of S is
   a. $(-2, 2)$ **b.** $(-\infty, -2) \cup (2, \infty)$
   c. $(-1, 1)$ **d.** $(-\infty, -1) \cup (1, \infty)$
6. If a, b, and c also represent the sides of a triangle, then the complete set of α^2 is
   a. $\left(\frac{1}{3}, 3\right)$ **b.** $(2, 3)$
   c. $\left[\frac{1}{3}, 2\right]$ **d.** $\left(\frac{\sqrt{5}+3}{2}, 3\right)$

For Problems 7–9

In a G.P., the sum of the first and last term is 66, the product of the second and the last but one is 128 and the sum of the terms is 126.

7. If an increasing G.P. is considered, then the number of terms in G.P. is
   a. 9 **b.** 8
   c. 12 **d.** 6
8. If the decreasing G.P. is considered, then the sum of infinite terms is
   a. 64 **b.** 128
   c. 256 **d.** 729
9. In any case, the difference of the least and greatest term is
   a. 78 **b.** 126
   c. 126 **d.** none of these

For Problems 10–12

Four different integers form an increasing A.P. One of these numbers is equal to the sum of the squares of the other three numbers. Then

10. The product of all numbers is
 a. –2 b. 1
 c. 0 d. 2
11. The sum of all the four numbers is
 a. 3 b. 0
 c. 4 d 2
12. The common difference of the four numbers is
 a. 1 b. 3
 c. 2 d. – 2

For Problems 13–15

Consider the sequence in the form of groups (1), (2, 2), (3, 3, 3), (4, 4, 4, 4), (5, 5, 5, 5, 5), ….

13. The 2000th term of the sequence is not divisible by
 a. 3 b. 9
 c. 7 d. none of these
14. The sum of first 2000 terms is
 a. 84336 b. 96324
 c. 78466 d. none of these
15. The sum of the remaining terms in the group after 2000th term in which 2000th term lies is
 a. 1088 b. 1008
 c. 1040 d. none of these

For Problems 16–18

There are two sets A and B each of which consists of three numbers in A.P. whose sum is 15 and where D and d are the common differences such that $D - d = 1$. If $\frac{p}{q} = \frac{7}{8}$, where p and q are the product of the numbers, respectively, and $d > 0$ in the two sets.

16. Sum of the product of the numbers in set A taken two at a time is
 a. 51 b. 71
 c. 74 d. 86
17. Sum of the product of the numbers in set B taken two at a time is
 a. 74 b. 64
 c. 73 d. 81
18. Value of $q - p$ is
 a. 20 b. 30
 c. 15 d. 25

For Problems 19–21

Let $A_1, A_2, A_3, \ldots, A_m$ be the arithmetic means between –2 and 1027 and $G_1, G_2, G_3, \ldots, G_n$ be the geometric means between 1 and 1024. The product of geometric means is 2^{45} and sum of arithmetic means is 1025×171.

19. The value of $\sum_{r=1}^{n} G_r$ is
 a. 512 b. 2046
 c. 1022 d. none of these
20. The number of arithmetic means is
 a. 442 b. 342
 c. 378 d. none of these
21. The numbers $2A_{171}, G_5^2 + 1, 2A_{172}$ are in
 a. A. P. b. G.P.
 c. H.P. d. none of these

For Problems 22–24

Two consecutive numbers from 1, 2, 3, …, n are removed. The arithmetic mean of the remaining numbers is 105/4.

22. The value of n lies in
 a. [45, 55] b. [52, 60]
 c. [41, 49] d. none of these
23. The removed numbers
 a. lie between 10 and 20 b. are greater than 10
 c. are less than 15 d. none of these
24. Sum of all numbers
 a. exceeds 1600 b. is less than 1500
 c. lies between 1300 and 1500 d. none of these

For Problems 25–27

Two arithmetic progressions have the same numbers. The ratio of the last term of the first progression to first term of the second progression is equal to the ratio of the last term of the second progression to the first term of the first progression and is equal to 4, the ratio of the sum of the n terms of the first progression to the sum of the n terms of the second progression is equal to 2.

25. The ratio of their common difference is
 a. 12 b. 24
 c. 26 d. 9
26. The ratio of their nth term is
 a. 6/5 b. 7/2
 c. 9/5 d. none of these
27. Ratio of their first term is
 a. 2/7 b. 3/5
 c. 4/7 d. 2/5

For Problems 28–30

The numbers a, b, and c are between 2 and 18, such that
 (i) their sum is 25
 (ii) the numbers 2, a, and b are consecutive terms of an A.P.
 (iii) the numbers b, c, 18 are consecutive terms of a G.P.

28. The value of abc is
 a. 500 b. 450
 c. 720 d. none of these
29. Roots of the equation $ax^2 + bx + c = 0$ are
 a. real and positive
 b. real and negative
 c. imaginary
 d. real and of opposite sign
30. If a, b, and c are roots of the equation $x^3 + qx^2 + rx + s = 0$, then the value of r is
 a. 184 b. 196
 c. 224 d. none of these

For Problems 31–33

Let $T_1, T_2, T_3, \ldots, T_n$ be the terms of a sequence and let $(T_2 - T_1) = T'_1$, $(T_3 - T_2) = T'_2, \ldots, (T_n - T_{n-1}) = T'_{n-1}$.

Case I:
If $T'_1, T'_2, \ldots, T'_{n-1}$ are in A.P., then T_n is quadratic in 'n'. If $T'_1 - T'_2, T'_2 - T'_3, \ldots$, are in A.P., then T_n is cubic in n.

Case II:
If $T'_1, T'_2, \ldots, T'_{n-1}$ are not in A.P., but in G.P., then $T_n = ar^n + b$, where r is the common ratio of the G.P. $T'_1, T'_2, T'_3, \ldots$ and $a, b \in R$.
Again, if $T'_1, T'_2, \ldots, T'_{n-1}$ are not in G.P. but $T'_2 - T'_1, T'_3 - T'_2, \ldots, T'_{n-2}$ are in G.P., then T_n is of the form $ar^n + bn + c$ and r is the common ratio of the G.P. $T'_2 - T'_1, T'_3 - T'_2, T'_4 - T_3, \ldots$ and $a, b, c \in R$.

31. The sum of 20 terms of the series $3 + 7 + 14 + 24 + 37 + \cdots$ is

a. 4010 **b.** 3860
c. 4240 **d.** none of these

32. The 100th term of the series $3 + 8 + 22 + 72 + 266 + 1036 + \cdots$ is divisible by 2^n, then maximum value of n is

a. 4 **b.** 2
c. 3 **d.** 5

33. For the series $2 + 12 + 36 + 80 + 150 + 252 + \cdots$, the value of $\lim_{n\to\infty} \frac{T_n}{n^3}$ is (where T_n is n^{th} term)

a. 2 **b.** 1/2
c. 1 **d.** none of these

Matrix-Match Type

Solutions on page 3.67

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a→p, a→s, a→q, b→r, c→p, c→q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1.

Column I	Column II
a. If a, b, c are in G.P., then $\log_a 10$, $\log_b 10$, $\log_c 10$ are in	**p.** A.P.
b. If $\frac{a+be^x}{a-be^x} = \frac{b+ce^x}{b-ce^x} = \frac{c+de^x}{c-de^x}$, then a, b, c, d are in	**q.** H.P.
c. If a, b, c are in A.P.; a, x, b are in G.P. and b, y, c are in G.P., then x^2, b^2, y^2 are in	**r.** G.P.
d. If x, y, z are in G.P., $a^x = b^y = c^z$, then $\log a, \log b, \log c$ are in	**s.** none of these

2.

Column I	Column II
a. If $\Sigma n = 210$, then Σn^2 is divisible by the greatest prime number which is greater than	**p.** 16
b. Between 4 and 2916 is inserted odd number $(2n + 1)$ G.M's. Then the $(n + 1)$th G.M. is divisible by greatest odd integer which is less than	**q.** 10
c. In a certain progression, four consecutive terms are 40, 30, 24, 20. Then the integral part of the next term of the progression is more than	**r.** 34
d. $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots$ to $\infty = \frac{a}{b}$, where H.C.F.$(a, b) = 1$, then $a - b$ is less than	**s.** 30

Integer Type

Solutions on page 3.68

1. If the roots of $10x^3 - nx^2 - 54x - 27 = 0$ are in harmonic progression, then 'n' equals.

2. The difference between the sum of the first k terms of the series $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3$ and the sum of the first k terms of $1 + 2 + 3 + \ldots\ldots + n$ is 1980. The value of k is.

3. The value of the $\sum_{n=0}^{\infty} \frac{2n+3}{3^n}$ is equal to.

4. Let $a_1, a_2, a_3, \ldots, a_{101}$ are in G.P. with $a_{101} = 25$ and $\sum_{i=1}^{201} a_i = 625$. Then the value of $\sum_{i=1}^{201} \frac{1}{a_i}$ equals.

5. Let $S = \sum_{n=1}^{9999} \frac{1}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt[4]{n}+\sqrt[4]{n+1}\right)}$, then S equals.

6. The 5th and 8th terms of a geometric sequence of real numbers are 7! and 8! respectively. If the sum to first n terms of the G.P. is 2205, then n equals.

7. Let a, b, c, d be four distinct real numbers in A.P. Then half of the smallest positive value of k satisfying $2(a - b) + k(b - c)^2 + (c - a)^3 = 2(a - d) + (b - d)^2 + (c - d)^3$ is.

8. Number of positive integral ordered pairs of (a, b) such that $6, a, b$ are in harmonic progression is.

9. For $a, b > 0$, let $5a - b, 2a + b, a + 2b$ be in A.P. and $(b + 1)^2, ab + 1, (a - 1)^2$ are in G.P., then the value of $(a^{-1} + b^{-1})$ is.

10. The coefficient of the quadratic equation $ax^2 + (a + d)x + (a + 2d) = 0$ are consecutive terms of a positively valued, increasing arithmetic sequence. Then the least integral value of $\frac{d}{a}$ such that the equation has real solutions is.

11. Let $a + ar_1 + ar_1^2 + \ldots + \infty$ and $a + ar_2 + ar_2^2 + \ldots + \infty$ be two infinite series of positive numbers with the same first term. The sum of the first series is r_1 and the sum of the second series is r_2. Then the value of $(r_1 + r_2)$ is.

12. If the equation $x^3 + ax^2 + bx + 216 = 0$ has three real roots in G.P., then b/a has the value equal to.

13. Let $a_n = 16, 4, 1, \ldots$be a geometric sequence. Define P_n as the product of the first n terms. Then the value of $\frac{1}{4}\sum_{n=1}^{\infty} \sqrt[n]{P_n}$ is.

14. The terms a_1, a_2, a_3 form an arithmetic sequence whose sum is 18. The terms $a_1 + 1$, a_2, $a_3 + 2$, in that order, form a geometric sequence. Then the absolute value of the sum of all possible common difference of the A.P. is.

15. Given a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in H.P. If $a = 2$ and $e = 18$, then the sum of all possible value of 'c' is.

16. Let sum of first three terms of G.P. with real terms is $\frac{13}{12}$ and their product is -1. If the absolute value of the sum of their infinite terms is S, then the value of $7S$ is.

17. Let S denote sum of the series $\frac{3}{2^3} + \frac{4}{2^4 \cdot 3} + \frac{5}{2^6 \cdot 3} + \frac{6}{2^7 \cdot 5} + \cdots \infty$
Then the value of S^{-1} is.

18. The first term of an arithmetic progression is 1 and the sum of the first nine terms equal to 369. The first and the ninth term of a geometric progression coincide with the first and the ninth term of the arithmetic progression. If the seventh term of the geometric progression is T_7, then the value of $T_7/9$ is.

Archives

Solutions on page 3.70

Subjective Type

1. The harmonic mean of two numbers is 4. Their arithmetic mean A and the geometric mean G satisfy the relation $2A + G^2 = 27$. Find the two numbers. **(IIT-JEE, 1997)**

2. The interior angles of a polygon are in arithmetic progression. The smallest angle is 120°, and the common difference is 5°. Find the number of sides of the polygon. **(IIT-JEE, 1980)**

3. If $a_1, a_2, \cdots, a_n$ are in arithmetic progression, where $a_i > 0$ for all i. Show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

(IIT-JEE, 1982)

4. Does there exist a geometric progression containing 27, 8 and 12 as three of its terms? If it exists, how many such progressions are possible? **(IIT-JEE, 1982)**

5. Find three numbers a, b, and c, between 2 and 18, such that
(i) their sum is 25
(ii) the numbers 2, a, b, are consecutive terms of an A.P. and
(iii) the numbers b, c, and 18 are consecutive terms of a G.P.
(IIT-JEE, 1983)

6. The sum of the squares of three distinct real numbers, which are in G.P., is S^2. If their sum is aS, show that $a^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$.
(IIT-JEE, 1986)

7. If $\log_3 2$, $\log_3 (2^x - 5)$, and $\log_3\left(2^x - \frac{7}{2}\right)$ are in arithmetic progression, determine the value of x. **(IIT-JEE, 1990)**

8. Let p be the first of the n arithmetic means between two numbers and q the first of n harmonic means between the same numbers.
Show that q does not lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$.
(IIT-JEE, 1991)

9. If $S_1, S_2, S_3, \ldots, S_n$ are the sums of an infinite geometric series whose first terms are $1, 2, 3, \ldots, n$ and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}$, respectively, then find the value of
$S_1^2 + S_2^2 + S_3^2 + \cdots + S_{2n-1}^2$. **(IIT-JEE, 1991)**

10. The real number x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P. Find the intervals in which β and γ lie.
(IIT-JEE, 1996)

11. Let a, b, c, and d be real numbers in a G.P. u, v, w, satisfy the system of equations
$u + 2v + 3w = 6$
$4u + 5v + 6w = 12$
$6u + 9v = 4$
Show that the roots of the equation $\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]\, x + u + v + w = 0$ and $20x^2 + 10(a-d)^2 x - 9 = 0$ are reciprocals of each other. **(IIT-JEE, 1999)**

12. The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive of it. Prove that the resulting sum is the squares of an integer. **(IIT-JEE, 2000)**

13. Let $a_1, a_2, \ldots$ be positive real numbers in a geometric progression. For each n, let A_n, G_n, H_n be, respectively, the arithmetic mean, geometric mean and harmonic mean of $a_1, a_2, \ldots, a_n$. Find an expression for the geometric mean of $G_1, G_2, \ldots, G_n$ in terms of $A_1, A_2, \ldots, A_n, H_1, H_2, \ldots, H_n$. **(IIT-JEE, 2005)**

14. Let a, b be positive real numbers. If a A_1, A_2, b be are in arithmetic progression, a, G_1, G_2, b are in geometric progression and a, H_1, H_2, b are in harmonic progression, show that

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2} = \frac{(2a+b)(a+2b)}{9ab}$$ **(IIT-JEE, 2002)**

15. If a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then prove that either $a = b = c$ or $a, b, -\frac{c}{2}$ form a G.P. **(IIT-JEE, 2003)**

16. If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$, then find the least natural number n_0 such that $b_n > a_n\ \forall\, n \geq n_0$.
(IIT-JEE, 2006)

Objective Type

Fill in the blanks

1. The sum of integers from 1 to 100 that are divisible by 2 or 5 is __________. **(IIT-JEE, 1984)**

2. The sum of the first n terms of the series $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \cdots$ is $n(n+1)^2/2$, when n is even. When n is odd, the sum is __________. **(IIT-JEE, 1988)**

3. Let the harmonic mean and geometric mean of two positive numbers be in the ratio 4 : 5. Then the two numbers are in the ratio __________. **(IIT-JEE, 1992)**

4. For any odd integer $n \geq 1$, $n^3 - (n-1)^3 + \cdots + (-1)^{n-1} 1^3 =$ __________. **(IIT-JEE, 1996)**

5. Let x be the arithmetic mean and y, z be the two geometric means between any two positive numbers. Then $\frac{y^3 + z^3}{xyz}$ = ________. **(IIT-JEE, 1997)**

6. Let p and q be roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmetic progression, then A = ________. **(IIT-JEE, 1997)**

Multiple choice questions with one correct answer

1. If x, y and z are p^{th}, q^{th} and r^{th} terms respectively of an A.P. and also of a G.P., then $x^{y-z} y^{z-x} z^{x-y}$ is equal to
   a. xyz b. 0
   c. 1 d. none of these

2. The third term of a geometric progression is 4. The product of the first five terms is
   a. 4^3 b. 4^5
   c. 4^4 d. none of these **(IIT-JEE, 1982)**

3. The rational number which equals the number $2.\overline{357}$ with recurring decimal is
   a. $\frac{2355}{1001}$ b. $\frac{2379}{997}$
   c. $\frac{2355}{999}$ d. none of these **(IIT-JEE, 1983)**

4. If a, b, and c are in G.P., then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in
   a. A.P. b. G.P.
   c. H. P. d. none of these **(IIT-JEE, 1985)**

5. Sum of the first n terms of the series $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \cdots$ is equal to
   a. $2^n - n - 1$ b. $1 - 2^{-n}$
   c. $n + 2^{-n} - 1$ d. $2^n + 1$ **(IIT-JEE, 1988)**

6. Find the sum $(x + 2)^{n-1} + (x + 2)^{n-2}(x + 1) + (x + 2)^{n-3}(x + 1)^2 + \cdots + (x + 1)^{n-1}$
   a. $(x + 2)^{n-2} - (x + 1)^n$ b. $(x + 2)^{n-1} - (x + 1)^{n-1}$
   c. $(x + 2)^n - (x + 1)^n$ d. none of these **(IIT-JEE, 1990)**

7. If $\ln(a + c)$, $\ln(a - c)$, and $\ln(a - 2b + c)$ are in A.P., then
   a. a, b, c are in A.P. b. a^2, b^2, c^2 are in A.P.
   c. a, b, c are in G.P. d. a, b, c are in H.P. **(IIT-JEE, 1994)**

8. Let $a_1, a_2, \ldots, a_{10}$ be in A.P. and $h_1, h_2, \ldots, h_{10}$ be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ is
   a. 2 b. 3
   c. 5 d. 6 **(IIT-JEE, 1999)**

9. The harmonic mean of the roots of the equation $(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + 8 + 2\sqrt{5} = 0$ is
   a. 2 b. 4
   c. 6 d. 8 **(IIT-JEE, 1999)**

10. Let the positive numbers a, b, c, and d be in A.P. Then abc, abd, acd, and bcd are
   a. not in A.P./G.P./H.P. b. in A.P.
   c. in G.P. d. in H.P. **(IIT-JEE, 2001)**

11. Consider an infinite geometric series with first term a and common ratio r. If its sum is 4 and the second term is 3/4, then
   a. $a = \frac{4}{7}, r = \frac{3}{7}$ b. $a = 2, r = \frac{3}{8}$
   c. $a = \frac{3}{2}, r = \frac{1}{2}$ d. $a = 3, r = \frac{1}{4}$ **(IIT-JEE, 2001)**

12. Let α, and β be the roots of $x^2 - x + p = 0$ and γ and δ be the root of $x^2 - 4x + q = 0$. If α, β, and γ, δ are in G.P., then the integral values of p and q, respectively, are
   a. $-2, -32$ b. $-2, 3$
   c. $-6, 3$ d. $-6, -32$

13. If the sum of the first $2n$ terms of the A.P. 2, 5, 8, …, is equal to the sum of the first n terms of A.P. 57, 59, 61, …, then n equals
   a. 10 b. 12
   c. 11 d. 13

14. Suppose a, b, and c are in A.P. and a^2, b^2, and c^2 are in G.P., if $a < b < c$ and $a + b + c = \frac{3}{2}$, then the value of a is
   a. $\frac{1}{2\sqrt{2}}$ b. $\frac{1}{2\sqrt{3}}$
   c. $\frac{1}{2} - \frac{1}{\sqrt{3}}$ d. $\frac{1}{2} - \frac{1}{\sqrt{2}}$ **(IIT-JEE, 2002)**

15. An infinite G.P. has first term as a and sum 5, then
   a. $a < -10$ b. $-10 < a < 10$
   c. $0 < a < 10$ and $a \neq 5$ d. $a > 10$ **(IIT-JEE, 2004)**

16. In the quadratic equation $ax^2 + bx + c = 0$, $\Delta = b^2 - 4ac$ and $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$ are in G.P. where α, β are the roots of $ax^2 + bx + c = 0$, then
   a. $\Delta \neq 0$ b. $b\Delta = 0$
   c. $c\,\Delta = 0$ d. $\Delta = 0$ **(IIT-JEE, 2005)**

Multiple choice questions with one or more than one correct answer

1. If the first and the $(2n - 1)^{st}$ terms of an A.P., a G.P. and a H.P. are equal and their n^{th} terms are a, b and c respectively, then
   a. $a = b = c$ b. $a \geq b \geq c$
   c. $a + b = b$ d. $ac - b^2 = 0$ **(IIT-JEE, 1988)**

2. For $0 < \phi < \pi/2$, if
   $x = \sum_{n=0}^{\infty} \cos^{2n}\phi$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$, and $z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$, then
   a. $xyz = xz + y$ b. $xyz = xy + z$
   c. $xyz = x + y + z$ d. $xyz = yz + x$ **(IIT-JEE, 1993)**

3. Let n be an odd integer. If $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r \theta$, for every value of θ, then
   a. $b_0 = 1, b_1 = 3$
   b. $b_0 = 0, b_1 = n$
   c. $b_0 = -1, b_1 = n$
   d. $b_0 = 0, b_1 = n^2 - 3n + 3$ **(IIT-JEE, 1998)**

4. Let T_r be the r^{th} term of an A.P., for $r = 1, 2, 3, \ldots$. If for some positive integers m, n, we have $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then T_{mn} equals
   a. $\frac{1}{mn}$
   b. $\frac{1}{m} + \frac{1}{n}$
   c. 1
   d. 0 **(IIT-JEE, 1998)**

5. If $x > 1$, $y > 1$, and $z > 1$ are in G.P., then $\frac{1}{1+\ln x}$, $\frac{1}{1+\ln y}$, and $\frac{1}{1+\ln z}$ are in
   a. A.P.
   b. H.P.
   c. G.P.
   d. none of these **(IIT-JEE, 1998)**

6. For a positive integer n, let $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots + \frac{1}{(2^n) - 1}$. Then,
   a. $a(100) \le 100$
   b. $a(100) > 100$
   c. $a(200) \le 100$
   d. $a(200) > 100$ **(IIT-JEE, 1999)**

Comprehension

For Problems 1–3

Let V_r denote the sum of first r terms of an arithmetic progression (A.P.) whose first term is r and the common difference is $2r - 1$. Let $T_r = V_{r+1} - V_r - 2$ and $Q_r = T_{r+1} - T_r$ for $r = 1, 2, \ldots$.
(IIT-JEE, 2007)

1. The sum $V_1 + V_2 + \cdots + V_n$ is
   a. $\frac{1}{12} n(n+1)(3n^2 - n + 1)$
   b. $\frac{1}{12} n(n+1)(3n^2 + n + 2)$
   c. $\frac{1}{2} n(2n^2 - n + 1)$
   d. $\frac{1}{3}(2n^3 - 2n + 3)$

2. T_r is always
   a. an odd number
   b. an even number
   c. a prime number
   d. a composite number

3. Which one of the following is a correct statement?
   a. $Q_1, Q_2, Q_3, \ldots$ are in A. P. with common difference 5
   b. $Q_1, Q_2, Q_3, \ldots$ are in A. P. with common difference 6
   c. $Q_1, Q_2, Q_3, \ldots$ are in A. P. with common difference 11
   d. $Q_1 = Q_2 = Q_3 = \cdots$

For Problems 4–6

Let A_1, G_1, and H_1 denote the arithmetic geometric and harmonic means, respectively, of two distinct positive numbers. For $n \ge 2$, let A_{n-1} and H_{n-1} have arithmetic and harmonic means as A_n, G_n, H_n, respectively.
(IIT-JEE, 2007)

4. Which one of the following statements is correct?
   a. $G_1 > G_2 > G_3 > \cdots$
   b. $G_1 < G_2 < G_3 < \cdots$
   c. $G_1 = G_2 = G_3 = \cdots$
   d. $G_1 < G_3 < G_5 < \cdots$ and $G_2 > G_4 > G_6 > \cdots$

5. Which one of the following statements is correct?
   a. $A_1 > A_2 > A_3 > \cdots$
   b. $A_1 < A_2 < A_3 < \cdots$
   c. $A_1 > A_3 > A_5 > \cdots$ and $A_2 < A_4 < A_6 >$
   d. $A_1 < A_3 < A_5 < \cdots$ and $A_2 > A_4 > A_6 > \cdots$

6. Which one of the following statements is correct?
   a. $H_1 > H_2 > H_3 > \cdots$
   b. $H_1 < H_2 < H_3 < \cdots$
   c. $H_1 > H_3 > H_5 > \cdots$ and $H_2 < H_4 < H_6 < \cdots$
   d. $H_1 < H_3 < H_6 < \cdots$ and $H_2 > H_4 > H_6 > \cdots$

Integer type

1. Let S_k, $k = 1, 2, \ldots, 100$, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$, then the value of $\frac{100^2}{100!} + \sum_{k=1}^{100}(k^2 - 3k + 1)S_k$ is.
   (IIT-JEE, 2010)

2. Let $a_1, a_2, a_3, \ldots, a_{11}$ be real numbers satisfying $a_1 = 15, 27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, \ldots, 11$. If $\frac{a_1^2 + a_2^2 + \cdots + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + \cdots + a_{11}}{11}$ is equal to.
   (IIT-JEE, 2010)

3. Let $a_1, a_2, a_3, \ldots, a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i, 1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is.
   (IIT-JEE, 2011)

ANSWERS AND SOLUTIONS

Subjective Type

1. $\sqrt{a-x}, \sqrt{x}, \sqrt{a+x}$ are in G. P.

$\Rightarrow \quad x = \sqrt{a^2-x^2}$

$\Rightarrow \quad x^2 = a^2 - x^2$

$\Rightarrow \quad x = \dfrac{a}{\sqrt{2}} \qquad (\because \quad a, x > 0)$

Let $\sqrt{x}$ be rational. Then,

$$\sqrt{x} = \frac{p}{q}, p, q \in Z, q \neq 0$$

$\Rightarrow \quad x = \dfrac{p^2}{q^2}$

$\Rightarrow \quad \sqrt{a-x} = \sqrt{\sqrt{2}x - x}$

$= \sqrt{\sqrt{2}-1}\sqrt{x}$

$= \sqrt{\sqrt{2}-1}\dfrac{p}{q}$ which is irrational.

Similarly, $\sqrt{a+x} = \sqrt{\sqrt{2}+1}\dfrac{p}{q}$

2. $\sqrt{a-x}$, $\sqrt{x}$, $\sqrt{a+x}$ are in A.P.

$\Rightarrow \quad 2\sqrt{x} = \sqrt{a-x} + \sqrt{a+x}$

$\Rightarrow \quad 4x = a - x + a + x + 2\sqrt{a^2-x^2}$ (squaring both sides)

$\Rightarrow \quad 2x - a = \sqrt{a^2-x^2}$

$\Rightarrow \quad 4x^2 - 4ax + a^2 = a^2 - x^2$

$\Rightarrow \quad 5x^2 = 4ax$

$\Rightarrow \quad a = \dfrac{5x}{4}$ (as $a, x > 0$)

Now, x must be a perfect square as $\sqrt{x}$ is an integer. Hence, $x = 1, 4, 9, 16, \ldots$, etc. For $x = 1$, $a = 5/4$ (rational number). For $x = 4$, $a = 5$ (prime number). For $x = 9$, $a = 45/4$ (rational number).For $x = 16$, $a = 20$ (composite number). Hence, the least composite value of a is 20.

3. Let the three digits be a, ar and ar^2. Each of the three quantities must lie between 1 and 9, and r must be rational. The three-digit number so formed can be written as $100a + 10ar + ar^2$. Now, from the given condition, the digits of the number $100a + 10ar + ar^2 - 100 = 100(a-1) + 10ar + ar^2$, i.e., $a-1$, ar, ar^2 are in A.P. Therefore,

$2ar = a - 1 + ar^2$

$\Rightarrow \quad a(r^2 + 1 - 2r) = 1$

$\Rightarrow \quad a(r-1)^2 = 1$

$\Rightarrow \quad r - 1 = \pm\dfrac{1}{\sqrt{a}}$

Since $(r-1)$ is rational, $\pm 1/\sqrt{a}$ must also be rational.Furthermore, $1 \le a - 1 \le 9$ so that $2 \le a \le 10$. Since $a \le 9$, we get $2 \le a \le 9$.

The only integer a between 2 and 9 such that $1/\sqrt{a}$ is rational is 4 or 9.

$\therefore \quad r - 1 = \pm\dfrac{1}{2}$ or $r - 1 = \pm\dfrac{1}{3}$

$\Rightarrow \quad r = \dfrac{3}{2}, \dfrac{1}{2}$ or $r = \dfrac{4}{3}, \dfrac{2}{3}$

Thus, $r = 3/2$ (rejecting $r = 1/2, 2/3, 4/3$). Hence, the required digits are 4, 6 and 9 forming the number 469.

4. Let there be $2n + 1$ stones. Clearly, one stone lies in the middle and n stones on each side of it in a row. Let P be the mid-stone and let A and B be the end stones on the left and right of P, respectively. Clearly, there are n intervals, each of length 10 m on both the sides of P. Now, suppose the man starts from A. He picks up the end stone on the left of mid-stone and goes to the mid-stone, drops it and goes to $(n-1)^{\text{th}}$ stone on left, picks it up, goes to the mid-stone and drops it. This process is repeated till he collects all stones on the left of the mid-stone at the mid-stone. So, the distance covered in collecting stones on the left of the mid-stones is

$$10 \times n + 2[10 \times (n-1) + 10 \times (n-2) + \cdots + 10 \times 2 + 10 \times 1]$$

After collecting all the stones on left of the mid-stone, the man goes to the stone B on the right side of the mid-stone, picks it up, goes to the mid-stone and drops it.

Then, he goes to n^{th} stone on the right and the process is repeated till he collects all stones at the mid-stone.

Distance covered in collecting the stones on the right side of the mid-stone is

$$2[10 \times n + 10 \times (n-1) + 10 \times (n-2) + \cdots + 10 \times 2 + 10 \times 1]$$

Therefore, total distance covered is

$$10 \times n + 2[10 \times (n-1) + 10 \times (n-2) + \cdots + 10 \times 2 + 10 \times 1] + 2[10 \times n + 10 \times (n-1) + \cdots + 10 \times 2 + 10 \times 1]$$

$= 4[10 \times n + 10 \times (n-1) + \cdots + 10 \times 2 + 10 \times 1] - 10 \times n$

$= 40[1 + 2 + 3 + \cdots + n] - 10n$

$= 40\left[\dfrac{n}{2}(1+n)\right] - 10n$

$= 20n(n+1) - 10n$

$= 20n^2 + 10n$

But the total distance covered is 3 km, i.e., 3000 m.

$\therefore \quad 20n^2 + 10n = 3000$

$\Rightarrow \quad 2n^2 + n - 300 = 0$

$\Rightarrow \quad (n-12)(2n+25) = 0$

$\Rightarrow \quad n = 12$

Hence, the number of stones is $2n + 1 = 25$.

5. Let r be the common ratio of the given G.P. Then,

$$b = n^{\text{th}} \text{ term} = ar^{n-1} \Rightarrow r^{n-1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n-1}}$$

Now, product of the first n terms is

$$\begin{aligned}
P &= a \times ar \times ar^2 \cdots ar^{n-1} \\
&= a^n\, r^{1+2+3+\cdots+(n-1)} \\
&= a^n r^{\frac{n(n-1)}{2}} \quad \left[\because 1+2+3+\cdots+(n-1) = \frac{n(n-1)}{2}\right] \\
&= a^n \left\{\left(\frac{b}{a}\right)^{\frac{1}{n-1}}\right\}^{\frac{n(n-1)}{2}} \\
&= a^n \left(\frac{b}{a}\right)^{n/2} \\
&= a^{n/2}\, b^{n/2} \\
&= (ab)^{n/2}
\end{aligned}$$

$\therefore \quad P^2 = [(ab)^{n/2}]^2 = (ab)^n$

6. $x = 1 + 3a + 6a^2 + 10a^3 + \cdots$ (1)

$\Rightarrow \quad ax = a + 3a^2 + 6a^3 + \cdots$ (2)

Subtracting (2) from (1), we have

$x(1-a) = 1 + 2a + 3a^2 + 4a^3 + \cdots$ (3)

Equation (3) is an arithmetico-geometric series. Therefore,

$$S_\infty = \frac{A}{1-R} + \frac{dR}{(1-R)^2}$$

$$\Rightarrow \quad x(1-a) = \frac{1}{1-a} + \frac{a}{(1-a)^2} = \frac{1}{(1-a)^2}$$

$$\Rightarrow \quad x = \frac{1}{(1-a)^3}$$

$\Rightarrow \quad (1-a)^3 = x^{-1}$

$\Rightarrow \quad a = 1 - x^{-1/3}$

Similarly,

$b = 1 - y^{-1/4}$

Now,

$$\begin{aligned}
S &= 1 + 3(ab) + 5(ab)^2 + \cdots \text{ infinity} \\
&= \frac{1}{1-ab} + \frac{2ab}{(1-ab)^2} \\
&= \frac{1+ab}{(1-ab)^2} \\
&= \frac{1 + \left(1-\frac{1}{x^{1/3}}\right)\left(1-\frac{1}{y^{1/4}}\right)}{\left(1-\left(1-\frac{1}{x^{1/3}}\right)\left(1-\frac{1}{y^{1/4}}\right)\right)^2}
\end{aligned}$$

7. Let $\dfrac{2n+1}{2n-1} = r$. Then, the given series is

$S = r + 3r^2 + 5r^3 + 7r^4 + \cdots + (2n-1)r^n$ (1)

$rS = r^2 + 3r^3 + 5r^4 + \cdots + (2n-3)r^n + (2n-1)r^{n+1}$ (2)

Subtracting (2) from (1), we get

$(1-r)\,S = r + 2r^2 + 2r^3 + \cdots + 2r^n - (2n-1)r^{n+1}$

$$\Rightarrow \quad 36(1-r) = r + \frac{2r^2(1-r^{n-1})}{1-r} - (2n-1)r^{n+1}$$

$$\begin{aligned}
\Rightarrow \quad 36(1-r)^2 &= r - r^2 + 2r^2 - 2r^{n+1} - (2n-1)r^{n+1} + (2n-1)r^{n+2} \\
&= r + r^2 - (2n+1)r^{n+1} + (2n-1)r^{n+2} \\
&= r + r^2 - (2n-1)\left[\frac{2n+1}{2n-1} r^{n+1} - r^{n+2}\right] \\
&= r + r^2 - (2n-1)\left[r\, r^{n+1} - r^{n+2}\right] \\
&= r(1+r)
\end{aligned}$$

$$\Rightarrow \quad 36\left[1 - \frac{2n+1}{2n-1}\right]^2 = \frac{2n+1}{2n-1}\left[1 + \frac{2n+1}{2n-1}\right]$$

$$\Rightarrow \quad 36\left[\frac{-2}{2n-1}\right]^2 = \frac{2n+1}{2n-1}\left[\frac{4n}{2n-1}\right]$$

$\Rightarrow \quad 36 = n(2n+1)$

$\Rightarrow \quad n = 4$

8.
$$\begin{aligned}
t_n &= \frac{n+2}{n(n+1)} \times \left(\frac{1}{2}\right)^n \\
&= \frac{2(n+1) - n}{n(n+1)} \times \left(\frac{1}{2}\right)^n \\
&= \frac{1}{n}\left(\frac{1}{2}\right)^{n-1} - \frac{1}{n+1} \times \left(\frac{1}{2}\right)^n
\end{aligned}$$

$$\begin{aligned}
S_n &= \sum_{n=}^{n} t_n \\
&= \left\{\frac{1}{1}\left(\frac{1}{2}\right)^0 - \frac{1}{2}\left(\frac{1}{2}\right)^1\right\} + \left[\frac{1}{2}\left(\frac{1}{2}\right)^1 - \frac{1}{3}\left(\frac{1}{2}\right)^2\right] \\
&\qquad + \cdots + \left\{\frac{1}{n}\left(\frac{1}{2}\right)^{n-1} - \frac{1}{n+1}\left(\frac{1}{2}\right)^n\right\} \\
&= 1 - \frac{1}{(n+1)\,2^n}
\end{aligned}$$

9. We have the following:

First term	Common difference	Sums of n terms
1	1	$S_1 = \frac{n}{2}[2 \times 1 + (n-1) \times 1]$
2	3	$S_2 = \frac{n}{2}[2 \times 2 + (n-1) \times 3]$
3	5	$S_3 = \frac{n}{2}[2 \times 3 + (n-1) \times 5]$
⋮	⋮	⋮
m	$2m-1$	$S_m = \frac{n}{2}[2m + (n-1)(2m-1)]$

Hence, $S_1+S_2+\cdots+S_m$

$$=\frac{n}{2}[2\times 1+(n-1)\times 1]+\frac{n}{2}[2\times 2+(n-1)\times 3]+\cdots+\frac{n}{2}[2m+(n-1)(2m-1)]$$

$$=\frac{n}{2}\,[2\times(1+2+3+\cdots+m)+(n-1)\,(1+3+5+\cdots+(2m-1)]$$

$$=\frac{n}{2}[2\times\frac{m}{2}(1+m)+(n-1)\frac{m}{2}(1+(2m-1))]$$

$$=\frac{n}{2}\,[m(m+1)+m^2(n-1)]$$

$$=\frac{mn}{2}(mn+1)$$

10. Let a be the first term and r the common ratio of the G.P. Then,

$$S_1=a\left(\frac{r^n-1}{r-1}\right),\ S_2=a\left(\frac{r^{2n}-1}{r-1}\right)\text{ and } S_3=a\left(\frac{r^{3n}-1}{r-1}\right)$$

Now,

$$S_1(S_3-S_2)=a\left(\frac{r^n-1}{r-1}\right)\left\{a\left(\frac{r^{3n}-1}{r-1}\right)-a\left(\frac{r^{2n}-1}{r-1}\right)\right\}$$

$$=\frac{a^2}{(r-1)^2}(r^n-1)\{(r^{3n}-1)-(r^{2n}-1)\}$$

$$=\frac{a^2}{(r-1)^2}(r^n-1)(r^{3n}-r^{2n})$$

$$=\frac{a^2}{(r-1)^2}(r^n-1)r^{2n}(r^n-1)$$

$$=\left[ar^n\left(\frac{r^n-1}{r-1}\right)\right]^2$$

$$(S_2-S_1)^2=\left[a\left(\frac{r^{2n}-1}{r-1}\right)-a\left(\frac{r^n-1}{r-1}\right)\right]^2$$

$$=\frac{a^2}{(r-1)^2}\{(r^{2n}-1)-(r^n-1)\}^2$$

$$=\frac{a^2}{(r-1)^2}\{r^n(r^n-1)\}^2$$

$$=\left[ar^n\left(\frac{r^n-1}{r-1}\right)\right]^2$$

$\therefore\quad S_1(S_3-S_1)=(S_2-S_1)^2$

11. Let the four numbers in G.P. be $\frac{a}{r^3},\frac{a}{r},ar$, and ar^3. The product is

$a^4=4096=8^4$

$\Rightarrow\ a=8$

The sum is

$$8\left(\frac{1}{r^3}+\frac{1}{r}+r+r^3\right)=85$$

$$\Rightarrow\ 8\left(r^3+\frac{1}{r^3}\right)+8\left(r+\frac{1}{r}\right)-85=0$$

$$\Rightarrow\ 8\left[\left(r+\frac{1}{r}\right)^3-3\left(r+\frac{1}{r}\right)\right]+8\left(r+\frac{1}{r}\right)-85=0 \qquad (1)$$

Let $r+1/r=t$. Hence, (1) becomes

$$8t^3-16t-85=0 \qquad (2)$$

Putting $2t=y$, we have

$y^3-8y-85=0$

$\Rightarrow\ (y-5)(y^2+5y+17)=0$

$\Rightarrow\ y=2t=5$

$$\Rightarrow\ 2\left(r+\frac{1}{r}\right)=5 \qquad (3)$$

The other factor gives imaginary values. From (3),

$2r^2-5r+2=0$

$\Rightarrow\ (r-2)\,(2r-1)=0$

$\Rightarrow\ r=2,\frac{1}{2}$ and $a=8$

Hence, the four numbers are 1, 4, 16, 64 or 64, 16, 4, 1.

12. $d=2,\ r=1/2$

There are $4n+1$ terms. Then the mid-term is $(2n+1)^{\text{th}}$ term. T_{n+1} and t_{n+1} are mid-terms of A.P. and G.P.

$T_{n+1}=a+nd=a+2n$

$$t_{n+1}=AR^n=T_{2n+1}\times\left(\frac{1}{2}\right)^n=(a+4n)\left(\frac{1}{2}\right)^n$$

By given condition,

$T_{n+1}=t_{n+1}$

$\Rightarrow\ a+2n=(a+4n)\frac{1}{2^n}$

$\Rightarrow\ (2^n-1)\,a=4n-2n\times 2^n$

$$\Rightarrow\ a=\frac{4n-n\times 2^{n+1}}{2^n-1}$$

Hence, the mid-term of the sequence is

$$a+4n=\frac{4n-n\times 2^{n+1}}{2^n-1}+4n$$

$$=\frac{-n\times 2^{n+1}+2n\times 2^{n+1}}{2^n-1}$$

$$=\frac{n\times 2^{n+1}}{2^n-1}$$

13. We have,

$(a_1+a_2+\cdots+a_m)^2=a_1^2+a_2^2+\cdots+a_m^2+2(a_1a_2+a_2a_3+\cdots)$

or

$$\left[\frac{a_1(1-r^m)}{1-r}\right]^2=\frac{a_1^2(1-r^{2m})}{1-r^2}+2\sum_{i<j}^{m}a_ia_j$$

$$\Rightarrow\ 2\sum_{i<j}^{m}a_ia_j=\frac{a_1^2(1-r^m)^2}{(1-r)^2}-\frac{a_1^2(1-r^{2m})}{1-r^2}$$

$$=\frac{2a_1^2}{(1-r)^2(1+r)}[r-r^m-r^{m+1}+r^{2m}]$$

$$=\frac{2r}{1+r}\left\{a_1\times\frac{(1-r^{m-1})}{1-r}\right\}\left\{\frac{a_1(1-r^m)}{1-r}\right\}$$

$$\Rightarrow\ \frac{r+1}{r}\sum_{i<j}^{m}a_ia_j=S_{m-1}\times S_m$$

14. $T_r = \tan\frac{x}{2^r}\sec\frac{x}{2^{r-1}}$

$$= \frac{\sin\frac{x}{2^r}}{\cos\frac{x}{2^r}\cos\frac{x}{2^{r-1}}}$$

$$= \frac{\sin\left(\frac{1}{2^{r-1}}-\frac{1}{2^r}\right)x}{\cos\frac{x}{2^r}\times\cos\frac{x}{2^{r-1}}}$$

$$= \frac{\sin\frac{x}{2^{r-1}}\cos\frac{x}{2^r}-\sin\frac{x}{2^r}\times\cos\frac{x}{2^{r-1}}}{\cos\frac{x}{2^r}\cos\frac{x}{2^{r-1}}}$$

$$= \tan\frac{x}{2^{r-1}}-\tan\frac{x}{2^r}$$

$$\Rightarrow \sum_{r=1}^{n} T_r = \tan\frac{x}{2^0}-\tan\frac{x}{2^n} = \tan x - \tan\frac{x}{2^n}$$

15. $\sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty}\sum\limits_{k=0}^{\infty}\frac{1}{3^i3^j3^k}$ $(i\neq j\neq k)$

= sum when i, j, k are independent
– 3 × (sum when any two of i, j, k are equal)
+ 2 × (sum when $i = j = k$)

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k} - 3\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{9^i3^k} + 2\sum_{i=0}^{\infty}\frac{1}{27^i}$$

$$= \left(\sum_{i=0}^{\infty}\frac{1}{3^i}\right)^3 - 3\left(\sum_{i=0}^{\infty}\frac{1}{9^i}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right) + 2\left(\sum_{i=0}^{\infty}\frac{1}{27^i}\right)$$

$$= \left(\frac{3}{2}\right)^3 - 3\left(\frac{9}{8}\right)\left(\frac{3}{2}\right) + 2\left(\frac{27}{26}\right)$$

$$= \frac{81}{208}$$

16. Let $\sqrt{a_1} = b_1;$

$\sqrt{a_2-1} = b_2;$

$\sqrt{a_3-2} = b_3;$

....

$\sqrt{a_n-(n-1)} = b_n$

$\therefore\ b_1 + b_2 + ... + b_n =$

$$\frac{1}{2}\left[b_1^2+(b_2^2+1)+(b_3^2+2)+\cdots+(b_n^2+(n-1))\right]-\frac{n(n-3)}{4}$$

$$\therefore\ \sum b_i = \frac{1}{2}[(b_1^2+b_2^2+b_3^2+\cdots+b_n^2)+$$

$$(1+2+3+\cdots+(n-1)]-\frac{n(n-3)}{4}$$

$$\Rightarrow\ 2\sum b_i = \sum b_i^2 + \frac{n(n-1)}{2} - \frac{n(n-3)}{2}$$

$$\Rightarrow\ 2\sum b_i = \sum b_i^2 + n$$

$$\therefore\ \sum b_i^2 - 2\sum b_i + \sum 1 = 0$$

$$\Rightarrow\ \sum_{i=1}^{n}(b_i-1)^2 = 0$$

$b_1 - 1 = 0 \Rightarrow b_1^2 = a_1 = 1$

$b_2 - 1 = 0 \Rightarrow b_2^2 = a_2 - 1 = 1 \Rightarrow a_2 = 2$

$b3 - 1 = 0 \Rightarrow b_3^2 = a3 - 2 = 1 \Rightarrow a_3 = 3$ and so on

Hence $a_n = n$

$$\therefore\ \sum_{i=1}^{100} a_i = 1 + 2 + 3 + ... + 100 = 5050$$

Objective Type

1. d. The given numbers are in A.P. Therefore,

$2\log_4(2^{1-x}+1) = \log_2(5\times 2^x+1)+1$

$\Rightarrow\ 2\log_{2^2}\left(\frac{2}{2^x}+1\right) = \log_2(5\times 2^x+1)+\log_2 2$

$\Rightarrow\ \frac{2}{2}\log_2\left(\frac{2}{2^x}+1\right) = \log_2(5\times 2^x+1)2$

$\Rightarrow\ \log_2\left(\frac{2}{2^x}+1\right) = \log_2(10\times 2^x+2)$

$\Rightarrow\ \frac{2}{2^x}+1 = 10\times 2^x+2$

$\Rightarrow\ \frac{2}{y}+1 = 10y+2$, where $2^x = y$

$\Rightarrow\ 10y^2+y-2=0$

$\Rightarrow\ (5y-2)(2y+1)=0$

$\Rightarrow\ y = 2/5$ or $y = -1/2$

$\Rightarrow\ 2^x = 2/5$ or $2^x = -1/2$

$\Rightarrow\ x = \log_2(2/5)$ [$\because 2^x$ cannot be negative]

$\Rightarrow\ x = \log_2 2 - \log_2 5$

$\Rightarrow\ x = 1 - \log_2 5$

2. b. Since a, b, c are in A.P., therefore, $b - a = d$ and $c - b = d$, where d is the common difference of the A.P.

$\therefore\ a = b - d$ and $c = b + d$

Now,

$abc = 4$

$\Rightarrow\ (b-d)\,b\,(b+d) = 4$

$\Rightarrow\ b(b^2-d^2) = 4$

But,

$b(b^2-d^2) < b\times b^2$

$\Rightarrow\ b(b^2-d^2) < b^3$

$\Rightarrow\ 4 < b^3$

$\Rightarrow\ b^3 > 4$

$\Rightarrow\ b > 2^{2/3}$

Hence, the minimum value of b is $2^{2/3}$.

3. a. n^{th} term of the series is $20 + (n-1)(-2/3)$.

For the sum to be maximum,

n^{th} term ≥ 0

$\Rightarrow$ $20 + (n-1)\left(-\frac{2}{3}\right) \geq 0$

$\Rightarrow$ $n \leq 31$

Thus, the sum of 31 terms is maximum and is

$$\frac{31}{2}\left[40+30\times\left(-\frac{2}{3}\right)\right]=310$$

4. d. 100^{th} term of 1, 11, 21, 31, ... is $1 + (100 - 1)10 = 991$.

100^{th} term of 31, 36, 41, 46, ... is $31 + (100 - 1)5 = 526$.

Let the largest common term be 526. Then,

$526 = 31 + (n-1)10$

$\Rightarrow$ $n = 50.5$

But n is an integer; hence $n = 50$. Hence, the largest common term is $31 + (50-1)10 = 521$.

5. d. Let a be the first term and d be the common difference of the given A.P. Then,

$S_m = S_n \Rightarrow \frac{m}{2}[2a+(m-1)d]=\frac{n}{2}[2a+(n-1)d]$

$\Rightarrow$ $2a(m-n) + \{m(m-1) - n(n-1)\}d = 0$

$\Rightarrow$ $2a(m-n) + \{(m^2-n^2) - (m-n)\}d = 0$

$\Rightarrow$ $(m-n)[2a + (m+n-1)d] = 0$

$\Rightarrow$ $2a + (m+n-1)d = 0$ $[\because m-n \neq 0]$ (1)

Now,

$S_{m+n}=\frac{m+n}{2}[2a+(m+n-1)d]=\frac{m+n}{2}\times 0=0$ [Using (1)]

6. a. Let $\angle C = 90°$ being greatest and $B = 90° - A$.

The sides are $a-d$, a and $a+d$

We have $(a+d)^2 = (a-d)^2 + a^2$ (using Pythagoras Theorem)

$\therefore$ $4ad - a^2 = 0 \Rightarrow a = 4d$

Hence the sides are $3d$, $4d$, $5d$

Clearly, $\sin A = \frac{BC}{AB}=\frac{a-d}{a+d}=\frac{3d}{5d}=\frac{3}{5}$

$\sin B = \frac{AC}{AB}=\frac{a}{a+d}=\frac{4d}{5d}=\frac{4}{5}$

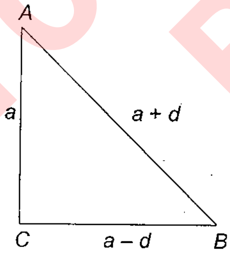


Fig. 3.4

7. b. $\dfrac{\frac{n}{2}(2a+(n-1)d)}{\frac{n}{2}(2a'+(n-1)d')}=\dfrac{5n+3}{3n+4}$

$\Rightarrow$ $\dfrac{(2a+(2n-2)d)}{(2a'+(2n-2)d')}=\dfrac{5(2n-1)+3}{3(2n-1)+4}$ (replace n by $2n-1$)

$\Rightarrow$ $\dfrac{(a+(n-1)d)}{(a'+(n-1)d')}=\dfrac{10n-2}{6n+1}$

$\Rightarrow$ $\dfrac{(a+(17-1)d)}{(a'+(17-1)d')}=\dfrac{168}{103}$

8. c. Suppose the work is completed in n days when the workers stopped working. Since four workers stopped working every day except the first day. Therefore, the total number of workers who worked all the n days is the sum of n terms of an A.P. with first term 150 and common difference –4, i.e.,

$\frac{n}{2}[2\times150+(n-1)\times-4]=n(152-2n)$

Had the workers not stopped working, then the work would have finished in $(n-8)$ days with 150 workers working on each day. Therefore, the total number of workers who would have worked all the n days is $150(n-8)$.

$\therefore$ $n(152-2n) = 150(n-8)$

$\Rightarrow$ $n^2 - n - 600 = 0$

$\Rightarrow$ $(n-25)(n+24) = 0$

$\Rightarrow$ $n = 25$

Thus, the work is completed in 25 days.

9. c. Given, $S_p = 0$. Therefore,

$\frac{p}{2}[2a+(p-1)d]=0 \Rightarrow d=\frac{-2a}{p-1}$ (1)

Sum of next q terms is sum of an A.P. whose first term will be

$T_{p+1} = a + pd$.

$\therefore$ $S=\frac{q}{2}[2(a+pd)+(q-1)d]$

$=\frac{q}{2}[2a+(p-1)d+(p+q)d]$

$=\frac{q}{2}\left[0-(p+q)\frac{2a}{p-1}\right]$

$=-a\frac{(p+q)q}{p-1}$ [Using (1)]

10. b. $S_{3n}=\frac{3n}{2}[2a+(3n-1)d]$

$S_{n-1}=\frac{n-1}{2}[2a+(n-2)d]$

$\Rightarrow$ $S_{3n}-S_{n-1}=\frac{1}{2}[2a(3n-n+1)]+\frac{d}{2}[3n(3n-1)-(n-1)(n-2)]$

$=\frac{1}{2}[2a(2n+1)+d(8n^2-2)]$

$= a(2n+1) + d(4n^2-1)$

$= (2n+1)[a + (2n-1)d]$

$S_{2n}-S_{2n-1}=T_{2n}=a+(2n-1)d$

$\Rightarrow$ $\frac{S_{3n}-S_{n-1}}{S_{2n}-S_{2n-1}}=(2n+1)$

Given,

$\frac{S_{3n}-S_{n-1}}{S_{2n}-S_{2n-1}}=31 \Rightarrow n=15$

11. d. $2b = a + c$

$\Rightarrow$ $8b^3 = (a+c)^3 = a^3 + c^3 + 3ac(a+c)$

$\Rightarrow$ $8b^3 = a^3 + c^3 + 3ac(2b)$

$\Rightarrow$ $a^3 + c^3 - 8b^3 = -6abc$

12. b. Let the series have $2n$ terms and the series is $a, a+d, a+2d, \ldots, a+(2n-1)d$.

According to the given conditions, we have

$[a + (a+2d) + (a+4d) + \cdots + (a+(2n-2)d)] = 24$

$\Rightarrow \quad \frac{n}{2}[2a+(n-1)2d]=24$

$\Rightarrow \quad n[a+(n-1)d]=24$ (1)

Also,

$[(a+d)+(a+3d)+\cdots+(a+(2n-1)d)]=30$

$\Rightarrow \quad \frac{n}{2}[2(a+d)+(n-1)2d]=30$

$\Rightarrow \quad n[(a+d)+(n-1)d]=30$ (2)

Also, the last term exceeds the first by 21/2. Therefore,

$a+(2n-1)d-a=21/2$

$\Rightarrow \quad (2n-1)d=21/2$ (3)

Now, subtracting (1) from (2),

$nd=6$ (4)

Dividing (3) by (4), we get

$$\frac{2n-1}{n}=\frac{21}{12}$$

$\Rightarrow \quad n=4$

13. b. Since a, q and c are in A.P., so

$2q=a+c$

$\Rightarrow \quad \frac{1}{p}+\frac{1}{r}=\frac{2}{b}$

$\Rightarrow \quad \frac{1}{p},\frac{1}{b},\frac{1}{r}$ are in A.P.

14. b. Given,

$$F(n+1)=\frac{2F(n)+1}{2}$$

$\Rightarrow \quad F(n+1)-F(n)=1/2$

Hence, the given series is an A.P. with common difference 1/2 and first term being 2. $F(101)$ is 101^{st} term of A.P. given by $2+(101-1)(1/2)=52$.

15. b. If t_r be the r^{th} term of the A.P., then

$t_r=S_r-S_{r-1}$

$=cr(r-1)-c(r-1)(r-2)$

$=c(r-1)(r-r+2)=2c(r-1)$

We have,

$$t_1^2+t_2^2+\cdots+t_n^2=4c^2(0^2+1^2+2^2+\cdots+(n-1)^2)$$

$$=4c^2\,\frac{(n-1)n(2n-1)}{6}$$

$$=\frac{2}{3}\,c^2n(n-1)(2n-1)$$

16. c. Given that

$a_3+a_5+a_8=11$

$\Rightarrow \quad a+2d+a+4d+a+7d=11$

$\Rightarrow \quad 3a+13d=11$ (1)

Given,

$a_4+a_2=-2$

$\Rightarrow \quad a+3d+a+d=-2$

$\Rightarrow \quad a=-1-2d$ (2)

Putting value of a from (2) in (1), we get

$3(-1-2d)+13d=11 \Rightarrow 7d=14 \Rightarrow d=2$ and $a=-5$

$\Rightarrow \quad a_1+a_6+a_7=7$

17. a. If p, q, r are in A.P., then in an A.P. or G.P. or an H.P. $a_1, a_2, a_3, \ldots$, etc., the terms a_p, a_q, a_r are in A.P., G.P. or H.P., respectively.

18. d. Given, $a_1, a_2, a_3, \ldots$ are terms of A.P.

$$\therefore \quad \frac{a_1+a_2+\cdots+a_p}{a_1+a_2+\cdots+a_q}=\frac{p^2}{q^2}$$

$$\Rightarrow \quad \frac{\frac{p}{2}[2a_1+(p-1)d]}{\frac{q}{2}[2a_1+(q-1)d]}=\frac{p^2}{q^2}$$

$$\Rightarrow \quad \frac{2a_1+(p-1)d}{2a_1+(q-1)d}=\frac{p}{q}$$

$\Rightarrow \quad [2a_1+(p-1)d]\,q=[2a_1+(q-1)d]\,p$

$\Rightarrow \quad 2a_1(q-p)=d[(q-1)p-(p-1)q]$

$\Rightarrow \quad 2a_1(q-p)=d(q-p)$

$\Rightarrow \quad 2a_1=d$

$$\therefore \quad \frac{a_6}{a_{21}}=\frac{a_1+5d}{a_1+20d}=\frac{a_1+10a_1}{a_1+40a_1}=\frac{11}{41}$$

19. b. $$\frac{S_{nx}}{S_x}=\frac{\frac{nx}{2}[2a+(nx-1)d]}{\frac{x}{2}[2a+(x-1)d]}$$

$$=\frac{n[(2a-d)+nxd]}{(2a-d)+xd}$$

For $\frac{S_{xn}}{S_x}$ to be independent of x,

$2a-d=0 \Rightarrow 2a=d$

Now,

$$S_p=\frac{p}{2}[2a+(p-1)d]=p^2a$$

20. b. Let the three numbers be a/r, a, ar. As the numbers form an increasing G.P., so, $r>1$. It is given that a/r, $2a$, ar are in A.P. Hence,

$$4a=\frac{a}{r}+ar$$

$\Rightarrow \quad r^2-4r+1=0$

$\Rightarrow \quad r=2\pm\sqrt{3}$

$=2+\sqrt{3} \quad [\because r>1]$

21. c. Let a_1, a_2, and a_3 be first three consecutive terms of G.P. with common ratio r. Then,

$a_2=a_1r$ and $a_3=a_1r^2$

Now,

$a_3>4a_2-3a_1$

$\Rightarrow \quad a_1r^2>4a_1r-3a_1$

$\Rightarrow \quad r^2>4r-3$

$\Rightarrow \quad r^2-4r+3>0$

$\Rightarrow \quad (r-1)(r-3)>0$

$\Rightarrow \quad r<1$ or $r>3$

22. d. We know that

$-1\le\cos x\le 1$

$\Rightarrow \quad |\cos x|\le 1$

But,

$x \in S \Rightarrow x \in (0, \pi) \Rightarrow |\cos x| < 1$

Now,

$$8^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\cdots \text{ to } \infty} = 4^3$$

$\Rightarrow$ $8^{1/(1-|\cos x|)} = 8^2$

$\Rightarrow$ $\dfrac{1}{1-|\cos x|} = 2$

$\Rightarrow$ $|\cos x| = \dfrac{1}{2}$

$\Rightarrow$ $\cos x = \pm\dfrac{1}{2}$

$\Rightarrow$ $x = \pi/3, 2\pi/3$

$\Rightarrow$ $S = \{\pi/3, 2\pi/3\}$

23. c. We have,

$1 + (1+a)b + (1+a+a^2)b^2 + (1+a+a^2+a^3)b^3 + \cdots \infty$

$$= \sum_{n=1}^{\infty}(1+a+a^2+\cdots+a^{n-1})b^{n-1}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1-a^n}{1-a}\right)b^{n-1}$$

$$= \sum_{n=1}^{\infty}\frac{b^{n-1}}{1-a} - \sum_{n=1}^{\infty}\frac{a^n b^{n-1}}{1-a}$$

$$= \frac{1}{1-a}\sum_{n=1}^{\infty}b^{n-1} - \frac{a}{1-a}\sum_{n=1}^{\infty}(ab)^{n-1}$$

$$= \frac{1}{1-a}[1+b+b^2+\cdots\infty] - \frac{a}{1-a}[1+ab+(ab)^2+\cdots\infty]$$

$$= \frac{1}{1-a}\times\frac{1}{1-b} - \frac{a}{(1-a)(1-ab)}$$

$$= \frac{1}{(1-ab)(1-b)}$$

24. c. Let 'A' be first term and 'r' be the common ratio. We have,

$a = Ar^{p+q-1}$, $b = Ar^{p-q-1}$

$\Rightarrow$ $ab = A^2 \times r^{2p-2}$

$\Rightarrow$ $\sqrt{ab} = Ar^{p-1} = p^{\text{th}}$ term

25. b. Let the sides of the triangle be a/r, a and ar, with $a > 0$ and $r > 1$. Let α be the smallest angle, so that the largest angle is 2α. Then α is opposite to the side a/r, and 2α is positive to the side ar. Applying sine rule, we get

$$\frac{a/r}{\sin\alpha} = \frac{ar}{\sin 2\alpha}$$

$\Rightarrow$ $\dfrac{\sin 2\alpha}{\sin\alpha} = r^2$

$\Rightarrow$ $2\cos\alpha = r^2$

$\Rightarrow$ $r^2 < 2$

$\Rightarrow$ $r < \sqrt{2}$

Hence, $1 < r < \sqrt{2}$.

26. a. We have,

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \frac{1/4}{1-1/2} = \frac{1}{2}$$

Hence,

$$0.2^{\log_{\sqrt{5}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)} = 0.2^{\log_{\sqrt{5}}\frac{1}{2}}$$

$$= \left(\frac{1}{5}\right)^{\log_{\sqrt{5}}\frac{1}{2}}$$

$$= (5^{-1})^{2\log_5\frac{1}{2}}$$

$$= (5)^{-2\log_5\frac{1}{2}}$$

$$= (5)^{\log_5 4}$$

$$= 4$$

27. b. Degree of x on L.H.S. is

$1 + 2 + 4 + \cdots + 128$

$= 1 + 2 + 2^2 + \cdots + 2^7$

$= \dfrac{2^8-1}{2-1}$

$= 255$

28. c. x, y, and z are in G.P. Hence,

$y^2 = xz$ (1)

We have,

$a^x = b^y = c^z = \lambda$ (say)

$\Rightarrow$ $x\log a = y\log b = z\log c = \log\lambda$

$\Rightarrow$ $x = \dfrac{\log\lambda}{\log a}$, $y = \dfrac{\log\lambda}{\log b}$, $z = \dfrac{\log\lambda}{\log c}$

Putting the values of x, y and z in (1), we get

$$\left(\frac{\log\lambda}{\log b}\right)^2 = \frac{\log\lambda}{\log a}\,\frac{\log\lambda}{\log c}$$

$\Rightarrow$ $(\log b)^2 = \log a \log c$

$\Rightarrow$ $\log_b a = \log_c b$

29. b. Required G.M. is $-\sqrt{-9\times-16} = -12$.

30. d. We have,

$$S = \frac{1}{1-\frac{1}{2}} = 2$$

$$S_n = \frac{(1-1/2^n)}{(1-1/2)} = 2\left(1-\frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}}$$

$\therefore$ $S - S_n < \dfrac{1}{1000} \Rightarrow \dfrac{1}{2^{n-1}} < \dfrac{1}{1000}$

$\Rightarrow$ $2^{n-1} > 1000$

$\Rightarrow$ $n - 1 \ge 10$

$\Rightarrow$ $n \ge 11$

Hence, the least value of n is 11.

31. d. Let the series be $21, 21r, 21r^2, \ldots$

Sum $= \dfrac{21}{1-r}$ is a positive integer

also $21r$ is a positive integer

$S = \dfrac{(21)(21)}{21-21r}$ as $21r \in N$ hence $21 - 21r$ must be an integer
also $21r < 21$
hence $21 - 21r$ may be equal to 1, 3, 7 or 9
i.e.must be a divisor of (21)(21)
hence $21 - 21r = 1$ or 3 or 7 or 9
$21r = 20, 18, 14$ or 12

32. c. For G.P., $t_n = 2^{n-1}$; for A.P., $T_m = 1 + (m-1)3 = 3m - 2$. They are common if $2^{n-1} = 3m - 2$. For G.P. 100th term is 2^{99}. For A.P. 100th term is $1 + (100-1)3 = 298$. Now we must choose value of m such that $3m - 2$ is of type 2^{n-1}. Hence, $3m - 2 = 1, 2, 4, 8, 16, 32, 64, 128, 256$ for which $m = 1, 4/3, 2, 10/3, 6, 34/2, 22, 130/3, 86$. Hence, possible values of m are 1, 2, 6, 22, 86. Hence, there are five common terms.

33. c. Initially the ball falls from a height of 120 m. After striking the floor, it rebounds and goes to a height of $\frac{4}{5} \times (120)$ m. Now, it falls from a height of $\frac{4}{5} \times (120)$ m and after rebounding goes to a height of $\frac{4}{5}\left(\frac{4}{5}(120)\right)$ m. This process is continued till the ball comes to rest.

Hence, the total distance travelled is

$$120 + 2\left[\frac{4}{5}(120) + \left(\frac{4}{5}\right)^2(120) + \cdots \infty\right]$$

$$= 120 + 2\left[\frac{\frac{4}{5}(120)}{1-\frac{4}{5}}\right] = 1080 \text{ m}$$

34. b. Let a be the first term and r the common ratio of the G.P. Then, the sum is given by

$$\frac{a}{1-r} = 57 \qquad (1)$$

Sum of the cubes is 9747. Hence,

$$a^3 + a^3r^3 + a^3r^6 + \cdots = 9747$$

$$\Rightarrow \frac{a^3}{1-r^3} = 9747 \qquad (2)$$

Dividing the cube of (1) by (2), we get

$$\frac{a^3}{(1-r)^3} \cdot \frac{(1-r^3)}{a^3} = \frac{(57)^3}{9747}$$

$$\Rightarrow \frac{1-r^3}{(1-r)^3} = 19$$

$$\Rightarrow \frac{1+r+r^2}{(1-r)^2} = 19$$

$\Rightarrow 18r^2 - 39r + 18 = 0$
$\Rightarrow (3r-2)(6r-9) = 0$
$\Rightarrow r = 2/3$ or $r = 3/2$

$= 2/3$ [$\because r \ne 3/2$, because $0 < |r| < 1$ for an infinite G.P.]

35. b. $a^2 + b^2, ab + bc, b^2 + c^2$ are in G.P.
$\Rightarrow (ab + bc)^2 = (a^2 + b^2)(b^2 + c^2)$
$\Rightarrow a^2b^2 + b^2c^2 + 2ab^2c = a^2b^2 + a^2c^2 + b^2c^2 + b^4$
$\Rightarrow b^4 + a^2c^2 - 2ab^2c = 0$
$\Rightarrow (b^2 - ac)^2 = 0$
$\Rightarrow b^2 = ac$
$\Rightarrow a, b,$ and c are in G.P.

36. b. Given $\dfrac{ar(r^{10}-1)}{r-1} = 18 \qquad (1)$

Also $\dfrac{\frac{1}{ar}\left(1-\frac{1}{r^{10}}\right)}{1-\frac{1}{r}} = 6$

$$\Rightarrow \frac{1}{ar^{11}} \cdot \frac{(r^{10}-1)r}{r-1} = 6$$

$$\Rightarrow \frac{1}{a^2r^{11}} \cdot \frac{ar(r^{10}-1)}{r-1} = 6 \qquad (2)$$

From (1) and (2),

$$\frac{1}{a^2r^{11}} \cdot 18 = 6$$

$\Rightarrow a^2r^{11} = 3$

Now $P = a^{10}r^{55} = (a^2r^{11})^5 = 3^5 = 243$

37. b. $a = 1 + 10 + 10^2 + \cdots + 10^{54}$

$$= \frac{10^{55}-1}{10-1} = \frac{10^{55}-1}{10^5-1} \times \frac{10^5-1}{10-1} = bc$$

38. a. Let a be the first term and r be the common ratio of the given G.P. Then,

$$\alpha = \sum_{n=1}^{100} a_{2n} \Rightarrow \alpha = a_2 + a_4 + \cdots + a_{200}$$
$$= ar + ar^3 + \cdots + ar^{199}$$
$$= ar(1 + r^2 + r^4 + \cdots + r^{198})$$
$$\beta = \sum_{n=1}^{100} a_{2n-1} \Rightarrow \beta = a_1 + a_3 + \cdots + a_{199}$$
$$= a + ar^2 + \cdots + ar^{198}$$
$$= a(1 + r^2 + \cdots + r^{198})$$

Clearly, $\alpha/\beta = r$.

39. c. The series is

$1 + 2 + 2 \times 3 + 2^2 \times 3 + 2^2 \times 3^2 + 2^3 \times 3^2 + \cdots$ to 20 terms
$= (1 + 2 \times 3 + 2^2 \times 3^2 + \cdots$ to 10 terms)
$+ (2 + 2^2 \times 3 + 2^3 \times 3^2 + \cdots$ to 10 terms)

$$= \frac{1(2^{10}3^{10}-1)}{6-1} + \frac{2(2^{10}3^{10}-1)}{6-1}$$

$$= \left(\frac{3}{5}\right)(6^{10}-1)$$

40. d. $a = 5, ar^2 = a + 3d, ar^4 = a + 15d$
$\therefore 5r^2 = 5 + 3d, 5r^4 = 5 + 15d$
$\Rightarrow r^4 = 1 + 3d$
$\Rightarrow 25r^4 = 25 + 75d$
$\Rightarrow (5 + 3d)^2 = 25 + 75d$
$\Rightarrow 25 + 30d + 9d^2 = 25 + 75d$
$\Rightarrow 9d^2 - 45d = 0$
$\Rightarrow d = 5, 0$

$\Rightarrow \quad T_4 = a + 3d = 5 + 15 = 20$

41. d. p^{th}, q^{th}, r^{th} terms of A.P. are

$$a + (p-1)d = x \quad (1)$$
$$a + (q-1)d = xR \quad (2)$$
$$a + (r-1)d = xR^2 \quad (3)$$

where R is common ratio of G.P.

Subtracting (2) from (3) and (1) from (2) and then dividing the former by the later, we have

$$\frac{q-r}{p-q} = \frac{xR^2 - xR}{xR - x} = R$$

42. b. Given that

$$a + (p-1)d = A$$
$$a + (q-1)d = AR$$
$$a + (r-1)d = AR^2$$
$$a + (s-1)d = AR^3$$

where R is common ratio of G.P. Now,

$$p - q = \frac{A - AR}{d}, \; q - r = R\left(\frac{A-AR}{d}\right),$$
$$r - s = R^2\left(\frac{A-AR}{d}\right)$$

Clearly, $p-q$, $q-r$, $r-s$ are in G.P.

43. a. Let r be the common ratio of the G.P., a, b, c, d. Then,

$b = ar$, $c = ar^2$ and $d = ar^3$

$$\therefore \quad (b-c)^2 + (c-a)^2 + (d-b)^2$$
$$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$$
$$= a^2r^2(1-r)^2 + a^2(r^2-1)^2 + a^2r^2(r^2-1)^2$$
$$= a^2(r^6 - 2r^3 + 1)$$
$$= a^2(1-r^3)^2$$
$$= (a - ar^3)^2$$
$$= (a-d)^2$$

44. d. Let $P = 0.cababab\ldots$

$\Rightarrow 10P = c.ababab\ldots \quad (1)$

and

$1000P = cab.ababab\ldots \quad (2)$

Subtracting Eq. (1) from Eq. (2), we have

$990P = cab - c$

or

$$P = \frac{100c + 10a + b - c}{990} = \frac{99c + 10a + b}{990}$$

45. c. $S_\infty = \dfrac{a}{1-r} = 162$

$$S_n = \frac{a(1-r^n)}{1-r} = 160$$

Dividing,

$$1 - r^n = \frac{160}{162} = \frac{80}{81}$$
$$\Rightarrow \quad 1 - \frac{80}{81} = r^n$$
$$\Rightarrow \quad r^n = \frac{1}{81} \text{ or } \left(\frac{1}{r}\right)^n = 81 \quad (1)$$

Now, it is given that $1/r$ is an integer and n is also an integer. Hence, the relation (1) implies that $1/r = 3, 9$ or 81 so that $n = 4, 2$ or 1.

$$\therefore \quad a = 162\left(1 - \frac{1}{3}\right) \text{ or } 162\left(1 - \frac{1}{9}\right) \text{ or } 162\left(1 - \frac{1}{81}\right)$$
$$= 108 \text{ or } 144 \text{ or } 160$$

46. d. $f(x) = 2x + 1$

$\Rightarrow \quad f(2x) = 2(2x) + 1 = 4x + 1$ and $f(4x) = 2(4x) + 1 = 8x + 1$

Now, $f(x), f(2x), f(4x)$ are in G.P. Hence,

$$(4x+1)^2 = (2x+1)(8x+1)$$
$$\Rightarrow \quad 2x = 0$$

Hence, $f(x), f(2x)$, and $f(4x)$ is equal to 1 which contradicts the given condition. Hence no such x exists.

47. b.

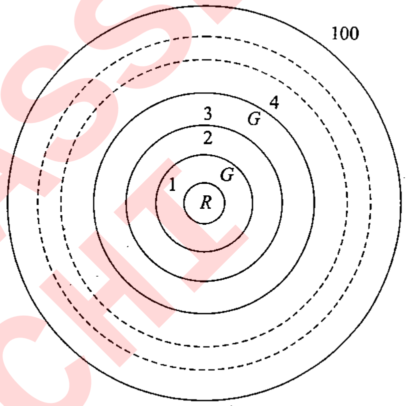


Fig. 3.5

$$\pi\left[(r_2^2 - r_1^2) + (r_4^2 - r_3^2) + \cdots + (r_{100}^2 - r_{99}^2)\right]$$
$$= \pi\,[r_1 + r_2 + r_3 + r_4 + \cdots + r_{100}] \;(\because r_2 - r_1 = r_4 - r_3 = \cdots = r_{100} - r_{99} = 1)$$
$$= \pi\,[1 + 2 + 3 + \cdots + r_{100}]$$
$$= 5050\pi \text{ sq. cm}$$

48. a. $\dfrac{t_4}{t_6} = \dfrac{1}{4} \Rightarrow \dfrac{ar^3}{ar^5} = \dfrac{1}{4} \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$

Also,

$$t_2 + t_5 = 216$$
$$\Rightarrow \quad ar + ar^4 = 216$$
$$\Rightarrow \quad a + 8a = 108$$
$$\Rightarrow \quad a = 12 \text{ (when } r = 2)$$

49. b. x, y, and z are in G.P. Hence,

$y = xr$, $z = xr^2$

Also, $x, 2y$, and $3z$ are in A.P. Hence,

$$4y = x + 3z$$
$$\Rightarrow \quad 4xr = x + 3xr^2$$
$$\Rightarrow \quad 3r^2 - 4r + 1 = 0$$
$$\Rightarrow \quad (3r-1)(r-1) = 0$$
$$\Rightarrow \quad r = 1/3 \quad (r \neq 1 \text{ is not possible as } x, y, z \text{ are distinct})$$

50. a. $S_p = \dfrac{1}{1-r^p}$, $s_p = \dfrac{1}{1+r^p}$, $S_{2p} = \dfrac{1}{1-r^{2p}}$

Clearly,

$$S_p + s_p = \frac{2}{1-r^{2p}} = 2S_{2p}$$

51. c. Multiplying the given expression by 2 and rewriting it, we have

$$(2x-3y)^2+(3y-4z)^2+(4z-2x)^2=0$$

$\Rightarrow$ $2x = 3y = 4z$

$\Rightarrow$ $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

$\Rightarrow$ x, y, z are in H.P.

52. c. $a_1, a_2, \ldots, a_n$ are in H.P.

$\Rightarrow$ $\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}$ are in A.P.

$\Rightarrow$ $\frac{a_1+a_2+a_3+\cdots+a_n}{a_1}, \frac{a_1+a_2+a_3+\cdots+a_n}{a_2}, \ldots, \frac{a_1+a_2+a_3+\cdots+a_n}{a_n}$ are in A.P.

$\Rightarrow$ $1+\frac{a_2+a_3+\cdots+a_n}{a_1}, 1+\frac{a_1+a_3+\cdots+a_n}{a_2}, \ldots, 1+\frac{a_1+a_2+\cdots+a_{n-1}}{a_n}$ are in A.P.

$\Rightarrow$ $\frac{a_2+a_3+\cdots+a_n}{a_1}, \frac{a_1+a_3+\cdots+a_n}{a_2}, \ldots, \frac{a_1+a_2+\cdots+a_{n-1}}{a_n}$ are in A.P.

$\Rightarrow$ $\frac{a_1}{a_2+a_3+\cdots+a_n}, \frac{a_2}{a_1+a_3+\cdots+a_n}, \ldots, \frac{a_n}{a_1+a_2+\cdots a_{n-1}}$ are in H.P.

53. c.

$$\frac{H_1+2}{H_1-2}+\frac{H_{20}+3}{H_{20}-3} = \frac{\frac{1}{2}+\frac{1}{H_1}}{\frac{1}{2}-\frac{1}{H_1}}+\frac{\frac{1}{3}+\frac{1}{H_{20}}}{\frac{1}{3}-\frac{1}{H_{20}}}$$

$$= \frac{\frac{1}{2}+\frac{1}{2}+d}{\frac{1}{2}-d-\frac{1}{2}}+\frac{\frac{1}{3}+\frac{1}{3}-d}{\frac{1}{3}+d-\frac{1}{3}}$$

$$= \frac{\frac{2}{2}+d}{-d}+\frac{\frac{2}{3}-d}{d}$$

$$= \frac{\frac{2}{3}-1}{d}-2$$

$$= 2\times 21-2 \quad \left[\text{as also, } \frac{1}{3}=\frac{1}{2}+21d\right]$$

$$= 40$$

54. b. We have,

a_1, a_2, a_3 are in A.P. $\Rightarrow 2a_2 = a_1 + a_3$ (1)

a_2, a_3, a_4 are in G.P. $\Rightarrow a_3{}^2 = a_2a_4$ (2)

a_3, a_4, a_5 are in H.P. $\Rightarrow a_4 = \frac{2a_3a_5}{a_3+a_5}$ (3)

Putting $a_2 = \frac{a_1+a_3}{2}$ and $a_4 = \frac{2a_3a_5}{a_3+a_5}$ in (2), we get

$$a_3^2 = \frac{a_1+a_3}{2}\times\frac{2a_3a_5}{a_3+a_5}$$

$\Rightarrow$ $a_3^2 = a_1a_5$

Hence, a_1, a_3, and a_5 are in G.P. So, $\log_e a_1$, $\log_e a_3$ and $\log_e a_5$ are in A.P.

55. d. a, b, and c are in A.P. Hence,

$2b = a + c$ (1)

$$\frac{a}{bc}+\frac{2}{b}=\frac{a+2c}{bc}\neq\frac{2}{c}$$

$\Rightarrow$ $\frac{a}{bc}, \frac{1}{c}, \frac{2}{b}$ are not in A.P.

$$\frac{bc}{a}+\frac{b}{2}=\frac{2bc+ab}{2a}\neq c$$

Hence, the given numbers are not in H.P. Again,

$$\frac{a}{bc}\frac{2}{b}=\frac{2a}{b^2c}\neq\frac{1}{c^2}$$

Therefore, the given numbers are not in G.P.

56. d. x, $2x + 2$, $3x + 3$ are in G.P. Hence,

$(2x + 2)^2 = x(3x + 3)$

$\Rightarrow$ $4x^2 + 8x + 4 = 3x^2 + 3x$

$\Rightarrow$ $x^2 + 5x + 4 = 0$

$\Rightarrow$ $x = -1, -4$

So, the G.P. is $-4, -6, -9, \ldots$ (considering $x = -4$, as for $x = -1$, $2x + 2 = 0$). Hence, the fourth term is $-9 \times 1.5 = -13.5$.

57. a. Let the numbers be a, ar, ar^2. Then,

$a + ar + ar^2 = 14$ (given) (1)

Now,

$a + 1$, $ar + 1$, $ar^2 - 1$ are in A.P.

$\Rightarrow$ $2(ar + 1) = a + 1 + ar^2 - 1$

$\Rightarrow$ $2ar + 2 = a + ar^2$ (2)

From (1) and (2),

$2ar + 2 = 14 - ar$

$\Rightarrow$ $3ar = 12$

$\Rightarrow$ $ar = 4$ (3)

From (1),

$a + 4 + 4r = 14$

$\Rightarrow$ $a + 4r = 10$ (4)

From (3) and (4),

$$a+\frac{16}{a}=10 \Rightarrow a = 2, 8$$

Hence, the smallest number is 2.

58. b. $\frac{p}{r}+\frac{r}{p}=\frac{p^2+r^2}{pr}=\frac{(p+r)^2-2pr}{pr}$

$$= \frac{\frac{4p^2r^2}{q^2}-2pr}{pr} \quad \left[\because p, q, r \text{ are in H.P.} \therefore q=\frac{2pr}{p+r}\right]$$

$$= \frac{4pr}{q^2}-2=\frac{4b^2}{ac}-2$$

$[\because ap, bq, cr$ are in A.P. $\Rightarrow b^2q^2 = acpr]$

$$= \frac{(a+c)^2}{ac} - 2 \qquad [a, b, c \text{ are in A.P.} \Rightarrow 2b = a + c]$$

$$= \frac{a}{c} + \frac{c}{a}$$

59. a. We have,

$2b = a + c$

$(c - b)^2 = (b - a)a$

$\Rightarrow (b - a)^2 = (b - a)a \quad [2b = a + c \Rightarrow b - a = c - b]$

$\Rightarrow b = 2a$

$\Rightarrow c = 3a \quad$ [Using $2b = a + c$]

$\Rightarrow a:b:c = 1:2:3$

60. a. Let $a = 1$, then $S_1 = 2008$

If $a \neq 1$ then $S = \dfrac{a^{2008} - 1}{a - 1}$

but $a^{2008} = 2a - 1$, therefore, $S_2 = \dfrac{2(a-1)}{a-1} = 2$

$\therefore \quad S = S_1 + S_2 = 2010$

61. b. For the equation $x^2 - px + 1 = 0$,

the product of roots, $\alpha\beta^2 = 1$

and for the equation $x^2 - qx + 8 = 0$,

the product of roots $\alpha^2\beta = 8$

Hence, $(\alpha\beta^2)(\alpha^2\beta) = 8$

$\Rightarrow \alpha^3\beta^3 = 8 \Rightarrow \alpha\beta = 2$

$\therefore$ From $\alpha\beta^2 = 1$, we have $\beta = \dfrac{1}{2}$ and from $\alpha^2 . \beta = 8$, we have $\alpha = 4$

Hence, from sum of roots $= -\dfrac{b}{a}$, we have

$$p = \alpha + \beta^2 = 4 + \frac{1}{4} = \frac{17}{4} \text{ and } q = \alpha^2 + \beta = 16 + \frac{1}{2} = \frac{33}{2}$$

$\dfrac{r}{8}$ is arithmetic mean of p and q

$$\therefore \quad \frac{r}{8} = \frac{p+q}{2}$$

$$\Rightarrow r = 4(p + q) = 4\left(\frac{17}{4} + \frac{33}{2}\right) = 17 + 66 = 83$$

62. c. $2b = a + c$, $c = \dfrac{2bd}{b+d}$

$\Rightarrow 2bd = c(b + d)$

$\Rightarrow (a + c)d = c(b + d) \quad$ [as $2b = a + c$]

$\Rightarrow ad + cd = bc + cd$

$\Rightarrow bc = ad$

63. c. $\dfrac{a_r - a_{r+1}}{a_r\, a_{r+1}} = k$ (constant)

$$\Rightarrow \frac{1}{a_{r+1}} - \frac{1}{a_r} = k$$

$$\Rightarrow \frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n} \text{ are in A.P.}$$

$\Rightarrow a_1, a_2, a_3, \ldots$, are in H.P.

64. c. Let $a = 1$, $b = 2$, $c = 4$ Then,

$a + b = 3$, $2b = 4$, $b + c = 6$

$$\Rightarrow \frac{1}{4} - \frac{1}{3} = -\frac{1}{12} \text{ and } \frac{1}{6} - \frac{1}{4} = -\frac{1}{12}$$

Hence, $a + b$, $2b$, $b + c$ are in H.P.

65. b. x is A.M. of a and b, y is G.M. of a and b, z is H.M. of a and b.

$y^2 = xz$

Also given,

$x = 9z$

$\Rightarrow x = 9y^2/x \Rightarrow 9y^2 = x^2 \Rightarrow x = 3|y|$

66. d. $A = \dfrac{25+n}{2}$, $G = 5\sqrt{n}$, $H = \dfrac{50n}{25+n}$

As A, G, H are natural numbers, n must be odd perfect square. Now, H will be a natural number, if we take $n = 225$.

67. a. Reciprocals of the terms of the series are 2/5, 13/20, 9/10, 23/20, … or 8/20, 13/20, 18/20, 23/20, …. Its n^{th} term is

$$\frac{8 + (n-1)5}{20} = \frac{5n+3}{20}$$

Therefore, the 15^{th} term is $\dfrac{20}{78} = \dfrac{10}{39}$.

68. b. Given, $b^2 = ac$ and $x = \dfrac{a+b}{2}$, $y = \dfrac{b+c}{2}$. Therefore,

$$\frac{a}{x} + \frac{c}{y} = \frac{2a}{a+b} + \frac{2c}{b+c}$$

$$= \frac{2a(b+c) + 2c(a+b)}{(a+b)(b+c)}$$

$$= 2\frac{2ac + ab + bc}{ab + ac + b^2 + bc}$$

$$= 2\frac{2ac + ab + bc}{2ac + ab + bc}$$

$$= 2$$

69. c. $2b = a + c$

a, p, b, q, c are in A.P. Hence,

$p = \dfrac{a+b}{2}$ and $q = \dfrac{b+c}{2}$

Again, a, p', b, q', and c are in G.P. Hence,

$p' = \sqrt{ab}$ and $q' = \sqrt{bc}$

$$\Rightarrow p^2 - q^2 = \frac{(a-c)(a+c+2b)}{4}$$

$= (a - c)b$

$= ab - bc$

$= p'^2 - q'^2$

70. d. As $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ are in A.P., hence

$d = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}$

$\sin d[\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \cdots + \sec a_{n-1} \sec a_n]$

$$= \frac{\sin(a_2 - a_1)}{\cos a_1 \cos a_2} + \frac{\sin(a_3 - a_2)}{\cos a_2 \cos a_3} + \cdots + \frac{\sin(a_n - a_{n-1})}{\cos a_{n-1} \cos a_n}$$

$= (\tan a_2 - \tan a_1) + (\tan a_3 - \tan a_2) + \cdots + (\tan a_n - \tan a_{n-1})$

$= \tan a_n - \tan a_1$

71. c. $S = [a - (a + d)] + [(a + 2d) - (a + 3d)] + \cdots$
$\quad + [(a + (2n - 2)d)] - a + (2n - 1)d] + (a + 2nd)$

$= [(-d) + (-d) + \cdots + n \text{ times}] + a + 2nd$

$= -nd + a + 2nd$

$= a + nd$

72. a. Let $1+1/50=x$. Let S be the sum of 50 terms of the given series. Then,

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots + 49x^{48} + 50x^{49} \quad (1)$$
$$xS = \quad x + 2x^2 + 3x^3 + \cdots \quad + 49x^{49} + 50x^{50} \quad (2)$$
$$(1-x)S = 1 + x + x^2 + x^3 + \cdots + x^{49} - 50x^{50}$$

[Subtracting (2) from (1)]

$$\Rightarrow \quad S(1-x) = \frac{1-x^{50}}{1-x} - 50x^{50}$$

$$\Rightarrow \quad S(-1/50) = -50(1-x^{50}) - 50x^{n}$$

$$\Rightarrow \quad \frac{1}{50}S = 50$$

$$\Rightarrow \quad S = 2500$$

73. a. Let T_r be the r^{th} term of the given series. Then,

$$T_r = \frac{2r+1}{1^2+2^2+\cdots+r^2}$$
$$= \frac{6(2r+1)}{(r)(r+1)(2r+1)}$$
$$= 6\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

So, sum is given by

$$\sum_{r=1}^{50} T_r = 6\sum_{r=1}^{50}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$
$$= 6\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{50}-\frac{1}{51}\right)\right]$$
$$= 6\left[1-\frac{1}{51}\right]$$
$$= \frac{100}{17}$$

74. a. $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$

$$= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\cdots\right)-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right)$$
$$= \frac{\pi^2}{6}-\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$$
$$= \frac{\pi^2}{6}-\frac{1}{4}\left(\frac{\pi^2}{6}\right)$$
$$= \frac{\pi^2}{8}$$

75. b. Coefficient of x^{18} in $(1 + x + 2x^2 + 3x^3 + \cdots + 18x^{18})^2$

$= $ Coefficient of x^{18} in $(1 + x + 2x^2 + 3x^3 + \cdots + 18x^{18}) \times (1 + x + 2x^2 + 3x^3 + \cdots + 18x^{18})$

$= 1\times 18 + 1\times 17 + 2\times 16 + \cdots + 17\times 1 + 18\times 1$

$$= 36 + \sum_{r=1}^{17} r(18-r)$$
$$= 36 + 18\sum_{r=1}^{17} r - \sum_{r=1}^{17} r^2$$
$$= 1005$$

76. c. $T(r) = \dfrac{r}{1\times3\times5\times\cdots\times(2r+1)}$

$$= \frac{2r+1-1}{2(1\times3\times5\cdots(2r+1))}$$
$$= \frac{1}{2}\left(\frac{1}{1\times3\times5\cdots(2r-1)}-\frac{1}{1\times3\times5\cdots(2r+1)}\right)$$
$$= -\frac{1}{2}[V(r)-V(r-1)]$$

$$\Rightarrow \quad \sum_{r=1}^{n} T(r) = -\frac{1}{2}(V(n)-V(0))$$
$$= \frac{1}{2}\left(1-\frac{1}{1\times3\times5\times\cdots\times(2n+1)}\right)$$

$$\Rightarrow \quad \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{1\times3\times5\times7\times9\times\cdots\times(2r+1)}$$
$$= \lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{1\times3\times5\times\cdots\times(2n+1)}\right) = \frac{1}{2}$$

77. d. $r\times r! = (r+1-1)\times r!$

$$= (r+1)! - r!$$
$$= V(r) - V(r-1)$$

$$\Rightarrow \quad \sum_{r=1}^{30} r(r!) = V(31) - V(0)$$
$$= (31)! - 1$$

$$\Rightarrow \quad 1 + \sum_{r=1}^{30} r(r!) = 31!$$

which is divisible by 31 consecutive integers which is a prime number.

78. b. $I(2n) = 1^4 + 2^4 + 3^4 + \cdots + (2n-1)^4 + (2n)^4$

$$= [(1^4 + 3^4 + 5^4 + \cdots + (2n-1)^4] + 2^4(1^4 + 2^4 + 3^4 + 4^4 + \cdots n^4)$$
$$= \sum_{r=1}^{n}(2r-1)^4 + 16\times I(n)$$

$$\Rightarrow \quad \sum_{r=1}^{n}(2r-1)^4 = I(2n) - 16I(n)$$

79. c. Consider the first product,

$$P = \left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right)$$

$$= \frac{\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right)}{\left(1-\frac{1}{3}\right)}$$

$$= \frac{\left(1-\frac{1}{3^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right)}{\left(1-\frac{1}{3}\right)}$$

$$= \frac{\left(1-\frac{1}{3^4}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right)}{\left(1-\frac{1}{3}\right)}$$

$$= \frac{1}{\left(1-\frac{1}{3}\right)}\left(1-\left(\frac{1}{3}\right)^{2^{n+1}}\right)$$

$$= \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{2^{n+1}}\right)$$

$$\Rightarrow \left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\ldots \text{ infinity}$$

$$= \lim_{n\to\infty}\frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{2^{n+1}}\right)$$

$$= \frac{3}{2}$$

80. a. Clearly, $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_{20}}$ will be in A.P. Hence,

$$\frac{1}{x_2}-\frac{1}{x_1} = \frac{1}{x_3}-\frac{1}{x_2} = \cdots = \frac{1}{x_{r+1}}-\frac{1}{x_r} = \cdots = \lambda \text{ (say)}$$

$$\Rightarrow \frac{x_r - x_{r+1}}{x_r x_{r+1}} = \lambda$$

$$\Rightarrow x_r x_{r+1} = -\frac{1}{\lambda}(x_{r+1}-x_r)$$

$$\Rightarrow \sum_{r=1}^{19} x_r x_{r+1} = -\frac{1}{\lambda}\sum_{r=1}^{19}(x_{r+1}-x_r)$$

$$= -\frac{1}{\lambda}(x_{20}-x_1)$$

Now,

$$\frac{1}{x_{20}} = \frac{1}{x_1}+19\lambda$$

$$\Rightarrow \frac{x_1 - x_{20}}{x_1 x_{20}} = 19\lambda$$

$$\Rightarrow \sum_{r=1}^{19} x_r x_{r+1} = 19x_1x_{20} = 19\times 4 = 76$$

($\because$ $x_1, 2, x_{20}$ are in G.P., then $x_1x_{20} = 4$)

81. b. $T_r = r(-a)^r + (r+1)a(-a)^r$
$= r(-a)^r - (r+1)(-a)^{r+1}$
$= v_r - v_{r+1}$ (say)

So,

$$\sum_{r=0}^{n} T_r = \sum_{r=0}^{n}(v_r - v_{r+1})$$
$$= v_0 - v_{n+1}$$
$$= -(n+1)(-a)^{n+1}$$

82. d. $\Sigma a_i b_i = \Sigma a_i(1-a_i)$
$= na - \Sigma a_i^2$
$= na - \Sigma(a_i - a + a)^2$
$= na - \Sigma[(a_i - a)^2 + a^2 + 2a(a_i - a)]$
$= na - \Sigma(a_i - a)^2 - \Sigma a^2 - 2a\,\Sigma(a_i - a)$

$\Rightarrow \Sigma a_i b_i + \Sigma(a_i - a)^2 = na - na^2 - 2a(na - na)$

$= na(1-a) = nab$ $\quad\left[\begin{array}{l}\because \Sigma b_i = \Sigma 1 - \Sigma a_i \\ \therefore nb = n - na \\ \text{or } a + b = 1\end{array}\right]$

83. a. The general term of the given series is

$$t_n = \frac{x^{2^{n-1}}}{1-x^{2^n}} = \frac{1+x^{2^{n-1}}-1}{\left(1+x^{2^{n-1}}\right)\left(1-x^{2^{n-1}}\right)}$$

$$\Rightarrow t_n = \frac{1}{1-x^{2^{n-1}}} - \frac{1}{1-x^{2^n}}$$

Now,

$$S_n = \sum_{n=1}^{n} t_n$$

$$= \left[\left\{\frac{1}{1-x}-\frac{1}{1-x^2}\right\}+\left\{\frac{1}{1-x^2}-\frac{1}{1-x^4}\right\} + \cdots + \left\{\frac{1}{1-x^{2^{n-1}}}-\frac{1}{1-x^{2^n}}\right\}\right]$$

$$= \frac{1}{1-x} - \frac{1}{1-x^{2^n}}$$

Therefore, the sum to infinite terms is

$$\lim_{n\to\infty} S_n = \frac{1}{1-x} - 1$$

$$= \frac{x}{1-x} \quad [\because \lim_{n\to\infty} x^{2n} = 0, \text{ as } |x| < 1]$$

84. a. The general term can be given by

$$t_{r+1} = \frac{a_{2n+1-r} - a_{r+1}}{a_{2n+1-r} + a_{r+1}}, \; r = 0, 1, 2, \ldots, n-1$$

$$= \frac{a_1 + (2n - r)d - \{a_1 + rd\}}{a_1 + (2n-r)d + \{a_1 + rd\}}$$

$$= \frac{(n-r)d}{a_1 + nd}$$

Therefore, the required sum is

$$S_n = \sum_{r=0}^{n-1} t_{r+1}$$

$$= \sum_{r=0}^{n-1} \frac{(n-r)d}{a_1 + nd}$$

$$= \left[\frac{n + (n-1) + (n-2) + \cdots + 1}{a_1 + nd}\right] d$$

$$= \frac{n(n+1)d}{2a_{n+1}}$$

$$= \frac{n(n+1)}{2}\,\frac{a_2 - a_1}{a_{n+1}} \quad [\because d = a_2 - a_1]$$

85. a. Let,

$S = i - 2 - 3i + 4 + 5i + \cdots + 100i^{100}$
$= i + 2i^2 + 3i^3 + 4i^4 + 5i^5 + \cdots + 100i^{100}$

$\Rightarrow iS = i^2 + 2i^3 + 3i^4 + \cdots + 99i^{100} + 100i^{101}$

$\Rightarrow S - iS = [i + i^2 + i^3 + i^4 + \cdots + i^{100}] - 100i^{101}$

$$\Rightarrow S(1-i) = \frac{i(i^{100}-1)}{i-1} - 100i^{101}$$

$$= -100i^{101}$$

$$\Rightarrow \quad S = \frac{-100i}{1-i} = -50i(1+i) = -50(i-1) = 50(1-i)$$

86. b. $S = \frac{4}{19} + \frac{44}{19^2} + \frac{444}{19^3} + \cdots \quad (1)$

$$\Rightarrow \quad \frac{1}{19}S = \frac{4}{19^2} + \frac{44}{19^3} + \cdots \quad (2)$$

Subtracting (2) from (1), we get

$$\frac{18}{19}S = \frac{4}{19} + \frac{40}{19^2} + \frac{400}{19^3} + \cdots$$

$$= \frac{\frac{4}{19}}{1 - \frac{10}{19}}$$

$$= 4/9$$

$$\Rightarrow \quad S = 38/81$$

87. b. $S_n = 1 + \frac{3}{2} + \frac{5}{3} + \cdots + \frac{99}{50}$

$$= (2-1) + \left(2 - \frac{1}{2}\right) + \left(2 - \frac{1}{3}\right) + \cdots + \left(2 - \frac{1}{50}\right)$$

$$= 100 - H_{50}$$

88. b. $S = 1 + 2r + 3r^2 + 4r^3 + \cdots$

$$rS = r + 2r^2 + 3r^3 + 4r^4 + \cdots$$

$$\Rightarrow \quad (1-r)S = 1 + r + r^2 + r^3 + \cdots$$

$$= \frac{1}{1-r}$$

$$\Rightarrow \quad S = \frac{1}{(1-r)^2}$$

Given, $S = 9/4 \Rightarrow \frac{1}{(1-r)^2} = 9/4$

$$\Rightarrow \quad 1 - r = \pm\frac{2}{3}$$

$$\Rightarrow \quad r = 1/3 \text{ or } 5/3$$

Hence, $r = 1/3$ as $0 < |r| < 1$.

89. d. Let,

$$S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots \infty$$

Then,

$$\frac{1}{5}S = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \cdots \infty$$

$$\Rightarrow \quad S\left(1 - \frac{1}{5}\right) = 1 + 3\left[\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots \infty\right]$$

$$\Rightarrow \quad \frac{4}{5}S = 1 + 3 \times \frac{1/5}{1-(1/5)} = 1 + \frac{3}{4} = \frac{7}{4}$$

$$\Rightarrow \quad S = \frac{35}{16}$$

90. c. $T_n = \frac{1^3 + 2^3 + 3^3 + \cdots + n^3}{1 + 3 + 5 + \cdots n \text{ terms}} = \frac{\Sigma n^3}{\frac{n}{2}[2\times 1 + (n-1)2]}$

$$= \frac{1}{4} \times \frac{n^2(n+1)^2}{n^2} = \frac{1}{4}(n^2 + 2n + 1) \quad (1)$$

Now,

$$S_n = \frac{1}{4}(\Sigma n^2 + 2\Sigma n + n)$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + 2 \times \frac{n(n+1)}{2} + n\right]$$

$$= \frac{n}{24}[2n^2 + 3n + 1 + 6n + 6 + 6]$$

$$= \frac{n}{24}[2n^2 + 9n + 13]$$

Putting $n = 16$, we get

$$S_{16} = \frac{16}{24}[2(256) + 144 + 13]$$

$$= \frac{2}{3}(669) = 446$$

91. c. Here the successive differences are 2, 4, 8, 16, … which are in G.P.

$$S = 1 + 3 + 7 + 15 + 31 + \cdots + T_{100}$$

$$S = (2^1 - 1) + (2^2 - 1) + (2^3 - 1) + \cdots + (2^{100} - 1)$$

$$= (2 + 2^2 + 2^3 + \cdots + 2^{100}) - 100$$

$$= 2\left(\frac{2^{100} - 1}{2 - 1}\right) - 100$$

$$= 2^{101} - 102$$

92. a. Series is $a, a+2, a+4, \ldots + a + 4n, (a+4n)0.5, (a+4n)(0.5)^2, \ldots, (a+4n)(0.5)^{2n-1}$

The middle term of A.P. and G.P. are equal

$$\Rightarrow \quad a + 2n = (a+4n)(0.5)^n$$

$$\Rightarrow \quad a.2^n + 2^{n+1}n = a + 4n$$

$$\Rightarrow \quad a = \frac{4n - n2^{n+1}}{2^n - 1}$$

$\Rightarrow$ The middle term of entire sequence

$$= (a + 4n)0.5 = \left(\frac{4n - n2^{n+1}}{2^n - 1} + 4n\right)\frac{1}{2} = \frac{n.2^{n+1}}{2^n - 1}$$

93. c. Here, number of factors is 50. Therefore, the coefficient of x^{49} is

$$-1 - 3 - 5 - \cdots - 99 = -\frac{50}{2}(1 + 99) = -2500$$

94. b. $T_r = (-1)^r \frac{r^2 + r + 1}{r!}$

$$= (-1)^r\left[\frac{r}{(r-1)!} + \frac{1}{(r-1)!} + \frac{1}{r!}\right]$$

$$= (-1)^r\left[\frac{1}{(r-2)!} + \frac{1}{(r-1)!} + \frac{1}{(r-1)!} + \frac{1}{r!}\right]$$

$$= \left[\frac{(-1)^r}{r!} + \frac{(-1)^r}{(r-1)!}\right] + \left[\frac{(-1)^r}{(r-1)!} + \frac{(-1)^r}{(r-2)!}\right]$$

$$= \left[\frac{(-1)^r}{r!} - \frac{(-1)^{r-1}}{(r-1)!}\right] - \left[\frac{(-1)^{r-1}}{(r-1)!} - \frac{(-1)^{r-2}}{(r-2)!}\right]$$

$$= V(r) - V(r-1)$$

$$\therefore \quad \sum_{r=1}^{n} T_r = V(n) - V(0) = \left[\frac{(-1)^n}{n!} - \frac{(-1)^{n-1}}{(n-1)!}\right] - 1$$

Therefore the sum of 20 terms is

$$\left[\frac{1}{20!} - \frac{-1}{19!}\right] - 1 = \frac{21}{20!} - 1$$

95. c. Let the 1025th term fall is in the n^{th} group. Then

$1 + 2 + 4 + \cdots + 2^{n-1} < 1025 \le 1 + 2 + 4 + \cdots + 2^n$

$\Rightarrow \quad 2^{n-1} < 1026 \le 2^{n+1}$

$\Rightarrow \quad n = 10$

$\Rightarrow$ 1025th term is 2^{10}

96. d. Let $t_n = \dfrac{1}{4(n+2)(n+3)}$. Then

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \cdots + \frac{1}{t_{2003}}$$

$$= 4\left[\frac{1}{3\times4} + \frac{1}{4\times5} + \frac{1}{5\times6} + \cdots + \frac{1}{2005\times2006}\right]$$

$$= 4\left[\frac{1}{3} - \frac{1}{2006}\right]$$

$$= 4 \times \frac{2003}{3(2006)} = \frac{4006}{3009}$$

97. d. $S = \dfrac{2}{10} + \dfrac{4}{10^3} + \dfrac{6}{10^5} + \dfrac{8}{10^7} + \cdots$ to ∞ (1)

$$= \frac{\frac{2}{10}}{1 - \frac{1}{10^2}} + \frac{2 \times \left(\frac{1}{10^2}\right)}{\left(1 - \frac{1}{10^2}\right)^2}$$

$$= \frac{20}{99} + \frac{200}{9801}$$

$$= \frac{2180}{9801}$$

98. c. The sum equals $\dfrac{n(n+1)(n+2)}{6} = 220$

which is true for $n = 10$

99. a. $S = (1)(2003) + (2)(2002) + (3)(2001) + \cdots + (2003)(1)$

$$= \sum_{r=1}^{2003} r(2003 - (r-1))$$

$$= \sum_{r=1}^{2003} r(2004 - r)$$

$$= \sum_{r=1}^{2003} 2004r - \sum_{r=1}^{2003} r^2$$

$$= \frac{2004 \times 2003 \times 2004}{2} - 2003 \times 4007 \times 334$$

$$= 2003 \times 334 \times (6012 - 4007)$$

$$= 2003 \times 334 \times 2005$$

Hence, $x = 2005$.

100. d. $2 + 3 + 6 + 11 + 18 + \cdots = (0^2 + 2) + (1^2 + 2) + (2^2 + 2) + (3^2 + 2) + \cdots$

Hence, $t_{50} = 49^2 + 2$.

101. d. We have,

$2^{n+10} = 2 \times 2^2 + 3 \times 2^3 + 4 \times 2^4 + \cdots + n \times 2^n$

$\Rightarrow \quad 2(2^{n+10}) = 2 \times 2^3 + 3 \times 2^4 + \cdots + (n-1) \times 2^n + n \times 2^{n+1}$

Subtracting, we get

$-2^{n+10} = 2 \times 2^2 + 2^3 + 2^4 + \cdots + 2^n - n \times 2^{n+1}$

$$= 8 + \frac{8(2^{n-2} - 1)}{2 - 1} - n \cdot 2^{n+1}$$

$$= 8 + 2^{n+1} - 8 - n \times 2^{n+1} = 2^{n+1} - (n)\, 2^{n+1}$$

$\Rightarrow \quad 2^{10} = 2n - 2 \Rightarrow n = 513$

102. a. We have,

$$\frac{\pi}{4} = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{9} - \frac{1}{11}\right) + \cdots$$

$$= \frac{2}{1\times3} + \frac{2}{5\times7} + \frac{2}{9\times1} + \cdots$$

$$\Rightarrow \quad \frac{1}{1\times3} + \frac{1}{5\times7} + \frac{1}{9\times11} + \cdots = \frac{\pi}{8}$$

103. b. The coefficient of x^{19} in the polynomial $(x-1)(x-2)(x-2^2)\cdots(x-2^{19})$ is

$$-(1 + 2 + 2^2 + \cdots + 2^{19}) = -1\left(\frac{2^{20} - 1}{2 - 1}\right)$$

$$= 1 - 2^{20}$$

104. b $b_2 = \dfrac{1}{1 - b_1}$

$$b_3 = \frac{1}{1 - b_2} = \frac{1}{1 - \frac{1}{1 - b_1}} = \frac{1 - b_1}{-b_1} = \frac{b_1 - 1}{b_1}$$

$b_1 = b_3 \Rightarrow b_1^2 - b_1 + 1 = 0$

$\Rightarrow \quad b_1 = -\omega$ or $\omega^2 \Rightarrow b_2 = \dfrac{1}{1+\omega} = -\omega$ or ω^2

$$\sum_{r=1}^{2001} b_r^{2001} = \sum_{r=1}^{2001} (-\omega)^{2001}$$

$$= -\sum_{r=1}^{2001} 1$$

$$= -2001$$

105. d. $(1^2 - t_1) + (2^2 - t_2) + \cdots + (n^2 - t_n) = \dfrac{1}{3} n(n^2 - 1)$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \cdots + n^2$

$$-\{t_1 + t_2 + \cdots + t_n\} = \frac{1}{3} n(n^2 - 1)$$

$$\Rightarrow \quad \frac{n(n+1)(2n+1)}{6} - S_n = \frac{1}{3} n(n^2 - 1)$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{6}[2n + 1 - 2(n-1)]$$

$$= \frac{n(n+1)}{6}[2n + 1 - 2n + 2]$$

$$= \frac{n(n+1)}{2}$$

$$\Rightarrow \quad S_{n-1} = \frac{n(n-1)}{2}$$

$\Rightarrow \quad T_n = S_n - S_{n-1} = n$

106. d. Since $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$, therefore, when $ax^3 + bx^2 + cx + d$ is divided by $ax^2 + c$ the remainder should be zero. Now when $ax^3 + bx^2 + cx + d$ is divided by $ax^2 + c$, then the remainder is $(bc/a) - d$.

$$\therefore \quad \frac{bc}{a} - d = 0$$

$$\Rightarrow \quad bc = ad$$

$$\Rightarrow \quad \frac{b}{a} = \frac{d}{c}$$

Hence, from this, a, b, c, d are not necessarily in G.P.

107. b. We know that $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. Thus, the given equation can be written as

$$\left(\frac{p+1}{2}\right)^2 + \left(\frac{q+1}{2}\right)^2 = \left(\frac{r+1}{2}\right)^2$$

$$\Rightarrow \quad (p + 1)^2 + (q + 1)^2 = (r + 1)^2$$

As $p > 6$, $p + 1 > 7$, we may take $p + 1 = 8$, $q + 1 = 6$, $r + 1 = 10$. Hence,

$$p + q + r = 21$$

108. c. $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$\Rightarrow \quad U_n = \sum_{n=1}^{n} \frac{a(r^n - 1)}{r - 1} = \frac{a}{r - 1} \sum_{n=1}^{n} (r^n - 1)$$

$$\Rightarrow \quad U_n = \frac{a}{r - 1}\{r + r^2 + \cdots + r^n - n\}$$

$$= \frac{a}{r - 1}\left\{\frac{r(r^n - 1)}{r - 1} - n\right\}$$

$$\Rightarrow \quad (r - 1)\, U_n = \frac{ar(r^n - 1)}{r - 1} - an$$

$$\Rightarrow \quad (r - 1)\, U_n = rS_n - an$$

$$\Rightarrow \quad rS_n + (1 - r)\, U_n = an$$

109. b. We have,

$$(OM_{n-1})^2 = (OP_n)^2 + (P_nM_{n-1})^2$$
$$= 2(OP_n)^2$$
$$= 2\alpha_n^2 \text{ (say)}$$

Also,

$$(OP_{n-1})^2 = (OM_{n-1})^2 + (P_{n-1}M_{n-1})^2$$

$$\Rightarrow \quad \alpha_{n-1}^{\ 2} = 2\alpha_n^2 + \frac{1}{2}\alpha_{n-1}^{\ 2}$$

$$\Rightarrow \quad \alpha_n = \frac{1}{2}\alpha_{n-1}$$

$$\Rightarrow \quad OP_n = \alpha_n = \frac{1}{2}\alpha_{n-1} = \frac{1}{2^2}\alpha_{n-1} = \cdots = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$$

110. b. $(1 - p)(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5) = 1 - p^6$

$$\Rightarrow \quad 1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 = \frac{1 - p^6}{1 - p}$$

$$\Rightarrow \quad 1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 = 1 + p + p^2 + p^3 + p^4 + p^5$$

Comparing, we get $p = 3x$ or $p/x = 3$.

111. c.

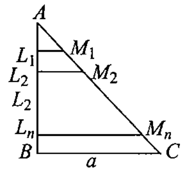


Fig. 3.6

$$\frac{AL_1}{AB} = \frac{L_1M_1}{BC}$$

$$\Rightarrow \quad \frac{1}{n+1} = \frac{L_1M_1}{a}$$

$$\Rightarrow \quad L_1M_1 = \frac{a}{n+1}$$

$$\frac{AL_2}{AB} = \frac{L_2M_2}{BC}$$

$$\Rightarrow \quad \frac{2}{n+1} = \frac{L_2M_2}{a} \Rightarrow L_2M_2 = \frac{2a}{n+1}, \quad \text{etc.}$$

Hence, the required sum is

$$\frac{a}{n+1} + \frac{2a}{n+1} + \frac{3a}{n+1} + \cdots + \frac{na}{n+1}$$

$$= \frac{a}{n+1}\,\frac{n(n+1)}{2} = \frac{an}{2}$$

112. d. $S_n - S_{n-2} = 2$

$$\Rightarrow \quad T_n + T_{n-1} = 2$$

Also,

$$T_n + T_{n-1} = \left(\frac{1}{n^2} + 1\right)T_{n-1} = 2$$

$$\Rightarrow \quad T_{n-1} = \frac{2}{1 + \dfrac{1}{n^2}} = \frac{2n^2}{1 + n^2}$$

So,

$$T_m = \frac{2(m+1)^2}{1 + (m+1)^2}$$

113. c.

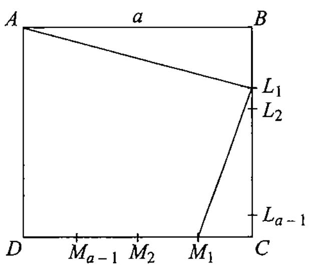


Fig. 3.7

$$(AL_1)^2 + (L_1M_1)^2 = (a^2 + 1^2) + \{(a - 1)^2 + 1^2\}$$
$$(AL_2)^2 + (L_2M_2)^2 = (a^2 + 2^2) + \{(a - 2)^2 + 2^2\}$$
$$\vdots$$
$$(AL_{a-1})^2 + (L_{a-1}M_{a-1})^2 = a^2 + (a - 1^2) + \{1^2 + (a - 1)^2\}$$

Therefore, the required sum is

$(a-1)a^2 + \{1^2 + 2^2 + \cdots + (a-1)^2\} + 2\{1^2 + 2^2 + \cdots + (a-1)^2\}$

$$= (a-1)a^2 + 3\frac{(a-1)a(2a-1)}{6}$$

$$= a(a-1)\left(a + \frac{2a-1}{2}\right)$$

$$= \frac{1}{2}(a-1)(4a-1)$$

114. b. x, y, z are in G.P. Hence,

$$y^2 = xz$$

Now, $x+3, y+3, z+3$ are in H.P. Hence,

$$y+3 = \frac{2(x+3)(z+3)}{(x+3)+(z+3)}$$

$$= \frac{2[xz+3(x+z)+9]}{[(x+z)+6]}$$

$$= \frac{2[y^2+3(x+z)+9]}{[x+z+6]}$$

Obviously, $y = 3$ satisfies it.

115. a. x, y, z are in G.P.

$\Leftrightarrow y^2 = xz$

$\Leftrightarrow x$ is a factor of y (not possible)

Taking $x = 3, y = 5, z = 7$, we have x, y, z are in A.P. Thus x, y, z may be in A.P. but not in G.P.

Multiple Correct Answers Type

1. a, c.

$$S = 1 + \frac{1}{(1+3)}(1+2)^2 + \frac{1}{(1+3+5)}(1+2+3)^2 + \frac{1}{(1+3+5+7)}(1+2+3+4)^2 + \cdots$$

The r^{th} term is given by

$$T_r = \frac{1}{r^2}(1+2+\cdots+r)^2$$

$$= \frac{1}{r^2}\left\{\frac{r(r+1)}{2}\right\}^2$$

$$= \frac{r^2+2r+1}{4}$$

$\therefore\ T_7 = 16$ and $S_{10} = \sum_{r=1}^{10} T_r$

$$= \frac{1}{4}\left\{\frac{(10)(10+1)(20+1)}{6} + (10)(10+1) + 10\right\} = \frac{505}{4}$$

2. b, c.

We have,

$$\frac{p}{1-1/p} = \frac{9}{2}$$

$\Rightarrow\ 2p^2 - 9p + 9 = 0$

$\Rightarrow\ p = 3/2, 3$

3. a, b, c, d.

$an^4 + bn^3 + cn^2 + dn + e$

$$= 2\sum_{r=1}^{n} r(r+1)(r+2) - \sum_{r=1}^{n} r(r+1)$$

$$= \frac{2}{4}n(n+1)(n+2)(n+3) - \frac{1}{3}n(n+1)(n+2)$$

$$= \frac{1}{6}(3n^4 + 16n^3 + 27n^2 + 14n)$$

4. a, b, c.

Given that $a = 4$, $T_3 - T_5 = 32/81$. Hence,

$a(r^2 - r^4) = 32/81$

or

$r^4 - r^2 + 8/81 = 0$

or

$81r^4 - 81r^2 + 8 = 0$

or

$(9r^2 - 8)(9r^2 - 1) = 0$

$\therefore\ r^2 = 8/9, 1/9$

Therefore, the value of r is to be +ve since all the terms are +ve.

For $r = 1/3$,

$$S_\infty = \frac{a}{1-r} = \frac{4}{1-\frac{1}{3}} = \frac{4\times 3}{2} = 6$$

Similarly, we can find S_∞ when $r = 2\sqrt{2}/3$.

5. a, b, c.

Let the three-digit number be xyz. According to given condition, we have

$y^2 = xz$ (1)

$2(y+2) = x + z$ (2)

$100x + 10y + z - 792 = 100z + 10y + x$

$\Rightarrow\ x - z = 8$ (3)

Squaring (2) and (3), and subtracting, we have

$4xz = 4(y+2)^2 - 64$ (4)

$\Rightarrow\ y^2 = (y+2)^2 - 16$ [Using (1)]

$\Rightarrow\ y = 3$

$\Rightarrow\ x + z = 10$ [Using (2)]

$\Rightarrow\ x = 9, z = 1$

Hence, the number is $931 = 7^2 \times 19$.

6. a, b, c, d.

Clearly, n^{th} term of the given series is negative or positive accordingly as n is even or odd, respectively.

Case I: When n is even: In this case, the given series is

$S_n = 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (n-1)^2 - n^2$

$= (1^2 - 2^2) + (3^2 - 4^2) + \cdots + ((n-1)^2 - n^2)$

$= (1-2)(1+2) + (3-4)(3+4) + \cdots + ((n-1)-(n))(n-1+n)$

$= -(1 + 2 + 3 + 4 + \cdots + (n-1) + n)$

$$= -\frac{n(n+1)}{2} \quad (1)$$

Case II: When n is odd: In this case, the given series is

$S_n = (1^2 - 2^2) + (3^2 - 4^2) + \cdots + \{(n-2)^2 - (n-1)^2\} + n^2$

$= (1-2)(1+2) + (3-4)(3+4) + \cdots + ((n-2) - (n-1)) \times ((n-2) + (n-1)) + n^2$

$= -(1 + 2 + 3 + 4 + \cdots + (n-2) + (n-1)) + n^2$

$$= -\frac{(n-1)(n-1+1)}{2} + n^2 = \frac{n(n+1)}{2} \quad (2)$$

$\Rightarrow\ S_{40} = -820$ [Using (1)]

$S_{51} = 1326$ [Using (2)]

Also,

$S_{2n} > S_{2n+2}$ [From (1)]

$S_{2n+1} > S_{2n-1}$ [From (2)]

7. a, b, d.

$$x + y + z = 3\left(\frac{a+b}{2}\right)$$

$$\Rightarrow \quad 15 = 3\frac{(a+b)}{2}$$

$$\Rightarrow \quad a + b = 10 \qquad (1)$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{3\left(\frac{1}{a}+\frac{1}{b}\right)}{2}$$

$$\Rightarrow \quad \frac{5}{3} = \frac{3(a+b)}{2ab} = \frac{3\times 10}{2ab}$$

$$\Rightarrow \quad ab = 9 \qquad (2)$$

From (1) and (2), $a = 9$, $b = 1$ or $a = 1$ and $b = 9$. Hence, G.M. $= \sqrt{ab} = 3$, $a + 2b = 11$ or 19.

8. b, d.

Given,

$$3a_1 + 7a_2 + 3a_3 - 4a_5 = 0$$

$$\Rightarrow \quad 7(a_1 + a_2 + a_3) = 4(a_1 + a_3 + a_5)$$

$$\Rightarrow \quad 7(1 + r + r^2) = 4(1 + r^2 + r^4)$$

$$\Rightarrow \quad 7 = 4(r^2 - r + 1)$$

$$\Rightarrow \quad 4r^2 - 4r + 1 = 4$$

$$\Rightarrow \quad (2r - 1)^2 = 4$$

$$\Rightarrow \quad 2r - 1 = \pm 2$$

$$\Rightarrow \quad r = 3/2, -1/2$$

9. a, b, c.

$$\frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \cdots n \text{ terms}$$

$$= \frac{\sqrt{5}-\sqrt{2}}{3} + \frac{\sqrt{8}-\sqrt{5}}{3} + \cdots + \frac{\sqrt{5+(n-1)3} - \sqrt{2+(n-1)3}}{3}$$

$$= \frac{\sqrt{3n+2}-\sqrt{2}}{3}$$

$$= \frac{3n+2-2}{3(\sqrt{3n+2}+\sqrt{2})}$$

$$= \frac{n}{\sqrt{3n+2}+\sqrt{2}}$$

$$= \frac{n}{\sqrt{2+3n}+\sqrt{2}} < \frac{n}{\sqrt{3n}} < n$$

10. a, b.

$$\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right) = \left(\frac{1}{b}+\frac{1}{c}-\frac{2}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{b}-\frac{1}{c}\right)$$

$$= \left(\frac{2}{c}-\frac{1}{b}\right)\frac{1}{b} = \frac{2}{bc}-\frac{1}{b^2}$$

Also by eliminating b, we get the given expression $\dfrac{(a+c)(3a-c)}{4a^2c^2}$.

11. a, d.

$$p(x) = \left(\frac{1-x^{2n}}{1-x^2}\right)\left(\frac{1-x}{1-x^n}\right) = \frac{1+x^n}{1+x}$$

As $p(x)$ is a polynomial, $x = -1$ must be a zero of $1 + x^n$. Hence, $1 + (-1)^n = 0$. So, n must be odd.

12. a, c, d.

$$a_1 + a_3 + a_5 = -12$$

$$a + a + 2d + a + 4d = -12 \quad (d > 0)$$

$$a + 2d = -4 \qquad (1)$$

$$a_1a_3a_5 = 80$$

$$a(a + 2d)(a + 4d) = 80$$

or

$$(-4 - 2d)(-4 + 2d) = -20 \Rightarrow d = \pm 3$$

Since A.P. is increasing, so $d = +3$; $a = -10$. Hence,

$$\left.\begin{aligned} &a_1 = -10;\ a_2 = -7\\ &a_3 = a + 2d = -10 + 6 = -4\\ &a_5 = a + 4d = -10 + 12 = 2\end{aligned}\right\}$$

13. a, b, c.

$$a = \frac{n^{64}-1}{n-1}$$

$$= (n + 1)(n^2 + 1)(n^4 + 1)(n^8 + 1)(n^{16} + 1)(n^{32} + 1)$$

14. a, b, c.

If p, q, r are in A.P., then p^{th}, q^{th} and r^{th} terms are equal distant terms which are always in the same series of which they are terms.

15. a, d.

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots\infty$$

$$\Rightarrow \quad xS = x + 2x^2 + 3x^3 + 4x^4 + \cdots\infty$$

$$\Rightarrow \quad (1 - x)S = 1 + x + x^2 + \cdots\infty = \frac{1}{1-x}$$

$$\Rightarrow \quad S = \frac{1}{(1-x)^2}$$

Now,

$$S \geq 4 \Rightarrow \frac{1}{(1-x)^2} > 4$$

$$\Rightarrow \quad (x-1)^2 \leq \frac{1}{4}$$

$$\Rightarrow \quad -\frac{1}{2} \leq x-1 \leq \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2} \leq x \leq \frac{3}{2}. \text{ Also } 0 < |x| < 1$$

$$\Rightarrow \quad \frac{1}{2} \leq x < 1$$

16. a, b, d.

$$E < 1 + \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \cdots$$

$$= 1 + \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots = 2$$

$$E > 1 + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \cdots$$

$$= 1 + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots = \frac{3}{2}$$

17. a, c, d.

$$S_n = \frac{n}{2}\left[2a' + (n-1)d\right] = a + bn + cn^2$$

$$\Rightarrow \quad na' + \frac{n(n-1)}{2}d = a + bn + cn^2$$

$$\Rightarrow \quad \left(a' - \frac{d}{2}\right)n + \frac{n^2 d}{2} = a + bn + cn^2$$

On comparing,

$$a = 0,\ b = a' - \frac{d}{2},\ c = \frac{d}{2} \quad \Rightarrow d = 2c$$

18. a, b.

$$x^2 + 9y^2 + 25z^2 = 15yz + 5zx + 3xy$$

$$\Rightarrow \quad (x)^2 + (3y)^2 + (5z)^2 - (x)(3y) - (3y)(5z) - (x)(5z) = 0$$

$$\Rightarrow \quad \frac{1}{2}\left[(x-3y)^2 + (3y-5z)^2 + (x-5z)^2\right] = 0$$

$$\Rightarrow \quad x - 3y = 0,\ 3y - 5z = 0,\ x - 5z = 0$$

$$x = 34 = 5z$$

$$\Rightarrow \quad x : y : z = \frac{1}{1} : \frac{1}{3} : \frac{1}{5}$$

Therefore, $1/x$, $1/y$, and $1/z$ are in A.P. and x, y, and z are in H.P.

19. a, c.

Let $b = a + p$, $c = a + 2p$, $d = a + 3p$ (where p is common difference). Then,

$$\frac{\frac{1}{a} + \frac{1}{d}}{\frac{1}{b} + \frac{1}{c}} = \frac{\frac{1}{a} + \frac{1}{a+3p}}{\frac{1}{a+p} + \frac{1}{a+2p}}$$

$$= \frac{(a+p)(a+2p)}{a(a+3p)}$$

$$= \frac{a^2 + 3ap + 2p^2}{a^2 + 3ap} > 1$$

$$\therefore \quad \frac{1}{a} + \frac{1}{d} > \frac{1}{b} + \frac{1}{c}$$

$$\left(\frac{1}{b} + \frac{1}{c}\right)(a+d) = \left(\frac{1}{a+p} + \frac{1}{a+2p}\right)(a + a + 3p)$$

$$= \frac{(2a+3p)^2}{a^2 + 3ap + 2p^2}$$

$$= 4 + \frac{p^2}{a^2 + 3ap + 2p^2} > 4$$

20. a, d.

x, $x^2 + 2$, $x^3 + 10$ are in G.P. Hence,

$x(x^3 + 10) = (x^2 + 2)^2 = x^4 + 4x^2 + 4$

$\Rightarrow \quad 4x^2 - 10x + 4 = 0$

$\Rightarrow \quad 2x^2 - 5x + 2 = 0$

$\Rightarrow \quad x = 2, \frac{1}{2}$

The 4th term of G.P. is

$$(x^3 + 10)r = (x^3 + 10)\left(\frac{x^2 + 2}{x}\right)$$

$$= \begin{cases} 54 \text{ when } x = 2 \\ \dfrac{729}{16} \text{ when } x = \dfrac{1}{2} \end{cases}$$

21. a, b, c.

Last term in n^{th} row is

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1) \qquad (1)$$

As terms in the n^{th} row forms an A.P. with common difference 1, so

First term = Last term – $(n-1)$ (1)

$$= \frac{1}{2}n(n+1) - n + 1$$

$$= \frac{1}{2}(n^2 - n + 2) \qquad (2)$$

$$\text{Sum of terms} = \frac{1}{2}n\left[\frac{1}{2}(n^2 - n + 2) + \frac{1}{2}(n^2 + n)\right]$$

$$= \frac{1}{2}n(n^2 + 1) \qquad (3)$$

Now, put $n = 20$ in (1), (2), (3) to get required answers.

22. a, b, c.

Since A_1, A_2 are two arithmetic means between a and b, therefore, a, A_1, A_2, b are in A.P. with common difference d given by

$$d = \frac{b-a}{2+1} = \frac{b-a}{3} \quad \left[\text{using } d = \frac{b-a}{n+1}\right]$$

Now,

$$A_1 = a + d = a + \frac{b-a}{3} = \frac{2a+b}{3}$$

and

$$A_2 = a + 2d = a + 2\left(\frac{b-a}{3}\right) = \frac{a+2b}{3}$$

It is given that $G_1 G_2$ are two geometric means between a and b. Therefore, a, G_1, G_2, b are in G.P. with common ratio r given by

$$r = \left(\frac{b}{a}\right)^{\frac{1}{2+1}} = \left(\frac{b}{a}\right)^{1/3} \quad \left[\because r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]$$

Now,

$$G_1 = ar = a\left(\frac{b}{a}\right)^{1/3} = a^{2/3}\, b^{1/3}$$

and

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/3} = a^{1/3}\, b^{2/3}$$

It is also given that H_1, H_2 are two harmonic means between a and b, therefore, a, H_1, H_2, b are in H.P. Hence, $1/a$, $1/H_1$, $1/H_2$, $1/b$, are in A.P. with common difference D given by

$$D = \frac{a-b}{(2+1)ab} = \frac{a-b}{3ab} \quad \left[\because D = \frac{a-b}{(n+1)ab}\right]$$

Now,

$$\frac{1}{H_1} = \frac{1}{a} + D = \frac{1}{a} + \frac{a-b}{3ab} = \frac{a+2b}{3ab}$$

$$\Rightarrow \quad H_1 = \frac{3ab}{a+2b}$$

$$\frac{1}{H_2} = \frac{1}{a} + 2D$$

$$= \frac{1}{a} + \frac{2(a-b)}{3ab}$$

$$= \frac{2a+b}{3ab}$$

$$\Rightarrow \quad H_2 = \frac{3ab}{2a+b}$$

We have,

$$A_1H_2 = \frac{2a+b}{3} \times \frac{3ab}{2a+b} = ab,$$

$$A_2H_1 = \frac{a+2b}{3} \times \frac{3ab}{a+2b} = ab,$$

$$G_1G_2 = (a^{2/3} b^{1/3})(a^{1/3} b^{2/3}) = ab$$

$$\therefore \quad A_1H_2 = A_2H_1 = G_1G_2 = ab$$

23. b, c, d.

We have, length of a side of S_n is equal to the length of a diagonal of S_{n+1}. Hence,

Length of a side of S_n = $\sqrt{2}$ (Length of a side of S_{n+1})

$$\Rightarrow \quad \frac{\text{Length of a side of } S_{n+1}}{\text{Length of side of } S_n} = \frac{1}{\sqrt{2}}, \text{ for all } n \geq 1$$

Hence, sides of $S_1, S_2, \ldots, S_n$ form a G.P. with common ratio $1/\sqrt{2}$ and first term 10.

$$\therefore \quad \text{Side of } S_n = 10\left(\frac{1}{\sqrt{2}}\right)^{n-1} = \frac{10}{2^{\frac{n-1}{2}}}$$

$$\Rightarrow \quad \text{Area of } S_n = (\text{side})^2 = \left(\frac{10}{2^{\frac{n-1}{2}}}\right)^2 = \frac{100}{2^{n-1}}$$

Now, area of $S_n < 1 \Rightarrow n = b, c, d$.

24. a, c.

$$\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$$

$$\Rightarrow \quad \frac{1}{b-a} - \frac{1}{c} = \frac{1}{a} - \frac{1}{b-c}$$

$$\Rightarrow \quad \frac{c-b+a}{c(b-a)} = \frac{b-c-a}{a(b-c)}$$

$$\Rightarrow \quad c-b+a = 0 \text{ or } \frac{1}{c(b-a)} = \frac{1}{a(c-b)}$$

$$\Rightarrow \quad b = a+c \quad \text{or} \quad bc - ac = ac - ab$$

$$\Rightarrow \quad b = a+c \quad \text{or} \quad b = \frac{2ac}{a+c}$$

25. a, c.

a, b, c are in G.P. Hence,

$b^2 = ac$ (1)

x is A.M. of a and b. Hence,

$2x = a + b$ (2)

y is A.M. of b and c. Hence,

$2y = b + c$ (3)

$$\therefore \quad \frac{a}{x} + \frac{c}{y} = a \times \frac{2}{a+b} + c \times \frac{2}{b+c} \quad \text{[Using (2) and (3)]}$$

$$= 2\left[\frac{ab+ac+ac+bc}{ab+ac+b^2+bc}\right]$$

$$= 2 \quad \text{[Using (i)]}$$

Again,

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{a+b} + \frac{2}{b+c}$$

$$= \frac{2(a+c+2b)}{ab+ac+b^2+bc}$$

$$= \frac{2(a+c+2b)}{ab+2b^2+bc} \quad (\because b^2 = ac)$$

$$= \frac{2(a+c+2b)}{b(a+c+2b)}$$

$$= \frac{2}{b}$$

26. a, c. Given $a_1 = 2$; $\dfrac{a_n}{a_{n-1}} = \dfrac{a_{n-1}}{a_{n-2}}$

$\Rightarrow \quad a_1, a_2, a_3, a_4, a_5, \ldots$ in G.P.

Let $a_2 = x$ then for $n = 3$ we have

$$\frac{a_3}{a_2} = \frac{a_2}{a_1} = \frac{x^2}{2}$$

$$\Rightarrow \quad a_1^2 = a_1a_3$$

$$\Rightarrow \quad a_3 = \frac{x^2}{2}$$

i.e. $2, x, \dfrac{x^2}{2}, \dfrac{x^3}{4}, \dfrac{x^4}{8}, \ldots$ with common ratio $r = \dfrac{x}{2}$

given $\dfrac{x^4}{8} \leq 162$

$\Rightarrow \quad x^4 \leq 1296 \leq \quad x \leq 6$

Also x $\dfrac{x^4}{8}$ and are integers

$\Rightarrow x$ must be even then only $\dfrac{x^4}{8}$ will be an integer.

hence possible values of x is 4 and 6. ($x \neq 2$ as terms are distinct)

hence possible value of $a_5 = \dfrac{x^4}{8}$ is $\dfrac{4^4}{8}, \dfrac{6^4}{8}$

27. a, b, c. Let a, b, c are pth, qth and rth terms of A.P.

then $a = A + (p-1)D$, $b = A + (q-1)D$, $c = A + (r-1)D$

$$\Rightarrow \quad \frac{r-q}{q-p} = \frac{c-b}{b-a} \text{ is rational number.}$$

Now for 1, 6, 19 $\dfrac{r-q}{q-p} = \dfrac{19-6}{6-1}$ is rational number.

For $\sqrt{2}, \sqrt{50}, \sqrt{98}$, $\dfrac{r-q}{q-p} = \dfrac{\sqrt{98}-\sqrt{50}}{\sqrt{50}-\sqrt{2}} = \dfrac{7\sqrt{2}-5\sqrt{2}}{5\sqrt{2}-\sqrt{2}}$

$= \dfrac{1}{2}$ is rational number.

For log2, log16, log128,

$$\frac{r-q}{q-p} = \frac{\log 128 - \log 16}{\log 16 - \log 2} = \frac{7\log 2 - 4\log 2}{4\log 2 - \log 2} = 1 \text{ is rational number.}$$

But for $\sqrt{2}, \sqrt{3}, \sqrt{7}$, $\frac{r-q}{q-p}$ is not rational number.

28. c, d. $4 = 1 + (n-1)d$, $16 = 1 + (m-1)d \Rightarrow \frac{15}{3} = \frac{m-1}{n-1}$ or

$\frac{n-1}{1} = \frac{m-1}{5} = p =$ positive integer.

$\therefore$ $n = p + 1$, $m = 5p + 1$. So, n m have infinite pairs of values.
Also, $4 = 1 . r^n$, $16 = 1 . r^m \Rightarrow rm^{-n} = 4 = r^n$. So, $m - n = n$

$\therefore$ $\frac{m}{2} = \frac{n}{1} = q$ = positive integer. So, m, m have infinite pairs of values.

Reasoning Type

1. a. Let p, q, r be the l^{th}, m^{th} and n^{th} terms of an A.P. Then

$p = (a + (l-1)d$, $q = a + (m-1)d$ and $r = a + (n-1)d$

Hence, $r - q = (n - m)d$ and $q - p = (m - l)d$, so that

$$\frac{r-q}{p-q} = \frac{(n-m)d}{(m-l)d} = \frac{n-m}{m-l} \quad (\because d \neq 0) \quad (1)$$

Since l, m, n are positive integers and $m \neq l$, $(n-m)/(m-l)$ is a rational number. From (1), using $p = \sqrt{2}, q = \sqrt{3}, r = \sqrt{5}$, we have

$$\frac{\sqrt{5}-\sqrt{3}}{\sqrt{3}-\sqrt{2}} = \frac{n-m}{m-l} \text{ (which is not possible.)}$$

Hence, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ cannot be the terms of an A.P.

2. a. Statement 2 is true as it is a property of sequence in G.P.

Now T_{m-n}, T_m and T_{m+n} are in G.P. ($\because$ T_m from T_{m-n} and T_{m+n} from T_m are at same distance)

$\therefore$ $T_m^2 = T_{m-n} T_{m+n}$

$\Rightarrow$ $T_m = \sqrt{pq}$

3. b. Let, if possible, 8 be the first term and 12 and 27 be m^{th} and n^{th} terms, respectively. Then,

$12 = ar^{m-1} = 8r^{m-1}$, $27 = 8r^{n-1}$

$\Rightarrow$ $\frac{3}{2} = r^{m-1}$, $\left(\frac{3}{2}\right)^3 = r^{n-1} = r^{3(m-1)}$

$\Rightarrow$ $n - 1 = 3m - 3$ or $3m = n + 2$

$\Rightarrow$ $\frac{m}{1} = \frac{n+2}{3} = k$ (say)

$\therefore$ $m = k$, $n = 3k - 2$

By giving k different values, we get the integral values of m and n.

Hence there can be infinite number of G.P.'s whose any three terms will be 8, 12, 27 (not consecutive). Obviously, statement 2 is not a correct explanation of statement 1.

4. a. $x^2 + 9y^2 + 25z^2 = xyz\left(\frac{15}{x} + \frac{5}{y} + \frac{3}{z}\right)$

$\Rightarrow$ $x^2 + 9y^2 + 25z^2 - 15yz - 5xz - 3xy = 0$

$\Rightarrow$ $2x^2 + 18y^2 + 50z^2 - 30yz - 10xz - 6xy = 0$

$\Rightarrow$ $(x - 3y)^2 + (3y - 5z)^2 + (5z - x)^2 = 0$

$\Rightarrow$ $x - 3y = 0$, $3y - 5z = 0$, $5z - x = 0$

$\Rightarrow$ $x = 3y = 5z = k$ (say)

$\Rightarrow$ $x = k$, $y = k/3$, $z = k/5$

Hence, x, y, z are in H.P. Hence option (a) is correct.

5. a. Coefficient of x^{14} in $(1 + 2x + 3x^2 + \cdots + 16x^{15})^2$

= Coefficient of x^{14} in $(1 + 2x + 3x^2 + \cdots + 16x^{15})(1 + 2x + 3x^2 + \cdots + 16x^{15})$

$= 1 \times 15 + 2 \times 14 + \cdots + 15 \times 1$

$= \sum_{r=1}^{15} r(16 - r)$

Also,

$$\sum_{r=1}^{n-1} r(n-r) = \sum_{r=1}^{n-1} nr - \sum_{r=1}^{n-1} r^2$$

$$= n\frac{n(n-1)}{2} - \frac{n(n-1)(2n-1)}{6}$$

$$= \frac{n(n-1)}{6}(3n - (2n-1))$$

$$= \frac{n(n^2-1)}{6}$$

$\Rightarrow$ $\sum_{r=1}^{15} r(16-r) = \frac{15(15^2-1)}{6} = 560$

Hence option (a) is correct.

6. b. $x = 1111\cdots91$ times

$= 1 + 10 + 10^2 + 10^3 + \cdots + 10^{90}$

$= \frac{1(10^{91}-1)}{10-1}$

$= \frac{(10^{13\times 7}-1)}{10-1}$

$= \frac{((10^{13})^7-1)}{10^{13}-1} \times \frac{(10^{13}-1)}{10-1}$

$= (1 + 10^{13} + 10^{26} + \cdots + 10^{78}) \times (1 + 10 + 10^2 + \cdots + 10^{12})$

= composite numbers

But statement 2 is not a correct explanation of statement 1 as 111 has 1 digit 3 times, and 3 is a prime number but $111 = 3 \times 37$ is a composite number. Hence (b) is the correct option.

7. a. We have,

$$a \times ar \times \cdots \times ar^{n-1} = a^n \times r^{1+2+\cdots+(n-1)} = a^n r^{\frac{n(n-1)}{2}} = (a^2 r^{n-1})^{n/2}$$

Hence, statement 1 is true.

Also, $(a \times r^{k-1})(a \times r^{n-k}) = a^2 \times r^{n-1}$, which is independent of k. Hence, statement 2 is a correct explanation for statement 1, as in the product of a, ar, ar^2, ... ar^{n-1}, there are $n/2$ groups of numbers, whose product is a^2r^{n-1}. Hence (a) is the correct option.

8. d. For odd integer n, we have

$S_n = n^3 - (n-1)^3 + \cdots + (-1)^{n-1} 1^3$

$= 1^3 - 2^3 + 3^3 - 4^3 + \cdots + n^3$

$= [1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3] - 2[2^3 + 4^3 + 6^3 + \cdots + (n-1)^3]$

$$= \frac{n^2(n+1)^2}{4} - 2 \times 2^3\left[1^3 + 2^3 + \cdots + \left(\frac{n-1}{2}\right)^3\right]$$

$$= \frac{n^2(n+1)^2}{4} - 2^4 \frac{\left(\frac{n-1}{2}\right)^2\left(\frac{n-1}{2}+1\right)^2}{4}$$

$$= \frac{n^2(n+1)^2}{4} - \frac{(n-1)^2(n+1)^2}{4}$$

$$= \frac{(n+1)^2}{4}\left[n^2 - (n-1)^2\right]$$

$$= \frac{1}{4}(2n-1)(n+1)^2$$

Now, putting $n = 11$ in above formula, $S_{11} = 756$. Hence statement 1 is false and statement 2 is correct.

9. d. Sum $= \frac{x/r}{1-r} = 4$ (where r is common ratio)

$$x = 4r(1-r) = 4(r-r^2)$$

For $r \in (-1, 1) - \{0\}$

$$r - r^2 \in \left(-2, \frac{1}{4}\right) - \{0\}$$

$\Rightarrow$ $x \in (-8, 1) - \{0\}$

10. b. The given inequality is

$(p_1^2 + p_2^2 + \cdots + p_{n-1}^2)x^2 + 2(p_1p_2 + p_2p_3 + \cdots + p_{n-1}p_n)x + (p_2^2 + \cdots + p_n^2) \le 0$

$\Rightarrow$ $(p_1x + p_2)^2 + (p_2x + p_3)^2 + \cdots + (p_{n-1}x + p_n)^2 \le 0$ (1)

But each one of the terms on the L.H.S. is a perfect square and hence is positive or zero.

Therefore (1) holds only if

$$p_1x + p_2 = 0 = p_2x + p_3 = p_3x + p_4 = \cdots = p_{n-1}x + p_n$$

$$\Rightarrow \quad -x = \frac{p_2}{p_1} = \frac{p_3}{p_2} = \cdots = \frac{p_n}{p_{n-1}}$$

Hence, $p_1, p_2, \ldots, p_n$ are in G.P.

11. a. Statement 2 is true as

$$\frac{a^n + b^n}{a+b} = \frac{a^n - (-b)^n}{a - (-b)} = a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots (-1)^{n-1}b^{n-1}$$

Now,

$1^{99} + 2^{99} + \cdots + 100^{99} = (1^{99} + 100^{99}) + (2^{99} + 99^{99}) + \cdots + (50^{99} + 51^{99})$

Each bracket is divisible by 101; hence the sum is divisible by 101. Also,

$1^{99} + 2^{99} + L \cdots 100^{99} = (1^{99} + 99^{99}) + (2^{99} + 98^{99}) + \cdots + (49^{99} + 51^{99}) + 50^{99} + 100^{99}$

Here, each bracket and 50^{99} and 100^{99} are divisible by 100. Hence sum is divisible by 100. Hence sum is divisible by $101 \times 100 = 10100$.

12. a. For two positive numbers $(\mathrm{GM})^2 = (\mathrm{AM}) \times (\mathrm{HM})$.

Linked Comprehension Type

For Problems 1–3

1. c, 2. b, 3. d.

Sol. Let the odd integers be $2m + 1, 2m + 3, 2m + 5, \ldots$ and let their number be n. Then,

$$57^2 - 13^2 = (n/2)[2(2m+1) + (n-1) \times 2]$$
$$= n(2m+n)$$
$$= 2mn + n^2$$

$\Rightarrow$ $57^2 - 13^2 = (n+m)^2 - m^2$

$\Rightarrow$ $m = 13$ and $n + m = 57$

$\Rightarrow$ $n = 57 - 13 = 44$

Hence, the required odd integers are 27, 29, 31, ..., 113.

For Problems 4–6

4. c, 5. c, 6. d.

Sol. 4. a, b, c are in G.P. Hence, a, ar, ar^2 are in G.P. So,

$$\frac{a^2 + a^2r^2 + a^2r^4}{(a + ar + ar^2)^2} = \frac{t^2}{\alpha^2 t^2} = \frac{1}{\alpha^2}$$

$$\alpha^2 = \frac{r^2 + r + 1}{r^2 - r + 1}$$

Let $\alpha^2 = y$.

$$\therefore \quad y = \frac{r^2 + r + 1}{r^2 - r + 1}$$

$$(y-1)r^2 - r(y+1) + (y-1) = 0$$

For real r,

$$(y+1)^2 - 4(y-1)^2 \ge 0$$

$\Rightarrow$ $\frac{1}{3} \le y \le 3$

But $y \ne 1/3, 1, 3$ $(\because r \ne 1, -1, 0)$

$\therefore$ $\frac{1}{3} < y < 3$ and $y \ne 1$

$$\alpha^2 \in \left(\frac{1}{3}, 3\right) - \{1\}$$

5. $S = r + \frac{1}{r}$

$S \in (-\infty, -2) \cup (2, \infty)$

6. Let $b = ar$, $c = ar^2$ and $r > 0$.

As sum of two sides is more than the third side, we have,

$$r \in \left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right) - \{1\}$$

$\Rightarrow$ $r + \frac{1}{r} - 1 \in (1, \sqrt{5} - 1)$

As $\alpha^2 = \frac{r^2 + r + 1}{r^2 - r + 1} = 1 + \frac{2}{r + \frac{1}{r} - 1}$

$$\therefore \quad \alpha^2 \in \left(\frac{\sqrt{5}+3}{2}, 3\right)$$

For Problems 7–9

7. d, 8. b, 9. d.

Sol. Let a be the first term and r the common ratio of the given G.P. Further, let there be n terms in the given G.P. Then,

$a_1 + a_n = 66 \Rightarrow a + ar^{n-1} = 66$ (i)

$a_2 \times a_{n-1} = 128$

$\Rightarrow$ $ar \times ar^{n-2} = 128$

$\Rightarrow$ $a^2 r^{n-1} = 128$

$\Rightarrow$ $a \times (ar^{n-1}) = 128 \Rightarrow ar^{n-1} = \frac{128}{a}$

Putting this value of ar^{n-1} in (i), we get

$$a + \frac{128}{a} = 66$$

$\Rightarrow$ $a^2 - 66a + 128 = 0$

$\Rightarrow$ $(a-2)(a-64) = 0$

$\Rightarrow$ $a = 2, 64$

Putting $a = 2$ in (1), we get

$$2 + 2\times r^{n-1} = 66 \Rightarrow r^{n-1} = 32$$

Putting $a = 64$ in (1), we get

$$64 + 64r^{n-1} = 66 \Rightarrow r^{n-1} = \frac{1}{32}$$

For an increasing G.P., $r > 1$. Now,

$$S_n = 126$$

$$\Rightarrow \quad 2\left(\frac{r^n - 1}{r-1}\right) = 126$$

$$\Rightarrow \quad \frac{r^n - 1}{r-1} = 63$$

$$\Rightarrow \quad \frac{r^{n-1}\times r - 1}{r-1} = 63$$

$$\Rightarrow \quad \frac{32r - 1}{r-1} = 63$$

$$\Rightarrow \quad r = 2$$

$$\therefore \quad r^{n-1} = 32 \Rightarrow 2^{n-1} = 32 = 2^5 \Rightarrow n-1 = 5 \Rightarrow n = 6$$

For decreasing G.P., $a = 64$ and $r = 1/2$. Hence, the sum of infinite terms is $64/\{1-(1/2)\} = 128$.

For $a = 2$, $r = 2$, terms are 2, 4, 8, 16, 32, 64. For $a = 64$, $r = 1/2$ terms are 64, 32, 16, 8, 4, 2 Hence difference is 62.

For Problems 10–12

10. c, 11. d, 12. a.

Sol. Let the four integers be $a - d$, a, $a + d$ and $a + 2d$, where a and d are integers and $d > 0$. Now,

$$a + 2d = (a-d)^2 + a^2 + (a+d)^2$$

$$\Rightarrow \quad 2d^2 - 2d + 3a^2 - a = 0 \qquad (1)$$

$$\therefore \quad d = \frac{1}{2}\left[1 \pm \sqrt{1 + 2a - 6a^2}\right] \qquad (2)$$

Since d is a positive integer, so

$$1 + 2a - 6a^2 > 0$$

$$\Rightarrow \quad 6a^2 - 2a - 1 < 0$$

$$\Rightarrow \quad \frac{1-\sqrt{7}}{6} < a < \frac{1+\sqrt{7}}{6}$$

$$\Rightarrow \quad a = 0 \qquad (\because a \text{ is an integer})$$

Hence from (2),

$$d = 1 \text{ or } 0$$

But since $d > 0$,

$$\therefore \quad d = 1$$

Hence, the four numbers are $-1, 0, 1, 2$.

For Problems 13–15

13. d, 14. a, 15. b.

Sol. 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ...

Let us write the terms in the groups as follows: 1, (2, 2), (3, 3, 3), (4, 4, 4, 4), ... consisting of 1, 2, 3, 4, ... terms. Let 2000th term fall in n^{th} group. Then,

$$\frac{(n-1)n}{2} < 2000 \le \frac{n(n+1)}{2}$$

$$\Rightarrow \quad n(n-1) < 4000 \le n(n+1)$$

Let us consider,

$$n(n-1) < 4000$$

$$\Rightarrow \quad n^2 - n - 4000 < 0$$

$$\Rightarrow \quad n < \frac{1+\sqrt{16001}}{2} \Rightarrow n < 64$$

We have,

$$n(n+1) \ge 4000 \Rightarrow n^2 + n - 4000 \ge 0 \Rightarrow n \ge 63$$

That means 2000th term falls in 63rd group. That also means that the 2000th term is 63. Now, total number of terms up to 62nd group is $(62 \times 63)/2 = 1953$. Hence, sum of first 2000 terms is

$$1^2 + 2^2 + \cdots + 62^2 + 63(2000-1953) = \frac{62(63)125}{6} + 63\times 47 = 84336$$

Sum of the remaining terms is $63 \times 16 = 1008$.

For Problems 16–18

16. b, 17. a, 18. c.

Sol. Let numbers in set A be $a-D, a, a+D$ and these in set B be $b-d, b, b+d$. Now,

$$3a = 3b = 15$$

$$\Rightarrow \quad a = b = 5$$

$$\text{Set } A = \{5-D, 5, 5+D\}$$

$$\text{Set } B = \{5-d, 5, 5+d\}$$

where $D = d + 1$

Also,

$$\frac{p}{q} = \frac{5(25-D^2)}{5(25-d^2)} = \frac{7}{8}$$

$$\Rightarrow \quad 25(8-7) = 8(d+1)^2 - 7d^2$$

$$\Rightarrow \quad d = -17, 1 \text{ but } d > 0 \Rightarrow d = 1$$

So, the numbers in set A are 3, 5, 7 and the numbers in set B are 4, 5, 6.

Now, sum of product of numbers in set A taken two at a time is $3\times5 + 3\times7 + 5\times7 = 71$. The sum of product of numbers in set B taken two at a time is $4\times5 + 5\times6 + 6\times4 = 74$. Also,

$$p = 3\times5\times7 = 105 \text{ and } q = 4\times5\times6 = 120$$

$$\Rightarrow \quad q - p = 15$$

For Problems 19–21

19. c, 20. b, 21. a.

Sol. 19. $G_1 G_2 \cdots G_n = (\sqrt{1\times1024})^n = 2^{5n}$

Given,

$$2^{5n} = 2^{45} \Rightarrow n = 9$$

Hence,

$$r = (1024)^{\frac{1}{9+1}} = 2$$

$$\Rightarrow \quad G_1 = 2, r = 2$$

$$\Rightarrow \quad G_1 + G_2 + \cdots + G_9 = \frac{2\times(2^9-1)}{2-1} = 1024 - 2 = 1022$$

20. b. $A_1 + A_2 + A_3 + \cdots + A_{m-1} + A_m = 1025 \times 771$

$$\Rightarrow m\left(\frac{-2+1027}{2}\right) = 1025 \times 171$$

$$\Rightarrow m = 342$$

21. a. We have,

$$A_{171} + A_{172} = -2 + 1027 = 1025$$

$$\Rightarrow \quad \frac{2A_{171} + 2A_{172}}{2} = 1025$$

Also,

$$G_5 = 1\times 2^5 = 32$$

$\Rightarrow \quad G_5^2 = 1024$

$\Rightarrow \quad G_5^2 + 1 = 1025$

$\Rightarrow \quad 2A_{171}, G_5^2 + 1, 2A_{172}$ are in A.P.

For Problems 22–24

22. a, 23. c, 24. b.

Sol. Let m and $(m + 1)$ be the removed numbers from 1, 2, ..., n.

Then, sum of the remaining numbers is $n(n + 1)/2 - (2m + 1)$.

From given condition,

$$\frac{105}{4} = \frac{\frac{n(n+1)}{2} - (2m+1)}{(n-2)}$$

$\Rightarrow \quad 2n^2 - 103n - 8m + 206 = 0$

Since n and m are integers, so n must be even. Let $n = 2k$. Then,

$$m = \frac{4k^2 + 103(1-k)}{4}$$

Since m is an integer, then $1 - k$ must be divisible by 4. Let $k = 1 + 4t$. Then we get $n = 8t + 2$ and $m = 16t^2 - 95t + 1$. Now,

$$1 \le m < n$$

$\Rightarrow \quad 1 \le 16t^2 - 95t + 1 < 8t + 2$

Solving, we get $t = 6$. Hence,

$$n = 50 \quad \text{and} \quad m = 7$$

Hence, the removed numbers are 7 and 8. Also, sum of all numbers is $50(50 + 1)/2 = 1275$.

For Problems 25–27

25. c, 26. b, 27. a.

Sol. Let the first term a and common difference d of the first A.P. and the first term b and common difference e of the second A.P. and let the number of terms be n. Then,

$$\frac{a + (n-1)d}{b} = \frac{b + (n-1)e}{a} = 4 \quad (1)$$

$$\frac{\frac{n}{2}[2a + (n-1)d]}{\frac{n}{2}[2b + (n-1)e]} = 2 \quad (2)$$

From (1) and (2), we get

$a - 4b + (n - 1)d = 0$ (3)

$b - 4a + (n - 1)e = 0$ (4)

$2a - 4b + (n - 1)d - 2(n - 1)e = 0$ (5)

$4 \times (3) + (4)$ gives

$-15b + 4(n - 1)d + (n - 1)e = 0$ (6)

$(4) + 2 \times (5)$ gives

$-7b + 2(n - 1)d - 3(n - 1)e = 0$ (7)

Further, $15 \times (7) - 7 \times (6)$ gives

$2(n - 1)d - 52(n - 1)e = 0$

or

$d = 26e \quad (\because n > 1)$

$\therefore \quad d/e = 26$

Putting $d = 26e$ in (3) and solving it with (4), we get

$a = 2(n - 1)e, b = 7(n - 1)e$

Then, the ratio of their n^{th} terms is

$$\frac{2(n-1)e + (n-1)26e}{7(n-1)e + (n-1)e} = \frac{7}{2}$$

For Problems 28–30

28. d, 29. c, 30. b.

Sol. We have,

$a + b + c = 25$ (1)

$2a = b + 2$ (2)

$c^2 = 18b$ (3)

Eliminating a from (1) and (2), we have

$$b = 16 - \frac{2c}{3}$$

Then from (3),

$$c^2 = 18\left(16 - \frac{2c}{3}\right)$$

$\Rightarrow \quad c^2 + 12c - 18 \times 16 = 0$

$\Rightarrow \quad (c - 12)(c + 24) = 0$

Now, $c = -24$ is not possible since it does not lie between 2 and 18. Hence, $c = 12$. Then from (3), $b = 8$ and finally from (2), $a = 5$.

Thus, $a = 5$, $b = 8$ and $c = 12$. Hence, $abc = 5 \times 8 \times 12 = 480$.

Also, equation $ax^2 + bx + c = 0$ is $5x^2 + 8x + 12 = 0$, which has imaginary roots.

If a, b, c are roots of the equation $x^3 + qx^2 + rx + s = 0$, then sum of product of roots taken two at a time is $r = 5 \times 8 + 5 \times 12 + 8 \times 12 = 196$.

For Problems 31–33

31. c, 32. c, 33. c.

Sol. 31. Clearly, here the differences between the successive terms are $7 - 3, 14 - 7, 24 - 14, \ldots$, i.e., 4, 7, 10, ... which are in A.P.

$\therefore \quad T_n = an^2 + bn + c$

Thus, we have

$3 = a + b + c$

$7 = 4a + 2b + c$

$14 = 9a + 3b + c$

Solving, we get $a = 3/2$, $b = -1/2$, $c = 2$. Hence,

$$T_n = \frac{1}{2}(3n^2 - n + 4)$$

$$\therefore \quad S_n = \frac{1}{2}\left[3\Sigma n^2 - \Sigma n + 4n\right]$$

$$= \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right]$$

$$= \frac{n}{2}(n^2 + n + 4)$$

$\Rightarrow \quad S_{20} = 4240$

32. The first differences are 5, 14, 50 194, 770, ...

The second differences are 9, 36, 144, 576, ...

They are in G.P. whose n^{th} term is $ar^{n-1} = a4^{n-1}$.

Therefore, T_n of the given series will be of the form

$T_n = a4^{n-1} + bn + c$

$T_1 = a + b + c = 3$

$T_2 = 4a + 2b + c = 8$

$T_3 = 16a + 3b + c = 22$

Solving, we have $a = 1, b = 2, c = 0$.

$\therefore \quad T_n = 4^{n-1} + 2n$

$\Rightarrow \quad T_{100} = 4^{99} + 200$

$= 2^{198} + 2^3 \times 25$

$= 8(2^{195} + 25)$ (which is divisible by 8)

33. Given series is $2 + 12 + 36 + 80 + 150 + 252 + \cdots$

The first differences are 10, 24, 44, 70, 102, ...

The second differences are 14, 20, 26, 32, ... which are in A.P.

Hence, general term of the series is

$T_n = an^3 + bn^2 + cn + d$

$\Rightarrow \quad 2 = a + b + c + d$

$12 = 8a + 4b + 2c + d$

$36 = 27a + 9b + 3c + d$

$80 = 64a + 16b + 4c + d$

Solving for a, we get $a = 1$.

$$\therefore \quad \lim_{n\to\infty} \frac{T_n}{n^3} = \lim_{n\to\infty}\left(1 + \frac{b}{n} + \frac{c}{n^2} + \frac{d}{n^3}\right) = 1$$

Matrix-Match Type

1. **a → q; b → r; c → p; d → r.**

a. a, b, c are in G.P. Hence,

$b^2 = ac$

$\Rightarrow \quad 2\log_{10} b = \log_{10} a + \log_{10} c$

$$\Rightarrow \quad \frac{2}{\log_b 10} = \frac{1}{\log_a 10} + \frac{1}{\log_c 10}$$

$$\Rightarrow \quad \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

Hence, x, y, z are in H.P.

b. $$\frac{a + be^x}{a - be^x} = \frac{b + ce^x}{b - ce^x} = \frac{c + de^x}{c - de^x}$$

$$\Rightarrow \quad \frac{2a}{a - be^x} - 1 = \frac{2b}{b - ce^x} - 1 = \frac{2c}{c - de^x} - 1$$

$$\Rightarrow \quad \frac{a - be^x}{a} = \frac{b - ce^x}{b} = \frac{c - de^x}{c}$$

$$\Rightarrow \quad 1 - \frac{b}{a}e^x = 1 - \frac{c}{b}e^x = 1 - \frac{d}{e}e^x$$

$$\Rightarrow \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

Hence, a, b, c, d are in G.P.

c. Given, $2b = a + c$, $x^2 = ab$, $y^2 = bc$. Now,

$x^2 + y^2 = b(a + c) = b\, 2b = 2b^2$

$\Rightarrow \quad x^2 + y^2 = 2b^2$

Hence, x^2, b^2, y^2 are in A.P.

d. $x \log a = y \log b = z \log c = k$ (say)

Also,

$y^2 = xz$

$$\Rightarrow \quad \frac{k^2}{(\log b)^2} = \frac{k^2}{\log a \log c}$$

Hence, $\log a$, $\log b$, $\log c$ are in G.P.

2. **a → p, q, r, s; b → r, s; c → p, q; d → r, s.**

a. $\Sigma n = 210$

$\Rightarrow \quad n(n + 1) = 420$

$\Rightarrow \quad (n - 20)(n + 21) = 0$

$\Rightarrow \quad n = 20$

Hence,

$$\Sigma n^2 = \frac{n}{6}(n + 1)(2n + 1)$$

$$= \frac{20}{6}(21)(41)$$

$= (10)(7)(41)$

Hence, the greatest prime number by which Σn^2 is divisible is 41.

b. $4, G_1, G_2, \ldots, G_{n+1}, \ldots, G_{2n}, G_{2n+1}, 2916$

G_{n+1} will be the middle mean of $(2n + 1)$ odd means and it will be equidistant from the first and last terms. Hence,

$4, G_{n+1}, 2916$ will also be in G.P. So,

$\Rightarrow \quad G_{n+1}^2 = 4 \times 2916$

$= 4 \times 9 \times 324$

$= 4 \times 9 \times 4 \times 81$

$\Rightarrow \quad G_{n+1} = 2 \times 3 \times 2 \times 9 = 108$

Hence, the greatest odd number by which G_{n+1} is divisible is 27.

c. Terms are 40, 30, 24, 20. Now,

$$\frac{1}{30} - \frac{1}{40} = \frac{1}{120}$$

$$\Rightarrow \quad \frac{1}{24} - \frac{1}{30} = \frac{6}{24 \times 30} = \frac{1}{120}$$

and

$$\frac{1}{20} - \frac{1}{24} = \frac{4}{20 \times 24} = \frac{1}{120}$$

Hence, 1/30, 1/24, 1/20 are in A.P. with common difference d = 1/120. Hence, the next term is 1/20 + 1/120 = 7/120. Therefore, the next term of given series is $\frac{120}{7} = 17\frac{1}{7}$. Hence, the integral part of $17\frac{1}{7}$ is 17.

d. $$S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots \infty$$

$$\Rightarrow \quad \frac{1}{5}S = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \cdots$$

$$\Rightarrow \quad S\left(1 - \frac{1}{5}\right) = 1 + 3\left[\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots \infty\right]$$

$$\Rightarrow \quad \frac{4}{5}S = 1 + 3\left[\frac{1/5}{1 - 1/5}\right] = 1 + \frac{3}{4} = \frac{7}{4}$$

$$\Rightarrow \quad S = \frac{35}{16}$$

$\Rightarrow \quad a = 35$ and $b = 16$

$\Rightarrow \quad a - b = 19$

Integer Type

1.(0) $10x^3 - nx^2 - 54x - 27 = 0$ has roots in H.P.

put $x = 1/t$

$27t^3 + 54t^2 + nt - 10 = 0$

This equation has roots in A.P., let the roots are $a - d$, a and $a + d$

$\therefore 3a = -\frac{54}{27} \Rightarrow a = -\frac{2}{3}$

Also $(a - d)\, a\, (a + d) = \frac{10}{27}$

$\therefore \quad \frac{2}{3}\left(\frac{4}{9} - d^2\right) = -\frac{10}{27} \Rightarrow \left(\frac{4}{9} - d^2\right) = -\frac{5}{9}$

$\therefore \quad d^2 = 1 \Rightarrow d = \pm 1$

For $d = 1$, roots are $-\frac{2}{3} + 1, -\frac{2}{3}, -\frac{2}{3} - 1 \Rightarrow \frac{1}{3}, -\frac{2}{3}, -\frac{5}{3}$,

for $d = -\frac{2}{3}, -1 - \frac{2}{3}, -\frac{2}{3} + 1 \Rightarrow -\frac{5}{3}, -\frac{2}{3}, \frac{1}{3}$

$\therefore \quad \frac{n}{27} = \frac{10}{9} - \frac{5}{9} - \frac{2}{9} \Rightarrow \frac{n}{27} = \frac{3}{9}$

$\Rightarrow n = 9$

2.(9) $\left[\frac{k(k+1)}{2}\right]^2 - \frac{k(k+1)}{2} = 1980$

$\Rightarrow \quad \frac{k(k+1)}{2}\left[\frac{k(k+1)}{2} - 1\right] = 1980$

$\Rightarrow \quad k(k+1)(k^2 + k - 2) = 1980 \times 4$

$\Rightarrow \quad (k-1)\, k(k+1)\,(k+2) = 8.\ 9.\ 10.\ 11$

$\therefore \quad k - 1 = 8 \Rightarrow k = 9$

3.(6) We have $S = 3 + \sum_{n=1}^{\infty} \frac{2n+3}{3^n} = 3 + \underbrace{\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}}_{S_1} + \underbrace{\sum_{n=1}^{\infty} \frac{2n}{3^n}}_{S_2}$

Now $S_1 = \sum_{n=1}^{\infty} \frac{1}{3^{n-1}} = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\infty$

$\therefore \quad S_1 = \frac{1}{1 - (1/3)} = \frac{3}{2}$

$S_2 = \sum_{n=1}^{\infty} \frac{2n}{3^n} = \frac{2}{3} + \frac{4}{3^2} + \frac{6}{3^3} + \frac{8}{3^4} + \cdots\infty$

$$S_2 = \frac{2}{3} + \frac{4}{3^2} + \frac{6}{3^3} + \frac{8}{3^4} + \cdots\infty$$

Now, $\frac{S_2}{3} = \quad + \frac{2}{3^2} + \frac{4}{3^3} + \frac{6}{3^4} + \cdots\infty$

$\frac{2S_2}{3} = \frac{2}{3}\left[1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\infty\right]$ [On subtracting]

$\therefore \quad S_2 = \frac{1}{1 - (1/3)} = \frac{3}{2}$

Hence, $S = 3 + \frac{3}{2} + \frac{3}{2} = 6.$

4.(1) Let a be the first term r be the common ratio of G.P.

$$\therefore \quad \frac{a(1 - r^{201})}{1 - r} = 625 \qquad (1)$$

Now $\sum_{i=1}^{201} \frac{1}{a_i} = \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{201}}\right)$

$= \frac{1}{a} + \frac{1}{ar} + \cdots + \frac{1}{ar^{200}}$

$= \frac{\frac{1}{a}\left(\left(\frac{1}{r}\right)^{201} - 1\right)}{\left(\frac{1}{r} - 1\right)}$

$= \frac{1}{a}\left(\frac{1 - r^{201}}{1 - r}\right)\frac{1}{r^{200}}$

$= \frac{1}{a} \times \frac{625}{a} \times \frac{1}{r^{200}}$ [from (1)]

$= \frac{625}{(ar^{100})^2} = \frac{625}{(a_{101})^2} = \frac{625}{625} = 1$

5.(9) Given $S = \sum_{n=1}^{9999} \frac{1}{\left(\sqrt{n} + \sqrt{n+1}\right)\left(\sqrt[4]{n} + \sqrt[4]{n+1}\right)}$

$= \sum_{n=1}^{9999} \frac{1}{\left(\sqrt{n} + \sqrt{n+1}\right)\left(\sqrt[4]{n} + \sqrt[4]{n+1}\right)}\left(\frac{\sqrt[4]{n} - \sqrt[4]{n+1}}{\sqrt[4]{n} - \sqrt[4]{n+1}}\right)$

$= \sum_{n=1}^{9999}\left((n+1)^{1/4} - n^{1/4}\right)$

$= \left(\left(2^{\frac{1}{4}} - 1\right) + \left(3^{\frac{1}{4}} - 2^{\frac{1}{4}}\right) + \left(4^{\frac{1}{4}} - 3^{\frac{1}{4}}\right) + \cdots + \left((9999+1)^{\frac{1}{4}} - (9999)^{\frac{1}{4}}\right)\right)$

$= (10^4)^{\frac{1}{4}} - 1 = 9$

6.(3) Let $a, ar, ar^2, ar^3, \ldots$ are in G.P.

Now $ar^4 = 7!$ and $ar^7 = 8!$

$\therefore$ On dividing, we get $r^3 = 8 \Rightarrow r = 2$

Hence, $a \,.\, 2^4 = 5040$

$\therefore a = \frac{5040}{16} = 315$

So 315, 630, 1260, ... are in G.P.

$\therefore \quad S_3 = 2205 \quad \Rightarrow \quad n = 3$

7.(8) Since a, b, c, d are in A.P.

$\therefore \quad b - a = c - b = d - c = D$ (let common difference)

$\Rightarrow \quad d = a + 3D$

$\Rightarrow \quad a - d = -3D$ and $d = b + 2D$

$\Rightarrow \quad b - d = -2D$

Also $c = a + 2D \quad \Rightarrow \quad c - a = 2D$

$\therefore$ Given equation $2(a - b) + k(b - c)^2 + (c - a)^3 = 2(a - d) + (b - d)^2 + (c - d)^3$

becomes $-2D + kD^2 + (2D)^3 = -6D + 4D^2 - D^3$

$\Rightarrow 9D^2 + (k-4)D + 4 = 0$

Since D is real $\Rightarrow (k-4)^2 - 4(4)(9) \geq 0$

$\Rightarrow k^2 - 8k - 128 \geq 0 \quad \Rightarrow \quad (k-16)(k+8) \geq 0$

$\therefore k \in (-\infty, -8] \cup [16, \infty)$

Hence, the smallest positive value of $k = 16$.

8.(7) 6, a, b in H.P.

$\Rightarrow \frac{1}{6}, \frac{1}{a}, \frac{1}{b}$ are in A.P.

$\Rightarrow \frac{2}{a} = \frac{1}{6} + \frac{1}{b}$

$\Rightarrow \frac{1}{b} = \frac{2}{a} - \frac{1}{6}$

$\Rightarrow \frac{1}{b} = \frac{12-a}{6a}$

$\Rightarrow b = \frac{6a}{12-a}$

$a \in \{3, 4, 6, 8, 9, 10, 11\}$

9.(6) 10 For the given A.P., we have $2(2a + b) = (5a - b) + (a + 2b)$

$\Rightarrow b = 2a$ (i)

Also for the given G.P., we have $(ab + 1)^2 = (a-1)^2(b+1)^2$ (ii)

$\therefore$ Putting $b = 2a$ from (i) in (ii), we get $a = 0, -2$ or $\frac{1}{4}$

But $a > 0$, so $a = \frac{1}{4}$ and $b = 2a = \frac{1}{2}$

Hence, $(a^{-1} + b^{-1}) = 2 + 4 = 6$.

10.(7) $ax^2 + (a+d)x + (a+2d) = 0$

$a, a+d, a+2d$ are in increasing A.P. $(d > 0)$

for real roots $D \geq 0$

$\Rightarrow (a+d)^2 - 4a(a+2d) \geq 0$

$\Rightarrow d^2 - 3a^2 - 6ad \geq 0$

$\Rightarrow (d-3a)^2 - 12a^2 \geq 0$

$\Rightarrow (d - 3a - \sqrt{12}\,a)(d - 3a + \sqrt{12}\,a) \geq 0$

$\Rightarrow \left[\frac{d}{a} - \left(3+2\sqrt{3}\right)\right]\left[\frac{d}{a} - \left(3-2\sqrt{3}\right)\right] \geq 0$

$\therefore \left.\frac{d}{a}\right|_{\text{Min}} = 3 + 2\sqrt{3}$

$\Rightarrow$ least integral value = 7

11.(1) $\frac{a}{1-r_1} = r_1$ and $\frac{a}{1-r_2} = r_2$

Hence, r_1 and r_2 are the roots of $\frac{a}{1-r} = r$

$\Rightarrow r^2 - r + a = 0$

$\Rightarrow r_1 + r_2 = 1$

12.(6) Let $\frac{\alpha}{r}$, α, αr be the roots.

$\therefore \alpha^3 = -216$ (1)

Again $\frac{\alpha^2}{r} + \alpha^2 r + \alpha^2 = b$

$\alpha^2\left(1 + r + \frac{1}{r}\right) = b$ (2)

and $\alpha\left(1 + r + \frac{1}{r}\right) = -a$ (3)

On dividing (2) by (3), we get

$\Rightarrow \alpha = -\frac{b}{a}$

$\Rightarrow \alpha^3 = -\frac{b^3}{a^3}$ (4)

From (1) and (4), $\left(\frac{b}{a}\right)^3 = 216$

$\Rightarrow \frac{b}{a} = 6$

13.(8) For the G.P. $a, ar, ar^2, \ldots$

$P_n = a(ar)(ar^2)\ldots(ar^{n-1}) = a^n . r^{n(n-1)/2}$

$\therefore S = \sum_{n=1}^{\infty} \sqrt[n]{P_n} = \sum_{n=1}^{\infty} ar^{(n-1)/2}$

Now, $\sum_{n=1}^{\infty} ar^{(n-1)/2} = a[1 + \sqrt{r} + r + r\sqrt{r} + \cdots + \infty] = \frac{a}{1-\sqrt{r}}$

Given $a = 16$ and $r = 1/4$

$\therefore S = \frac{16}{1-(1/2)} = 32$

14.(1) Let $a_1 = a - d$; $a_2 = a$; $a_3 = a + d$

$\therefore 3a = 18 \Rightarrow a = 6$

Hence, the number in A.P.

$6-d, d, 6+d$

$a_1 + 1, a_2, a_3 + 2$ in G.P.

i.e. $7-d, 6\ 8+d$ in G.P.

$\therefore 36 = (7-d)(8+d)$

$36 = 56 - d - d^2$

$d^2 + d - 20 = 0$

Hence, the sum of all possible common different is –1.

15.(0) a, b, c are in A.P. $\Rightarrow b = \frac{a+c}{2}$ (1)

b, c, d are in G.P. $\Rightarrow c^2 = bd$ (2)

and c, d, e are in H.P. $\Rightarrow d = \frac{2ce}{c+e}$ (3)

Now $c^2 = bd \quad \Rightarrow \quad c^2 = \left(\frac{a+c}{2}\right)\left(\frac{2ce}{c+e}\right)$ [using (1) and (3)]

$\therefore c^2 + ce = ae + ce$

$\Rightarrow c^2 = ae$

Now given $a = 2$ and $e = 18$

$\therefore c^2 = ae \quad \Rightarrow c^2 = 2 \times 18 = 36 \quad \Rightarrow c = 6$ or -6

16.(4) Let $\frac{a}{r}$, a, ar be three terms in G.P.

$\therefore$ Product of terms $= a^3 = -1$ (Given)

$\Rightarrow a = -1$

Now, sum of terms $= \frac{a}{r} + a + ar = \frac{13}{12}$ (Given)

$$\Rightarrow \frac{-1}{r} - 1 - r = \frac{13}{12}$$

$$\Rightarrow 12r^2 + 25r + 12 = 0$$

$$\therefore (3r + 4)(4r + 3) = 0$$

$$\Rightarrow r = \frac{-4}{3}, \frac{-3}{4}$$

But $r \neq \frac{-4}{3}$

$$\therefore |S| = \left|\frac{a}{1-r}\right| = \left|\frac{-1}{1-\left(\frac{-3}{4}\right)}\right| = \left|\frac{-1}{1+\frac{3}{4}}\right| = \left|\frac{-4}{7}\right| = \frac{4}{7}$$

17.(2) Let $S = \sum_{r=1}^{\infty} \frac{r+2}{2^{r+1}\cdot r\cdot(r+1)}$

$$= \sum_{r=1}^{\infty} \frac{2(r+1)-r}{2^{r+1}\cdot r\cdot(r+1)}$$

$$= \sum_{r=1}^{\infty} \frac{1}{2^{r+1}}\left(\frac{2}{r} - \frac{1}{r+1}\right)$$

$$= \sum_{r=1}^{\infty} \left(\frac{1}{2^r\cdot r} - \frac{1}{2^{r+1}(r+1)}\right)$$

$$= \lim_{n\to\infty}\left[\left(\frac{1}{2^1\cdot 1} - \frac{1}{2^2\cdot 2}\right) + \left(\frac{1}{2^2\cdot 2} - \frac{1}{2^3\cdot 3}\right)\right] + \left(\frac{1}{2^3\cdot 3} - \frac{1}{2^4\cdot 4}\right)$$

$$= + \cdots + \left(\frac{1}{2^n\cdot n} - \frac{1}{2^{n+1}\cdot(n+1)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2^{n+1}(n+1)}\right)$$

$$\therefore \quad S = \frac{1}{2}$$

Hence, $S^{-1} = 2$.

18.(3) $369 = \frac{9}{2}[2 + (9-1)d]$

$$\Rightarrow 82 = 2 + 8d$$

$$\Rightarrow d = 10$$

Now $ar^8 = a + 8d = 1 + 8 \times 80 = 81$

$$\Rightarrow r^8 = 81$$

$$\Rightarrow r = \sqrt{3}$$

$$\Rightarrow ar^{(7-1)} = 1 \times (\sqrt{3})^6 = 27$$

Archives

Subjective Type

1. Let a and b be the two numbers and let H be the harmonic mean between them. Then, $H = 4$ (given). Since A, G, H are in G.P., therefore,

$$G^2 = AH$$

$$\Rightarrow \quad G^2 = 4A$$

But

$$2A + G^2 = 27 \text{ (given)}$$

$$\therefore \quad 6A = 27 \quad [\because G^2 = 4A]$$

$$\Rightarrow \quad A = \frac{9}{2}$$

$$\Rightarrow \quad \frac{a+b}{2} = \frac{9}{2}$$

$$\Rightarrow \quad a + b = 9$$

Now, $G^2 = 4A$ and $A = 9/2 \Rightarrow G^2 = 18 \Rightarrow ab = 18$. The quadratic equation having a, b as its roots is

$$x^2 - (a+b)x + ab = 0$$

$$\Rightarrow \quad x^2 - 9x + 18 = 0 \quad [\because a + b = 9 \text{ and } ab = 18]$$

$$\Rightarrow \quad x = 3, 6$$

Hence, the two numbers are 3 and 6.

2. Let there be n sides in the polygon. Then by geometry, sum of all n interior angles of polygon is $(n-2) \times 180°$. Also the angles are in A.P. with the smallest angle 120° and common difference 5°. Therefore, sum of all interior angles of polygon is

$$\frac{n}{2}[2 \times 120 + (n-1) \times 5]$$

Thus, we must have

$$\frac{n}{2}[2 \times 120 + (n-1)5] = (n-2) \times 180$$

$$\Rightarrow \quad \frac{n}{2}[5n + 235] = (n-2) \times 180$$

$$\Rightarrow \quad 5n^2 + 235n = 360n - 720$$

$$\Rightarrow \quad 5n^2 - 125n + 720 = 0$$

$$\Rightarrow \quad n^2 - 25n + 144 = 0$$

$$\Rightarrow \quad (n-16)(n-9) = 0$$

$$\Rightarrow \quad n = 16, 9$$

But if $n = 16$, then 16^{th} angle $= 120 + 15 \times 5 = 195 > 180°$ which is not possible. Hence $n = 9$.

3. Given, $a_1, a_2, \ldots, a_n$ are in A.P., $\forall\, a_i > 0$.

$$\therefore \quad a_1 - a_2 = a_2 - a_3 = \cdots = a_{n-1} - a_n = d \text{ (a constant)}$$

Now,

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

$$= \frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2} + \frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3} + \cdots + \frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$$

$$= \frac{1}{-d}\left[\sqrt{a_1} - \sqrt{a_n}\right]$$

$$= \frac{a_1 - a_n}{-d(\sqrt{a_1}+\sqrt{a_n})}$$

$$= \frac{(n-1)d}{d(\sqrt{a_1}+\sqrt{a_n})}$$

$$= \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

4. See problem 3 in Reasoning Type section.
5. See problems 28–30 in the Linked Comprehension Type section.
6. Let the three distinct real numbers be α/r, α, αr. As sum of squares of three numbers is S^2,

$$\therefore \quad \frac{\alpha^2}{r^2} + \alpha^2 + \alpha^2 r^2 = S^2$$

$$\Rightarrow \frac{\alpha^2(1+r^2+r^4)}{r^2} = S^2 \quad (1)$$

Sum of numbers is aS. Hence,

$$\frac{\alpha}{r} + \alpha + \alpha r = aS$$

$$\Rightarrow \frac{\alpha(1+r+r^2)}{r} = aS \quad (2)$$

Dividing Eq. (1) by the square of Eq. (2), we get

$$\frac{\alpha^2(1+r^2+r^4)}{r^2} \times \frac{r^2}{\alpha^2(1+r+r^2)^2} = \frac{S^2}{a^2S^2}$$

$$\Rightarrow \frac{(1+2r^2+r^4)-r^2}{(1+r+r^2)^2} = \frac{1}{a^2}$$

$$\Rightarrow \frac{(1+r+r^2)(1-r+r^2)}{(1+r+r^2)^2} = \frac{1}{a^2}$$

$\Rightarrow a^2r^2 - a^2r + a^2 = 1 + r + r^2$

$\Rightarrow (a^2-1)r^2 - (a^2+1)r + (a^2-1) = 0$

$$\Rightarrow r^2 + \left(\frac{1+a^2}{1-a^2}\right)r + 1 = 0 \quad (3)$$

For real values of r,

$D \geq 0$

$$\Rightarrow \left(\frac{1+a^2}{1-a^2}\right)^2 - 4 \geq 0$$

$\Rightarrow 1 + 2a^2 + a^4 - 4 + 8a^2 - 4a^4 \geq 0$

$\Rightarrow 3a^4 - 10a^2 + 3 \leq 0$

$\Rightarrow (3a^2-1)(a^2-3) \leq 0$

$$\Rightarrow \left(a^2 - \frac{1}{3}\right)(a^2-3) \leq 0$$

Clearly, the above inequality holds for $1/3 \leq a^2 \leq 3$.

But from Eq. (3), $a \neq 1$.

$$\therefore a^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$$

7. Given that $\log_3 2$, $\log_3(2^x-5)$, $\log_3(2^x-7/2)$ are in A.P. Hence,

$$2\log_3(2^x-5) = \log_3\left(2^x - \frac{7}{2}\right) + \log_3 2$$

$$\Rightarrow (2^x-5)^2 = 2\left(2^x - \frac{7}{2}\right)$$

$\Rightarrow (2^x)^2 - 10 \times 2^x + 25 - 2 \times 2^x + 7 = 0$

$\Rightarrow (2^x)^2 - 12 \times 2^x + 32 = 0$

Let $2^x = y$. Then we get

$y^2 - 12y + 32 = 0$

$\Rightarrow (y-4)(y-8) = 0$

$\Rightarrow y = 4$ or 8

$\Rightarrow 2^x = 2^2$ or 2^3

$\Rightarrow x = 2$ or 3

But for $\log_3(2^x-5)$ and $\log_3(2^x-7/2)$ to be defined,

$$2^x - 5 > 0 \text{ and } 2^x - \frac{7}{2} > 0$$

$$\Rightarrow 2^x > 5 \text{ and } 2^x > \frac{7}{2}$$

$\Rightarrow 2^x > 5$

$\Rightarrow x \neq 2$

and therefore, $x = 3$.

8. Let a and b be two numbers and $A_1, A_2, A_3, \ldots, A_n$ be n A.M.'s between a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ are in A.P. There are $n+2$ terms in the series. Now,

$$a + (n+1)d = b \Rightarrow d = \frac{b-a}{n+1}$$

$$\Rightarrow A_1 = p = a + \frac{b-a}{n+1} = \frac{an+b}{n+1} \quad (1)$$

The first H.M. between a and b, when n H.M.'s are inserted between a and b can be obtained by replacing a by $1/a$ and b by $1/b$ in Eq. (1) and then taking its reciprocal. Therefore,

$$q = \frac{1}{\dfrac{\left(\dfrac{1}{a}\right)n + \dfrac{1}{b}}{n+1}} = \frac{(n+1)ab}{bn+a} \quad (2)$$

Substituting $b = p(n+1) - an$ [from (1)] in Eq. (2), we get

$aq + nq[p(n+1) - an] = (n+1)a[p(n+1) - an]$

$\Rightarrow a^2n(n+1) + a[q(1-n^2) - p(n+1)^2] + npq(n+1) = 0$

$\Rightarrow na^2 - [(n+1)p + (n-1)q]a + npq = 0$

$\Rightarrow D \geq 0$ ($\because a$ is real)

$\Rightarrow [(n+1)p + (n-1)q]^2 - 4n^2pq \geq 0$

$\Rightarrow (n-1)^2q^2 + \{2(n^2-1) - 4n^2\}pq + (n+1)^2p^2 \geq 0$

$$\Rightarrow q^2 - 2\frac{n^2+1}{(n-1)^2}pq + \left(\frac{n+1}{n-1}\right)^2 p^2 \geq 0$$

$$\Rightarrow \left[q - p\left(\frac{n+1}{n-1}\right)^2\right][q-p] \geq 0$$

[On factorizing by discriminent method]

Hence, q cannot lie between p and $p\left(\dfrac{n+1}{n-1}\right)^2$.

9. According to the question, we have

$$S_1 = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \infty = \frac{1}{1-\frac{1}{2}} = 2$$

$$S_2 = 2 + 2 \times \frac{1}{3} + 2 \times \left(\frac{1}{3}\right)^2 + \cdots \infty = \frac{2}{1-\frac{1}{3}} = 3$$

$$S_3 = 3 + 3 \times \frac{1}{4} + 3 \times \left(\frac{1}{4}\right)^2 + \cdots \infty = \frac{3}{1-\frac{1}{4}} = 4$$

$$S_n = n + n \times \frac{1}{n+1} + n \times \left(\frac{1}{n+1}\right)^2 + \cdots \infty = \frac{n}{1-\frac{1}{n+1}} = (n+1)$$

$$\therefore S_1^2 + S_2^2 + S_3^2 + \cdots + S_{2n-1}^2$$

$$= 2^2 + 3^2 + 4^2 + \cdots + (n+1)^2 + \cdots + (2n)^2$$

$$= \left(\sum_{r=1}^{2n} r^2\right) - 1$$

$$= \frac{2n(2n+1)(4n+1)}{6} - 1^2$$

$$= \frac{n(2n+1)(4n+1)-3}{3}$$

10. Since x_1, x_2, x_3 are in A.P., let $x_1 = a - d$, $x_2 = a$ and $x_3 = a + d$. And x_1, x_2, x_3 are the roots of $x^3 - x^2 + \beta x + \gamma = 0$. Now, the sum of roots is

$$\Sigma\alpha = a - d + a + a + d = 1 \quad (1)$$

Sum of product of roots taken two at a time is

$$\Sigma\alpha\beta = (a-d)\,a + a\,(a+d) + (a-d)(a+d)$$
$$= \beta \quad (2)$$

Product of roots is $\alpha\beta\gamma = (a-d)\,a\,(a+d)$

$$= -\gamma \quad (3)$$

From (1), we get

$$3a = 1 \Rightarrow a = 1/3$$

From (2), we get

$$3a^2 - d^2 = \beta$$

$$\Rightarrow 3(1/3)^2 - d^2 = \beta \Rightarrow 1/3 - \beta = d^2$$

$$\Rightarrow \frac{1}{3} - \beta \geq 0 \ (\because d^2 \geq 0)$$

$$\Rightarrow \beta \leq \frac{1}{3}$$

$$\Rightarrow \beta \in (-\infty, 1/3]$$

From (3),

$$a(a^2 - d^2) = -\gamma$$

$$\Rightarrow \frac{1}{3}\left(\frac{1}{9} - d^2\right) = -\gamma$$

$$\Rightarrow \frac{1}{27} - \frac{1}{3}d^2 = -\gamma$$

$$\Rightarrow \gamma + \frac{1}{27} = \frac{1}{3}d^2$$

$$\Rightarrow \gamma + \frac{1}{27} \geq 0$$

$$\Rightarrow \gamma \geq -\frac{1}{27}$$

$$\Rightarrow \gamma \in \left[-\frac{1}{27}, \infty\right)$$

Hence, $\beta \in (-\infty, 1/3]$ and $\gamma \in [-1/27, \infty)$.

11. Solving the system of equations, $u + 2v + 3w = 6$, $4u + 5v + 6w = 12$ and $6u + 9v = 4$, we get

$$u = -1/3,\ v = 2/3,\ w = 5/3$$

$$\therefore \ u + v + w = 2 \text{ and } \frac{1}{u} + \frac{1}{v} + \frac{1}{w} = -\frac{9}{10}$$

Let r be the common ratio of the G.P. a, b, c, d. Then, $b = ar$, $c = ar^2$, $d = ar^3$. Then the first equation

$$\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]\,x + (u+v+w) = 0 \text{ becomes}$$

$$-\frac{9}{10}x^2 + [(ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2]\,x + 2 = 0$$

$$\Rightarrow 9x^2 - 10a^2(1-r)^2(r^4 + 2r^3 + 3r^2 + 2r + 1)\,x - 20 = 0$$

$$\Rightarrow 9x^2 - 10a^2(1-r)^2(1 + r + r^2)^2\,x - 20 = 0$$

$$\Rightarrow 9x^2 - 10a^2(1 - r^3)^2\,x - 20 = 0 \quad (1)$$

The second equation is

$$20x^2 + 10(a - ar^3)^2\,x - 9 = 0$$

$$\Rightarrow 20x^2 + 10a^2(1 - r^3)^2\,x - 9 = 0 \quad (2)$$

Since (2) can be obtained by changing x to $1/x$, so Eqs. (1) and (2) have reciprocal roots.

12. Let $a - 3d$, $a - d$, $a + d$ and $a + 3d$ be any consecutive terms of an A.P. with common difference $2d$. Hence,

$$P = (2d)^4 + (a-3d)(a-d)(a+d)(a+3d)$$
$$= 16d^4 + (a^2 - 9d^2)(a^2 - d^2)$$
$$= (a^2 - 5d^2)^2$$

which is an integer.

13. $G_k = (a_1 a_2 \cdots a_k)^{1/k}$

$$= a_1\,(r^{1+2+\cdots+(k-1)})^{1/k}$$

$$= a_1 r^{\frac{k-1}{2}} \quad (1)$$

$$A_k = \frac{a_1 + a_2 + \cdots + a_k}{k}$$

$$= \frac{a_1(1 + r + \cdots + r^{k-1})}{k}$$

$$= \frac{a_1(r^k - 1)}{(r-1)\times k} \quad (2)$$

$$H_k = \frac{k}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}}$$

$$= \frac{a_1 k}{\left(1 + \frac{1}{r} + \cdots + \frac{1}{r^{k-1}}\right)}$$

$$= \frac{a_1 k\,(r-1)\,r^{k-1}}{r^k - 1} \quad (3)$$

From (1), (2) and (3), we get

$$G_k = (A_k H_k)^{1/2}$$

$$\Rightarrow \prod_{k=1}^{n} G_k = \prod_{k=1}^{n} (A_k H_k)^{1/2}$$

$$\Rightarrow \left(\prod_{k=1}^{n} G_k\right)^{1/n} = (A_1 A_2 \cdots A_n \times H_1 H_2 \cdots H_n)^{1/2n}$$

14. Clearly, $A_1 + A_2 = a + b$. Now,

$$\frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b}$$

$$\Rightarrow \frac{H_1 + H_2}{H_1 H_2} = \frac{a+b}{ab} = \frac{A_1 + A_2}{G_1 G_2}$$

$$\Rightarrow \frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2}$$

Also,

$$\frac{1}{H_1} = \frac{1}{a} + \frac{1}{3}\left(\frac{1}{b} - \frac{1}{a}\right) \Rightarrow H_1 = \frac{3ab}{2b+a}$$

and

$$\frac{1}{H_2} = \frac{1}{a} + \frac{2}{3}\left(\frac{1}{b} - \frac{1}{a}\right) \Rightarrow H_2 = \frac{3ab}{2a+b}$$

$$\Rightarrow \frac{A_1 + A_2}{H_1 + H_2} = \frac{a+b}{3ab\left(\frac{1}{2b+a} + \frac{1}{2a+b}\right)}$$

$$= \frac{(2b+a)(2a+b)}{9ab}$$

15. Given that a, b, and c are in A.P. Hence,

$$2b = a + c \quad (1)$$

a^2, b^2, c^2 are in H.P. Hence,

$$\frac{1}{b^2} - \frac{1}{a^2} = \frac{1}{c^2} - \frac{1}{b^2}$$

$$\Rightarrow \frac{(a-b)(a+b)}{b^2a^2} = \frac{(b-c)(b+c)}{b^2c^2}$$

$$\Rightarrow ac^2 + bc^2 = a^2b + a^2c \; [\because a - b = b - c]$$

$$\Rightarrow ac\,(c-a) + b(c-a)(c+a) = 0$$

$$\Rightarrow (c-a)(ab + bc + ca) = 0$$

$$\Rightarrow c - a = 0 \text{ or } ab + bc + ca = 0$$

For $c = a$, from (1), $a = b = c$. For $(a + c)b + ca = 0$, from (1),

$$2b^2 + ca = 0$$

$$\Rightarrow b^2 = a\left(\frac{-c}{2}\right)$$

Hence, a, b, $-c/2$ are in G.P.

16. $$a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^3 + \cdots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$$

$$= \frac{\frac{3}{4}\left(1 - \left(-\frac{3}{4}\right)^n\right)}{1 - \left(-\frac{3}{4}\right)}$$

$$= \frac{3}{7}\left(1 - \left(-\frac{3}{4}\right)^n\right)$$

Now, $b_n = 1 - a_n$ and $b_n > a_n$ for $n \ge n_0$.

$$\therefore \quad 1 - a_n > a_n \Rightarrow 2a_n < 1$$

$$\Rightarrow \frac{6}{7}\left[1 - \left(-\frac{3}{4}\right)^n\right] < 1$$

$$\Rightarrow -\left(-\frac{3}{4}\right)^n < \frac{1}{6}$$

$$\Rightarrow (-3)^{n+1} < 2^{2n-1}$$

For n to be even, inequality always holds. For n to be odd, it holds for $n \ge 7$. Therefore,the least natural number for which it holds is 6.

Objective Type

Fill in the blanks

1. The sum of integers from 1 to 100 that are divisible by 2 or 5 is

S = sum of integers from 1 to 100 divisible by 2

+ sum of integers from 1 to 100 divisible by 5

– sum of integers from 1 to 100 divisible by 10

$$= (2 + 4 + 6 + \cdots + 100) + (5 + 10 + 15 + \cdots + 100) - (10 + 20 + \cdots + 100)$$

$$= \frac{50}{2}[2\times2+49\times2] + \frac{20}{2}[2\times5+19\times5] - \frac{10}{2}[2\times10+9\times10]$$

$$= 2550 + 1050 - 550 = 3050$$

2. When n is odd, last term is n^2. Hence, the required sum is

$$S = [1^2 + 2\times2^2 + 3^2 + 2\times4^2 + \cdots + 2\times(n-1)^2] + n^2$$

$$= \frac{(n-1)\,n^2}{2} + n^2 \quad \text{[Using sum for } (n-1) \text{ to be even]}$$

$$= \frac{n^2(n+1)}{2}$$

3. Let a and b be two positive numbers. Then, H.M. $= \frac{2ab}{a+b}$ and G.M. $= \sqrt{ab}$. According to question, H.M.:G.M. = 4:5

$$\therefore \frac{2ab}{(a+b)\sqrt{ab}} = \frac{4}{5}$$

$$\Rightarrow \frac{2\sqrt{ab}}{a+b} = \frac{4}{5}$$

$$\Rightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{5+4}{5-4}$$

$$\Rightarrow \left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)^2 = 9$$

$$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})}{\sqrt{a}-\sqrt{b}} = 3, -3$$

$$\Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{3+1}{3-1}, \frac{-3+1}{-3-1}$$

$$\Rightarrow \frac{\sqrt{a}}{\sqrt{b}} = 2, \frac{1}{2} \Rightarrow \frac{a}{b} = 4, \frac{1}{4}$$

$$\Rightarrow a{:}b = 4{:}1 \text{ or } 1{:}4$$

4. Since n is an odd integer, $(-1)^{n-1} = 1$ and $n-1, n-3, n-5, \ldots$ are even integers. The given series is

$$n^3 - (n-1)^3 + (n-2)^3 - (n-3)^3 + \cdots + (-1)^{n-1}\,1^3$$
$$= [n^3 + (n-1)^3 + (n-2)^3 + \cdots + 1^3] - 2\,[(n-1)^3 + (n-3)^3 + \cdots + 2^3]$$

$$= \left[\frac{n(n+1)}{2}\right]^2 - 16\left[\left\{\frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)\right\}\right]^2$$

$$= \frac{1}{4} n^2 (n+1)^2 - 16 \frac{(n-1)^2 (n+1)^2}{16 \times 4}$$

$$= \frac{1}{4} (n+1)^2 [n^2 - (n-1)^2]$$

$$= \frac{1}{4} (n+1)^2 (2n-1)$$

5. Given that x is the A.M. between a and b and y, and z are the G.M.'s between a and b where a and b are positive. Then a, x, b are in A.P. So,

$$x = \frac{a+b}{2}$$

a, y, z, b are in G.P. So,

$y = ar$, and $z = ar^2$, where $r = \sqrt[3]{\frac{b}{a}}$. Also, $yz = ab$. Now,

$$\frac{y^3 + z^3}{xyz} = \frac{a^3 r^3 + a^3 r^6}{ab\left(\frac{a+b}{2}\right)}$$

$$= \frac{a^3 \times \frac{b}{a} + a^3 \times \frac{b^2}{a^2}}{ab\left(\frac{a+b}{2}\right)}$$

$$= \frac{2(a^2 b + ab^2)}{a^2 b + ab^2}$$

$= 2$

6. Let p and q be roots of the equation $x^2 - 2x + A = 0$. Then,

$p + q = 2, pq = A$

Let r and s be the roots of the equation $x^2 - 18x + B = 0$. Then,

$r + s = 18, rs = B$

And it is given that p, q, r, and s are in A.P. Let $p = a - 3b$, $q = a - d$, $r = a + d$ and $s = a + 3d$. As $p < q < r < s$, we have $d > 0$. Now,

$2 = p + q = a - 3d + a - d = 2a - 4d$

$\Rightarrow \quad a - 2d = 1 \qquad (1)$

and

$18 = r + s = a + d + a + 3d$

$\Rightarrow \quad a + 2d = 9 \qquad (2)$

Solving (1) and (2), $a = 5, d = 2$

$\therefore \quad p = -1, q = 3, r = 7, s = 11$

Therefore, $A = pq = -3$ and $B = rs = 77$.

Multiple choice questions with one correct answer

1. **c.** Given that x, y, z are the p^{th}, q^{th} and r^{th} terms of an A.P.

$\therefore \quad x = A + (p-1) D$

$y = A + (q-1) D$

$z = A + (r-1) D$

$\Rightarrow \quad x - y = (p-q) D$

$y - z = (q-r) D$

$z - x = (r-p) D$

where A is the first term and D is the common difference. Also x, y, z are the p^{th}, q^{th} and r^{th} terms of a G.P.

$\therefore \quad x = aR^{p-1}, y = aR^{q-1}, z = aR^{r-1}$

$\therefore \quad x^{y-z} y^{z-x} z^{x-y} = (aR^{p-1})^{y-z} (aR^{q-1})^{z-x} (aR^{r-1})^{x-y}$

$= a^{y-z+z-x+x-y} R^{(p-1)(y-z)+(q-1)(z-x)+(r-1)(x-y)}$

$= A^\circ R^{(p-1)(q-r)D+(q-1)(r-p)D+(r-1)(p-q)D}$

$= A^\circ R^\circ = 1$

2. **b.** Given

$ar^2 = 4$

$\Rightarrow \quad a \times ar \times ar^2 \times ar^3 \times ar^4 = a^5 r^{10} = (ar^2)^5 = 4^5$

3. **c.** $2.\overline{357} = 2 + 0.357 + 0.000357 + \cdots \infty$

$$= 2 + \frac{357}{10^3} + \frac{357}{10^6} + \cdots \infty$$

$$= 2 + \frac{\frac{357}{10^3}}{1 - \frac{1}{10^3}}$$

$$= 2 + \frac{357}{999} = \frac{2355}{999}$$

Alternative solution:

Let,

$x = 2.\overline{357}$

$\Rightarrow \quad 1000x = 2357.\overline{357}$

On subtracting, we get

$$999x = 2355 \Rightarrow x = \frac{2355}{999}$$

4. **a.** For first equation $D = 4b^2 - 4ac = 0$ (as given a, b, c are in G.P.)

$\Rightarrow$ equation has equal roots which are equal to $-\frac{b}{a}$ each.

Thus it should also be the root of the second equation.

$$\text{Thus, } d\left(\frac{-b}{a}\right)^2 + 2e\left(\frac{-b}{a}\right) + f = 0$$

$$\Rightarrow \quad d\frac{b^2}{a^2} - 2\frac{be}{a} + f = 0$$

$$\Rightarrow d\frac{ac}{a^2} - 2\frac{be}{a} + f = 0 \quad (\text{as } b^2 = ac)$$

$$\Rightarrow \frac{d}{a} + \frac{f}{c} = 2\frac{eb}{ac} = 2\frac{e}{b}$$

5. **c.** Let,

$$S = \frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \cdots n \text{ terms}$$

$$= \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{8}\right) + \left(1 - \frac{1}{16}\right) + \cdots n \text{ terms}$$

$$= (1 + 1 + 1 + \cdots n \text{ times}) - \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^n}\right)$$

$$= n - \left[\frac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}\right] = n - 1 + 2^{-n}$$

6. **c.** We have,

$$\frac{(x+2)^n - (x+1)^n}{(x+2) - (x+1)} = (x+2)^{n-1} + (x+2)^{n-2}(x+1)$$
$$+ (x+2)^{n-3}(x+1)^2 + \cdots + (x+1)^{n-1}$$

Hence, the required sum is

$(x+2)^n-(x+1)^n$ [$\because (x+2)-(x+1)=1$]

7. d. $\ln(a+c)$, $\ln(c-a)$, $\ln(a-2b+c)$ are in A.P. Hence, $a+c$, $c-a$, $a-2b+c$ are in G.P. Therefore,

$(c-a)^2=(a+c)(a-2b+c)$

$\Rightarrow (c-a)^2=(a+c)^2-2b(a+c)$

$\Rightarrow 2b(a+c)=(a+c)^2-(c-a)^2$

$\Rightarrow 2b(a+c)=4ac$

$$\Rightarrow b=\frac{2ac}{a+c}$$

Hence, a, b, and c are in H.P.

8. d. $a_1=h_1=2$, $a_{10}=h_{10}=3$

$3=a_{10}=2+9d \Rightarrow d=1/9$

$\therefore a_4=2+3d=7/3$

Also,

$$3=h_{10}\Rightarrow \frac{1}{3}=\frac{1}{h_{10}}=\frac{1}{2}+9D$$

$$\Rightarrow D=-\frac{1}{54}$$

$$\Rightarrow \frac{1}{h_7}=\frac{1}{2}+6D=\frac{1}{2}-\frac{1}{9}=\frac{7}{18}$$

$$\therefore a_4h_7=\frac{7}{3}\times\frac{18}{7}=6$$

9. b. Harmonic mean H of roots α and β is

$$H=\frac{2\alpha\beta}{\alpha+\beta}=\frac{2\dfrac{8+2\sqrt{5}}{5+\sqrt{2}}}{\dfrac{4+\sqrt{5}}{5+\sqrt{2}}}=4$$

10. d. a, b, and c, d are in A.P. Therefore, d, c, b and a are also in A.P. Hence,

$$\frac{d}{abcd},\frac{c}{abcd},\frac{b}{abcd},\frac{a}{abcd} \text{ are also in A.P.}$$

$$\Rightarrow \frac{1}{abc},\frac{1}{abd},\frac{1}{acd},\frac{1}{bcd} \text{ are in A.P.}$$

$\Rightarrow abc, abd, acd, bcd$ are in H.P.

11. d. Sum is 4 and second term is 3/4. It is given that first term is a and common ratio is r. Hence,

$$\frac{a}{1-r}=4 \text{ and } ar=3/4 \Rightarrow r=\frac{3}{4a}$$

Therefore,

$$\frac{a}{1-\dfrac{3}{4a}}=4\Rightarrow \frac{4a^2}{4a-3}=4$$

$\Rightarrow a^2-4a+3=0$

$\Rightarrow (a-1)(a-3)=0$

$\Rightarrow a=1$ or 3

When $a=1$, $r=3/4$ and when $a=3$, $r=1/4$.

12. a. α,β are the roots of $x^2-x+p=0$. Hence,

$\alpha+\beta=1$ (1)

$\alpha\beta=p$ (2)

γ,δ are the roots of $x^2-4x+q=0$. Hence,

$\therefore \gamma+\delta=4$ (3)

$\gamma\delta=q$ (4)

$\alpha,\beta,\gamma,\delta$ are in G.P. Let $\alpha=a$, $\beta=ar$, $\gamma=ar^2$, $\delta=ar^3$. Substituting these values in Eqs. (1), (2), (3) and (4), we get

$a+ar=1$ (5)

$a^2r=p$ (6)

$ar^2+ar^3=4$ (7)

$a^2r^5=q$ (8)

Dividing (7) by (5), we get

$$\frac{ar^2(1+r)}{a(1+r)}=\frac{4}{1}\Rightarrow r^2=4\Rightarrow r=2,-2$$

$$(5)\Rightarrow a=\frac{1}{1+r}=\frac{1}{1+2} \text{ or } \frac{1}{1-2}=\frac{1}{3} \text{ or } -1$$

As p is an integer (given), r is also an integer (2 or –2). Therefore, from (6), $a\neq 1/3$. Hence, $a=-1$ and $r=-2$.

$\therefore p=(-1)^2\times(-2)=-2$

$q=(-1)^2\times(-2)^5=-32$

13. c. Given,

$2+5+8+\cdots 2n$ terms $=57+59+61+\cdots n$ terms

$$\Rightarrow \frac{2n}{2}[4+(2n-1)3]=\frac{n}{2}[114+(n-1)2]$$

$\Rightarrow 6n+1=n+56$

$\Rightarrow 5n=55$

$\Rightarrow n=11$

14. d. Given that a, b, and c are in A.P. Hence,

$2b=a+c$

But given,

$a+b+c=3/2$

$\Rightarrow 3b=3/2$

$\Rightarrow b=1/2$

Hence,

$a+c=1$

Again, a^2, b^2, c^2 are in G.P. Hence,

$b^4=a^2c^2$

$\Rightarrow b^2=\pm ac$

$$\Rightarrow ac=\frac{1}{4} \text{ or } -\frac{1}{4} \text{ and } a+c=1 \quad (1)$$

Now,

$$a+c=1 \text{ and } ac=\frac{1}{4}$$

$\Rightarrow (a-c)^2=(a+c)^2-4ac=1-1=0$

$\Rightarrow a=c$

But $a\neq c$ as given that $a<b<c$. We consider $a+c=1$ and $ac=-1/4$. Hence,

$(a-c)^2=1+1=2$

$\Rightarrow a-c=\pm\sqrt{2}$

But

$a<c\Rightarrow a-c=-\sqrt{2}$ (2)

Solving (1) and (2), we get

$$a=\frac{1}{2}-\frac{1}{\sqrt{2}}$$

15. c. $S_\infty=\dfrac{a}{1-r}=5$ (given)

$$\Rightarrow r=\frac{5-a}{5}$$

But

$$0 < |r| < 1$$

$$\Rightarrow \quad 0 < \left|\frac{5-a}{5}\right| < 1$$

$$\Rightarrow \quad -1 < \frac{5-a}{5} < 1 \text{ and } a \neq 5$$

$$\Rightarrow \quad -5 < 5-a < 5 \text{ and } a \neq 5$$

$$\Rightarrow \quad -10 < -a < 0 \text{ and } a \neq 5$$

$$\Rightarrow \quad 10 > a > 0 \text{ and } a \neq 5$$

$$\Rightarrow \quad 0 < a < 10 \text{ and } a \neq 5$$

16. c. $\alpha+\beta, \alpha^2+\beta^2, \alpha^3+\beta^3$ are in G.P. Hence,

$$(\alpha^2+\beta^2)^2 = (\alpha+\beta)(\alpha^3+\beta^3)$$

$$\Rightarrow \alpha^4+\beta^4+2\alpha^2\beta^2 = \alpha^4+\beta^4+\alpha\beta^3+\beta\alpha^3$$

$$\Rightarrow \quad \alpha\beta(\alpha^2+\beta^2-2\alpha\beta) = 0$$

$$\Rightarrow \quad \alpha\beta(\alpha-\beta)^2 = 0$$

$$\Rightarrow \quad \alpha = 0 \text{ or } \alpha-\beta = 0$$

$$\Rightarrow \quad c/a = 0 \text{ or } \alpha = \beta$$

$$\Rightarrow \quad c = 0 \text{ or } \Delta = 0 \text{ (equal roots)}$$

$$\therefore \quad c\Delta = 0$$

Multiple choice questions with one or more than one correct answer

1. b, d. Let x be the first term and y be the $(2n-1)^{\text{th}}$ term of A.P., G.P. and H.P. whose n^{th} terms are a, b, c, respectively. Now according to the property of A.P., G.P. and H.P., x, a, y are in A.P.; x, b, y are in G.P. and x, c, y are in H.P. Hence,

$$a = \frac{x+y}{2} = \text{A.M.}$$

$$b = \sqrt{xy} = \text{G.M.}$$

$$c = \frac{2xy}{x+y} = \text{H.M.}$$

Now, A.M, G.M. and H.M. are in G.P. Hence,

$$b^2 = ac$$

Also, A.M. ≥ G.M. ≥ H.M. Hence,

$$a \geq b \geq c$$

2. b, c. We have, for $0 < \phi < \pi/2$

$$x = \sum_{n=0}^{\infty} \cos^{2n}\phi$$

$$= 1 + \cos^2\phi + \cos^4\phi + \cdots \infty$$

$$= \frac{1}{1-\cos^2\phi}$$

$$= \frac{1}{\sin^2\phi} \qquad (1)$$

$$y = \sum_{n=0}^{\infty} \sin^{2n}\phi$$

$$= 1 + \sin^2\phi + \sin^4\phi + \cdots \infty$$

$$= \frac{1}{1-\sin^2\phi}$$

$$= \frac{1}{\cos^2\phi} \qquad (2)$$

$$z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$$

$$= 1 + \cos^2\phi\sin^2\phi + \cos^4\phi\sin^4\phi + \cdots \infty$$

$$= \frac{1}{1-\cos^2\phi\sin^2\phi} \qquad (3)$$

Substituting the values of $\cos^2\phi$ and $\sin^2\phi$ in (3), from (1) and (2), we get

$$z = \frac{1}{1-\frac{1}{x}\frac{1}{y}}$$

$$\Rightarrow \quad z = \frac{xy}{xy-1}$$

$$\Rightarrow \quad xyz - z = xy$$

$$\Rightarrow \quad xyz = xy + z$$

Also,

$$x+y+z = \frac{1}{\cos^2\phi} + \frac{1}{\sin^2\phi} + \frac{1}{1-\cos^2\phi\sin^2\phi}$$

$$= \frac{\sin^2\phi(1-\cos^2\phi\sin^2\phi) + \cos^2\phi(1-\cos^2\phi\sin^2\phi) + \cos^2\phi\sin^2\phi}{\cos^2\phi\sin^2\phi(1-\cos^2\phi\sin^2\phi)}$$

$$= \frac{(\sin^2\phi+\cos^2\phi)(1-\cos^2\phi\sin^2\phi)+\cos^2\phi\sin^2\phi}{\cos^2\phi\sin^2\phi(1-\cos^2\phi\sin^2\phi)}$$

$$= \frac{1}{\cos^2\phi\sin^2\phi(1-\cos^2\phi\sin^2\phi)} = xyz$$

Thus, (b) and (c) both are correct.

3. b. Putting $\theta = 0$, we get $b_0 = 0$.

$$\therefore \quad \sin n\theta = \sum_{r=1}^{n} b_r \sin^r\theta$$

$$\Rightarrow \quad \frac{\sin n\theta}{\sin\theta} = \sum_{r=1}^{n} b_r(\sin\theta)^{r-1}$$

$$= b_1 + b_2\sin\theta + b_3\sin^2\theta + \cdots + b_n\sin^{n-1}\theta$$

Taking limit as $\theta \to 0$, we obtain

$$\lim_{\theta\to0}\frac{\sin n\theta}{\sin\theta} = b_1 \Rightarrow b_1 = n$$

4. c. $T_m = a+(m-1)d = 1/n$

$$T_n = a+(n-1)d = 1/m$$

$$\Rightarrow \quad (m-n)d = 1/n - 1/m = (m-n)/mn$$

$$\Rightarrow \quad d = 1/mn$$

$$\Rightarrow \quad a = \frac{1}{mn}$$

$$\therefore \quad T_{mn} = a + (mn-1)d$$

$$= \frac{1}{mn} + (mn-1)\frac{1}{mn}$$

$$= \frac{1}{mn} + 1 - \frac{1}{mn}$$

$$= 1$$

5. b. If x, y, and z are in G.P. ($x, y, z > 1$), then $\log x$, $\log y$, $\log z$ are in A.P. Hence,

$1+\log x$, $1+\log y$, $1+\log z$ will also be in A.P.

$$\Rightarrow \quad \frac{1}{1+\log x}, \frac{1}{1+\log y}, \frac{1}{1+\log z} \text{ will be in H.P.}$$

6. a, d. We have

$$a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{2^n - 1}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{15}\right) + \cdots + \frac{1}{2^n - 1}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{2^2 - 1}\right) + \left(\frac{1}{2^2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{2^3 - 1}\right) + \left(\frac{1}{2^3} + \cdots + \frac{1}{2^4 - 1}\right) + \cdots$$

$$< 1 + 1 + \cdots + 1 = n$$

Thus,

$$a(100) < 100$$

Also,

$$a(n) = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \frac{1}{2^n - 1}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2^1 + 1} + \frac{1}{2^2}\right) + \left(\frac{1}{2^2 + 1} + \frac{1}{2^3}\right) + \cdots + \left(\frac{1}{2^{n-1} + 1}\right)$$

$$> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^{n-1}}{2^n} - \frac{1}{2^n}$$

$$> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^{n-1}}{2^n} - \frac{1}{2^n}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}\right) - \frac{1}{2^n}$$

$$= 1 + \frac{n}{2} - \frac{1}{2^n} = \left(1 - \frac{1}{2^n}\right) + \frac{n}{2}$$

Thus,

$$a > \left(1 - \frac{1}{2^{200}}\right) + \frac{200}{2} > 100$$

i.e.,

$$a(200) > 100$$

Comprehension

1. b. $V_1 + V_2 + \cdots + V_n$

$$= \sum_{r=1}^{n} V_r$$

$$= \sum_{r=1}^{n} \left(\frac{r}{2}(2r + (r-1)(2r-1))\right)$$

$$= \sum_{r=1}^{n} \left(r^3 - \frac{r^2}{2} + \frac{r}{2}\right)$$

$$= \Sigma n^3 - \frac{\Sigma n^2}{2} + \frac{\Sigma n}{2}$$

$$= \frac{n^2 (n+1)^2}{4} - \frac{n (n+1)(2n+1)}{12} + \frac{n (n+1)}{4}$$

$$= \frac{n (n+1)}{4} \left[n (n+1) - \frac{2n+1}{3} + 1\right]$$

$$= \frac{n (n+1)(3n^2 + n + 2)}{12}$$

2.d. $T_r = r + (r-1)(2r-1)$

$= (r+1)(3r-1)$

For each r, T_r has two different factors other than 1 and itself.

Therefore, T_r is always a composite number.

3. b. Since $Q_{r+1} - Q_r = 6(r+1) + 5 - 6r - 5 = 6$ (constant), therefore, $Q_1, Q_2, Q_3, \ldots$ are in A.P. with common difference 6.

4. c. Given,

$$A_1 = \frac{a+b}{2}, G_1 = \sqrt{ab}, H_1 = \frac{2ab}{a+b}$$

Also,

$$A_n = \frac{A_{n-1} + H_{n-1}}{2}$$

$$G_n = \sqrt{A_{n-1} H_{n-1}}$$

$$H_n = \frac{2 A_{n-1} \times H_{n-1}}{A_{n-1} + H_{n-1}}$$

$$\Rightarrow \quad G_n^2 = A_n H_n \Rightarrow A_n H_n = A_{n-1} H_{n-1}$$

Similarly, we can prove

$$A_n H_n = A_{n-1} H_{n-1} = A_{n-2} H_{n-2} = \cdots = A_1 H_1$$

$$\Rightarrow \quad A_n H_n = ab$$

$$\Rightarrow \quad G_1^2 = G_2^2 = G_3^2 = \cdots = G_n^2 = ab$$

$$\Rightarrow \quad G_1 = G_2 = G_3 = \cdots = \sqrt{ab}$$

5. a. We have,

$$A_n = \frac{A_{n-1} + H_{n-1}}{2}$$

$$\therefore \quad A_n - A_{n-1} = \frac{A_{n-1} + H_{n-1}}{2} - A_{n-1}$$

$$= \frac{H_{n-1} - A_{n-1}}{2} < 0 \; (\because A_{n-1} > H_{n-1})$$

$$\Rightarrow \quad A_n < A_{n-1} \text{ or } A_{n-1} > A_n$$

Hence, we can conclude that $A_1 > A_2 > A_3 > \cdots$

6. b. We have,

$$A_n H_n = ab \Rightarrow H_n = \frac{ab}{A_n}$$

$$\frac{1}{A_{n-1}} < \frac{1}{A_n} \Rightarrow H_{n-1} < H_n$$

$$\therefore \quad H_1 < H_2 < H_3 < \cdots$$

Integer type

1.(3) $S_k = \dfrac{\dfrac{k-1}{k!}}{1 - \dfrac{1}{k}} = \dfrac{1}{(k-1)!}$

$$\sum_{k=2}^{100} \left|(k^2 - 3k + 1)\frac{1}{(k-1)!}\right|$$

$$= \sum_{k=2}^{100} \left|\frac{(k-1)^2 - k}{(k-1)!}\right|$$

$$= \Sigma \left|\frac{k-1}{(k-2)!} - \frac{k}{(k-1)!}\right|$$

$$= \left|\frac{1}{0!} - \frac{2}{1!}\right| \left|\frac{2}{1!} - \frac{3}{2!}\right| + \left|\frac{3}{2!} - \frac{4}{3!}\right| + \cdots$$

$$= \frac{2}{1!} - \frac{1}{0!} + \frac{2}{1!} - \frac{3}{2!} + \frac{3}{2!} - \frac{4}{3!} + \cdots + \frac{99}{98!} - \frac{100}{99!}$$

$$= 3 - \frac{100}{99!}$$

2.(0) $a_k = 2a_{k-1} - a_{k-2} \Rightarrow a_1, a_2, \ldots, a_{11}$ are in A.P.

$$\therefore \quad \frac{a_1^2 + a_2^2 + \cdots + a_{11}^2}{11} = \frac{11a^2 + 35 \times 11d^2 + 10ad}{11} = 90$$

$\Rightarrow \quad 225 + 35d^2 + 150d = 90$

$35d^2 + 150d + 135 = 0 \quad \Rightarrow \quad -3, -9/7$

Given $a_2 < \frac{27}{2}$, we get $d = -3$ and $d = -9/7$

$$\Rightarrow \quad \frac{a_1 + a_2 + \cdots + a_{11}}{11} = \frac{11}{2}[30 - 10 \times 3] = 0$$

3.(6) $a_1, a_2, a_3, \ldots, a_{100}$ is an A.P.

$a_1 = 3, S_p = \sum_{i=1}^{p} a_i, 1 \le p \le 100$

$$\frac{S_m}{S_n} = \frac{S_{5n}}{S_n} = \frac{\frac{5n}{2}(6 + (5n-1)d)}{\frac{n}{2}(6 - d + nd)}$$

$\frac{S_m}{S_n}$ is independent of n of $6 - d = 0 \quad \Rightarrow \quad d = 6.$

CHAPTER 4

Inequalities Involving Means

- Inequalities Involving Simple A.M., G.M., H.M.
- Inequalities Involving Arithmetic Mean of m^{th} Power
- Inequalities Involving Weighted Means

INEQUALITIES INVOLVING SIMPLE A.M., G.M., H.M.

Let A, G and H be arithmetic, geometric and harmonic means of two positive numbers a and b. Then,

$$A = \frac{a+b}{2},\ G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b}$$

These three means possess the following property:
$A \geq G \geq H$.

Proof: We have

$$A = \frac{a+b}{2},\ G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b}$$

$$\Rightarrow \quad A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0$$

$$\Rightarrow \quad A \geq G \tag{1}$$

$$G - H = \sqrt{ab} - \frac{2ab}{a+b}$$

$$= \sqrt{ab}\left\{\frac{a+b-2\sqrt{ab}}{a+b}\right\}$$

$$= \frac{\sqrt{ab}}{a+b}(\sqrt{a}-\sqrt{b})^2 \geq 0$$

$$\Rightarrow \quad G \geq H \tag{2}$$

From Eqs. (1) and (2), we get $A \geq G \geq H$.

Note: *The equality holds in the above result only when $a = b$.*

In general, if $a_i > 0$, $i = 1, 2, \ldots, n$, then we have

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 \times a_2 \times \cdots \times a_n)^{1/n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

$\Rightarrow$ A.M. $\geq$ G.M. $\geq$ H.M.

This result can be proved by the principle of mathematical induction.

Example 4.1 **Prove that $(ab + xy)(ax + by) > 4abxy$ $(a, b, x, y > 0)$.**

Sol. Using A.M. > G.M., we have

$$\frac{ab+xy}{2} > \sqrt{abxy} \Rightarrow ab + xy > 2\sqrt{abxy}$$

Similarly,

$$ax + by > 2\sqrt{abxy}$$

Multiplying, we get

$$(ab + xy)(ax + by) > 4abxy$$

Example 4.2 **Prove that $b^2c^2 + c^2a^2 + a^2b^2 > abc \times (a + b + c)$. $(a, b, c > 0)$**

Sol. $b^2c^2 + c^2a^2 > 2(a^2b^2c^4)^{1/2}$ (Using A.M. > G.M.)

$\Rightarrow$ $b^2c^2 + c^2a^2 > 2abc^2$

Similarly,

$c^2a^2 + a^2b^2 > 2bca^2$ and $a^2b^2 + b^2c^2 > 2cab^2$

Adding, we get

$2b^2c^2 + 2c^2a^2 + 2a^2b^2 > 2abc^2 + 2bca^2 + 2cab^2$

$\Rightarrow$ $b^2c^2 + c^2a^2 + a^2b^2 > abc(a + b + c)$

Example 4.3 **Find the minimum value of $4^{\sin^2 x} + 4^{\cos^2 x}$.**

Sol. Using A.M. $\geq$ G.M., we have

$$4^{\sin^2 x} + 4^{\cos^2 x} \geq 2\sqrt{4^{\sin^2 x}4^{\cos^2 x}}$$

$$= 2\sqrt{4^{\sin^2 x + \cos^2 x}}$$

$$= 2\sqrt{4} = 4$$

$$\Rightarrow \left(4^{\sin^2 x} + 4^{\cos^2 x}\right)_{\min} = 4.$$

Example 4.4 **If $a + b + c = 1$, then prove that**

$$\frac{8}{27abc} > \left\{\frac{1}{a} - 1\right\}\left\{\frac{1}{b} - 1\right\}\left\{\frac{1}{c} - 1\right\} > 8$$

Sol. On multiplying both sides by abc, we have to prove that $8/27 > (1 - a)(1 - b)(1 - c) > 8abc$. Now,

$$\frac{(1-a)+(1-b)+(1-c)}{3} > [(1-a)(1-b)(1-c)]^{1/3}$$

$$\Rightarrow \frac{3-(a+b+c)}{3} > [(1-a)(1-b)(1-c)]^{1/3}$$

$$\Rightarrow \left(\frac{2}{3}\right)^3 > (1-a)(1-b)(1-c) \quad (\because a + b + c = 1)$$

Also, $(a + b)/2 > (ab)^{1/2}$, $(c + b)/2 > (cb)$ and $(a + c)/2 > (ac)^{1/2}$. Multiplying these three inequalities, we have

$$\frac{a+b}{2}\frac{b+c}{2}\frac{a+c}{2} > abc$$

$$\Rightarrow (a + b)(b + c)(c + a) > 8abc$$

$$\Rightarrow (1 - a)(1 - b)(1 - c) > 8abc$$

Example 4.5 **If a, b, c are positive, then prove that $a/(b + c) + b/(c + a) + c/(a + b) \geq 3/2$.**

Sol. Adding 3 to both sides, we have

$$\frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b} \geq \frac{3}{2} + 3$$

$$\Rightarrow (b + c)^{-1} + (c + a)^{-1} + (a + b)^{-1} \geq \frac{9}{2(a+b+c)}$$

Now,

A.M. $\geq$ H.M.

$$\Rightarrow \frac{\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}}{3} \geq \frac{3}{a+b+b+c+c+a}$$

$$\Rightarrow \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \geq \frac{9}{2(a+b+c)}$$

Example 4.6 **In a triangle ABC prove that $3/2 \leq a/(a + c) + b/(c + a) + c/(a + b) < 2$ (equality when $a = b = c$).**

Sol. We know that $b + c > a$ in a triangle.

Adding $(b + c)$ to both sides, we have

$$2(b + c) > (a + b + c)$$

$$\therefore \frac{1}{2(b+c)} < \frac{1}{a+b+c}$$

or $\dfrac{a}{b+c}<\dfrac{2a}{a+b+c}$

Similarly, $\dfrac{b}{c+a}<\dfrac{2b}{a+b+c}$

$\dfrac{c}{a+b}<\dfrac{2c}{a+b+c}$

Adding, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2\frac{a+b+c}{a+b+c}=2$$

Example 4.7 **Prove that $2^n > 1 + n\sqrt{2^{n-1}}$, $\forall\, n > 2$ where n is a positive integer.**

Sol. $2^n > 1+n\sqrt{2^{n-1}}$

$$\Rightarrow \frac{2^n-1}{2-1}>n\times 2^{(n-1)/2}$$

Now, $(2^n - 1)/(2 - 1)$ is the sum of a G.P. whose first term is 1 and common ratio is 2.

We have to prove that $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} > n \times 2^{(n-1)/2}$.

Now,

$$\frac{1+2+2^2+\cdots+2^{n-1}}{n} > (1\times 2\times 2^2\times 2^3 \cdots 2^{n-1})^{1/n}$$

($\because$ A.M. > G.M.)

Now R.H.S. $= (2^{1+2+3+\cdots+n-1})^{1/n}$

$= [2^{(n-1)n/2}]^{1/n} = 2^{(n-1)/2}$

$$\Rightarrow 1+2+2^2+\cdots+2^{n-1}>n\times 2^{(n-1)/2}$$

Example 4.8 **Prove that $[(n + 1)/2]^n > (n!)$.**

Sol. $$\frac{1+2+3+4+\cdots+n}{n}>(1\times2\times3\times4\times\cdots\times n)^{1/n}$$

(Using A.M. > G.M.)

$$\Rightarrow \frac{\frac{n(n+1)}{2}}{n}>(n!)^{1/n}$$

$$\Rightarrow \left(\frac{n+1}{2}\right)^n>(n!)$$

Example 4.9 **Find the least value of sec A + sec B + sec C in acute-angled triangle.**

Sol. In acute-angled triangle sec A, sec B, sec C are positive. Now,

A.M. ≥ H.M.

$$\Rightarrow \frac{\sec A+\sec B+\sec C}{3}\ge\frac{3}{\cos A+\cos B+\cos C}$$

But in ΔABC, $\cos A + \cos B + \cos C \le 3/2$

$$\Rightarrow \frac{\sec A+\sec B+\sec C}{3}\ge 2$$

$$\Rightarrow \sec A+\sec B+\sec C\ge 6$$

Example 4.10 **If $S = a_1 + a_2 + a_3 + \cdots + a_n$, $a_i \in R^+$ for $i = 1$ to n, then prove that**

$$\frac{S}{S-a_1}+\frac{S}{S-a_2}+\cdots+\frac{S}{S-a_n}\ge\frac{n^2}{n-1},\ \forall\, n\ge 2$$

Sol. We have,

A.M. ≥ H.M.

$$\Rightarrow \frac{\frac{S}{S-a_1}+\frac{S}{S-a_2}+\cdots+\frac{S}{S-a_n}}{n}\ge\frac{n}{\frac{S-a_1}{S}+\frac{S-a_2}{S}+\cdots+\frac{S-a_n}{S}}$$

$$\Rightarrow \frac{S}{S-a_1}+\frac{S}{S-a_2}+\cdots+\frac{S}{S-a_n}\ge\frac{n^2S}{nS-(a_1+a_2+\cdots+a_n)}$$

$$\Rightarrow \frac{S}{S-a_1}+\frac{S}{S-a_2}+\cdots+\frac{S}{S-a_n}\ge\frac{n^2S}{nS-S}$$

$$\Rightarrow \frac{S}{S-a_1}+\frac{S}{S-a_2}+\cdots+\frac{S}{S-a_n}\ge\frac{n^2}{n-1}$$

Example 4.11 **If $yz + zx + xy = 12$, where x, y, z are positive values, find the greatest value of xyz.**

Sol. Using A.M. ≥ G.M.

$$\Rightarrow \frac{xy+yz+zx}{3}\ge(x^2y^2z^2)^{1/3}$$

$$\Rightarrow \frac{12}{3}\ge(x^2y^2z^2)^{1/3}\Rightarrow(xyz)_{\max}=\left(\frac{12}{3}\right)^{3/2}$$

INEQUALITIES INVOLVING ARITHMETIC MEAN OF m^{th} POWER

If $a_i > 0$, $i = 1, 2, \ldots, n$, which are not identical, then

(i) $\dfrac{a_1^m+a_2^m+\cdots+a_n^m}{n}>\left(\dfrac{a_1+a_2+\cdots+a_n}{n}\right)^m$ if $m < 0$ or $m > 1$

(ii) $\dfrac{a_1^m+a_2^m+\cdots+a_n^m}{n}<\left(\dfrac{a_1+a_2+\cdots+a_n}{n}\right)^m$ if $0 < m < 1$

Example 4.12 **If $a, b > 0$ such that $a^3 + b^3 = 2$, then show that $a + b \le 2$.**

Sol. We know that A.M. of $(1/3)^{th}$ power ≤ $(1/3)^{th}$ power of A.M.

$$\therefore \frac{(a^3)^{1/3}+(b^3)^{1/3}}{2}\le\left(\frac{a^3+b^3}{2}\right)^{1/3}$$

$$\Rightarrow \frac{a+b}{2}\le 1$$

$$\Rightarrow a+b\le 2$$

Example 4.13 **If $m > 1$, $n \in N$ show that $1^m + 2^m + 2^{2m} + 2^{3m} + \cdots + 2^{nm-m} > n^{1-m}(2^n - 1)^m$.**

Sol. Since, $m > 0$, so

A.M. of m^{th} power > m^{th} power of A.M.

$$\Rightarrow \frac{1^m+2^m+4^m+8^m+\cdots+(2^{n-1})^m}{n}>\left(\frac{1+2+4+\cdots+2^{n-1}}{n}\right)^m$$

$$\Rightarrow 1^m+2^m+4^m+\cdots+2^{(n-1)m}>n\left(\frac{2^n-1}{n}\right)^m$$

$$\Rightarrow 1^m+2^m+4^m+\cdots+2^{(n-1)m}>n^{1-m}(2^n-1)^m$$

Example 4.14 **Prove that**

$$\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+\frac{a^2+b^2}{a+b}>a+b+c$$

Sol. $\frac{b^2+c^2}{2}>\left(\frac{b+c}{2}\right)^2 \Rightarrow \frac{b^2+c^2}{b+c}>\frac{1}{2}(b+c)$

Similarly,

$$\frac{a^2+b^2}{a+b}>\frac{1}{2}(a+b) \text{ and } \frac{c^2+a^2}{c+a}>\frac{1}{2}(c+a)$$

Adding, we get

$$\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+\frac{a^2+b^2}{a+b}>a+b+c$$

Example 4.15 **Prove that**

$$\frac{a^8+b^8+c^8}{a^3b^3c^3}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Sol. We have to prove that $a^8+b^8+c^8>a^2b^2c^2(bc+ca+ab)$.

Now,

$$\frac{a^8+b^8+c^8}{3}>\left(\frac{a+b+c}{3}\right)^8$$

$$\Rightarrow \frac{a^8+b^8+c^8}{3}>\left(\frac{a+b+c}{3}\right)^6\left(\frac{a+b+c}{3}\right)^2$$

$$>\left[(abc)^{1/3}\right]^6\left[\frac{a^2+b^2+c^2+2ab+2bc+2ca}{9}\right]$$

$$(\because \text{A.M.} > \text{G.M.})$$

But

$$a^2+b^2+c^2>ab+bc+ca$$

$$\Rightarrow \frac{a^8+b^8+c^8}{3}>a^2b^2c^2\frac{(3ab+3bc+3ca)}{9}$$

$$\Rightarrow a^8+b^8+c^8>a^2b^2c^2(ab+bc+ca)$$

$$\Rightarrow \frac{a^8+b^8+c^8}{a^3b^3c^3}>\frac{ab+bc+ca}{abc}$$

$$\Rightarrow \frac{a^8+b^8+c^8}{a^3b^3c^3}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Example 4.16 **Show that** $C_0^4+C_1^4+C_2^4+\cdots+C_n^4>\frac{2^{4n}}{n^3}$, **where** ${}^nC_r=n!/[r!(n-r)!]$.

Sol. A.M. of 4th power > 4th power of A.M.

$$\Rightarrow \frac{C_0^4+C_1^4+\cdots+C_n^4}{n}>\left(\frac{C_0+C_1+C_2+\cdots+C_n}{n}\right)^4$$

$$\Rightarrow C_0^4+C_1^4+C_2^4+\cdots+C_n^4>n\left(\frac{2^n}{n}\right)^4$$

$$\Rightarrow C_0^4+C_1^4+C_2^4+\cdots+C_n^4>\frac{2^{4n}}{n^3}$$

Example 4.17 **If a, b and c are positive and $a+b+c=6$, show that $(a+1/b)^2+(b+1/c)^2+(c+1/a)^2\geq 75/4$.**

Sol. A.M. $\geq$ H.M.

$$\Rightarrow \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\geq\frac{3}{a+b+c}=\frac{3}{6}=\frac{1}{2}$$

So,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{3}{2}$$

Now,

$$\frac{\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2}{3}\geq\left[\frac{\left(a+\frac{1}{b}\right)+\left(b+\frac{1}{c}\right)+\left(c+\frac{1}{a}\right)}{3}\right]^2$$

$$\Rightarrow \sqrt{\frac{\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2}{3}}\geq\frac{\left(a+\frac{1}{b}\right)+\left(b+\frac{1}{c}\right)+\left(c+\frac{1}{a}\right)}{3}\geq\frac{6+\frac{3}{2}}{3}=\frac{5}{2}$$

Squaring both sides, we get

$$\frac{\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2}{3}\geq\frac{25}{4}$$

$$\Rightarrow \left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq\frac{75}{4}$$

INEQUALITIES INVOLVING WEIGHTED MEANS

If $a_i>0$, $i=1, 2, \ldots, n$ and $w_i\geq 0$, where $i=1, 2, \ldots, n$, then weighted arithmetic mean,

$$A_w=\frac{w_1a_1+w_2a_2+\cdots+w_na_n}{w_1+w_2+\cdots+w_n}$$

Weighted geometric mean,

$$G_w=\left(a_1^{w_1}a_2^{w_2}\cdots a_n^{w_n}\right)^{\frac{1}{w_1+w_2+\cdots+w_n}}$$

Weighted harmonic mean,

$$H_w=\frac{w_1+w_2+\cdots+w_n}{\frac{w_1}{a_1}+\frac{w_2}{a_2}+\cdots+\frac{w_n}{a_n}}$$

From A.M. $\geq$ G.M. $\geq$ H.M., we have $A_w\geq G_w\geq H_w$.

Example 4.18 **Prove that**

$$\left[\frac{x^2+y^2+z^2}{x+y+z}\right]^{x+y+z}>x^xy^yz^z>\left[\frac{x+y+z}{3}\right]^{x+y+z}\quad (x, y, z>0)$$

Sol. $\dfrac{x^2+y^2+z^2}{x+y+z}$

$$=\frac{(x+x+x+\cdots x\text{ times})+(y+y+y+\cdots y\text{ times})}{x+y+z}+\frac{z+z+z+\cdots z\text{ times}}{x+y+z}$$

$$>[(x\,x\cdots x\text{ factors})\,(y\,y\cdots y\text{ factors})\,(z\,z\cdots z\text{ factors})]^{1/(x+y+z)}$$

$$\Rightarrow \frac{x^2+y^2+z^2}{x+y+z}>(x^xy^yz^z)^{1/(x+y+z)}$$

$$\Rightarrow \left(\frac{x^2+y^2+z^2}{x+y+z}\right)^{x+y+z}>x^xy^yz^z \quad (1)$$

Now,

$$\frac{\left(\frac{1}{x}+\frac{1}{x}+\cdots x\text{ times}\right)+\left(\frac{1}{y}+\frac{1}{y}+\cdots y\text{ times}\right)}{x+y+z}+\frac{\left(\frac{1}{z}+\frac{1}{z}+\cdots z\text{ times}\right)}{x+y+z}$$

$$>\left[\left(\frac{1}{x}\,\frac{1}{x}\cdots x\text{ factors}\right)\left(\frac{1}{y}\,\frac{1}{y}\cdots y\text{ factors}\right)\left(\frac{1}{z}\,\frac{1}{z}\cdots z\text{ factors}\right)\right]^{1/(x+y+z)}$$

$$\Rightarrow \frac{\frac{1}{x}x+\frac{1}{y}y+\frac{1}{z}z}{x+y+z}>\left[\frac{1}{x^x}\,\frac{1}{y^y}\,\frac{1}{z^z}\right]^{1/(x+y+z)}$$

$$\Rightarrow \left(\frac{3}{x+y+z}\right)^{x+y+z}>\frac{1}{x^xy^yz^z}$$

$$\Rightarrow \left(\frac{x+y+z}{3}\right)^{x+y+z}<x^xy^yz^z$$

$$\Rightarrow x^xy^yz^z>\left(\frac{x+y+z}{3}\right)^{x+y+x} \quad (2)$$

From Eqs. (1) and (2), we get

$$\left[\frac{x^2+y^2+z^2}{x+y+z}\right]^{x+y+z}>x^xy^yz^z>\left[\frac{x+y+z}{3}\right]^{x+y+z}$$

Example 4.19 **Prove that $1^1\times2^2\times3^3\times\cdots\times n^n\le[(2n+1)/3]^{n(n+1)/2}, n\in N$.**

Sol. For $1^1\times2^2\times3^3\times\ldots\times n^n$, consider 1, 1 times; 2, 2 times; …; n, n times. Now,

$$\frac{1+(2+2)+(3+3+3)+\cdots+(n+n+\cdots+n\text{ times})}{1+2+3+\cdots+n}\ge(1^1\times2^2\times\cdots\times n^n)^{\frac{1}{1+2+3+\cdots+n}}$$

$$\Rightarrow \frac{1+2^2+3^2+\cdots+n^2}{\frac{n(n+1)}{2}}\ge(1^1\times2^2\times\cdots\times n^n)^{\frac{2}{n(n+1)}}$$

$$\Rightarrow \frac{\frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)}{2}}\ge(1^1\times2^2\times\cdots\times n^n)^{\frac{2}{n(n+1)}}$$

$$\Rightarrow \frac{(2n+1)}{3}\ge(1^1\times2^2\times\cdots\times n^n)^{\frac{2}{n(n+1)}}$$

$$\Rightarrow 1^1\times2^2\times3^3\times\cdots\times n^n\le\left(\frac{2n+1}{3}\right)^{\frac{n(n+1)}{2}}$$

Example 4.20 **Find the greatest value of x^2y^3 where x and y lie in the first quadrant on the line $3x+4y=5$.**

Sol. $x^2y^3=x\cdot x\cdot y\cdot y\cdot y$

Now $3x+4y=5$

Consider two parts of $3x$ and three parts of $4y$

so we have $\dfrac{3x}{2}+\dfrac{3x}{2}+\dfrac{4y}{3}+\dfrac{4y}{3}+\dfrac{4y}{3}=5$

Using A.M. $\ge$ G.M., we have

$$\frac{\frac{3x}{2}+\frac{3x}{2}+\frac{4y}{3}+\frac{4y}{3}+\frac{4y}{3}}{5}\ge\left(\left(\frac{3x}{2}\right)^2\left(\frac{4y}{3}\right)^3\right)^{1/5}$$

$$\Rightarrow \left(\left(\frac{3x}{2}\right)^2\left(\frac{4y}{3}\right)^3\right)^{1/5}\le1$$

$$\Rightarrow x^2y^3\le\left(\frac{2}{3}\right)^2\left(\frac{3}{4}\right)^3$$

$$\Rightarrow x^2y^3\le\frac{3}{16}\Rightarrow(x^2y^3)_{\max}=\frac{3}{16}$$

Example 4.21 **Find the maximum value of $(7-x)^4(2+x)^5$ when x lies between -2 and 7.**

Sol. Since $-2<x<7$, therefore $x+2$ and $7-x$ are both positive. We have to find the maximum value of $(7-x)^4(2+x)^5$ or p^4q^5 where $p+q=9$.

Using A.M. $\ge$ G.M., we get

$$\frac{\frac{p}{4}+\frac{p}{4}+\frac{p}{4}+\frac{p}{4}+\frac{q}{5}+\frac{q}{5}+\frac{q}{5}+\frac{q}{5}+\frac{q}{5}}{9}\ge\left(\left(\frac{p}{4}\right)^4\left(\frac{q}{5}\right)^5\right)^{1/9}$$

$$\Rightarrow \frac{p+q}{9}\ge\left(\left(\frac{p}{4}\right)^4\left(\frac{q}{5}\right)^5\right)^{1/9}$$

$$\Rightarrow \left(\left(\frac{p}{4}\right)^4\left(\frac{q}{5}\right)^5\right)^{1/9}\le1$$

$$\Rightarrow p^4q^5\le(4^45^5)^9.$$

Concept Application Exercise 4.1

1. If $a_1, a_2, \ldots, a_n > 0$, then prove that

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\frac{a_3}{a_4}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}>n$$

2. Prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{1}{\sqrt{(bc)}}+\frac{1}{\sqrt{(ca)}}+\frac{1}{\sqrt{(ab)}}$, where $a, b, c > 0$.

3. If $a > b$ and n is a positive integer, then prove that $a^n - b^n > n(ab)^{(n-1)/2}(a-b)$.

4. Prove that $\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}<\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, where a, b, c > 0.

5. Find the minimum value of $2^{\sin x} + 2^{\cos x}$.

6. If $a^2 + b^2 + c^2 = x^2 + y^2 + z^2 = 1$, then show that $ax + by + cz < 1$.

7. If a, b, c are real numbers such that $0 < a < 1$, $0 < b < 1$, $0 < c < 1$, $a + b + c = 2$, then prove that

$$\frac{a}{1-a}\frac{b}{1-b}\frac{c}{1-c}\ge 8$$

8. In ΔABC, prove that $\tan A + \tan B + \tan C \ge 3\sqrt{3}$, where A, B, C are acute angles.

9. In ΔABC, prove that $\operatorname{cosec}(A/2) + \operatorname{cosec}(B/2) + \operatorname{cosec}(C/2) \ge 6$.

10. If $a_i > 0$ $(i = 1, 2, 3, \ldots, n)$, prove that

$$\sum_{1\le i<j\le n}\sqrt{a_i a_j}\le\frac{n-1}{2}(a_1+a_2+\cdots+a_n)$$

11. If n is a positive integer ≥ 1, then prove that $\frac{3^n}{2^n + n\,6^{\frac{n-1}{2}}} > 1$.

12. Prove that the greatest value of xy is $c^3/\sqrt{2ab}$, if $a^2x^4 + b^2y^4 = c^6$.

13. Prove that $a^4 + b^4 + c^4 > abc(a + b + c)$, where $a, b, c > 0$.

14. If $C_r = \frac{n!}{[r!(n-r)]}$, then prove that

$$\sqrt{C_1}+\sqrt{C_2}+\cdots+\sqrt{C_n}<\sqrt{n(2^n-1)}$$

15. If $a + b = 1$, $a > 0$, $b > 0$, prove that $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\ge\frac{25}{2}$.

16. Prove that $\left[\frac{a^2+b^2}{a+b}\right]^{a+b}>a^ab^b>\left\{\frac{a+b}{2}\right\}^{a+b}$.

17. Prove that $a^pb^q>\left(\frac{ap+bp}{p+q}\right)^{p+q}$.

18. Prove that $px^{q-r} + qx^{r-p} + rx^{p-q} > p + q + r$, where p, q, r are distinct and $x \ne 1$.

19. Given positive rational numbers a, b, c such that $a + b + c = 1$, then prove that $a^ab^bc^c + a^bb^cc^a + a^cb^ac^b \le 1$.

20. Find the greatest value of $x^2y^3z^4$ if $x^2 + y^2 + z^2 = 1$, where x, y, z are positive.

EXERCISES

Subjective Type

Solutions on page 4.9

1. Prove that $^nC_1(^nC_2)^2(^nC_3)^3\cdots(^nC_n)^n\le\left(\frac{2^n}{n+1}\right)^{^{n+1}C_2}$, $\forall n \in N$.

2. Let $x_1, x_2, \ldots, x_n$ be positive real numbers and we define $S = x_1 + x_2 + \cdots + x_n$. Prove that

$$(1+x_1)(1+x_2)\cdots(1+x_n)\le 1+S+\frac{S^2}{2!}+\frac{S^3}{3!}+\cdots+\frac{S^n}{n!}$$

3. If $2x^3 + ax^2 + bx + 4 = 0$ (a and b are positive real numbers) has three real roots, then prove that $a + b \ge 6(2^{1/3} + 4^{1/3})$.

4. Calculate the greatest and least values of the function

$$f(x)=\frac{x^4}{x^8+2x^6-4x^4+8x^2+16}$$

5. If a, b, c are three distinct positive real numbers in G.P., then prove that $c^2 + 2ab > 3ac$.

6. In ΔABC internal angle bisectors AI, BI and CI are produced to meet opposite sides in A', B', C', respectively. Prove that the maximum value of $\frac{AI\times BI\times CI}{AA'\times BB'\times CC'}$ is $\frac{8}{27}$.

7. In how many parts an integer $N \ge 5$ should be dissected so that the product of the parts is maximized.

8. If $x + y + z = 1$ and x, y, z are positive, then show that

$$\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2>\frac{100}{3}$$

Objective Type

Solutions on page 4.10

Each question has four choices a, b, c and d, out of which *only one* is correct.

1. The minimum value of $\frac{x^4+y^4+z^2}{xyz}$ for positive real numbers x, y, z is

a. $\sqrt{2}$ **b.** $2\sqrt{2}$

c. $4\sqrt{2}$ **d.** $8\sqrt{2}$

2. A rod of fixed length k slides along the coordinate axes. If it meets the axes at $A(a, 0)$ and $B(0, b)$, then the minimum value of $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2$ is
   a. 0 b. 8
   c. $k^2-4+\frac{4}{k^2}$ d. $k^2+4+\frac{4}{k^2}$

3. The least value of $6\tan^2\phi + 54\cot^2\phi + 18$ is
(I) 54 when A.M. $\geq$ G.M. is applicable for $6\tan^2\phi$, $54\cot^2\phi$, 18
(II) 54 when A.M. $\geq$ G.M. is applicable for $6\tan^2\phi$, $54\cot^2\phi$ and 18 is added further (III) 78 when $\tan^2\phi = \cot^2\phi$
   a. (I) is correct, II is false b. (I) and (II) are correct
   c. (III) is correct d. none of the above are correct

4. If ab^2c^3, $a^2b^3c^4$, $a^3b^4c^5$ are in A.P. $(a, b, c > 0)$, then the minimum value of $a + b + c$ is
   a. 1 b. 3
   c. 5 d. 9

5. If $y = 3^{x-1} + 3^{-x-1}$, then the least value of y is
   a. 2 b. 6
   c. 2/3 d. 3/2

6. Minimum value of $(b+c)/a+(c+a)/b+(a+b)/c$ (for real positive numbers a, b, c) is
   a. 1 b. 2
   c. 4 d. 6

7. If the product of n positive numbers is n^n, then their sum is
   a. a positive integer b. divisible by n
   c. equal to $n + 1/n$ d. never less than n^2

8. The minimum value of $P = bcx + cay + abz$, when $xyz = abc$, is
   a. $3abc$ b. $6abc$
   c. abc d. $4abc$

9. If l, m, n be the three positive roots of the equation $x^3 - ax^2 + bx - 48 = 0$, then the minimum value of $(1/l) + (2/m) + (3/n)$ equals
   a. 1 b. 2
   c. 3/2 d. 5/2

10. If positive numbers a, b, c be in H.P., then equation $x^2 - kx + 2b^{101} - a^{101} - c^{101} = 0$ $(k \in R)$ has
   a. both roots positive
   b. both roots negative
   c. one positive and one negative root
   d. both roots imaginary

11. For $x^2-(a+3)|x|+4=0$ to have real solutions, the range of a is
   a. $(-\infty, -7] \cup [1, \infty)$ b. $(-3, \infty)$
   c. $(-\infty, -7]$ d. $[1, \infty)$

12. If a, b, c are the sides of a triangle, then the minimum value of $\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}$ is equal to
   a. 3 b. 6
   c. 9 d. 12

13. If $a, b, c, d \in R^+ - \{1\}$, then the minimum value of $\log_d a + \log_b d + \log_a c + \log_c b$ is
   a. 4 b. 2
   c. 1 d. none of these

14. If $a, b, c \in R^+$, then $\frac{bc}{b+c}+\frac{ac}{a+c}+\frac{ab}{a+b}$ is always
   a. $\leq \frac{1}{2}(a+b+c)$ b. $\geq \frac{1}{3}\sqrt{abc}$
   c. $\leq \frac{1}{3}(a+b+c)$ d. $\geq \frac{1}{2}\sqrt{abc}$

15. If $a, b, c \in R^+$, then $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ is always
   a. ≥ 12 b. ≥ 9
   c. ≤ 12 d. none of these

16. If $a, b, c \in R^+$, then the minimum value of $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2)$ is equal to
   a. abc b. $2abc$
   c. $3abc$ d. $6abc$

17. If $a, b, c, d \in R^+$ and a, b, c, d are in H.P., then
   a. $a + d > b + c$ b. $a + b > c + d$
   c. $a + c > b + d$ d. none of these

18. If $a, b, c \in R^+$ such that $a + b + c = 18$, then the maximum value of $a^2 b^3 c^4$ is equal to
   a. $2^{18} \times 3^2$ b. $2^{18} \times 3^3$
   c. $2^{19} \times 3^2$ d. $2^{19} \times 3^3$

19. $f(x)=\frac{(x-2)(x-1)}{(x-3)}, \forall x > 3$. The minimum value of $f(x)$ is equal to
   a. $3+2\sqrt{2}$ b. $3+2\sqrt{3}$
   c. $3\sqrt{2}+2$ d. $3\sqrt{2}-2$

20. If $a > 0$, then least value of $(a^3 + a^2 + a + 1)^2$ is
   a. $64a^2$ b. $16a^4$
   c. $16a^3$ d. none of these

Multiple Correct Answers Type

Solutions on page 4.12

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. If A is the area and $2s$ the sum of the sides of a triangle, then
   a. $A \leq \frac{s^2}{4}$ b. $A \leq \frac{s^2}{3\sqrt{3}}$
   c. $A < \frac{s^2}{\sqrt{3}}$ d. none of these

2. If x, y, z are positive numbers in A.P., then
   a. $y^2 \geq xz$ b. $xy + yz \geq 2xz$
   c. $\frac{x+y}{2y-x}+\frac{y+z}{2y-z} \geq 4$ d. none of these

3. For positive real numbers a, b, c such that $a + b + c = p$, which one holds?
   a. $(p-a)(p-b)(p-c) \leq \frac{8}{27}p^3$
   b. $(p-a)(p-b)(p-c) \geq 8abc$

c. $\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\le p$

d. none of these

4. If first and $(2n-1)^{th}$ terms of an A.P., G.P. and H.P. are equal and their n^{th} terms are a, b, c, respectively, then

a. $a=b=c$ b. $a+c=b$

c. $a>b>c$ d. $ac-b^2=0$

Linked Comprehension Type

Solutions on page 4.13

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

If roots of the equation $f(x) = x^6 - 12x^5 + bx^4 + cx^3 + dx^2 + ex + 64 = 0$ are positive, then

1. which has the greatest absolute value?

a. b b. c c. d d. e

2. which has the least absolute value?

a. b b. c c. d d. e

3. remainder when $f(x)$ is divided by $x-1$ is

a. 2 b. 1 c. 3 d. 10

For Problems 4–6

Equation $x^4 + ax^3 + bx^2 + cx + 1 = 0$ has real roots (a, b, c are non-negative).

4. Minimum non-negative real value of a is

a. 10 b. 9 c. 6 d. 4

5. Minimum non-negative real value of b is

a. 12 b. 15 c. 6 d. 10

6. Minimum non-negative real value of c is

a. 10 b. 9 c. 6 d. 4

Integer Type

Solutions on page 4.13

1. For $x\ge 0$, the smallest value of the function $f(x)=\frac{4x^2+8x+13}{6(1+x)}$, is.

2. Let $x^2-3x+p=0$ has two positive roots 'a' and 'b', then minimum value of $\left(\frac{4}{a}+\frac{1}{b}\right)$ is.

3. If x, y and z are positive real numbers and $x=\frac{12-yz}{y+z}$. The maximum value of (xyz) equals.

4. If a, b are c are positive and $9a+3b+c=90$ then the maximum value of $(\log a+\log b+\log c)$ is (base of the logarithm is 10).

5. Given that x, y, z are positive reals such that $xyz = 32$. If the minimum value of $x^2+4xy+4y^2+2z^2$ is equal m then the value of $m/16$ is.

6. If $x, y \in R^+$ satisfying $x+y=3$, then the maximum value of x^2y is.

7. For any $x, y \in R$, $xy > 0$ then the minimum value of $\frac{2x}{y^3}+\frac{x^3y}{3}+\frac{4y^2}{9x^4}$ is.

Archives

Solutions on page 4.14

Subjective Type

1. If a, b, c are positive real numbers. Then prove that $(a+1)^7(b+1)^7(c+1)^7 > 7^7a^4b^4c^4$.

(IIT-JEE, 2003)

Objective Type

True or false

1. For every integer $n>1$, the inequality $(n!)^{1/n}<\frac{n+1}{2}$ holds.

(IIT-JEE, 1981)

2. If x and y are positive real numbers and m, n are any positive integers, then $\frac{x^ny^m}{(1+x^{2n})(1+y^{2m})}>\frac{1}{4}$. **(IIT-JEE, 1989)**

Multiple choice questions with one correct answer

1. The least value of the expression $2\log_{10}x-\log_x(0.01)$, for $x>1$, is

a. 10 b. 2 c. -0.01 d. none of these

(IIT-JEE, 1980)

2. The product of n positive numbers is unity. Then their sum is

a. a positive integer b. divisible by n

c. equals to $n+1/n$ d. never less than n

(IIT-JEE, 1991)

3. If a, b, c are different positive real numbers such that $b+c-a$, $c+a-b$ and $a+b-c$ are positive, then $(b+c-a)(c+a-b)(a+b-c)-abc$ is

a. positive b. negative

c. non-positive d. non-negative

4. If a, b, c, d are positive real numbers such that $a+b+c+d=2$, then $M=(a+b)(c+d)$ satisfies the relation

a. $0\le M\le 1$ b. $1\le M\le 2$

c. $2\le M\le 3$ d. $3\le M\le 4$

(IIT-JEE, 2000)

5. If $a_1, a_2, \ldots, a_n$ are positive real numbers whose product is a fixed number c, then the minimum value of $a_1+a_2+\cdots+a_{n-1}+2a_n$ is

a. $n(2c)^{1/n}$ b. $(n+1)c^{1/n}$

c. $2nc^{1/n}$ d. $(n+1)(2c)^{1/n}$

(IIT-JEE, 2002)

6. If $\alpha\in\left(0,\frac{\pi}{2}\right)$, then $\sqrt{x^2+x}+\frac{\tan^2\alpha}{\sqrt{x^2+x}}$ is always greater than or equal to

a. $2\tan\alpha$ b. 1 c. 2 d. $\sec^2\alpha$

(IIT-JEE, 2004)

Multiple choice questions with one or more than one correct answer

1. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then

a. $\frac{1}{PS}+\frac{1}{ST}<\frac{2}{\sqrt{QS\times SR}}$ **b.** $\frac{1}{PS}+\frac{1}{ST}>\frac{2}{\sqrt{QS\times SR}}$

c. $\frac{1}{PS}+\frac{1}{ST}<\frac{4}{QR}$ **d.** $\frac{1}{PS}+\frac{1}{ST}>\frac{4}{QR}$

(IIT-JEE, 2008)

Integer type

1. The minimum value of the sum of real numbers a^{-5}, a^{-4}, $3a^{-3}$, 1, a^8 and a^{10} with $a > 0$ is

(IIT-JEE, 2011)

ANSWERS AND SOLUTIONS

Subjective Type

1. Consider the series $S = {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \cdots + n\,{}^nC_n$

For the series, the general term is $T_r = r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$. So,

$$S = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} n\,{}^{n-1}C_{r-1} = n\times 2^{n-1} \quad (1)$$

Now,

A.M. $\geq$ G.M.

$$\Rightarrow \frac{{}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \cdots + n\,{}^nC_n}{\frac{n(n+1)}{2}}$$

$$\geq\left[({}^nC_1)({}^nC_2)^2({}^nC_3)^3\cdots({}^nC_n)^n\right]^{\frac{1}{\frac{n(n+1)}{2}}}$$

$$\Rightarrow \frac{n\times 2^{n-1}}{\left(\frac{n(n+1)}{2}\right)} \geq \left[({}^nC_1)({}^nC_2)^2({}^nC_3)^3\cdots({}^nC_n)^n\right]^{\frac{2}{n(n+1)}}$$

$$\Rightarrow ({}^nC_1)\,({}^nC_2)^2\,({}^nC_3)^3\cdots({}^nC_n)^n \leq \left(\frac{2^n}{n+1}\right)^{\frac{n(n+1)}{2}}$$

$$\Rightarrow ({}^nC_1)({}^nC_2)^2({}^nC_3)^3\cdots({}^nC_n)^n \leq \left(\frac{2^n}{n+1}\right)^{{}^{n+1}C_2}$$

2. We have to prove that

$$(1+x_1)(1+x_2)\cdots(1+x_n) \leq 1 + S + \frac{S^2}{2!}+\frac{S^3}{3!}+\cdots+\frac{S^n}{n!}$$

Product on left-hand side suggests that we must consider G.M. of $(1+x_1)$, $(1+x_2)$, ..., $(1+x_n)$. Also, $(1+x_1)$, $(1+x_2)$, ..., $(1+x_n)$ are positive. Now,

A.M. $\geq$ G.M.

$$\Rightarrow \frac{(1+x_1)+(1+x_2)+\cdots+(1+x_n)}{n} \geq [(1+x_1)(1+x_2)\cdots(1+x_n)]^{\frac{1}{n}}$$

$$\Rightarrow [(1+x_1)(1+x_2)\cdots(1+x_n)]^{1/n} \leq \frac{n+(x_1+x_2+\cdots+x_n)}{n}$$

$$\Rightarrow (1+x_1)(1+x_2)\cdots(1+x_n) \leq \left(\frac{n+S}{n}\right)^n$$

$$\Rightarrow \text{Now } \left(1+\frac{S}{n}\right)^n$$

$$= 1 + S + \frac{\left(1-\frac{1}{n}\right)}{2!}S^2 + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}S^3 + \cdots$$

Now,

$$\frac{1-\frac{1}{n}}{2!}<1,\ \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}<1$$

and so on. Hence,

$$(1+x_1)(1+x_2)\cdots(1+x_n) \leq 1 + S + \frac{S^2}{2!}+\frac{S^3}{3!}+\cdots$$

3. Let α, β, γ be the roots of $2x^3 + ax^2 + bx + 4 = 0$. Given that all the coefficients are positive, so all the roots will be negative. Let $\alpha_1 = -\alpha$, $\alpha_2 = -\beta$, $\alpha_3 = -\gamma$. Then,

$$\alpha_1 + \alpha_2 + \alpha_3 = a/2$$

$$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = b/2$$

and

$$\alpha_1\alpha_2\alpha_3 = 2$$

Now,

A.M. $\geq$ G.M.

$$\Rightarrow \frac{\alpha_1+\alpha_2+\alpha_3}{3} \geq (\alpha_1\alpha_2\alpha_3)^{1/3}$$

$$\Rightarrow \frac{a^3}{216} \geq 2$$

$$\Rightarrow a \geq 6\times 2^{1/3} \quad (1)$$

Also,

$$\frac{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3}{3} \geq (\alpha_1\alpha_2\alpha_3)^{2/3}$$

$$\Rightarrow \frac{b^3}{216} \geq 4$$

$$\Rightarrow b \geq 6\times 4^{1/3} \quad (2)$$

Adding Eqs. (1) and (2), we get

$$a + b \geq 6(2^{1/3} + 4^{1/3})$$

4. $$\frac{1}{f(x)} = \left(x^4+\frac{16}{x^4}\right)+2\left(x^2+\frac{4}{x^2}\right)-4$$

Now,

A.M. $\geq$ G.M.

$$\Rightarrow x^4 + \frac{16}{x^4} \geq 8 \text{ and } x^2 + \frac{4}{x^2} \geq 4$$

$$\Rightarrow \frac{1}{f(x)} \geq 12 \Rightarrow f(x) \leq \frac{1}{12}$$

Again using A.M. $\geq$ G.M., we have

$$\frac{2x^6+8x^2}{2} \geq 4x^4$$

$\Rightarrow \quad 2x^6 + 8x^2 - 4x^4 \ge 4x^4 \ge 0$

$\Rightarrow \quad x^8 + 2x^6 - 4x^4 - 8x^2 + 16 > 0$

Also,

$x^4 \ge 0$

$$\Rightarrow \quad \frac{x^4}{x^8+2x^6-4x^4+8x^2+16} \ge 0$$

$\Rightarrow \quad f(x) \ge 0$

Hence, the greatest value is 1/12 and the least value is 0.

5. Since a, b, c are three distinct positive real numbers, so applying A.M. > G.M. in c^2, ab, ab, we get

$$\frac{c^2+ab+ab}{3} > (c^2b^2a^2)^{1/3}$$

$$\Rightarrow \quad \frac{c^2+2ab}{3} > (c^2aca^2)^{1/3} \quad (\because b \text{ is G.M. of } a \text{ and } c)$$

$\Rightarrow \quad c^2 + 2ab > 3ac$

6.

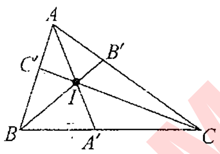


Fig. 4.1

Since angle bisector divides opposite sides in ratio of sides containing the angle, hence

$$BA' = \frac{ac}{b+c},\ CA' = \frac{ab}{b+c}$$

Now BI is also an angle bisector of $\angle B$ for triangle ABA'.

$$\therefore \quad \frac{AI}{AI'} = \frac{b+c}{a} \Rightarrow \frac{AI}{AA'} = \frac{b+c}{a+b+c}$$

Similarly,

$$\frac{BI}{BB'} = \frac{a+c}{a+b+c} \text{ and } \frac{CI}{CC'} = \frac{a+b}{a+b+c}$$

$$\Rightarrow \quad \frac{AI \times BI \times CI}{AA' \times BB' \times CC'} = \frac{(a+b)(b+c)(c+a)}{(a+b+c)^3}$$

Using A.M. ≥ GM., we get

$$\frac{2(a+b+c)}{3(a+b+c)} \ge \left[\frac{(a+b)(b+c)(c+a)}{(a+b+c)^3}\right]^{1/3}$$

$$\Rightarrow \quad \frac{(a+b)(b+c)(c+a)}{(a+b+c)^3} \le \frac{8}{27}$$

7. Using A.M. ≥ G.M, we get

$$\frac{x_1+x_2+\cdots+x_n}{n} \ge (x_1x_2\cdots x_n)^{1/n}$$

$$\Rightarrow \quad x_1x_2\cdots x_n \le \left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n$$

Therefore, maximum value of $x_1x_2x_3\cdots x_n$ is obtained when $x_1 = x_2 = x_3 = \cdots = x_n$, i.e., the parts are all equal. Now,

$$x_1+x_2+x_3+\cdots+x_n = N$$

Now, function to be maximized is $\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n$ which is a discrete function of n.

In order to arrive at some possible neighbourhood we make it continuous first. Thus changing the variable n to x, we have

$$f(x) = \left(\frac{N}{x}\right)^x$$

For maxima, $f'(x) = 0$, i.e.,

$$f'(x) = f(x)\left(\ln\left(\frac{N}{x}\right) - 1\right)$$

$\therefore \quad f'(x) = 0$ for $x = N/e$

Hence, the nearest integer is $[N/e]$ or $[N/e] + 1$ where $[x]$ denotes the greatest integer less than or equal to x.

8. A.M. of 2nd power > 2nd power of A.M.

$$\Rightarrow \quad \frac{\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2}{3}$$

$$> \left[\frac{\left(x+\frac{1}{x}\right)+\left(y+\frac{1}{y}\right)+\left(z+\frac{1}{z}\right)}{3}\right]^2$$

$$\Rightarrow \quad \frac{\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2}{3}$$

$$> \frac{1}{3}\left(x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2$$

$$\Rightarrow \quad \frac{\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2}{3} > \frac{1}{9}\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2$$

$[\because \ x+y+z=1]$

Again,

$$\frac{x^{-1}+y^{-1}+z^{-1}}{3} > \left(\frac{x+y+z}{3}\right)^{-1}$$

or

$$x^{-1}+y^{-1}+z^{-1} > 9$$

$$\Rightarrow \quad \left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2 > \frac{100}{3}$$

Objective Type

1. b. Using A.M. ≥ G.M., we have

$x^4 + y^4 \ge 2x^2y^2$ and $2x^2y^2 + z^2 \ge \sqrt{8}xyz$

$$\Rightarrow \quad \frac{x^4+y^4+z^2}{xyz} \ge \sqrt{8}$$

2. d. $a^2 + b^2 = k^2$

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 = a^2+b^2+4+\frac{1}{a^2}+\frac{1}{b^2}$$

$$= k^2+4+\frac{k^2}{a^2b^2}$$

$$\geq k^2+4+\frac{k^2}{\left(\frac{a^2+b^2}{2}\right)^2}$$

$$=k^2+4+\frac{4}{k^2}$$

3. b. Applying A.M. $\geq$ G.M. in $6\tan^2\phi$, $54\cot^2\phi$, 18, we get

$$\frac{6\tan^2\phi+54\cot^2\phi+18}{3}\geq(6\times54\times18)^{1/3}\geq 18$$

Now equality holds when

$$6\tan^2\phi=54\cot^2\phi=18$$

$$\Rightarrow\quad \tan^2\phi=3 \text{ and } \cot^2\phi=\frac{1}{3}$$

Hence, the statements I and II are correct.

4. b. A.M. $\geq$ G.M.

$$\Rightarrow\quad \frac{a+b+c}{3}\geq(abc)^{1/3}$$

$$\Rightarrow\quad a+b+c\geq 3\,(abc)^{1/3} \qquad \text{(i)}$$

But given ab^2c^3, $a^2b^3c^4$, $a^3b^4c^5$ are in A.P. ($\because abc\neq 0$). Hence,

$$2abc=1+a^2b^2c^2$$

$$\Rightarrow\quad (abc-1)^2=0$$

$$\therefore\quad abc=1$$

Now from Eq. (i), we get

$$a+b+c\geq 3(1)^{1/3}$$

$$\Rightarrow\quad (a+b+c)\geq 3$$

Hence, minimum value of $a+b+c$ is 3.

5. c. $y=\frac{3^x}{3}+\frac{1}{3^x\,3}$

Now,

A.M. $\geq$ G.M.

$$\Rightarrow\quad \frac{\frac{3^x}{3}+\frac{1}{3^x3}}{2}\geq\sqrt{\frac{3^x}{3}\frac{1}{3^x3}}$$

$$\Rightarrow\quad 3^{x-1}+3^{-x-1}\geq\frac{2}{3}$$

6. d. We have A.M. $\geq$ G.M.

$$\therefore\quad \frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\geq 6$$

$$\Rightarrow\quad \frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq 6$$

Hence, the least value is 6.

7. d. Let $a_1, a_2, \ldots, a_n$ be n positive numbers such that $a_1a_2\cdots a_n=n^n$. Since A.M. $\geq$ G.M., hence,

$$\frac{a_1+a_2+\cdots+a_n}{n}\geq(a_1a_2\cdots a_n)^{1/n}$$

$$\Rightarrow\quad \frac{a_1+a_2+\cdots+a_n}{n}\geq n$$

$$\Rightarrow\quad a_1+a_2+\cdots+a_n\geq n^2$$

8. a. A.M. $\geq$ G.M.

$$\Rightarrow\quad \frac{bcx+cay+abz}{3}\geq(a^2b^2c^2xyz)^{1/3}$$

$$\Rightarrow\quad bcx+cay+abz\geq 3xyz$$

$$\text{or}\quad bcx+acy+abz\geq 3abc$$

9. c. Consider $1/l$, $2/m$, $3/n$ and use A.M. $\geq$ G.M. Then,

$$\frac{1}{3}\left(\frac{1}{l}+\frac{2}{m}+\frac{3}{n}\right)\geq\left(\frac{1}{l}\frac{2}{m}\frac{3}{n}\right)^{1/3}=\left(\frac{6}{lmn}\right)^{1/3}$$

But $lmn=48$.

$$\therefore\quad \frac{1}{3}\left(\frac{1}{l}+\frac{2}{m}+\frac{3}{n}\right)\geq\left(\frac{6}{48}\right)^{1/3}=\frac{1}{2}$$

$$\therefore\quad \left(\frac{1}{l}+\frac{2}{m}+\frac{3}{n}\right)_{\min}=\frac{3}{2}$$

10. c. a, b, c are in H.P. Hence, H.M. of a and c is b.

$$\therefore\quad \sqrt{ac}>b \qquad (\because \text{G.M.} > \text{H.M.})$$

Since A.M. > G.M., so

$$\frac{a^{101}+c^{101}}{2}>(\sqrt{ac})^{101}>b^{101} \qquad (\because \sqrt{ac}>b)$$

$$\Rightarrow\quad 2b^{101}-a^{101}-c^{101}<0$$

Let,

$$f(x)=x^2-kx+2b^{101}-a^{101}-c^{101}$$

$$\therefore\quad f(0)=2b^{101}-a^{101}-c^{101}<0$$

Hence equation $f(x)=0$ has one root in $(-\infty, 0)$ and other in $(0, \infty)$.

11. d. $a=\frac{x^2+4}{|x|}-3$

$$=|x|+\frac{4}{|x|}-3$$

$$\geq 2\sqrt{|x|\times\frac{4}{|x|}}-3 \qquad (\because \text{A.M.}\geq\text{G.M.})$$

$$\Rightarrow a\geq 1$$

12. a.

$$2E=\frac{2a}{b+c-a}+\frac{2b}{c+a-b}+\frac{2c}{a+b-c}$$

$$=\frac{2a}{b+c-a}+1+\frac{2b}{c+a-b}+1+\frac{2c}{a+b-c}+1-3$$

$$=(a+b+c)\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)-3$$

Using A.M. $\geq$ H.M. we have

$$\frac{\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}}{3}\geq\frac{3}{a+b+c}$$

$$\Rightarrow\quad (a+b+c)\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)\geq 9$$

$$\Rightarrow\quad (a+b+c)\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)-3\geq 6$$

$$\Rightarrow\quad E\geq 3$$

13. a. A.M. $\geq$ G.M.

$$\Rightarrow\quad \frac{\log_d a+\log_c b+\log_a c+\log_b d}{4}$$

$$\geq\sqrt[4]{\frac{\log a}{\log d}\times\frac{\log b}{\log c}\times\frac{\log c}{\log a}\times\frac{\log d}{\log b}}$$

$\Rightarrow \quad \log_d a + \log_b d + \log_a c + \log_c b \geq 4$

14. a. Using A.M. and H.M. inequality, we get

$$\frac{2bc}{b+c} \leq \frac{b+c}{2}, \frac{2ac}{a+c} \leq \frac{a+c}{2}, \frac{2ab}{a+b} \leq \frac{a+b}{2}$$

$$\Rightarrow \quad \frac{bc}{b+c} + \frac{ac}{a+c} + \frac{ab}{a+b} \leq \frac{1}{2}(a+b+c)$$

15. b. $\dfrac{a+b+c}{3} \geq \dfrac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$ ($\because$ A.M. $\geq$ H.M.)

$$\Rightarrow \quad (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

16. d. $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$

$= ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2$

Using A.M. $\geq$ G.M., we get

$$\frac{ab^2+ac^2+bc^2+ba^2+ca^2+cb^2}{6} \geq (a^6b^6c^6)^{1/6}$$

$$\Rightarrow \quad a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2) \geq 6abc$$

17. a. a, b, c, d are in H.P. Hence, b is H.M. of a and c; c is H.M. of b and d. Using A.M. $\geq$ H.M., we get

$$\frac{a+c}{2} > b \Rightarrow a+c > 2b$$

and

$$\frac{b+d}{2} > c \Rightarrow b+d > 2c$$

$\Rightarrow \quad a+c+b+d > 2b+2c$

$\Rightarrow \quad a+d > b+c$

18. d. $a+b+c = 18$

$$\Rightarrow \quad 2\times\frac{a}{2}+3\times\frac{b}{3}+4\times\frac{c}{4} = 18$$

Using weighted A.M. and G.M. inequality, we get

$$\frac{2\times\frac{a}{2}+3\times\frac{b}{3}+4\times\frac{c}{4}}{9} \geq \left(\left(\frac{a}{2}\right)^2\left(\frac{b}{3}\right)^3\left(\frac{c}{4}\right)^4\right)^{1/9}$$

$$\Rightarrow \quad 2^9 \geq \frac{a^2}{2^2}\times\frac{b^3}{3^3}\times\frac{c^4}{4^4}$$

$\Rightarrow \quad a^2b^3c^4 \leq 3^3 \times 2^{19}$

19. a. Let,

$x-3 = t$

$\Rightarrow \quad x-2 = (t+1)$ and $x-1 = t+2$

$$\Rightarrow \quad f(x) = \frac{(x-2)(x-1)}{(x-3)}$$

$$= \frac{(t+1)(t+2)}{t}$$

$$= \frac{(t^2+3t+2)}{t}$$

$$= t + \frac{2}{t} + 3$$

$\geq 3 + 2\sqrt{2}$ (using A.M. $\geq$ G.M., as $t > 0$)

20. c. We have,

$$\frac{a^3+1}{2} \geq \sqrt{a^3 \times 1}$$

and

$$\frac{a^2+a^1}{2} \geq \sqrt{a^2a^1}$$

Adding, we get

$$\frac{a^3+a^2+a+1}{2} \geq 2\sqrt{a^3}$$

Multiple Correct Answers Type

1. a, b.

We have,

$2s = a+b+c$

$A^2 = s(s-a)(s-b)(s-c)$

Now, A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{s+(s-a)+(s-b)+(s-c)}{4} \geq [s(s-a)(s-b)(s-c)]^{1/4}$$

$$\Rightarrow \quad \frac{4s-2s}{4} \geq [A^2]^{1/4}$$

$\Rightarrow \quad s/2 \geq A^{1/2} \Rightarrow A \leq s^2/4$

Also,

$$\frac{(s-a)+(s-b)+(s-c)}{3} \geq [(s-a)(s-b)(s-c)]^{1/3}$$

$$\Rightarrow \quad \frac{s}{3} \geq \left[\frac{A^2}{s}\right]^{1/3}$$

$$\Rightarrow \quad \frac{A^2}{s} \leq \frac{s^3}{27}$$

$$\Rightarrow \quad A \leq \frac{s^2}{3\sqrt{3}}$$

2. a, c.

A.M. of x and z is y. G.M. of x and z is $\sqrt{xz}$.

Now, A.M. $\geq$ G.M. $\Rightarrow y^2 \geq xz$

Also, A.M $\geq$ H.M. $\Rightarrow y \geq \dfrac{2xz}{x+z}$

$$\frac{x+y}{2y-x} = \frac{x+y}{x+z-x} = \frac{x+y}{z} \text{ and } \frac{y+z}{2y-z} = \frac{y+z}{x}$$

$$\therefore \quad \frac{\frac{x+y}{2y-x}+\frac{y+z}{2y-z}}{2} \geq \sqrt{\frac{x+y}{z}\,\frac{y+z}{x}}$$

$$= \sqrt{1+\frac{y(x+y+z)}{xz}} \quad [\because x+z = 2y]$$

$$= \sqrt{1+\frac{3y^2}{xz}}$$

$$\therefore \quad \frac{x+y}{2y-x}+\frac{y+z}{2y-z} \geq 2\sqrt{1+3\frac{y^2}{xz}} \geq 4 \quad [\because y^2 \geq xz]$$

3. a, b.

Using A.M. $\geq$ G.M. one can show that

$(b+c)(c+a)(a+b) \geq 8abc$

$\Rightarrow\quad (p-a)(p-b)(p-c) \geq 8abc$

Therefore, (b) holds. Also,

$$\frac{(p-a)+(p-b)+(p-c)}{3} \geq [(p-a)(p-b)(p-c)]^{1/3}$$

$$\Rightarrow\quad \frac{3p-(a+b+c)}{3} \geq [(p-a)(p-b)(p-c)]^{1/3}$$

$$\Rightarrow\quad \frac{2p}{3} \geq [(p-a)(p-b)(p-c)]^{1/3}$$

$$\Rightarrow\quad (p-a)(p-b)(p-c) \leq \frac{8p^3}{27}$$

Therefore, (a) holds. Again,

$$\frac{1}{2}\left(\frac{bc}{a}+\frac{ca}{b}\right) \geq \sqrt{\left(\frac{bc}{a}\frac{ca}{b}\right)}$$

and so on. Adding the inequalities, we get

$$\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c} \geq a+b+c = p$$

Therefore, (c) does not hold.

4. c, d.

Consider the A.P. Since a is equidistant from the first term α and the last term β of the A.P., therefore, α, a, β are in A.P. Hence, a is the A.M. of α and β. So,

$$a = \frac{\alpha+\beta}{2}$$

Similarly, b and c are the geometric and harmonic means, i.e.,

$$b = \sqrt{\alpha\beta} \text{ and } c = \frac{2\alpha\beta}{\alpha+\beta}$$

Since A.M., G.M. and H.M. are in G.P. and A.M. $\geq$ G.M. $\geq$ H.M., therefore, a, b, c are in G.P. and $a \geq b \geq c$.

Linked Comprehension Type

For Problems 1–3

1. c, 2. a, 3. b.

Sol. Let roots of equation $x^6 - 12x^5 + bx^4 + cx^3 + dx^2 + ex + 64 = 0$ be x_i, $i = 1, 2, \ldots, 6$. Now,

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 12$$

and

$$x_1 x_2 x_3 x_4 x_5 x_6 = 64$$

Thus,

$$\frac{x_1+x_2+\cdots+x_6}{6} = 2 \text{ and } (x_1 x_2 x_3 x_4 x_5 x_6)^{1/6} = 2$$

$\Rightarrow\quad$ A.M. = G.M.

$\Rightarrow\quad x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 2$

Hence, the given equation is equivalent to

$$(x-2)^6 = 0$$

or

$$x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x - 64 = 0$$

$\therefore\quad f(1) = 1 - 12 + 60 - 160 + 240 - 192 + 64 = 1$

For Problems 4–6

4. d, 5. c, 6. d.

Sol. For non-negative values of a, roots must be negative. Let the roots be x_1, x_2, x_3, x_4 (< 0). Then,

$$x_1 + x_2 + x_3 + x_4 = -a$$

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = b$$

$$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 = -c$$

$$x_1x_2x_3x_4 = 1$$

Now,

$$\frac{(-x_1)+(-x_2)+(-x_3)+(-x_4)}{4} \geq [(-x_1)(-x_2)(-x_3)(-x_4)]^{1/4}$$

(∵ A.M. ≥ G.M.)

$$\Rightarrow\quad \frac{a}{4} \geq 1$$

$\Rightarrow\quad a \geq 4$

Hence, the minimum value of a is 4. Similarly,

$$\frac{x_1x_2 + x_1x_3 + \cdots + x_3x_4}{6} \geq \left[x_1{}^3x_2{}^3x_3{}^3x_4{}^3\right]^{1/4}$$

$$\Rightarrow\quad \frac{b}{6} \geq 1$$

$\Rightarrow\quad b \geq 6$

Hence, the minimum value of b is 6. Finally,

$$\frac{-x_1x_2x_3 - x_2x_3x_4 - x_3x_4x_1 - x_4x_1x_2}{4} \geq \left[x_1{}^3x_2{}^3x_3{}^3x_4{}^3\right]^{1/4}$$

$$\Rightarrow\quad \frac{c}{4} \geq 1$$

$\Rightarrow\quad c \geq 4$

Hence, the minimum value of c is 4.

Integer Type

1.(2)
$$f(x) = \frac{4x^2+8x+13}{6(1+x)}$$

$$= \frac{4(x+1)^2+9}{6(1+x)}$$

$$= \frac{2}{3}(x+1) + \frac{3}{2(x+1)}$$

$$\geq 2\sqrt{\frac{2}{3}\cdot\frac{3}{2}} = 2 \qquad \text{(A.M. ≥ G.M.)}$$

Therefore, minimum value of $f(x)$ is 2.

2.(3) $a + b = 3$

HM ≤ AM for 3 numbers $\frac{a}{2}, \frac{a}{2}, b$ we have

$$\frac{3}{\frac{2}{a}+\frac{2}{a}+\frac{1}{b}} \leq \frac{\frac{a}{2}+\frac{a}{2}+b}{3} = 1;$$

$$\therefore\ 1 \geq \frac{3}{\frac{2}{a}+\frac{2}{a}+\frac{1}{b}} \Rightarrow \frac{2}{a}+\frac{2}{a}+\frac{1}{b} \geq 3$$

$\therefore \frac{4}{a}+\frac{1}{b} \ge 3$

3.(8) Consider values xy, yz, zx

Now A.M. $\ge$ G.M.

$\Rightarrow \frac{xy+yz+zx}{3} \ge (x^2y^2z^2)^{1/3}$

$\Rightarrow 4^{3/2} \ge xyz$

$\Rightarrow xyz \le 8$

4.(3) $9a+3b+c=90$

$\Rightarrow 3a+b+\frac{c}{3}=30$

Now consider numbers $3a$, b and $\frac{c}{3}$

$\left(3a \times b \times \frac{c}{3}\right)^{\frac{1}{3}} \le \frac{3a+b+\frac{c}{3}}{3}$ (as GM $\le$ AM)

$\Rightarrow (abc)^{1/3} \le \frac{30}{3}=10$

$\Rightarrow abc \le 1000$

$\Rightarrow \log a+\log b+\log c \le 3$

5.(6) Using AM $\ge$ GM for $x^2, 2xy, 2xy, 4y^2, z^2, z^2$

$\therefore \frac{x^2+2xy+2xy+4y^2+z^2+z^2}{6} \ge [16(xyz)^4]^{1/6}$

$= [16(32)^4]^{1/4} = (224)^{1/6} = 16$

$\Rightarrow m/16 = 6$

6.(4) We have $\frac{2\left(\frac{x}{2}\right)+y}{3} \ge \left(\left(\frac{x}{2}\right)^2 y\right)^{1/3}$

$\Rightarrow \left(\frac{3}{3}\right)^3 \ge \frac{x^2y}{4} \Rightarrow x^2y \le 4$

Therefore, maximum value of x^2y is 4.

7.(2) As $x, y \in R$ and $xy > 0$, so x and y will be of same sign.

Therefore, all the quantities $\frac{2x}{y^3}, \frac{x^3y}{3}, \frac{4y^2}{9x^4}$ are positive.

Now A.M. $\ge$ G.M.

$\Rightarrow \frac{2x}{y^3}+\frac{x^3y}{3}+\frac{4y^2}{9x^4} \ge 3\left(\left(\frac{2x}{y^3}\right)\left(\frac{x^3y}{3}\right)\left(\frac{4y^2}{9x^4}\right)\right)^{1/3}$

$= 3 \times \frac{2}{3} = 2$

Archives

Subjective Type

1. Given that a, b, c are positive real numbers. To prove that $(a+1)^7 (b+1)^7 (c+1)^7 > 7^7 a^4 b^4 c^4$.

L.H.S. $= (1+a)^7 (1+b)^7 (1+c)^7$

$= [(1+a)(1+b)(1+c)]^7$

$= [1+a+b+c+ab+bc+ca+abc]^7$ (1)

Now,

A.M. $\ge$ G.M.

$\Rightarrow \frac{a+b+c+ab+bc+ca+abc}{7} \ge \left(a^4b^4c^4\right)^{1/7}$

$\Rightarrow (a+b+c+ab+bc+ca+abc)^7 \ge 7^7 (a^4 b^4 c^4)$ (2)

From Eqs. (1) and (2), we get

$[(1+a)(1+b)(1+c)]^7 > 7^7 a^4b^4c^4$

Objective Type

True or false

1. Consider n numbers, namely 1, 2, 3, 4, ..., n. Now, using A.M. > G.M. for distinct numbers, we get

$\frac{1+2+3+4+\cdots+n}{n} > (1 \times 2 \times 3 \times 4 \times \cdots \times n)^{1/n}$

$\Rightarrow \frac{n(n+1)}{2n} > (n!)^{1/n}$

$\Rightarrow (n!)^{1/n} < \frac{n+1}{2}$

Hence, the statement is true.

2. As x and y are positive real numbers and m and n are positive integers, therefore,

$\frac{1+x^{2n}}{2} \ge \left(1 \times x^{2n}\right)^{1/2}$

and

$\frac{1+y^{2m}}{2} \ge \left(1 \times y^{2m}\right)^{1/2}$

(since for two positive numbers A.M. $\ge$ G.M.)

$\therefore \left(\frac{1+x^{2n}}{2}\right) \ge x^n$ (1)

and

$\left(\frac{1+y^{2m}}{2}\right) \ge y^m$ (2)

Multiplying Eqs. (1) and (2), we get

$\frac{\left(1+x^{2n}\right)\left(1+y^{2m}\right)}{4} \ge x^n y^m$

$\Rightarrow \frac{1}{4} \ge \frac{x^n y^m}{\left(1+x^{2n}\right)\left(1+y^{2m}\right)}$

Hence, the statement is false.

Multiple choice questions with one correct answer

1. b. $2\log_{10} x - \log_x 0.01 = 2\log_{10} x - \frac{\log_{10} 0.01}{\log_{10} x}$

$= 2\log_{10} x + \frac{2}{\log_{10} x}$

$$= 2\left[\log_{10} x + \frac{1}{\log_{10} x}\right]$$

[Here $x > 1 \Rightarrow \log_{10} x > 0$]

Now,

A.M. $\geq$ G.M.

$$\Rightarrow \frac{\log_{10} x + \frac{1}{\log_{10} x}}{2} \geq \left(\log_{10} x \frac{1}{\log_{10} x}\right)^{1/2}$$

$$\Rightarrow \log_{10} x + \frac{1}{\log_{10} x} \geq 2$$

2. d. Let $x_1, x_2, \ldots, x_n$ be the n positive numbers. Given that

$$x_1 x_2 x_3 \cdots x_n = 1 \quad (1)$$

We know for positive numbers,

A.M. $\geq$ G.M.

$$\Rightarrow \frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

$$\Rightarrow \frac{x_1 + x_2 + \cdots + x_n}{n} \geq 1 \quad \text{[Using Eq. (1)]}$$

$$\Rightarrow x_1 + x_2 + \cdots + x_n \geq n$$

3. b. Since A.M. > G.M. for different numbers, so

$$\frac{(b+c-a)+(c+a-b)}{2} > [(b+c-a)(c+a-b)]^{1/2}$$

$$\Rightarrow c > [(b+c-a)(c+a-b)]^{1/2}$$

Similarly,

$$b > [(b+c-a)(a+b-c)]^{1/2}$$

and

$$a > [(a+b-c)(c+a-b)]^{1/2}$$

Multiplying, we get

$$abc > (b+c-a)(c+a-b)(a+b-c)$$

$$\Rightarrow (b+c-a)(c+a-b)(a+b-c) - abc < 0$$

4. a. As A.M. $\geq$ G.M. for positive real numbers, we get

$$\frac{(a+b)+(c+d)}{2} \geq \sqrt{(a+b)(c+d)}$$

$$\Rightarrow M \leq 1$$

Also,

$$(a+b)(c+d) > 0 \quad [\because a, b, c, d > 0]$$

$$\therefore \quad 0 \leq M \leq 1$$

5. a. From A.M. $\geq$ G.M., we have

$$\frac{(a_1 + a_2 + \cdots + a_{n-1} + 2a_n)}{n} \geq (a_1 a_2 \cdots a_{n-1} 2a_n)^{1/n}$$

$$\Rightarrow \frac{(a_1 + a_2 + \cdots + a_{n-1} + 2a_n)}{n} \geq (2c)^{1/n}$$

$$\Rightarrow a_1 + a_2 + \cdots + a_{n-1} + 2a_n \geq n(2c)^{1/n}$$

6. a. Using A.M. $\geq$ G.M., we have

$$\frac{\sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}}}{2} \geq \left(\sqrt{x^2+x}\,\frac{\tan^2\alpha}{\sqrt{x^2+x}}\right)^{1/2}$$

$$\Rightarrow \sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}} \geq 2\tan\alpha$$

Multiple choice questions with one or more than one correct answer

1. b, d.

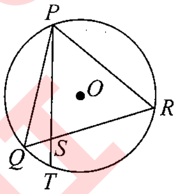


Fig. 4.2

We have $PS \times ST = QS \times SR$ (property of circle). Now,

A.M. > G.M.

$$\Rightarrow \frac{\frac{1}{PS} + \frac{1}{ST}}{2} > \sqrt{\frac{1}{PS} \times \frac{1}{ST}}$$

$$\Rightarrow \frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$$

Also,

$$\frac{QS + SR}{2} > \sqrt{QS \times SR}$$

$$\Rightarrow \frac{1}{\sqrt{QS \times SR}} > \frac{2}{QR}$$

$$\Rightarrow \frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$$

Integer type

1. (8) Using A.M. $\geq$ G.M.

$$\Rightarrow \frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + a^8 + a^{10} + 1}{8} \geq 1$$

$$\Rightarrow a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + a^8 + a^{10} + 1 \geq 8$$

$$\Rightarrow \left(a^{-5} + a^{-4} + 3a^{-3} + a^8 + a^{10} + 1\right)_{\min} = 8$$

CHAPTER 5

Permutation and Combination

FUNDAMENTAL PRINCIPLE OF COUNTING

Multiplication Rule

If a work A can be done in m ways and another work B can be done in n ways and C is a work, which is done only when both A and B are completed, the number of ways of doing the work C is $m \times n$. In other words, if an operation can be performed in m different ways and corresponding to each of these there are n different ways of performing another operation, then both the operations can be performed in $m \times n$ different ways.

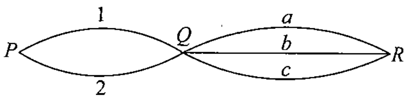


Fig. 5.1

First of all, we give an example to show the validity of the above principle. Suppose, there are three stations P, Q and R and we have three routes to go from P to Q and two routes to go from Q to R. We want to know the number of routes to go from P to R.

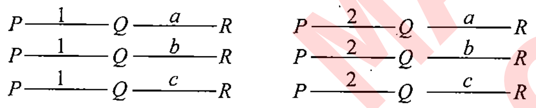


Fig. 5.2

For each path to go from P to Q, there are three paths to go from Q to R.

Thus, for going from P to R via Q, there will be $2 \times 3 = 6$ paths.

Proof of the Multiplication Rule of Fundamental Principle of Counting

The first operation can be performed in any one of the m ways and for each of these ways of performing the first operation, there are n ways of performing the second operation. Thus, if the first operation could be performed in one such way, there would have been $1 \times n = n$ ways of performing both the operations. But it is given that first operation can be performed in m ways and for each way of performing the first operation, second can be performed in n ways.

Therefore, the total number of ways of performing both the operations is $n + n + n + \cdots$ to m terms $= n \times m$.

Note: *If three operations can be separately performed in m, n and p ways, respectively, then the three operations together can be performed in $m \times n \times p$ ways. Similar result holds for any number of operations.*

Addition Rule

If a work A can be done in m ways and another work B can be done in n ways and C is a work which is done only when either A or B is completed, then number of ways of doing the work C is $m + n$.

Example:

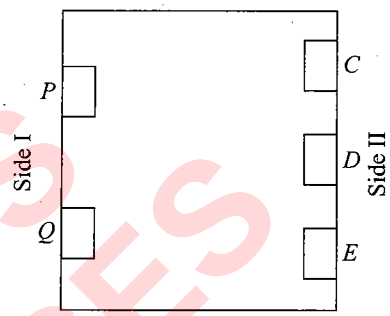


Fig. 5.3

Suppose, there are 5 doors in a room: 2 on one side and 3 on the other. A man has to go out of the room. The man can go out from any one of the 5 doors. Thus, the number of ways in which the man can go out is 5. Here, the work of going out through the doors on one side will be done in 2 ways and the work of going out through the doors on other side will be done in 3 ways. The work of going out will be done when the man goes out from either side I or side II. Thus, the work of going out can be done in $2 + 3 = 5$ ways.

Example 5.1 **Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word MAKE, where the repetition of the letters is not allowed.**

Sol. There are as many words as there are ways of filling in 4 vacant places XXXX by the 4 letters, keeping in mind that the repetition is not allowed. The first place can be filled in 4 different ways by anyone of the 4 letters M, A, K, E.

Following which, the second place can be filled in by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways; following which, the fourth place can be filled in 1 way.

Thus, the number of ways in which the4 places can be filled, by the multiplication principle, is $4 \times 3 \times 2 \times 1 = 24$.

Hence, the required number of words is 24.

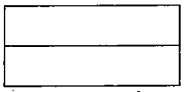
Example 5.2 **How many 2 digit even numbers can be formed from the digits1, 2, 3, 4, 5 if the digits can be repeated?**

Sol. There will be as many ways as there are ways of filling 2 vacant places XX in succession by the five given digits. Here, we start filling in unit's place, because the options for this place are 2 and 4 only and this can be done in 2ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated.

Therefore, by the multiplication principle, the required number of two digits even numbers is 2×5, i.e., 10.

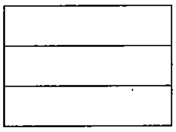
Example 5.3 **Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.**

Sol. A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 2 flags, 3 flags, 4 flags and 5 flags separately and then add the respective numbers.



There will be as many 2 flag signals as there are ways of filling in 2 vacant places in succession by the 5 flags available. By Multiplication rule, the number of ways is $5 \times 4 = 20$.

Similarly, there will be as many 3 flag signals as there are ways of filling in 3vacant places in succession by the 5 flags.



The number of ways is $5 \times 4 \times 3 = 60$. Continuing the same way, we find that the number of 4 flag signals $= 5 \times 4 \times 3 \times 2 = 120$ and the number of 5 flag signals $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Therefore, the required no of signals $= 20 + 60 + 120 + 120 = 320$.

Example 5.4 **Poor Dolly's T.V. has only 4 channels; all of them quite boring, hence it is not surprising that she desires to switch (change) channel after every one minute. Then find the number of ways in which she can change the channels so that she is back to her original channel for the first time after 4 minutes.**

Sol. Let there be 4 channels C_1 C_2 C_3 and C_4

at $t = 0$ minute she is watching channel 1

$\therefore$ after 1st minute she has 3 choices to switch the channel (C_2, C_3, C_4)

after 2nd minute she has 2 choices to switch the channel

after 3rd minute she has 2 choices to switch the channel

but after the 4th minute she has only 1 choice to switch the channel i.e. C_1

$\therefore$ Total number of ways $= 3 \times 2 \times 2 = 12$

Example 5.5 **There are 'n' locks and 'n' matching keys. If all the locks and keys are to be perfectly matched, find the maximum number of trials required to open a lock.**

Sol. The maximum number of trials needed for the first key is n. For second key, it will be $n - 1$.

Now, for the r^{th} key, the maximum number of trials needed is $n - r + 1$. Thus, the required answer is

$$n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}$$

Example 5.6 **Find the 2-digit number (having different digits), which is divisible by 5.**

Sol. Any number of required type ends in either 5 or 0. Hence, the two-digit number (with different digits) that ends in 5 is 8 and that of 0 is 9. Therefore, by addition principle, the required number is $8 + 9 = 17$.

Example 5.7 **Find the total number of ways in which n distinct objects can be put into two different boxes.**

Sol. Let the two boxes be B_1 and B_2. For each of the n objects, there are two choices, it is put in either box B_1 or box B_2. Therefore, by fundamental principle of counting, the total number of ways is $2 \times 2 \times \cdots \times 2$ (n times) $= 2^n$.

Example 5.8 **Three dice are rolled. Find the number of possible outcomes in which at least one dice shows 5.**

Sol. When a dice is rolled, there are six possible outcomes. So, the total number of outcomes when three dice are rolled is $6 \times 6 \times 6 = 6^3$.

Now, the number of possible outcomes in which at least one dice shows 5 is as follows.

Total number of possible outcomes – Number of possible outcomes in which 5 does not appear on any dice $= 6^3 - 5^3 = 91$

Example 5.9 **Find the number of distinct rational numbers x such that $0 < x < 1$ and $x = p/q$, where $p, q \in \{1, 2, 3, 4, 5, 6\}$.**

Sol. As $0 < x < 1$, we have $p < q$.

p	q
1	2, 3, 4, 5, 6
2	3, 4, 5, 6
3	4, 5, 6
4	5, 6
5	6

Thus, the number of rational numbers is $5+4+3+2+1=15$.

When p and q have a common factor, we get some rational numbers, which are not different from those already counted. Here, there are four such numbers: 2/4, 2/6, 3/6, 4/6.

Therefore, the required number of rational numbers is $15-4=11$.

Example 5.10 **Find the total number of integer 'n' such that $2 \le n \le 2000$ and H.C.F. of n and 36 is 1.**

Sol. $36 = 2^2 \times 3^2$

If H.C.F. of integer 'n' and 36 is 1, then n should not be divisible by 2 or 3.

Let us first find the numbers that are divisible by 2 or 3.

The number of integers in the range [2, 2000] that are divisible by 2 is 1000 (2, 4, 6,..., 1998, 2000).

The number of integers in the range [2, 2000] that are divisible by 3 is 666 (3, 6, 9, ..., 1995, 1998).

The number of integers in the range [2, 2000] that are divisible by 6 is 333 (6, 12, 18,..., 1992, 1998).

Total number of integers divisible by 2 or 3 is $1000 + 666 - 33 = 1333$.

Thus, the total number of integers that are divisible by neither 2 nor 3 is $1999 - 1333 = 666$.

Example 5.11 **Find the number of polynomials of the form $x^3 + ax^2 + bx + c$ that are divisible by $x^2 + 1$, where $a, b, c \in \{1, 2, 3, \ldots, 9, 10\}$.**

Sol.

$$\begin{array}{r|l} & x+a \\ \hline x^2+1 & x^3+ax^2+bx+c \\ & x^3+x \\ \hline & ax^2+(b-1)x+c \\ & ax^2+a \\ \hline & (b-1)x+c-a \end{array}$$

Now, remainder $(b-1)x+c-a$ must be zero for any x. Then,

$b - 1 = 0$ and $c - a = 0$

$\Rightarrow$ $b = 1$ and $c = a$

Now, c or a can be selected in 10 ways. Hence, number of polynomials are 10.

Example 5.12 **Find the number of diagonals in the polygon of n sides.**

Sol.

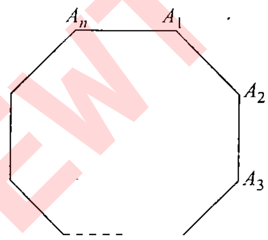


Fig. 5.4

For diagonal, we have to select any two vertices. The first vertex can be selected in n ways. Let A_1 be chosen as first vertex. Now, diagonal cannot be formed if any of A_2 and A_n is chosen. Hence, for A_1 another vertex can be selected in $n - 3$ ways from remaining $n - 3$ vertices.

Again, by principle of counting, the number of ways two vertices can be selected is $n(n - 3)$.

Now, when A_1 is chosen as the first vertex, sometimes A_4 is chosen as the second vertex.

Similarly, when A_4 is chosen as the first vertex, sometimes A_1 is chosen as the second vertex.

Hence, each pair is selected twice. Therefore, the total number of diagonals is $n(n - 3)/2$.

Example 5.13 **Find the total number of 'n'-digit numbers ($n > 1$), having the property that no two consecutive digits are same.**

Sol.

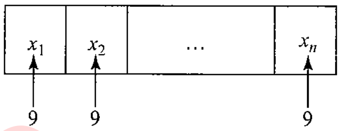


Fig. 5.5

The digit 'x_1' can be selected in 9 ways as '0' cannot be selected.

The digit 'x_2' can be selected in 9 ways as '0' can be selected but digit in position x_1 cannot be selected.

Similarly, each of the remaining digits can also be selected in 9 ways.

Thus, the total number of such numbers is 9^n.

Concept Application Exercise 5.1

1. Four buses run between Bhopal and Gwalior. If a man goes from Gwalior to Bhopal by a bus and comes back to Gwalior by another bus, find the total possible ways.
2. A gentleman wants to invite six friends. In how many ways can he send invitation cards to them, if he has three servants to carry the cards.
3. Find the total number of ways of answering five objective type questions, each question having four choices.
4. In how many ways first and second rank in Mathematics, first and second rank in Physics, first rank in Chemistry and first rank in English be given away to a class of 30 students.
5. Five persons entered the lift cabin on the ground floor of an 8-floor building. If, each of them can leave the cabin independently at any floor beginning with the first; find the total number of ways in which each of the five persons can leave the cabin: (i) at any one of the 7 floors and (ii) at different floors.
6. If $p, q \in \{1, 2, 3, 4\}$, then find the number of equations of the form $px^2 + qx + 1 = 0$ having real roots.
7. Find the number of non-zero determinant of order 2 with elements 0 or 1 only.
8. Find the number ordered pairs (x, y) if $x, y \in \{0, 1, 2, 3, \ldots, 10\}$ and if $|x - y| > 5$.
9. **(a)** If $a, b \in \{1, 2, 3, 4, 5, 6\}$, find the number of ways a and b can be selected if

$$\lim_{x \to 0} \left(\frac{a^x + b^x}{2} \right)^{\frac{2}{x}} = 6$$

(b) If $a, b, c \in \{1, 2, 3, 4, 5, 6\}$, find the number of ways a, b, c can be selected if $f(x) = x^3 + ax^2 + bx + c$ is an increasing function.

10. Find the number of ways in which two small squares can be selected on the normal chessboard if they are not in same row or same column.

FACTORIAL NOTATION

The product of first n natural number is denoted by $n!$ and is read as 'factorial n'. Thus,

$$n! = 1 \times 2 \times 3 \times 4 \cdots (n-1) \times n$$
$$= n(n-1)(n-2) \cdots 3 \times 2 \times 1$$

For example $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$, $4! = 1 \times 2 \times 3 \times 4 = 24$.

Some Results Related to Factorial *n*

- $n! = 1 \times 2 \times 3 \cdots (n-1)n = \{1 \times 2 \times 3 \cdots (n-1)\}n$

  Hence,
  $$n! = (n-1)!n = n(n-1)!$$
  Similarly,
  $$(n-1)! = (n-1)(n-2)!$$
  Thus,
  $$n! = n(n-1)!$$
  $$= n(n-1)(n-2)!$$
  $$= n(n-1)(n-2)(n-3)!$$
  and so on

- If n and r are positive integers, then
  $$\frac{n!}{r!} = \frac{1 \times 2 \times 3 \times \cdots \times n}{1 \times 2 \times 3 \times \cdots \times r}$$
  $$= (r+1)(r+2)\cdots(n-1)n$$
  $$= n(n-1)(n-2)\cdots(r+1)$$

- $$\frac{n!}{(n-r)!} = \frac{1 \times 2 \times 3 \times \cdots \times (n-1)n}{1 \times 2 \times \cdots \times (n-r)}$$
  $$= \frac{1 \times 2 \times 3 \times \cdots \times (n-r)(n-r+1)(n-r+2) \times \cdots \times (n-1)n}{1 \times 2 \times 3 \times \cdots \times (n-r)}$$
  $$= (n-r+1)(n-r+2)\cdots(n-1)n$$
  $$= n(n-1)(n-2)\cdots(n-r+2)(n-r+1)$$
  $$= n(n-1)(n-2)\cdots \text{ to } r \text{ factors}$$

Exponent of Prime in *n!*

Let p be a given prime and n any positive integer. Then the maximum power of p present in $n!$ is $[n/p] + [n/p^2] + [n/p^3] + \cdots$ where $[x]$ denotes the greatest integer function. The proof of the above formula can be obtained using the fact that $[n/m]$ gives the number of integral multiples of m in $1, 2, \ldots, n$ for any positive integers n and m. The above formula does not work for composite numbers. For example, if we find the maximum power of 6 present in 32!, we find that the answer is not $[32/6] + [32/6^2] + \cdots = 5$, as 5 is the number of integral multiples of 6 in $1, 2, \ldots, 32$ and 6 can also be obtained by multiplying 2 and 3.

Hence, for the required number, we find the maximum powers of 2 and 3 (say r and s) present in 32! using the above formula $r = 31$ and $s = 14$. Hence, 2 and 3 will be combined 14 times (to form 6). Thus, maximum power of 6 present in 32! is 14.

Example 5.14 **Find n, if $(n+1)! = 12 \times (n-1)$.**

Sol. $(n+1)! = 12 \times (n-1)!$

$\Rightarrow (n+1) \times n \times (n-1)! = 12 \times (n-1)!$

$\Rightarrow n(n+1) = 12$

$\Rightarrow n^2 + n - 12 = 0$

$\Rightarrow (n+4)(n-3) = 0$

$\Rightarrow n = 3$

Example 5.15 **Prove that $(n!)^2 < n^n\, n! < (2n)!$ for all positive integers n.**

Sol. We have,

$(n!)^2 = (n!)(n!) = (1 \times 2 \times 3 \times 4 \times \cdots \times (n-1)n)(n!)$

Now, $1 \le n, 2 \le n, 3 \le n, \ldots, n \le n$

$\Rightarrow 1 \times 2 \times 3 \cdots (n-1)n \le n \times n \times n \cdots n$

$\Rightarrow n! \le n^n$

$\Rightarrow (n!)(n!) \le (n!)\,n^n$

$\Rightarrow (n!)^2 \le n^n (n!)$ (1)

Also,

$(2n)! = 1 \times 2 \cdots n \times (n+1) \cdots (2n-1) \times (2n)$

Now,

$n+1 > n, n+2 > n, n+3 > n, \ldots, n+n > n$

$\Rightarrow (n+1)(n+2)(n+3)\cdots(2n-1)(2n) > n^n$

$\Rightarrow n!\,(n+1)(n+2)\cdots(2n-1)(2n) > n!\,n^n$

$\Rightarrow (2n)! > n!\,n^n \Rightarrow n!\,n^n < (2n)!$ (2)

From (1) and (2), we get $(n!)^2 \le n^n (n!) < (2n)!$

Example 5.16 **Find the sum of the series $\sum_{r=1}^{n} r \times r!$**

Sol. Here, the general term of the series is

$T_r = r \times r! = (r+1-r)r! = (r+1)r! - r! = (r+1)! - r!$

Hence,

$T_1 = 2! - 1!$

$T_2 = 3! - 2!$

$T_3 = 4! - 3!$

$T_n = (n+1)! - n!$

Adding all the above terms, we have the sum of n, terms, i.e.,

$S_n = (n+1)! - 1$

Example 5.17 **Find the exponent of 3 in 100!**

Sol. $100! = 1 \times 2 \times 3 \times \cdots \times 98 \times 99 \times 100$

$= (1 \times 2 \times 4 \times 5 \times \cdots \times 98 \times 100)$
$(3 \times 6 \times 9 \times \cdots \times 96 \times 99)$

$= K \times 3^{33}(1 \times 2 \times 3 \times \cdots \times 32 \times 33)$

$= K \times 3^{33}(1 \times 2 \times 4 \times \cdots \times 31 \times 32)$
$(3 \times 9 \times 12 \times \cdots \times 30 \times 33)$

$= [K(1 \times 2 \times 4 \times \cdots \times 31 \times 32)] \times 3^{33}$
$\times (3 \times 9 \times 12 \times \cdots \times 30 \times 33)$

$= K_1 \times 3^{33} \times 3^{11}(1 \times 2 \times 3 \times \cdots \times 10 \times 11)$

$= K_1 \times (1 \times 2 \times 4 \times \cdots \times 10 \times 11)\, 3^{33} \times 3^{11}$
$(3 \times 6 \times 9 \times 12)$

$= K_2 \times 3^{33} \times 3^{11} \times 3^4 \times (1 \times 2 \times 3 \times 4)$

$= K_3 \times 3^{33} \times 3^{11} \times 3^4 \times 3$

$= K_3 \times 3^{49}$

Hence, exponent of 3 is 49.

Alternative solution:

Exponent of 3 in 100! is

$$\left[\frac{100}{3}\right]+\left[\frac{100}{3^2}\right]+\left[\frac{100}{3^3}\right]+\left[\frac{100}{3^4}\right]=33+11+4+1=49$$

Example 5.18 **Find the number of zeroes at the end of 130.**

Sol. Number of zeroes at the end of 130! is equal to the exponent of 10 in 130. Now, exponent of 10 is equal to exponent of 5 as exponent of 2 is higher than exponent of 5. Now, exponent of 5 is

$$\left[\frac{130}{5}\right]+\left[\frac{130}{5^2}\right]+\left[\frac{130}{5^3}\right]=26+5+1=32$$

Also, exponent of 10 is 32 and hence, there are 32 zeros at the end of 130. It should be noted that exponent of 2 is

$$\left[\frac{130}{2}\right]+\left[\frac{130}{2^2}\right]+\left[\frac{130}{2^3}\right]+\left[\frac{130}{2^4}\right]+\left[\frac{130}{2^5}\right]+\left[\frac{130}{2^6}\right]+\left[\frac{130}{2^7}\right]$$

$$=65+32+16+8+4+2+1=128$$

Hence, exponent of 10 is equal to exponent of 5.

Concept Application Exercise 5.2

1. Prove that $(2n)!/n = \{1\times 3\times 5 \cdots (2n-1)\}\ 2^n$.
2. Show that $1!+2!+3!+\cdots+n!$ cannot be a perfect square for any $n \in N, n \geq 4$.
3. Prove that $(n!+1)$ is not divisible by any natural number between 2 and n.
4. Find the remainder when $1!+2!+3!+4!+\cdots+n!$ is divided by 15, if $n \geq 5$.
5. Find the exponent of 80 in 200!

PERMUTATION

Each of the different arrangements that can be made by taking some or all of a number of given things or objects at a time is called a permutation. In permutation, order of appearance of things is taken into account.

Example:

The following six arrangements can be made with three distinct objects a, b, c taking two at a time: ab, ba, bc, cb, ac, and ca. Each of these arrangements is called a permutation.

Number of Permutations of n Different Things Taken r at a Time

To establish the formula $^nP_r = n!/(n-r)!$

Proof:

^nP_r is number of permutations of r things out of n different things, i.e., number of ways of filling up r vacant places with n different things. (In each place, exactly one object is put.)

Let the n different things be $a_1, a_2, a_3, \ldots, a_n$.

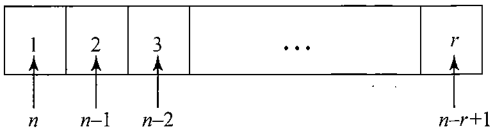


Fig. 5.6

First place can be filled up by any one of the n things $a_1, a_2, a_3, \ldots, a_n$ in n ways.

Number of things left after filling up the first place is $n-1$.

Second place can be filled up by any one of the remaining $n-1$ things in $n-1$ ways.

Number of things left after filling up the first and second places is $n-2$.

Third place can be filled up by any one of the remaining $n-2$ things in $n-2$ ways.

The number of ways of filling up the third place is $n-2$ and so on.

Finally, the number of ways of filling up the r^{th} place is $n-(r-1)=n-r+1$.

By the multiplication rule of counting, first, second, third, ..., r^{th} places can together be filled up in $n(n-1)(n-2)\cdots(n-r+1)$ ways. Hence,

$$^nP_r = n(n-1)(n-2)\cdots(n-r+1)$$

$$=\frac{[n(n-1)(n-2)\cdots(n-r+1)](n-r)\cdots 3\times 2\times 1}{(n-r)(n-r-1)\ 3\times 2\times 1}$$

$$=\frac{n!}{(n-r)!}$$

Thus,

$$^nP_r = \frac{n!}{(n-r)!}$$

Number of Permutations of n Different Things Taken All at a Time Is $n!$

Proof:

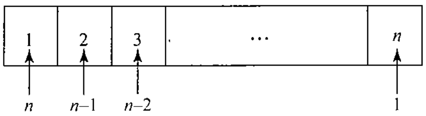


Fig. 5.7

By multiplication rule, number of ways of filling up the first, second, third, ... n^{th} places together is

$$n(n-1)(n-2)\cdots 2\times 1 = n!$$

Thus, $^nP_n = n!$

Note:

$$^nP_n = {^nP_{n-1}} = n!$$

Factorial Zero

From the formula ^nP_r, we have

$$^nP_n = \frac{n!}{0!} \quad (1)$$

Also, number of permutations of n different things taken all at a time is $n!$ (2)

From Eq. (1), number of permutations of n different things taken all at a time,

$$^nP_n = \frac{n!}{0!} \quad (3)$$

From Eq. (2) and (3),

$$n! = \frac{n!}{0!} \quad (4)$$

Again, Eq. (4) will be valid only when 0! is taken as 1.

Thus, 0! has no meaning from the definition of factorial.

But in order to make the formula for $^nP_r = n!/(n-r)!$ valid for $r = n$, 0! is taken as 1.

Meaning of* 1/(–k)! *where k Is a Positive Integer

$$^nP_r = \frac{n!}{(n-r)!} \quad (1)$$

Putting $r = n + k$, we have

$$^nP_{n+k} = \frac{n!}{(-k)!} \quad (2)$$

But, number of ways of arranging $n + k$ out of n different things is 0.

$$\therefore \quad \frac{n!}{(-k!)} = 0, \text{ i.e., } \frac{1}{(-k)!} = 0$$

Note: Although $(-k)!$ has no meaning by the definition of factorial but if $1/(-k)!$ is taken as 0 (zero), then the formula $^nP_r = n!/(n-r)!$ will become valid even for $r > n$.

Example 5.19 **If $^{10}P_r = 5040$, find the value of r.**

Sol. $^{10}P_r = 5040$

$= 10 \times 504$

$= 10 \times 9 \times 8 \times 7$

$= {}^{10}P_4$

$\Rightarrow \quad r = 4$

Example 5.20 **If $^9P_5 + 5\,{}^9P_4 = {}^{10}P_r$, find the value of r.**

Sol. $^{10}P_r = {}^9P_5 + 5\,{}^9P_4$

$$= \frac{9!}{(9-5)!} + 5 \times \frac{9!}{(9-4)!}$$

$$= \frac{9!}{4!} + 5 \times \frac{9!}{5!}$$

$$= \frac{9!}{4!} + \frac{9!}{4!}$$

$$= 2 \times \frac{9!}{4!}$$

$$= \frac{5 \times 2 \times 9!}{5 \times 4!}$$

$$= \frac{10 \times 9!}{5!}$$

$$= \frac{10!}{5!}$$

$= {}^{10}P_5$

$\Rightarrow r = 5$

Example 5.21 **If $^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$, then find the value of n.**

Sol. $^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$

$$\Rightarrow \frac{^{2n+1}P_{n-1}}{^{2n-1}P_n} = \frac{3}{5}$$

$$\Rightarrow \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \frac{(2n+1)\,(2n)}{(n+2)\,(n+1)n} = \frac{3}{5}$$

$\Rightarrow 10\,(2n + 1) = 3\,(n + 2)\,(n + 1)$

$\Rightarrow 3n^2 - 11n - 4 = 0$

$\Rightarrow (n - 4)\,(3n + 1) = 0$

$\Rightarrow n = 4$

Example 5.22 **Prove that if $r \le s \le n$, then ^nP_s is divisible by ^nP_r.**

Sol. Let $s = r + k$ where $0 \le k \le s - r$. Then,

$$^nP_s = \frac{n!}{(n-s)!}$$

$= n\,(n-1)\,(n-2) \cdots (n-(s-1))$

$= n\,(n-1)\,(n-2) \cdots (n-(r+k-1))$

$= n\,(n-1)\,(n-2) \cdots (n-(r-1))\,(n-r)\,(n-(r+1)) \cdots (n-(r+k-1))$

$= \{n\,(n-1)\,(n-2) \cdots n-(r-1)\}\,\{(n-r)\,(n-(r+1)) \cdots (n-(r+k-1))\}$

$= {}^nP_r\,\{(n-r)\,(n-(r+1)) \cdots (n-(r+k-1))\}$

$= {}^nP_r \times \text{Integer}$

Hence, ^nP_s is divisible by ^nP_r.

Example 5.23 **Seven athletes are participating in a race. In how many ways can the first three athletes win the prizes?**

Sol. It is equivalent to filling 3 places (as prizes) with 7 persons The number of permutations of 7 objects taken three at a time is

$$^7P_3 = 7 \times 6 \times 5 = 210$$

Example 5.24 **In how many ways can 6 persons stand in a queue?**

Sol. The number of ways in which 6 persons can stand in a queue is same as filling six places with six persons. The number of permutations of six objects taken all at a time is

$$^6P_6 = 6! = 720$$

Example 5.25 **How many different signals can be given using any number of flags from 5 flags of different colours?**

Sol. The signals can be made by using one or more flags at a time.

The total number of signals when r flags are used at a time from 5 flags is equal to the number of arrangements of 5, taking r at a time, i.e., ${}^5P_r$.

Since r can take the values 1, 2, 3, 4, 5, hence, by the fundamental principle of addition, the total number of signals is

$$\begin{aligned}{}^5P_1+{}^5P_2+{}^5P_3+{}^5P_4+{}^5P_5 &= 5+(5\times4)+(5\times4\times3)+(5\times4\times3\times2)\\ &\quad+(5\times4\times3\times2\times1)\\ &= 5+20+60+120+120\\ &= 325\end{aligned}$$

Example 5.26 **Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.**

Sol. Total number of numbers formed with the digits 2, 3, 4, 5 taken all at a time is equal to the number of arrangements of 4 digits, taken all at a time, i.e., ${}^4P_4 = 4! = 24$.

To find the sum of these 24 numbers, we have to find the sum of the digits at unit's, ten's, hundred's and thousand's places in all these numbers.

Consider the digits in the unit's places in all these numbers.

If 2 is the digit in unit's place, remaining three places can be filled in 3! ways or we can say 2 occurs in unit's place 3! (= 6) times. Similarly, each digit occurs six times.

So, the total sum of the digits in the unit's place in all these numbers is $(2 + 3 + 4 + 5) \times 3! = 84$.

Similarly, sum of digits is 84 in ten's, hundred's and thousand's places.

Hence, the sum of all the numbers is $84\,(10^0 + 10^1 + 10^2 + 10^3) = 93324$.

Example 5.27 **How many 4-letter words, with or without meaning, can be formed out of the letters in the word 'LOGARITHMS', if repetition of letters is not allowed?**

Sol. There are 10 letters in the word 'LOGARITHMS'. So, the number of 4-letter words is equal to number of arrangements of 10 letters, taken 4 at a time, i.e., ${}^{10}P_4 = 5040$.

Example 5.28 **Eleven animals of a circus have to be placed in eleven cages (one in each cage). If 4 of the cages are too small for 6 of the animals, then find the number of the ways of caging all the animals.**

Sol. Let the 6 animals be placed in 7 of larger cages. This can be done in ${}^7P_6$ ways. In each of these ways, one larger cage is left vacant. The remaining five animals can be placed in the remaining five cages in 5! ways. Hence, by the fundamental theorem, the required number of ways is ${}^7P_6 \times 5! = 604800$.

Example 5.29 **If $A = \{x \mid x$ is a prime number and $x < 30\}$, find the number of different rational numbers whose numerator and denominator belong to A.**

Sol. Here, $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$. A rational number is made by taking any two numbers in any order. Therefore, the required number of rational numbers is ${}^{10}P_2 + 1$ (including 1).

Example 5.30 **How many different numbers of 4 digits can be formed from the digits 0, 1, 2, ..., 9 if repetition is**
(i) allowed,
(ii) not allowed.

Sol. (i) Repetition is allowed.

First place is filled by any number from 1 to 9 as 0 cannot occur at first place and each of the remaining 3 places can be filled by any one of the digits from 0, 1, ..., 9, i.e., in 10 different ways.

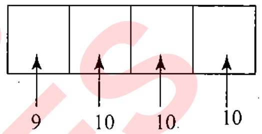


Fig. 5.8

Hence, the total number of 4-digit numbers that can be formed is $9 \times 10 \times 10 \times 10 = 9 \times 10^4$.

(ii) Repetition is not allowed.

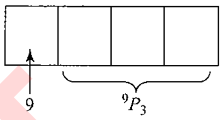


Fig. 5.9

The total number is $9 \times 9 \times 8 \times 7 = 4536$. Hence, the total number of 4-digit numbers that can be formed is $9 \times {}^9P_3$.

Example 5.31 **How many six-digit odd numbers, greater than 6,00,000, can be formed from the digits 5, 6, 7, 8, 9 and 0 if**
(i) repetition of digits is allowed,
(ii) repetition of digits is not allowed.

Sol. We have 6 digits, viz., 5, 6, 7, 8, 9 and 0 and we have to form numbers (integers) greater than 6,00,000, which are odd.

So the first place (lakh's position) should be ≥ 6 and the last position (i.e. unit) must be odd, i.e., 5, 7 or 9.

(i) When repetitions are allowed.

First place can be filled by 6, 7, 8 or 9 in 4 ways, last place can be filled by 5, 7 or 9 in 3 ways and each of the remaining 4 places (i.e. 2nd, 3rd, 4th, 5th) can be filled by any of the 6 digits in 6 ways. Hence, the total number will be $4 \times 6 \times 6 \times 6 \times 6 \times 3 = 15552$.

Note: *The above discussion can also be shown as follows:*

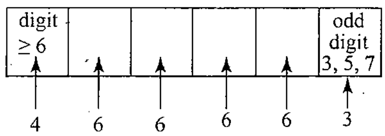


Fig. 5.10

The total number of numbers that can be formed is $4 \times 6 \times 6 \times 6 \times 6 \times 3 = 15552$.

(ii) When repetitions are not allowed.

Since we have restrictions on first and last places and no digit can be repeated, we have the following system: 1st place having numbers ≥ 6 and last place having odd numbers.

Digits in first place	Digits in last place
6	5, 7, 9
7	5, 9
8	5, 7, 9
9	5, 7
Total number of ways of filling last place	10 ways

Thus, the first and the last place can be filled in 10 ways, and the remaining four places can be filled by the remaining 4 digits in 4! = 24 ways. Hence, the total number of numbers that can be formed is $10 \times 24 = 240$.

Example 5.32 **A shelf contains 20 books of which 4 are single volume and the other form sets of 8, 5 and 3 volumes, respectively. Find the number of ways in which the books may be arranged on the shelf so that**

(i) volumes of each set will not be separated,

(ii) volumes of each set remain in their due order.

Sol. (i) Considering each set as single unit, permutations of 7 units is 7!.

Permutations of books of the set of 8 volumes among themselves is 8!.

Respective permutations of books of the set of 5 volumes is 5! and that of books of 3 volumes is 3!.

By the product rule, total number of permutations is

$7!\,8!\,5!\,3!$.

(ii) Since the books in a set of books containing any number of volumes can be arranged in due order in 2 ways, the total number of permutations is $7! \times 2 \times 2 \times 2 = 8 \times 7! = 8$.

Number of Permutations of *n* Things Taken All Together When the Things Are Not All Different

To find the number of permutations of things taken all at a time when p of them are similar and are of one type, q of them are similar and are of second type, r of them are similar and are of third type and rest are all different.

Proof: Total number of things is n. p things are identical and are of one type, q things are identical and are of second type, r things are identical and are of third type, and rest are all different.

Let the required number of permutations be x.

Since p different things can be arranged among themselves in $p!$ ways, therefore, if we replace p identical things by p different things, which are also different from the rest of things, the number of permutations will become $x \times p!$.

Again, if we replace q identical things by q different things, the number of permutations will become $(x \times p! \times q!)$.

Again, if we replace r identical things by r different things, which are different from the rest, the number of permutations will become $(x\,p!\,q!)\,r!$.

Now, all the n things are different and therefore, number of permutations should be $n!$. Thus,

$$x\,p!\,q!\,r! = n!$$

$$\therefore x = \frac{n!}{p!\,q!\,r!}$$

Example 5.33 **How many words can be formed with the letters of the word 'MATHEMATICS' by rearranging them.**

Sol. Since there are 2 M's, 2 A's and 2 T's, the required number of ways is $11!/(2!2!2!)$.

Example 5.34 **Find the total number of nine-digit numbers that can be formed using the digits 2, 2, 3, 3, 5, 5, 8, 8, 8 so that the odd digit occupy the even places.**

Sol. Odd digits 3, 3, 5, 5 occupy four even places in $4!/(2!2!) = 6$ ways. Rest five digits 2, 2, 8, 8, 8 occupy rest five places in $5!/(2!3!) = 10$ ways. Hence, total number of ways is $6 \times 10 = 60$.

Example 5.35 **Find the number of permutation of all the letters of the word "MATHEMATICS" which starts with consonants only.**

Sol. (M M), (A A), (T T), H, E, I, C, S

Words starting with M or A or T are $\frac{10!}{2!2!}$

Words starting with H, E, I, C, S are $\frac{10!}{2!2!2!}$

Hence number of words are

$$3\frac{10!}{2!2!} + 5\frac{10!}{2!2!2!} = \frac{10!}{2!2!}\left(3+\frac{5}{2}\right) = \frac{11!}{8}$$

Example 5.36 **There are six periods in each working day of a school. Find the number of ways in which 5 subjects can be arranged if each subject is allotted at least one period and no period remains vacant.**

Sol. Let the five subjects are a, b, c, d, e.

Since number of subjects are less than the number of periods, any one of the five subjects will occur twice.

If subject 'a' occur twice (a, a, b, c, d, e), then six subjects can be arranged in $\frac{6!}{2!}$ ways.

Similar number of ways when subject b, c, d and e occur twice.

Hence total number of ways are $5 \times \frac{6!}{2!} = 1800$

Number of Permutations of *n* Different Things Taken *r* at a Time When Each Thing Can Be Repeated Any Number of Times

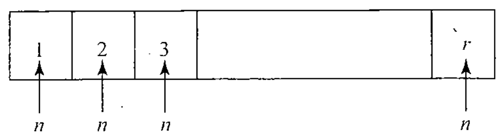


Fig. 5.11

By multiplication rule of fundamental principle of counting, number of ways in which first, second, third, …, r^{th} places can together be filled up is $n \times n \times n \times \cdots r$ times $= n^r$.

Example 5.37 **How many 4-digit numbers can be formed by using the digits 1, 2, 3, 4, 5, 6, 7 if at least one digit is repeated.**

Sol. The numbers that can be formed when repetition of digits is allowed are 7^4.

The numbers that can be formed when all the digits are distinct or when repetition is not allowed are ${}^7P_4$.

Therefore, the numbers that can be formed when at least one digit is repeated are $7^4 - {}^7P_4$.

Example 5.38 **Find the total number of permutations of n different things taken not more than r at a time, when each thing may be repeated any number of times.**

Sol. Here, we have to arrange p things out of n, $1 \le p \le r$, and repetition is allowed. When $p = 1$, the number of permutations is n. When $p = 2$, the number of permutations is $n \times n = n^2$.

(Since repetition is allowed, first thing can be taken in n ways and the second thing can also be taken in n ways.)

When $p = 3$, the number of permutations is $n \times n \times n = n^3$. When $p = r$, the number of permutations is $n \times n \times n \cdots r$ times $= n^r$.

Hence, total number of permutations is

$$n + n^2 + n^3 + \cdots + n^r = \frac{n(n^r - 1)}{(n-1)} \quad \text{[sum of G.P.]}$$

Permutations Under Restrictions

When Particular Objects Are Never Together (Gap Method)

Example 5.39 **Number of ways in which 5 girls and 5 boys can be arranged in a row if no two boys are together.**

Sol. In the question, there is no condition for arranging the girls. Now, 5 girls can be arranged in 5! ways.

$$\times G \times G \times G \times G \times G \times$$

When girls are arranged, six gaps are generated as shown above with '×'.

Now, boys must occupy the places with '×' marked so that no two boys are together.

Therefore, five boys can be arranged in these six gaps in ${}^6P_5$ ways.

Hence, total number of arrangement is $5! \times {}^6P_5$.

Example 5.40 **Number of ways in which 5 girls and 5 boys can be arranged a in row if boys and girls are alternate.**

Sol. First five girls can be arranged in 5! ways, i.e.,

$$\times G \times G \times G \times G \times G$$

or, $\quad G \times G \times G \times G \times G \times$

Now, if girls and boys are alternate, then boys can occupy places with '×' mark in the diagram.

Hence, total number of arrangements is $5! \times 5! + 5! \times 5! = 2 \times 5! \times 5!$

When Particular Objects Are Always Together

Example 5.41 **If the best and the worst papers never appear together, find in how many ways six examination papers can be arranged.**

Sol. If the best and worst papers appear always together, the number of ways is $5! \times 2$. Therefore, required number of ways is as follows.

Total number of ways without any restrictions − number of ways when best and worst paper are together $= 6! - 5! \times 2 = 480$.

Example 5.42 **Find the number of arrangements of the letters of the word 'SALOON', if the two O's do not come together.**

Sol. The total number of arrangements is $6!/2! = 360$. The number of ways in which O's come together is $5! = 120$. Hence, the required number of ways is $360 - 120 = 240$.

Example 5.43 **Find the number of seven letter words that can be formed by using the letters of the word SUCCESS so that the two C are together but no two S are together.**

Sol. Considering CC as single object, U, CC, E can be arranged in 3! ways

$$\times U \times CC \times E \times$$

Now the three S are to be placed in the four available places (×)

Hence required no. of ways $= 3! \, . \, {}^4C_3 = 24$.

Example 5.44 **There are six teachers. Out of them two are primary teachers, two are middle teachers and two secondary teaches. They are to stand in a row, so as the primary teaches, middle teaches and secondary teaches are always in a set. Find the number of ways in which they can do so.**

Sol. There are 2 primary teachers. They can stand in a row in $2! = 2$ ways

There are 2 middle teachers. They can stand in a row in $2! = 2$ ways.

There are 2 secondary teachers. They can stand in a row in $2! = 2$ ways.

These three sets can be arranged In themselves in $= 3! = 6$ ways

Hence the required number of ways $= 2 \times 2 \times 2 \times 6 = 48$

Example 5.45 **There are 2 identical white balls, 3 identical red balls and 4 green balls of different shades. Find the number of ways in which they can be arranged in a row so**

that at least one ball is separated from the balls of the same colour.

Sol. Total number of arrangements without any restrictions = $\frac{9!}{2!\ 3!}$

Now number of ways when balls of the same color are together $= 3!\,4!$

Now required number of ways

= Total number of arrangements

– number of ways when balls of the same colour are together

$= \frac{9!}{2!\ 3!} - 3!4! = 6\,(7! - 4!)$

Example 5.46 **Find the number of ways in which 6 boys and 6 girls can be seated in a row so that**

(i) all the girls sit together and all the boys sit together,

(ii) all the girls are never together.

Sol. (i) $b_1 b_2 b_3 b_4 b_5 b_6$ $g_1 g_2 g_3 g_4 g_5 g_6$

Fig. 5.12

Considering boys and girls as two units, the number of permutations is $2! \times 6! \times 6! = 2 \times (6!)^2$.

(ii) The total arrangements where all girls are not together is as follows: Total arrangement without any restrictions – arrangement when all girls are together = $(12)! - 7!\,6!$.

Example 5.47 **The numbers ways in which the letters of the word 'ARRANGE' be arranged so that**

(i) the two R's are never together,

(ii) the two A's are together but not two R's,

(iii) neither two A's nor two R's are together.

Sol. The letters of word ARRANGE can be rewritten as

A R N G E

A R

So we have 2 A's and 2 R's, i.e., total 7 letters.

(i) Total number of words is $1/x + 1/y = 1/n$.

The number of words in which 2 R's are together [consider (R R) as one unit] is $6!/2!$. e.g.,

(R R), A, A, N, G, E

Note that permutations of R R give nothing extra. Therefore, number of words in which the two R's are never together is

$\frac{7!}{2!2!} - \frac{6!}{2!} = 900$

(ii) The number of words in which both as are together is $6!/2! = 360$, e.g.,

(A A), R, R, N, G, E

The number of words in which both A's and both R's are together is $5! = 120$, e.g.,

(A A), (R R), N, G, E

Therefore, the number of words in which both A's are together but the two R's are not together is $360 - 120 = 240$.

(iii) There are in all 900 words in each of which the two R's are never together. Consider any such word. Either the two A's are together or the two A's are not together. But the number of all such arrangements in which the two A's are together is 240. Hence, the number of all such arrangements in which the two A's are not together is $900 - 240 = 660$.

Concept Application Exercise 5.3

1. Prove that ${}^nP_r - 5\,{}^{n-1}P_r + r\,{}^{n-1}P_{r2-1}$.
2. If ${}^nP_5 = 20\,{}^nP_3$, find the value of n.
3. a. If ${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$, find r.

   b. If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800:1$, find r.
4. How many numbers can be formed from the digits 1, 2, 3, 4 when repetition is not allowed?
5. Find the 3-digit odd numbers that can be formed by using the digits 1, 2, 3, 4, 5, 6 when the repetition is allowed.
6. If the 11 letters $A, B, \ldots, K$ denote an arbitrary permutation of the integers $(1, 2, \ldots, 11)$, then $(A - 1)(B - 2)(C - 3) \cdots (K - 11)$ will be

   a. necessarily zero b. always odd

   c. always even d. none of these
7. In how many ways can 5 boys and 3 girls sit in a row so that no two girls are together?
8. In how many ways can 5 girls and 5 boys be arranged in row if all boys are together.
9. Find the number of words that can be made out of the letters of the word 'MOBILE' when consonants always occupy odd places.
10. Find the number of positive integers, which can be formed by using any number of digits from 0, 1, 2, 3, 4, 5 but using each digit not more than once in each number. How many of these integers are greater than 3000? What happened when repetition is allowed?

COMBINATION

Each of the different groups or selections, which can be made by taking some or all of a number of given things or objects at a time, is called a combination. In combination, order of appearance of things is not taken into account.

Example 1:

Three groups can be made with three different objects a, b, c taking two at a time, i.e., ab, bc, ac.

Here, ab and ba are the same group. It is also clear that for each combination (selection or group) of two things, number of permutations (arrangements) is 2!. For example, for combination ab, there are two permutations, i.e., ab and ba.

Example 2:

Four groups can be made with 4 different things a, b, c, d taking three at a time, i.e., abc, abd, acd, and bcd. Now, for each combination (group) of three things, number of permutations is 3!, i.e., 6. For example, for the group *abc*, there are 6 permutations (arrangements): *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*.

Number of Combinations of *n* Different Things Taking *r* at a Time ($r < n$)

To establish the formula

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Proof: Let the number of combinations of n different things taken r at a time be ^nC_r.

Now, each combination consists of r different things and these r things can be arranged among themselves in $r!$ ways.

Thus, for one combination of r different things, the number of arrangements is $r!$.

Hence, for ^nC_r combinations, number of arrangements is

$$r!\,^nC_r \quad (1)$$

But number of permutations of n different things taken r at a time is

$$^nP_r \quad (2)$$

From Eqs. (1) and (2), we get

$$r!\,^nC_r = {^nP_r} = \frac{n!}{(n-r)!}$$

$$\therefore \quad ^nC_r = \frac{n!}{r!\,(n-r)!}$$

Properties of ^nC_r

1. $^nC_r = {^nC_{n-r}}$

Proof:

$$^nC_r = \frac{n!}{r!(n-r)!} \quad (1)$$

$$^nC_{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{r!(n-r)!} \quad (2)$$

From Eqs. (1) and (2), it follows that $^nC_r = {^nC_{n-r}}$.

2. If $^nC_x = {^nC_y}$, then either $x = y$ or $x + y = n$.

Proof:

$$^nC_x = {^nC_y} = {^nC_{n-y}} \quad [\because {^nC_r} = {^nC_{n-r}}]$$

From ^nC_x (i) and (ii),

$$x = y$$

From ^nC_x (i) and (iii),

$$x = n - y \text{ or } x + y = n$$

3. $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

Proof:

L.H.S. $= {^nC_r} + {^nC_{r-1}}$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\,(n-(r-1))!}$$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n+1}{r(n-r+1)}\right]$$

$$= \frac{(n+1)n!}{r(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{(n+1)!}{r!(n-r+1)!} = {^{n+1}C_r} = \text{R.H.S.}$$

4. $r\,^nC_r = n\,^{n-1}C_{r-1}$

Proof:

L.H.S. $= r\,^nC_r$

$$= r\frac{n!}{r!(n-r)!}$$

$$= r\frac{n!}{r(r-1)!(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}$$

$$= n\frac{(n-1)!}{(r-1)!(n-r)!}$$

$$= n\,^{n-1}C_{r-1} = \text{R.H.S.}$$

5. $\dfrac{^nC_r}{r+1} = \dfrac{^{n+1}C_{r+1}}{n+1}$

Proof:

L.H.S. $= \dfrac{^nC_r}{r+1}$

$$= \frac{\dfrac{n!}{r!(n-r)!}}{r+1}$$

$$= \frac{n!}{(r+1)r!(n-r)!}$$

$$= \frac{n!}{(r+1)!(n-r)!}$$

$$= \frac{(n+1)\,n!}{(n+1)(r+1)!(n-r)!}$$

$$= \frac{1}{(n+1)}\,\frac{(n+1)!}{(r+1)!(n-r)!}$$

$$= \frac{^{n+1}C_{r+1}}{n+1} = \text{R.H.S.}$$

6. $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

7. Maximum value of ^nC_r

We can observe that in the list of $^6C_0$, $^6C_1$, $^6C_2$, $^6C_3$, $^6C_4$, $^6C_5$, and $^6C_6$, the maximum value is $^6C_3$.

Also, in the list of $^7C_0$, $^7C_1$, $^7C_2$, $^7C_3$, $^7C_4$, $^7C_5$, $^7C_6$, and $^7C_7$, the maximum value is $^7C_3$ or $^7C_4$.

In general, when n is even, maximum value of ^nC_r is $^nC_{n/2}$ and when n is odd, maximum value of ^nC_r is $^nC_{(n-1)/2}$ or $^nC_{(n+1)/2}$.

8. The product of k consecutive positive integers is divisible by $k!$

Let the k consecutive integers be m, $m+1$, $m+2$, ..., $m+k-1$. Then,

$$m(m+1)(m+2)\cdots(m+k-1)$$
$$= \frac{(m-1)!m(m+1)\cdots(m+k-1)}{(m-1)!}$$
$$= \frac{(m+k-1)!}{(m-1)!}$$
$$= k!\frac{(m+k-1)!}{(m-1)!k!}$$
$$= (k!)\,^{(m+k-1)}C_k$$

Since $^{(m+k-1)}C_k$ is an integer, it follows that $k!$ divides $m(m+1)\cdots(m+k-1)$.

9. ^nC_r is divisible by n only if n is a prime number $(1 \le r \le n-1)$

For example, $^6C_2$ is not divisible by 6, but $^7C_4$ is divisible by 7.

Restricted Combinations

Number of Combinations of n Different Things Taken r at a Time when p Particular Things Are Always Included

Already p things are selected. The remaining $r-p$ things from the remaining $n-p$ things can be selected in $^{n-p}C_{r-p}$ ways.

Number of Combinations of n Different Things Taken r at a Time when p Particular Things Are Always to Be Excluded

Since p particular things are always to be excluded, therefore, we have to select r things out of remaining $n-p$ different things. This can be done in $^{n-p}C_r$ ways.

Example 5.48 **Prove that $^nC_r + 2\,^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$.**

Sol. $^nC_r + 2\,^nC_{r-1} + {}^nC_{r-2}$
$$= {}^nC_r + {}^nC_{r-1} + {}^nC_{r-1} + {}^nC_{r-2}$$
$$= {}^{n+1}C_r + {}^{n+1}C_{r-1} = {}^{n+2}C_r$$

Example 5.49 **Prove that $^nC_r + {}^{n-1}C_r + \cdots + {}^rC_r = {}^{n+1}C_{r+1}$.**

Sol. $^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \cdots + {}^{n-1}C_r + {}^nC_r$
$$= {}^{r+1}C_{r+1} + {}^{r+1}C_r + {}^{r+2}C_r + \cdots + {}^{n-1}C_r + {}^nC_r$$
$$= {}^{r+2}C_{r+1} + {}^{r+2}C_r + \cdots + {}^{n+1}C_r + {}^nC_r$$
$$= {}^{r+3}C_{r+1} + \cdots + {}^{n-1}C_r + {}^nC_r$$

On adding similar way, we get

$$\text{L.H.S.} = {}^{n-1}C_{r+1} + {}^{n-1}C_r + {}^nC_r$$
$$= {}^nC_{r+1} + {}^nC_r$$
$$= {}^{n+1}C_{r+1} = \text{R.H.S.}$$

Example 5.50 **If $^{15}C_{3r} = {}^{15}C_{r+3}$, then find r.**

Sol. $^{15}C_{3r} = {}^{15}C_{r+3}$

$\Rightarrow$ $3r = r+3$ or $3r + r + 3 = 15$

$\Rightarrow$ $r = 3/2$ or $r = 3$

$\Rightarrow$ $r = 3$ (as r is positive integer)

Example 5.51 **If $^nC_r = 84$, $^nC_{r-1} = 36$ and $^nC_{r+1} = 126$, then find the value of n.**

Sol. $\dfrac{n-r+1}{r} = \dfrac{84}{36} = \dfrac{7}{3}$ and $\dfrac{n-r}{r+1} = \dfrac{126}{84} = \dfrac{3}{2}$

$\therefore$ $\dfrac{7}{3}r - 1 = n - r = \dfrac{3}{2}(r+1)$

or

$14r - 6 = 9r + 9$ or $r = 3$

$\therefore$ $n = 9$

Example 5.52 **If $^nC_8 = {}^nC_6$, then find $^nC_2$.**

Sol. If $^nC_x = {}^nC_y$ and $x \ne y$, then $x + y = n$. Hence,

$^nC_8 = {}^nC_6$

$\Rightarrow$ $n = (8+6) = 14$

Now,

$$^nC_2 = {}^{14}C_2 = \frac{14 \times 13}{2} = 91$$

Example 5.53 **If the ratio $^{2n}C_3 : {}^nC_3$ is equal to 11:1, find n.**

Sol. We have,

$$^{2n}C_3 : {}^nC_3 = 11:1$$

$$\Rightarrow \frac{^{2n}C_3}{^nC_3} = \frac{11}{1}$$

$$\Rightarrow \frac{\dfrac{(2n)!}{(2n-3)!(3!)}}{\dfrac{n!}{(n-3)!(3!)}} = \frac{11}{1}$$

$$\Rightarrow \frac{(2n)!}{(2n-3)!} \times \frac{(n-3)!}{n!} = \frac{11}{1}$$

$$\Rightarrow \frac{(2n)(2n-1)(2n-2)}{n(n-1)(n-2)} = \frac{11}{1}$$

$$\Rightarrow \frac{4(2n-1)}{n-2} = \frac{11}{1}$$

$$\Rightarrow 8n - 4 = 11n - 22$$

$$\Rightarrow 3n = 18 \Rightarrow n = 6$$

Example 5.54 **If $^{15}C_{3r}:{}^{15}C_{r+1} = 11:3$, find the value of '$r$'.**

Sol. $^{15}C_{3r}:{}^{15}C_{r+1} = 11:3$

Clearly, 'r' can be 0, 1, 2, 3, 4, 5 but possibilities of $r = 0$ or 5 are clearly ruled out (as $^{15}C_0 = {}^{15}C_{15} = 1$).

For $r = 1$,

$$^{15}C_{3r} = {}^{15}C_3 = \frac{15\times14\times13}{6} \text{ and } {}^{15}C_{r+1} = {}^{15}C_2 = \frac{15\times14}{2}$$

$\therefore$ $^{15}C_{3r}:{}^{15}C_{r+1} = 13:3$

For $r = 2$,

$$^{15}C_{3r} = {}^{15}C_6 = \frac{15\times14\times13\times12\times11\times10}{6\times5\times4\times3\times2\times1}$$

$$^{15}C_{r+1} = {}^{15}C_3 = \frac{15\times14\times13}{6}$$

$\therefore$ $^{15}C_{3r}:{}^{15}C_{r+1} \neq 11:3$

For $r = 3$,

$$^{15}C_{3r} = {}^{15}C_9 = \frac{15\times14\times13\times12\times11\times10}{6\times5\times4\times3\times2\times1}$$

$$^{15}C_{r+1} = {}^{15}C_4 = \frac{15\times14\times13\times12}{4\times3\times2\times1}$$

$\therefore$ $^{15}C_{3r}:{}^{15}C_{r+1} = 11:3$

For $r = 4$,

$$^{15}C_{3r} = {}^{15}C_{12} = {}^{15}C_3 = \frac{15\times14\times13}{3\times2\times1}$$

$$^{15}C_{r+1} = {}^{15}C_5 = \frac{15\times14\times13\times12\times11}{5\times4\times3\times2\times1}$$

$\therefore$ $^{15}C_{3r}:{}^{15}C_{r+1} = 5:33$

Thus, $r = 3$.

Example 5.55 **Twenty-eight games were played in a football tournament with each team playing once against each other. How many teams were there?**

Sol. Let the number of teams be n. Then number of matches to be played is $^nC_2 = 28$.

$\therefore$ $\frac{n(n-1)}{2} = 28$

$\Rightarrow$ $n^2 - n - 56 = 0$

$\Rightarrow$ $(n-8)(n+7) = 0$

$\Rightarrow$ $n = 8$ as $n \neq -7$

Example 5.56 **In a network of railways, a small island has 15 stations. Find the number of different types of tickets to be printed for each class, if every stations must have tickets for other stations.**

Sol. For each pair of stations, two different types of tickets are required.

Now, the number of selections of 2 stations from 15 stations $= {}^{15}C_2$.

$\therefore$ Required number of types of tickets $= 2\ {}^{15}C_2 = 2\frac{15!}{2!13!}$ $= 15\times14 = 210$.

Example 5.57 **In a certain algebraical exercise book there are 4 examples on arithmetical progressions, 5 examples on permutation and combination and 6 examples on binomial theorem. Find the number of ways a teacher can select for his pupils at least one but not more than 2 examples from each of these sets.**

Sol. Number of ways teacher can select examples from arithmetic progression $= ({}^4C_1 + {}^4C_2)$

Number of ways teacher can select examples from permutation and combinations $= ({}^5C_1 + {}^5C_2)$

Number of ways teacher can select examples from binomial theorem $= ({}^6C_1 + {}^6C_2)$

Hence total number of ways $= ({}^4C_1 + {}^4C_2)({}^5C_1 + {}^5C_2)({}^6C_1 + {}^6C_2)$

Example 5.58 **A person tries to form as many different parties as he can, out of his 20 friends. Each party should consist of the same number. How many friends should be invited at a time? In how many of these parties would the same friends be found?**

Sol. Let the person invite r number of friends at a time. Then, the number of parties is $^{20}C_r$, which is maximum when $r = 10$.

If a particular friend will be found in x parties, then x is the number of combinations out of 20 in which this particular friend must be included. Therefore, we have to select 9 more from 19 remaining friends. Hence, $x = {}^{19}C_9$.

Example 5.59 **In how many of the permutations of n things taken r at a time will three given things occur?**

Sol. According to the condition of the problem, we have to select $r - 3$ things from remaining $n - 3$ things and permute these r things. So the number of permutations is

$$^{(n-3)}C_{(r-3)}\ r! = \frac{(n-3)!\ r!}{(r-3)!\ (n-r)!}$$

Example 5.60 **Out of 10 consonants and 4 vowels, how many words can be formed each containing 3 consonants and 2 vowels?**

Sol. The number of ways of selection of three consonants from 10 is $^{10}C_3$. The number of ways of selection of two vowels from 4 is $^4C_2$. Permutation of these 5 letters (all distinct) is 5! Therefore, number of words that can be formed is $^{10}C_3 \times {}^4C_2 \times 5! = 86400$.

Example 5.61 **Find the maximum number of points of intersection of 6 circles.**

Sol. Two circles intersect maximum at two distinct points. Now, two circles can be selected in $^6C_2$ ways. Again, each selection of two circles gives two points of intersection. Therefore, the total number of points of intersection is $^6C_2 \times 2 = 30$.

Example 5.62 **There are 10 points on a plane of which no three points are collinear. If lines are formed joining**

these points, find the maximum points of intersection of these lines.

Sol. Two points are required to form a line. Then, the number of lines is equal to the number of ways two points are selected, i.e., ${}^{10}C_2 = 45$.

Now, two lines intersect at one point. Hence, the number of points of intersection of lines is ${}^{45}C_2$.

Example 5.63 **There are 10 points on a plane of which 5 points are collinear. Also, no three of the remaining 5 points are collinear. Then find (i) the number of straight lines joining these points; (ii) the number of triangles formed joining these points.**

Sol. (i) Line is formed joining two points. Hence, number of lines is ${}^{10}C_2$. But joining any points from 5 collinear points gives the same line. Again, 2 points are selected from 5 in ${}^5C_2$ ways or lines joining collinear points is taken ${}^5C_2$ (= 10) times. Then the number of straight lines = ${}^{10}C_2 - 10 + 1 = 36$.

(ii) For a triangle, three non-collinear points are required. Three points can be selected in ${}^{10}C_3$ ways. Now, the selection of three points from 5 collinear points does not form triangle. Hence, number of triangles is ${}^{10}C_3 - {}^5C_3$.

Example 5.64 **Find the total number of rectangles on the normal chessboard.**

Sol. To form a rectangle on a chessboard two vertical lines and two horizontal lines should be selected. There are 9 vertical lines and 9 horizontal lines found on the chessboard. Selection of 2 vertical and 2 horizontal lines can be done in ${}^9C_2 \times {}^9C_2$ ways, which is equivalent to the number of rectangles.

Example 5.65 **A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?**

Sol. The selection of 6 balls, consisting of at least two balls of each colour from 5 red and 6 white balls, can be made in the following ways:

Red balls (5)	White balls (6)	Number of ways
2	4	${}^5C_2 \times {}^6C_4 = 150$
3	3	${}^5C_3 \times {}^6C_3 = 200$
4	2	${}^5C_4 \times {}^6C_2 = 75$
	Total	425

Example 5.66 **In a conference 10 speakers are present. If S_1 wants to speak before S_2 and S_2 wants to speak after S_3, then find number of ways all the 10 speakers can give their speeches with the above restriction if the remaining seven speakers have no objection to speak at any number.**

Sol. According to question the order of speakers S_1, S_2 and S_3 can be (not necessarily consecutive)

$S_1S_2S_3$ or $S_3S_1S_2$

For each order we can select three slots out of ten in ${}^{10}C_3$ ways.

After selecting these three slots in which speakers S_1, S_2, S_3 have only one way of arrangement as said, the remaining seven speakers can be arranged in seven slots in 7! ways.

Hence total number of arrangements = $2\ {}^{10}C_3 7!$

Example 5.67 **A delegation of four students is to be selected from a total of 12 students. In how many ways can the delegation be selected**

(i) if all the students are equally willing?

(ii) if two particular students have to be included in the delegation?

(iii) if two particular students do not wish to be together in the delegation?

(iv) if two particular students wish to be included together only in the delegation?

(v) if two particular students refuse to be together and two other particular students wish to be together only in the delegation?

Sol. (i) Formation of delegation means selection of 4 out of 12. Hence, the number of ways is ${}^{12}C_4 = 495$.

(ii) If two particular students are already selected, we need to select only 2 out of the remaining 10. Hence, the number of ways is ${}^{10}C_2 = 45$.

(iii) The number of ways in which both are selected is 45. Hence, the number of ways in which the two are not included together is 495 – 45 = 450.

(iv) There are two possible cases:

(a) Both are selected. In this case, the number of ways in which the selection can be made is 45.

(b) Both are not selected. In this case, all the four students are selected from the remaining ten students. This can be done in ${}^{10}C_4 = 210$ ways.

Hence, the total number of ways of selection is 45 + 210 = 255.

(v) We assume that students A and B wish to be selected together and students C and D do not wish to be together.

Cases	Number of selection
A, B always selected	${}^{10}C_2 = 45 = s_1$
A, B always excluded	${}^{10}C_4 = 210 = s_2$
A, B, C, D always selected	$1 = s_3$
A, B excluded and C, D included	${}^8C_2 = 28 = s_4$

The total number of ways is

$s_1 - s_3 + s_2 - s_4$

$= 45 - 1 + 210 - 28$

$= 216$

Example 5.68 **Find the total number of ways of selecting five letters from the word 'INDEPENDENT'.**

Sol. Given letters are I, (N, N, N), (D, D), (E, E, E), P, T
The choices are as follows:

Choice	Ways
All the letters are distinct [different letters are I, N, D E, P, T]	${}^6C_5 = 6$
3 distinct, 2 alike	${}^3C_1 \times {}^5C_3 = 30$
2 distinct, 3 alike	${}^2C_1 \times {}^5C_2 = 20$
2 alike, 2 alike, 1 distinct	${}^3C_2 \times {}^4C_1 = 12$
3 alike, 2 alike	${}^2C_1 \times {}^2C_1 = 4$
	Total = 72

Example 5.69 **In a plane, there are 5 straight lines which will pass through a given point, 6 others which all pass through another given point and 7 others which all pass through a third given point. Supposing no three lines intersect at any point and no two are parallel, find the number of triangles formed by the intersection of the straight line.**

Sol. Let 5 straight lines be passing through A, 6 passing through B and 7 passing through C. In all, there are 18 straight lines. To find the number of triangles equivalent, we have to find the number of selection of 3 lines from these 18 lines, keeping in mind that selection of 3 lines from the lines passing through A, B or C will not give any triangle.

Hence, the required number of triangles is ${}^{18}C_3 - ({}^5C_3 + {}^6C_3 + {}^7C_3) = 751$.

Example 5.70 **A regular polygon of 10 sides is constructed. In how many ways can 3 vertices be selected so that no two vertices are consecutive?**

Sol. The required number of selections is given as
The number of selections without restriction
(the number of selections when 3 vertices are consecutive) – (the number of selections when 2 vertices are consecutive)

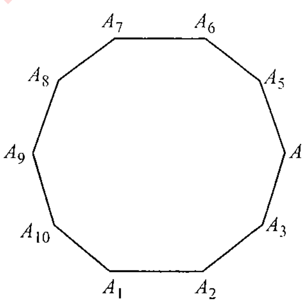


Fig. 5.13

Now, the number of selections of 3 vertices without restriction is ${}^{10}C_3$.

The number of selections of 3 consecutive vertices is 10 (by observation: $A_1A_2A_3, A_2A_3A_4, \ldots, A_{10}A_1A_2$).

The number of selections when two vertices are consecutive is $10 \times {}^6C_1$.

(After selecting two consecutive vertices in 10 ways, the third can be selected from 6 vertices.)

Therefore, the required number of selections is

$${}^{10}C_3 - 10 - 10 \times {}^6C_1 = \frac{10 \times 9 \times 8}{6} - 10 - 60 = 120 - 70 = 50$$

Concept Application Exercise 5.4

1. If ${}^{n+2}C_8 : {}^{n-2}P_4 = 57:16$, find n.
2. Find the ratio of ${}^{20}C_r$ and ${}^{25}C_r$ when each of them has the greatest possible value.
3. If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party?
4. On the occasion if Deepawali festival, each student in a class sends greeting cards to others. If there are 20 students in the class, find the total number of greeting cards exchanged by the students?
5. Out of 15 balls, of which some are white and the rest are black, how many should be white so that the number of ways in which the balls can be arranged in a row may be the greatest possible? It is assumed that the balls of same colour are alike.
6. A committee of 6 is chosen from 10 men and 7 women so as to contain at least 3 men and two women. In how many ways can this be done if two particular women refuse to serve on the same committee?

   a. 7850 **b.** 8700
   c. 7800 **d.** none of these
7. Find the number of ways in which the birthdays of six different persons will fall in exactly two calendar months.
8. A bag contains 50 tickets numbered 1, 2, 3, …, 50. Find the number of set of five tickets x_1, x_2, x_3, x_4, x_5 one has if $x_1 < x_2 < x_3 < x_4 < x_5$ and $x_3 = 30$.
9. Four visitors A, B, C, D arrived at a town that has 5 hotels. In how many ways, can they disperse themselves among 5 hotels.
10. In how many shortest ways can we reach from the point (0, 0, 0) to point (3, 7, 11) in space where the movement is possible only along the x-axis, y-axis and z-axis or parallel to them and change of axes is permitted only at integral points. (An integral point is one, which has its coordinate as integer.)
11. Out of 8 sailors on a boat, 3 can work only on one particular side and 2 only on the other side. Find the number of ways in which the sailors can be arranged on the boat.
12. For an examination, a candidate has to select 7 subjects from 3 different groups A, B, C, which contain 4, 5, 6 subjects, respectively. The number of different ways in which a candidate can make his selection if he has to select at least 2 subjects from each group is

    a. 2500 **b.** 2600
    c. 2700 **d.** 2800

CIRCULAR PERMUTATIONS

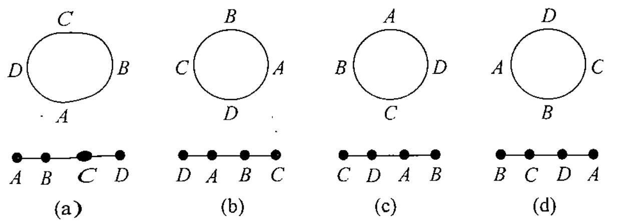


Fig. 5.14

Let us consider that persons A, B, C, D are sitting around a round table. If all of them (A, B, C, D) are shifted at one place in anticlockwise order, then we will get Fig. 5.13(b) from Fig. 5.14(a). Now, if we shift A, B, C, D in anticlockwise order, we will get Fig. 5.13(c). Again, if we shift them we will get Fig. 5.13(d); and in the next time, Fig. 5.13(a).

Thus, we see that if 4 persons are sitting at a round table, they can be shifted four times and the four different arrangements thus obtained will be the same, because anticlockwise order of A, B, C, D does not change.

But if A, B, C, D are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the four arrangements will be different.

Thus, if there are 4 things, then for each circular arrangement number of linear arrangements is 4.

Similarly, if n different things are arranged along a circle, for each circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different things is $n \times$ (number of circular arrangements of n different things). Hence, the number of circular arrangements of n different things is

$(1/n) \times$ (number of linear arrangements of n different things)

$= n!/n = (n-1)!$

Clockwise and Anticlockwise Arrangements

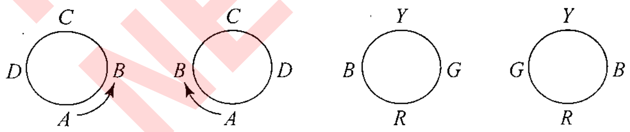


Fig. 5.15

Let the four persons A, B, C, D sit at a round table in anticlockwise as well as clockwise directions. These two arrangements are different. But if four flowers R (red), G (green), Y (yellow) and B (blue) be arranged to form a garland in anticlockwise and in clockwise order, then the two arrangements are same because if we see the garland from one side the four flowers R, G, Y, B will appear in anticlockwise direction and if seen from the other side the four flowers will appear in the clockwise direction. Here, the two arrangements will be considered as one arrangement because the order of flowers is not changing rather only the side of observation is changing. Here, two permutations will be counted as one.

Therefore, when clockwise and anticlockwise arrangements are not different, i.e., when observation can be made from both sides, the number of circular arrangements of n different things is $(n-1)!/2$.

Example 5.71 **Five boys and 5 girls sit alternately around a round table. In how many ways can this be done?**

Sol.

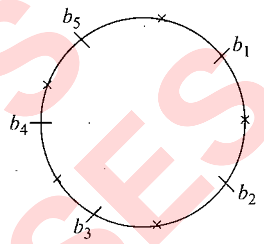


Fig. 5.16

Five boys can be arranged in a circle in 4! ways.

After that girls can be arranged in the five gaps shown as '×' in 5! ways. Hence, total number of ways is $4! \times 5! = 2880$.

Example 5.72 **A round-table conference is to be held among 20 delegates belonging from 20 different countries. In how many ways can they be seated if two particular delegates are (i) always to sit together; (ii) never to sit together.**

Sol. **(i)** Let the two particular delegates who wish to sit together be treated as one delegate. So we have 19 delegates who can be arranged on a round table in $(19-1)!$, i.e., 18! ways.

After this, the two particular delegates can be permuted between themselves in $2! = 2$ ways. Hence, by product rule, number of required arrangements is $2 \times (18)!$.

(ii) The total number of arrangements of 20 delegates on a round table is 19!.

Hence, the number of arrangements in which the two particular delegates never sit together is $19! - 2 \times 18! = 18!\,(19-2) = 17 \times 18!$.

Example 5.73 **A person invites a group of 10 friends at dinner and sits**

(i) 5 on a round table and 5 more on another round table,

(ii) 4 on one round table and 6 on the other round table.

Find the number of ways in each case in which he can arrange the guests.

Sol. **(i)** The number of ways of selection of 5 friends for first table is ${}^{10}C_5$. Remaining 5 friends are left for second table.

The total number of permutations of 5 guests on each table is 4!. Hence, the total number of arrangements is ${}^{10}C_5 \times 4! \times 4! = 10!/(5! \times 5!)4! \times 4! = 10!/25$.

(ii) The number of ways of selection of 6 guests is ${}^{10}C_6$. The number of ways of permutations of 6 guests on round table is 5!. The number of permutation of 4 guests on round table is 3!.

Therefore, total number of arrangements is

$$^{10}C_6 \times 5!\,3! = \frac{(10)!}{6! \times 4!}\,5!\,3! = \frac{(10)!}{24}$$

Example 5.74 **Find the number of ways in which 10 different diamonds can be arranged to make a necklace.**

Sol. Since diamonds do not have natural order of left and right so clockwise and anticlockwise arrangements are taken as identical. Therefore, the number of arrangements of 10 different diamonds to make a necklace is $1/2 \times 9 = 181440$.

Example 5.75 **Six persons *A, B, C, D, E, F* are to be seated at a circular table. In how many ways can this be done if *A* should have either *B* or *C* on his right and *B* must always have either *C* or *D* on his right.**

Sol. Let the seat occupied by A be numbered as 1 and the remaining 5 seats be numbered as 2, 3, 4, 5, 6 in anticlockwise direction. There arise two cases:

Case I: B is on right of A, i.e., at number 2.
Then, seat number 3 can be occupied by C or D in $^2C_1$ ways and remaining 3 persons can have remaining 3 seats in 3! ways. Hence, the number of arrangements in this case is $2 \times 6 = 12$.

Case II: C is on the right of A, i.e., at number 2.
Then, B can occupy any seat from number 3 or 4 or 5. Then, D must be on the right of B, so we are left with two persons and 2 seats, which can be occupied in 2! ways. Hence, the number of arrangements in this case is $^3C_1 \times 2! = 6$. These cases are exclusive. So by sum rule total number of arrangements is $12 + 6 = 18$.

Example 5.76 **Find the number of ways in which six persons can be seated at a round table, so that all shall not have the same neighbours in any two arrangements.**

Sol. In this case, anticlockwise and clockwise arrangements are the same.

Hence, the number of ways of arrangements is $5!/2 = 60$.

Concept Application Exercise 5.5

1. In how many ways can 3 ladies and 3 gentlemen be seated around a round table so that any two and only two of the ladies sit together?
2. In how many ways can 15 members of a council sit along a circular table, when the secretary is to sit on one side of the chairman and the deputy secretary on the other side?
3. Find the number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together.
4. Find the number of ways that 8 beads of different colours be strung as a necklace.
5. Find the number of ways in which 8 different flowers can be strung to form a garland so that four particular flowers are never separated.

ALL POSSIBLE SELECTIONS

Total Number of Combinations of *n* Different Things Taken One or More at a Time

Method 1

For each thing there are two possibilities, whether it is selected or not selected.

Hence, the total number of ways is given by total possibilities of all the things which is equal to $2 \times 2 \times 2 \times \cdots n$ times $= 2^n$.

But this includes one case in which nothing is selected.

Hence, the total number of ways of selecting one or more of n different things is $2^n - 1$.

Method 2

Number of ways of selecting one, two, three, …, n things from n different things is $^nC_1, {}^nC_2, {}^nC_3, \ldots, {}^nC_n$, respectively.

Hence, the total number of ways of selecting at least one thing is

$$^nC_1 + {}^nC_2 + \cdots + {}^nC_n = ({}^nC_0 + {}^nC_1 \cdots + {}^nC_n) - {}^nC_0 = 2^n - 1$$

Total Number of Selections of One or More Things from *p* Identical Things of One Type, *q* Identical Things of Another Type, *r* Identical Things of the Third Type and *n* Different Things

Since, the number of ways of selecting r things out of n identical things is 1 for all $r \le n$.

Hence, the number of ways of selecting zero or more things out of p identical things is

$$1 + 1 + 1 + \cdots (p + 1) \text{ times} = p + 1$$

Similarly, the number of ways of selecting zero or more things out of q and r identical things is $q + 1$ and $r + 1$, respectively.

Also the number of ways of selecting zero or more things out of n different things is $2 \times 2 \times 2 \times \cdots n$ times $= 2^n$.

Therefore, the number of ways of selecting zero or more things out of given things is $(p + 1)(q + 1)(r + 1)\,2^n$.

But the number of ways of selecting zero thing out of given things is $1 \times 1 \times 1 \times 1^n = 1$.

Thus, the total number of ways of selecting one or more things out of given things is $(p + 1)(q + 1)(r + 1)\,2^n - 1$.

Number of Divisors of *N*

- Every natural number N can always be put in the form $N = p_1^{\alpha_1}\; p_2^{\alpha_2}\; \ldots\; p_k^{\alpha_k}$ where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are non-negative integers.

- If $N = p_1^{\alpha_1}\; p_2^{\alpha_2}\; \cdots\; p_k^{\alpha_k}$ then number of divisor of N is equivalent of number of ways of selecting zero or more objects from the groups of identical objects, $(p_1, p_1, \ldots, \alpha_1$ times$)$, $(p_2, p_2, \ldots, \alpha_2$ times$)$, $(p_k, p_k \ldots \alpha_k$ times$) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$ which includes 1 and N also.

Note:

All the divisors excluding 1 and N are called proper divisors.

- *Also number of divisors of N can been seen as number of different terms in the expansion of*

$$(p_1^0 + p_1^1 + p_1^2 \cdots + p_1^{\alpha_1}) \times (p_2^0 + p_2^1 + p_2^2 + \cdots + p_2^{\alpha_2}) \times \cdots \times (1 + p_k + p_k^2 + \cdots + p_k^{\alpha_k})$$

Hence, sum of the divisors of N is

$$(1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1})(1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}) \cdots (1 + p_k + p_k^2 + \cdots + p_k^{\alpha_k})$$

$$= \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \cdots \frac{p_k^{\alpha_k+1} - 1}{p_k - 1}$$

- *The number of ways of putting N as a product of two natural numbers is (1/2) $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$ if N is not a perfect square.*

If N is a perfect square, then this is (1/2) $[(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1) + 1]$.

Example 5.77 **There are p copies each of n different books. Find the number of ways in which a non-empty selection can be made from them.**

Sol. Number of selections of any number of copies of a book is $p + 1$ (because copies of the same book are identical things). Similar is the case for each book. Therefore, total number of selections is $(p + 1)^n$.

But this includes a selection, which is empty, i.e., zero copy of each book. Excluding this, the required number of non-empty selections is $(p + 1)^n - 1$.

Example 5.78 **A person is permitted to select at least one and at most n coins from a collection of $(2n + 1)$ distinct coins. If the total number of ways in which he can select coins is 255, find the value of n.**

Sol. We have,

$$^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n = 255 \quad (1)$$

Also the sum of binomial coefficients is

$$^{2n+1}C_0 + {}^{2n+1}C_1 + \cdots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + \cdots + {}^{2n+1}C_{2n+1} = (1 + 1)^{2n+1} = 2^{2n+1}$$

$$\Rightarrow \; ^{2n+1}C_0 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n) + {}^{2n+1}C_{2n+1} = 2^{2n+1}$$

$$\Rightarrow \; 1 + 2(255) + 1 = 2^{2n+1}$$

$$\Rightarrow \; 1 + 255 = 2^{2n}$$

$$\Rightarrow \; 2^{2n} = 2^8 \Rightarrow n = 4$$

Example 5.79 **Nishi has 5 coins each of the different denomination. Find the number different sums of money she can form.**

Sol. Number of different sums of money she can form is equal to number of ways she select one or more coins

$\therefore$ Required no. of ways $= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 2^5 - 1 = 31$.

Example 5.80 **Find the number of groups that can be made from 5 different green balls, 4 different blue balls and 3 different red balls, if at least 1 green and 1 blue ball is to be included.**

Sol. At least, one green ball can be selected out of 5 green balls in $2^5 - 1$, i.e., in 31 ways.

Similarly, at least one blue ball can be selected from 4 blue balls in $2^4 - 1 = 15$ ways. And at least one red or no red ball can be selected in $2^3 = 8$ ways.

Hence, the required number of ways is $31 \times 15 \times 8 = 3720$.

Example 5.81 **There are 3 books of mathematics, 4 of science, and 5 of literature. How many different collections can be made such that each collection consists of**

(i) one book of each subject,
(ii) at least one book of each subject,
(iii) at least one book of literature.

Sol. (i) $^3C_1 \times {}^4C_1 \times {}^5C_1 = 3 \times 4 \times 5 = 60$

(ii) $(2^3 - 1)(2^4 - 1)(2^5 - 1) = 7 \times 15 \times 31 = 3255$

(iii) $(2^5 - 1) \times 2^7 = 31 \times 128 = 3968$

Example 5.82 **Find the total number of proper factors of the number 35700. Also find**

(i) sum of all these factors,
(ii) sum of the odd proper divisors,
(iii) the number of proper divisors divisible by 10 and the sum of these divisors.

Sol. $35700 = 5^2 \times 2^2 \times 3^1 \times 7^1 \times 17^1$

The total number of factors is equal to the total number of selections from (5, 5), (2, 2), (3), (7) and (17), which is given by $3 \times 3 \times 2 \times 2 \times 2 = 72$.

These include 1 and 35700. Therefore, the number of proper divisors (excluding 1 and 35700) is $72 - 2 = 70$

Sum of all these factors (proper) is

$$(5^0 + 5^1 + 5^2)(2^0 + 2^1 + 2^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 1 - 35700$$
$$= 31 \times 7 \times 4 \times 8 \times 18 - 1 - 35700 = 89291$$

Now, the sum of odd proper divisors is

$$(5^0 + 5^1 + 5^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 1$$
$$= 31 \times 4 \times 8 \times 18 - 1 = 17856 - 1 = 17855$$

(Here, 2 as a factor and 1 as a divisor or are to be excluded.)

The number of proper divisors divisible by 10 is equal to number of selections from (5, 5), (2, 2), (3), (7), (17) consisting of at least one 5 and at least one 2 and 35700 is to be excluded and is given by $2 \times 2 \times 2 \times 2 \times 2 - 1 = 31$.

Sum of these divisors is

$$(5^1 + 5^2)(2^1 + 2^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 35700$$
$$= 30 \times 6 \times 4 \times 8 \times 18 - 35700 = 67980$$

Example 5.83 **Find the number of ways in which the number 94864 can be resolved as a product of two factors.**

Sol. $94864 = 2^4 \times 7^2 \times 11^2$

Hence, the number of ways is

$$\frac{1}{2}[(4+1)(2+1)(2+1)+1]=23$$

Example 5.84 **Find the number of divisors of the number $N = 2^3 \cdot 3^5 \cdot 5^7 \cdot 7^9$ which are perfect square.**

Sol. Since the divisor is perfect square each prime factor must occur even number of times.

2 can be taken in 2 ways (2^0or 2^2)

3 can be taken in 3 ways (3^0 or 3^2 or 3^4)

Similarly 5 can be taken in 4 ways (5^0 or 5^2 or 5^4 or 5^6)

and 7 can be taken in 5 ways (7^0 or 7^2 or 7^4 or 7^6 or 7^8)

hence total divisors which are perfect squares

$$= 2\cdot3\cdot4\cdot5 = 120$$

Example 5.85 **Find the number of ways in which the number 300300 can be split into 2 factors which are relatively prime.**

Sol. $300300 = 2^2 3^1 5^2 7^1 11^1 13^1$

Now we have to make factors which are relative prime.

$\Rightarrow$ $2^2, 3^1, 5^2, 7^1, 11^1, 13^1$ should behave as single identities.

So no. of divisors $(1+1)(1+1)(1+1)(1+1)(1+1)(1+1)$

$$= 2^6 = 64$$

No. of ways of splitting into 2 factors $= \frac{64}{2} = 32$

Concept Application Exercise 5.6

1. Out of 10 white, 9 black and 7 red balls, find the number of ways in which selection of one or more balls can be made (balls of the same color are identical).
2. In an election, number of candidates exceeds the number to be elected by 2. A man can vote in 56 ways. Find the number of candidates.
3. There are 5 historical monuments, 6 gardens and 7 shopping malls in a city. In how many ways a tourist can visit the city if he visits at least one shopping mall.
4. Find the number of divisors of 720. How many of these are even? Also find the sum of divisors.
5. Find the number of odd proper divisors of $3^p \times 6^m \times 21^n$.
6. In how many ways the number 7056 can be resolved as a product of 2 factors.

DIVISION AND DISTRIBUTION

Distinct Objects

Division of $m + n$ Distinct Objects into Two Groups of the Size m and n ($m \neq n$)

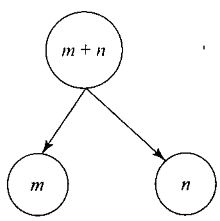


Fig. 5.17

The number of ways $m + n$ distinct objects are divided into two groups of the size m and n is equivalent to the number of ways m objects are selected out of $m + n$ objects to form one of the groups, which can be done in ${}^{m+n}C_m$ ways. The other group of n objects is formed by the remaining n objects. So, the number of ways is

$${}^{m+n}C_m = \frac{(m+n)!}{m!n!}$$

Now, distribution of $m + n$ distinct objects between two persons (if one gets m and the other gets n objects) is equal to number of ways to divide $m + n$ objects into two groups $\times$ number of ways in which these two groups can be given to two persons, which is equal to $\frac{(m+n)!}{m!n!}2!$

Division of $m + n + p$ Distinct Objects Into Three Groups of the Size m, n and p ($m \neq n \neq p$)

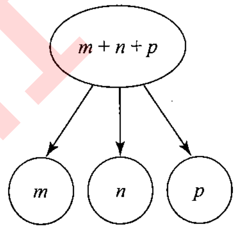


Fig. 5.18

For this division, let us first select m objects for first group which can be done in ${}^{m+n+p}C_m$ ways.

From the remaining $n + p$ objects, select n objects for second group, which can be done in ${}^{n+p}C_n$ ways.

The third group is formed from the remaining p objects. Hence, the total number of ways is

$$({}^{m+n+p}C_m)({}^{n+p}C_n) = \frac{(m+n+p)!}{m!\,n!\,p!}$$

Now, distribution of $m + n + p$ objects among three persons if they get m, n and p objects is equal to number of ways to divide $(m + n + p) \times$ (number of ways in which these three groups can be given to three persons) which is equal to

$$\frac{(m+n+p)!}{m!n!p!}3!$$

In general, division of $x_1 + x_2 + x_3 + \cdots + x_n$ into n groups of the size $x_1, x_2, x_3, \ldots, x_n$ $(x_1 \neq x_2 \neq \cdots \neq x_n)$

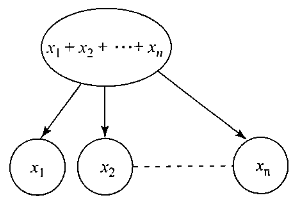


Fig. 5.19

The number of division ways is

$$\frac{(x_1+x_2+\cdots+x_n)!}{x_1!x_2!\cdots x_n!}$$

Now, distribution ways of these n groups among n persons is

$$\frac{(x_1+x_2+\cdots+x_n)!}{x_1!x_2!\cdots x_n!}n!$$

Division of Objects into Two Groups of Equal Size n

Consider the distribution of 4 objects a, b, c, d into two groups each of size 2. The number of ways of selection is $4!/(2!2!) = 6$.

But this answer is more similar to the one given in the division chart below:

No.	Group 1	Group 2
1	*ab*	*cd*
2	*ac*	*bd*
3	*ad*	*bc*
4	*bc*	*ad*
5	*bd*	*ac*
6	*cd*	*ab*

Clearly, in the above chart way number 1 and 6, 2 and 5, 3 and 4 are same division.

Then, the actual number of ways is

$$\frac{\frac{4!}{2!2!}}{2!}=3$$

as in each division way, say ab and cd, these groups can be arranged in 2! ways.

The division of $2n$ objects into two groups of equal size is

$$\frac{\frac{(2n)!}{n!n!}}{2!}=\frac{(2n)!}{n!n!2!}$$

Now, the distribution ways of these 2 groups between 2 persons is

$$\frac{(2n)!}{n!n!2!}2!=\frac{(2n)!}{n!n!}$$

Again, division of $3n$ objects into three groups of equal size n is

$$\frac{\frac{(3n)!}{n!n!n!}}{3!}=\frac{(3n)!}{n!n!n!3!}$$

as in each division way, say ab, cd and ef three groups can be arranged in 3! ways.

Now, distribution ways of these 3 groups among 3 persons is

$$\frac{(3n)!}{n!n!n!3!}3!=\frac{(3n)!}{n!n!n!}$$

Division of Distinct Objects into Multiple Equal Groups

Division of $12n$ objects into 5 groups of $2n$, $2n$, $2n$, $3n$, $3n$ size is

$$\frac{\frac{(12n)!}{(2n)!(2n)!(2n)!(3n)!(3n)!}}{3!2!}$$

Now, the distribution ways of these 5 groups among 5 persons is

$$\frac{\frac{(12n)!}{(2n)!(2n)!(2n)!(3n)!(3n)!}}{3!2!}5!$$

Distribution of n Distinct Objects in r Different Boxes, if Any Number of Objects Are Placed in Any Box (Empty Boxes Are Allowed)

Consider distribution of n distinct objects into two different boxes of any size

Number of objects in box 1	Number of objects in box 2	Number of ways
0	n	${}^nC_0$
1	$n-1$	${}^nC_1$
2	$n-2$	${}^nC_2$
…	…	…
n	0	${}^nC_n$

The total number of ways is ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$. Alternatively, each object has two possibilities; it can enter either into box 1 or box 2. The total number of possibilities for n objects is $2 \times 2 \times 2 \times \cdots$ n times $= 2^n$.

Consider distribution of n distinct objects into three groups of any size. For each object, there are three possibilities. Then the total number of possibilities for n objects is $3 \times 3 \times 3 \ldots$ n times $= 3^n$.

Similarly, for distribution of n distinct objects into r different boxes, if in any box any number of objects can be placed for each object, there are r possibilities. Then total number of possibilities for n objects is $r \times r \times r \cdots$ n times $= r^n$.

Distribution of n Distinct Objects Into r Different Boxes if Empty Boxes Are Not Allowed or in Each Box at Least One Object is Put (n > r)

The number of ways is given by

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \cdots + (-1)^{r-1}\cdot {}^rC_{r-1}\cdot 1$$

This formula can be derived from the principle of inclusion and exclusions using set theory.

Example 5.86 **Find the number of ways of dividing 52 cards among four players equally.**

Sol. We can divide 52 cards equally into four groups in $52!/(13!)^4(4!)$ ways.

Now, these four groups can be distributed among four players as $4!$ ways.

Therefore, the total number of ways of dividing the cards among four players equally is

$$\frac{52!}{(13!)^4(4!)} \times 4! = \frac{52!}{(13!)^4}$$

Example 5.87 **Find the number of ways to give 16 different things to three persons A, B, C so that B gets 1 more than A and C gets 2 more than B.**

Sol. Let A get n objects, then B gets $n + 1$ and C gets $n + 3$

Now $n + n + 1 + n + 3 = 16 \Rightarrow n = 4$

$\Rightarrow$ A, B, C gets 4, 5 and 7 objects respectively.

$\Rightarrow$ Number of ways of distribution is equal to number of ways 16 objects can be divided into three groups of size 4, 5 and 7.

Hence, number of ways $= \dfrac{16!}{4!5!7!}$.

Example 5.88 **In how many ways can 8 different books be distributed among 3 students if each receives at least 2 books?**

Sol. If each receives at least two books, then the division trees would be as shown below:

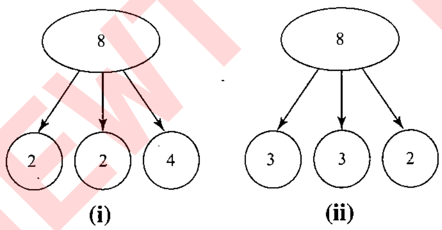


Fig. 5.20

The number of division ways for tree in Fig. 5.20 (i) is $8!/(2!)^2 4!2!$. The number of division ways for tree in Fig. 5.20 (ii) is $8!/(3!)^2 2!2!$. The total number of ways of distribution of these groups among 3 students is

$$\left[\frac{8!}{(2!)^2\,4!\,2!} + \frac{8!}{(3!)^2\,2!\,2!}\right] \times 3!$$

Example 5.89 ***n* different toys have to be distributed among *n* children. Find the number of ways in which these toys can be distributed so that exactly one child gets no toy.**

Sol. If exactly one child gets no toy, then exactly one child must get two toys and rest $n - 2$ gets one toy each.

The division tree will be as follows:

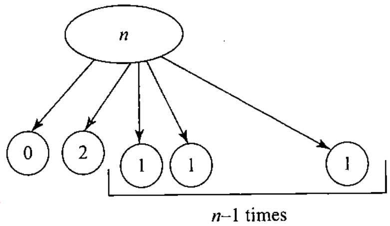


Fig. 5.21

The number ways of division in the groups as shown in the tree is

$$\frac{n!}{0!2!(1!)^{n-2}\,(n-2)!} = \frac{n!}{2!(n-2)!} = {}^nC_2$$

The number of ways of distribution of these n groups among n children is $n!$. Then, the total number of ways of distributions is ${}^nC_2 \times n!$.

Example 5.90 **Find the number of ways in which n different prizes can be distributed among m ($< n$) persons if each is entitled to receive at most $n - 1$ prizes.**

Sol. The total number of ways is $m \times m \times \cdots n$ times $= m^n$. The number of ways in which one gets all the prizes is m. Therefore, the required number of ways is $m^n - m$.

Distribution of Identical Objects

When identical objects are distributed for different boxes, the number of objects is only important. It is of no use to consider about the boxes that what objects they hold.

Suppose 4 identical objects are distributed in 2 distinct boxes if empty boxes are allowed.

Number of objects in box 1	Number of objects in box 2	Number of ways
0	4	1
1	3	1
2	2	1
3	1	1
4	0	1

Here, the total number of ways is 5.

Distribution of n Identical Objects in r Different Boxes if Empty Boxes Are Not Allowed

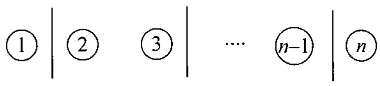


Fig. 5.22

Let n identical objects be put on the floor in line. Now, to form r groups, we require to put $r - 1$ identical partitions. If empty boxes are not allowed, we cannot put partition to the left of the first object and to the right of n^{th} object. Hence, there are $n-1$ gaps available to put $r-1$ partitions. Now, $r-1$ gaps can be selected from $n-1$ gaps in ${}^{n-1}C_{r-1}$ ways. This is equivalent to number of ways of distributing n objects in r boxes if empty boxes are not allowed.

Distribution of n Identical Objects in r Different Boxes if Empty Boxes Are Allowed

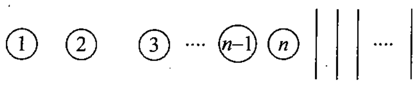


Fig. 5.23

Here, the number of ways are equivalent to arranging n identical objects and $r-1$ identical partitions, which automatically takes care of empty boxes. The number of ways is

$$\frac{(n+r-1)!}{n!(r-1)!} = {}^{n+r-1}C_{r-1}$$

Number of Non-Negative Integral Solutions of the Equation $x_1 + x_2 + \cdots + x_r = n$

This is equivalent to the number of ways of distributing n identical objects into r different boxes if empty boxes are allowed which is ${}^{n+r-1}C_{r-1} = {}^{n+r-1}C_n$.

Number of Positive Integral Solutions of the Equation $x_1 + x_2 + \cdots + x_r = n$

This is equivalent to the number of ways of distributing n identical objects into r different boxes if empty boxes are not allowed which is ${}^{n-1}C_{r-1}$.

Example 5.91 **Find the number of non-negative integral solutions of the equations $x + y + z = 10$.**

Sol. Here, the number of solutions is equivalent to the number of ways. Ten identical objects are distributed in 3 distinct boxes if empty boxes are allowed, which is ${}^{10+3-1}C_3 = {}^{12}C_3$.

Example 5.92 **Find the number of positive integral solutions of the equations $x + y + z = 12$.**

Sol. Here, the number of solutions is equivalent to number of ways. Twelve identical objects are distributed in 3 distinct boxes if empty boxes are not allowed, which is ${}^{12-1}C_{3-1} = {}^{11}C_2 = 55$.

Example 5.93 **Find the number of non-negative integral solutions of the equation $x + y + z + 2w = 20$.**

Sol. Let $w = 0$. Then, the equation reduces to $x + y + z = 20$. Number of non-negative integral solutions is ${}^{20+4-1}C_{4-1} = {}^{23}C_3$. If $w = 1$, then the equation reduces to $x + y + z = 18$. Number of non-negative integral solutions is ${}^{18+4-1}C_{4-1} = {}^{21}C_3$.

Similarly, we have $w = 2, 3, \ldots, 10$.

Therefore, the total number of solutions is ${}^{23}C_3 + {}^{21}C_3 + {}^{19}C_3 + \cdots + {}^{5}C_3 + {}^{3}C_3$.

Example 5.94 **Find the number of non-negative integral solutions of $x + y + z + w \le 20$.**

Sol. Let,

$$x + y + z + w + t = 20 \qquad (1)$$

where $t \ge 0$.

Now, we find the non-negative integral solutions of Eq. (1). The total number of such solutions is ${}^{20+5-1}C_{5-1}$

Example 5.95 **In how many ways can a party of 6 men be selected out of 10 Hindus, 8 Muslims and 6 Christians. If the party consists of at least one person of each religion, find the number of ways of selection. (Consider only the religion of the person.)**

Sol. Let x, y and z be the number of Hindus, Muslims and Christians, respectively, who are selected. Then,

$$x + y + z = 6 \qquad (1)$$

Also, at least one person must be selected from each religion when we have to find non-negative integral solutions of Eq. (1).

Therefore, the number of non-negative solutions of Eq. (1) is ${}^{6-1}C_{3-1} = {}^{5}C_2 = 10$.

Example 5.96 **Find the total number of positive integral solutions for (x, y, z) such that $xyz = 24$. Also find out the total number of integral solutions.**

Sol. $24 = 2^3 \times 3$

Now, consider three boxes x, y, z. 3 can be put in any of the three boxes.

Also, 2, 2, 2 can be distributed in the three boxes in ${}^{3+3-1}C_{3-1} = {}^{5}C_2$ ways. Hence, the total number of positive integral solutions is equal to the number of distributions which is given by $3 \times {}^{5}C_2 = 30$.

Note: *If any box remains empty, say x, than $x = 1$. To find integral solutions where negative integers are also allowed.*

Any two of the factors in each factorization may be negative. Therefore, the number of ways to associate negative sign in each case is ${}^{3}C_2 = 3$. Hence, the total number of integral solutions is $30 + 3 \times 30 = 120$.

Example 5.97 **In how many ways can 10 persons take seats in a row of 24 fixed seats so that no two persons take consecutive seats.**

Sol. As no two persons take consecutive seats, there will be at least one vacant seat between any two persons sitting before the first person and after the last person. Let the number of vacant seats before the first person $= x_0$, and the number of vacant seats between the first and the second persons be x_1, etc., as shown in the figure.

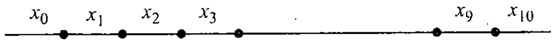


Fig. 5.24

Clearly, the total number of vacant seats is 24 – 10 = 14.

$$\therefore \quad x_0 + x_1 + x_2 + \cdots + x_9 + x_{10} = 14$$

where $x_0 \geq 0, x_1 \geq 1, x_2 \geq 1, x_3 \geq 1, \ldots, x_9 \geq 1, x_{10} \geq 0$.

Let $x_0 = y_0, x_1 = y_1 + 1, x_2 = y_2 + 1, \ldots, x_9 = y_9 + 1$, $x_{10} = y_{10}$. Then, the equation becomes

$$y_0 + (y_1 + 1) + (y_2 + 1) + \cdots + (y_9 + 1) + y_{10} = 14$$

$$\Rightarrow \quad y_0 + y_1 + y_2 + \cdots + y_9 + y_{10} = 14 - 9 = 5$$

Therefore, the required number of ways = the number of non-negative integral solutions of the equation $y_0 + y_1 + y_2 + \cdots + y_9 + y_{10} = 5$, i.e., ${}^{5+11-1}C_{11-1} = {}^{15}C_{10}$.

But 10 persons can arrange among themselves in 10! ways. Hence, the required number of ways is

$${}^{15}C_{10} \times 10! = \frac{15!}{10!5!} \times 10! = \frac{15!}{5!}.$$

Example 5.98 **Find the number of distinct throws which can be thrown with n six-faced normal dice, which are indistinguishable among themselves.**

Sol. Consider six faces as six beggars and n identical dice to be identical coins.

Now, number of distribution is ${}^{n+6-1}C_{6-1} = {}^{n+5}C_5$. If a beggar (say face 6) gets no coin, then it is equivalent to 6, which does not appear on the dice.

Example 5.99 **In how many ways 3 boys and 15 girls can sit together in a row such that between any 2 boys at least 2 girls sit.**

Sol.

x	B	y	B	z	B	w

Fig. 5.25

First three boys can be arranged in 3! ways. After arranging the boys, four gaps are created. Let in these gaps x, y, z and w girls sit as shown in the diagram. Let us first find out the distribution ways of girls in the four gaps. As given in question, $y, z \geq 2$ and $x, w \geq 0$, we have to find the integral solutions of the equation $x + y + z + w = 15$ with the above condition. Let,

$$y = y_1 + 2 \text{ and } z = z_1 + 2 \text{ (where } y_1, z_1 \geq 0)$$

$$\Rightarrow \quad x + y_1 + z_1 + w = 11$$

Number of solutions of above equation is ${}^{11+4-1}C_{4-1} = {}^{14}C_3$. After it is decided as in which gap how many girls will sit, they can be arranged in 15! ways.

Hence, the total number of ways is 3! 15! ${}^{14}C_3$.

Concept Application Exercise 5.7

1. Find the number of ways in which 22 different books can be given to 5 students, so that two students get 5 books each and all the remaining students get 4 books each.
2. Find the number of ways in which 16 constables can be assigned to patrol 8 villages, 2 for each.
3. In how many ways can 10 different prizes be given to 5 students if one particular boy must get 4 prizes and rest of the students can get any number of prizes?
4. In how many different ways can a set A of $3n$ elements be partitioned into 3 subsets of equal number of elements? (The subsets P, Q, R form a partition if $P \cup Q \cup R = A$, $P \cap R = \phi$, $Q \cap R = \phi$, $R \cap P = \phi$.)
5. In how many ways can Rs. 16 be divided into 4 persons when none of them gets less than Rs. 3?
6. Find the number of ways of selecting 10 balls out of an unlimited number of identical white, red and blue balls.
7. If a, b, c, d are odd natural numbers such that $a + b + c + d = 20$, then find the number of values of the ordered quadruplet (a, b, c, d).
8. Roorkee University has to send 10 professors to 5 centres for its entrance examination, 2 to each centre. Two of the centres are in Roorkee and the others are outside. Two of the professors prefer to work in Roorkee while three prefer to work outside. In how many ways can this be made if the preferences are to be satisfied?
9. In how many ways, two different natural numbers can be selected, which are less than or equal to 100 and differ by almost 10.

MULTINOMIAL THEOREM

Consider the equation $x_1 + x_2 + \cdots + x_r = n$, where $a_i \leq x_i \leq b_i$; $x_i \in 1$; $i = 1, 2, \ldots, r$.

In order to find the number of solutions of the given equation satisfying the given conditions, we observe that the number of solutions is the same as the coefficient of x^n in the product

$$(x^{a_1} + x^{a_1+1} + x^{a_1+2} + \cdots + x^{b_1}) \times (x^{a_2} + x^{a_2+1} + x^{a_2+2} + \cdots + x^{b_2})$$

$$\times (x^{a_3} + x^{a_3+1} + x^{a_3+2} + \cdots + x^{b_3}) \times \cdots \times (x^{a_r} + x^{a_r+1} + x^{a_r+2} + \cdots + x^{b_r})$$

For example, if we have to find the number of non-negative integral solutions of $x_1 + x_2 + \cdots + x_r = n$, then as above, the required number is the coefficient of x^n in $(x^0 + x^1 + \cdots + x^n)(x^0 + x^1 + \cdots + x^n) \ldots (x^0 + x^1 + \cdots + x^n)$ (r – brackets)

$$= \text{Coefficient of } x^n \text{ in } (1 + x + x^2 + \cdots + x^n)^r$$

$$= \text{Coefficient of } x^n \text{ in } (1 + x + x^2 + \cdots)^r$$

$$= \text{Coefficient of } x^n \text{ in } (1 - x)^{-r}$$

= Coefficient of x^n in

$$\left[1 + (-r)(-x) + \frac{(-r)(-r-1)}{2!}(-x)^2\right.$$

$$\left. + \frac{(-r)(-r-1)(-r-2))3!}{3!}(-x)^3 \right]$$

$= \text{Coefficient of } x^n \text{ in } [1 + {}^rC_1 x + {}^{r+1}C_2 x^2 + {}^{r+2}C_3 x^3 + \cdots]$

$= {}^{n+r-1}C_n$

$= {}^{n+r-1}C_{r-1}$

Note: *If there are l objects of one kind, m objects of second kind, n objects of third kind and so on, then the number of ways of choosing r objects out of these objects is the coefficient of x^r in the expansion of $(1 + x + x^2 + x^3 + \cdots + x^l) \times (1 + x + x^2 + \cdots + x^m) \times (1 + x + x^2 + \cdots + x^n)$.*

Further, if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects is the coefficient of x^r in the expansion of $(x + x^2 + x^3 + \cdots + x^l) \times (x + x^2 + x^3 + x^3 + \cdots + x^m) \times (x + x^2 + x^3 + \cdots + x^n)$.

Different Cases of Multinomial Theorem

Case I

If upper limit of a variable is more than or equal to the sum required, than the upper limit of that variable can be taken as infinite.

Example 5.100 **In how many ways the sum of upper faces of four distinct dices can be six.**

Sol. Here, the number of required ways will be equal to the number of solutions of $x_1 + x_2 + x_3 + x_4 = 6$, i.e., $1 \le x_1, x_2, x_3, x_4 \le 6$. Since the upper limit is six, which is equal to the sum required, so upper limit can be taken as infinite. So, number of solutions is equal to the coefficient of t^5 in $(1 + t + t^2 + \cdots)^4$

$= \text{Coefficient of } t^5 \text{ in } (1-t)^{-4}$

$= {}^{6+4-1}C_{4-1}$

$= {}^9C_3 = 84$

Case II

If the upper limit of a variable is less than the sum required and the lower limit of all the variables is non-negative, then the upper limit of that variable is that given in the problem.

Example 5.101 **In how many different ways can 3 persons A, B, C having 6 one-rupee coin, 7 one-rupee coin, 8 one-rupee coin, respectively, donate 10 one-rupee coin collectively?**

Sol. The number of ways in which they can donate Rs. 10 is same as the number of solutions to the equation $x_1 + x_2 + x_3 = 10$ subject to the condition $0 \le x_1 \le 6$, $0 \le x_2 \le 7$, $0 \le x_3 \le 8$.

Therefore, the required number of ways = Coefficient of x^{10} in $(1 + x + x^2 + \cdots + x^6)(1 + x + x^2 + \cdots + x^7)(1 + x + x^2 + \cdots + x^8)$

$= \text{Coefficient of } x^{10} \text{ in } (1-x^7)(1-x^8)(1-x^9)(1-x)^{-3}$

$= \text{Coefficient of } x^{10} \text{ in } (1 - x^7 - x^8 - x^9)(1 + {}^3C_1 x + {}^4C_2 x^2 + {}^5C_2 x^2 + {}^5C_3 x^3 + \cdots + {}^{12}C_{10} x^{10})$ (ignoring powers higher than 10).

$= {}^{12}C_2 - {}^5C_3 - {}^4C_2 - {}^3C_1$

$= 66 - 10 - 6 - 3 = 47$

Example 5.102 **In an examination, the maximum mark for each of the three papers is 50 and the maximum mark for the fourth paper is 100. Find the number of ways in which the candidate can score 605 marks in aggregate.**

Sol. Let the marks scored by the candidate in four papers be x_1, x_2, x_3, x_4. Then, $x_1 + x_2 + x_3 + x_4 = 150$ (i.e., 60% of 250 is 150), where $0 \le x_1, x_2, x_3 \le 50$, $0 \le x_4 \le 100$.

The number of solutions of the equation is same as the coefficient of x^{150} in $(1 + x + x^2 + \cdots + x^{50})^3 (1 + x + x^2 + \cdots + x^{100})$ which is given by

$$\text{Coefficient of } x^{150} \text{ in } \left(\frac{1-x^{51}}{1-x}\right)^3 \left(\frac{1-x^{101}}{1-x}\right)$$

$= \text{Coefficient of } x^{150} \text{ in } (1-x^{51})^3 (1-x^{101})(1-x)^{-4}$

$= \text{Coefficient of } x^{150} \text{ in } (1 - 3x^{51} + 3x^{102} - x^{101})(1-x)^{-4}$

$= {}^{153}C_3 - 3 \times {}^{102}C_3 + 3 \times {}^{51}C_3 - {}^{52}C_3$

$= 110551$

Case III

When coefficients of variables are not uniform. The number of solutions of $ax_1 + bx_2 + cx_3 = n$ is subject to this condition in $a_1 \le x_1 \le b_1$, $a_2 \le x_2 \le b_2$, $a_3 \le x_3 \le b_3$. The coefficient of t^n is

$$\{(t^a)^{a_1} + (t^a)^{a_1+1} + \cdots + (t^a)^{b_1}\} \times \{(t^b)^{a_2} + (t^b)^{a_2+1} + \cdots + (t^b)^{b_2}\}$$
$$\times \{(t^c)^{a_3} + (t^c)^{a_3+1} + \cdots + (t^c)^{b_3}\}$$

Example 5.103 **Find the number of non-negative integral solutions of $x_1 + x_2 + x_3 + 4x_4 = 20$.**

Sol. The number of non-negative integral solutions of the given equation is equal to the coefficient of x^{20} in $(1-x)^{-1}(1-x)^{-1}(1-x)^{-1} \times (1-x^4)^{-1}$, which is given by

Coefficient of x^{20} in $(1-x)^{-3}(1-x^4)^{-1}$

$= \text{Coefficient of } x^{20} \text{ in } (1 + {}^3C_1 x + {}^4C_2 x^2 + {}^5C_3 x^3 + {}^6C_4 x^4 + \cdots)(1 + x^4 + x^8 + \cdots)$

$= 1 + {}^6C_4 + {}^{10}C_8 + {}^{14}C_{12} + {}^{18}C_{16} + {}^{22}C_{20} = 536$

Example 5.104 **In how many ways can 15 identical blankets be distributed among six beggars such that everyone gets at least one blanket and two particular beggars get equal blankets and another three particular beggars get equal blankets.**

Sol. The number of ways of distributing blankets is equal to the number of solutions of the equation $3a + 2b + c = 15$ $a, b, c \geq 1$, which is equal to coefficient of t^{15} in $(t^3 + t^6 + t^9 + t^{12} + \cdots) \times (t^2 + t^4 + \cdots)(t + t^2 + \cdots)$, which is given by

Coefficient of t^9 in $(1 + t^3 + t^6 + t^9)(1 + t^2 + t^4 + t^6 + t^8) \times (1 + t + t^2 + \cdots + t^9)$ (neglecting higher powers)

$=$ Coefficient of t^9 in $(1 + t^2 t^3 + t^4 + t^5 + 2t^6 + t^7 + 2t^8 + 2t^9)(1 + t + t^2 + \cdots + t^9) = 1 + 1 + 1 + 1 + 1 + 2 + 1 + 2 + 2 = 12$

Case IV

In the case when variables are distinct, we introduce some new variables to remove the condition of distinctness.

Example 5.105 **In how many ways can 14 identical toys be distributed among three boys so that each one gets at least one toy and no two boys get equal number of toys.**

Sol. Let the boys get a, b, c toys. Now, $a + b + c = 14$, $a, b, c \geq 1$ and a, b and c are distinct.

Let $a < b < c$ and $x_1 = a$, $x_2 = b - a$, $x_3 = c - b$. So,

$$3x_1 + 2x_2 + x_3 = 13, \quad x_1, x_2, x_3 \geq 1$$

Therefore, the number of solutions is equal to the coefficient of t^{14} in $(t^3 + t^6 + t^9 + \cdots)(t^2 + t^4 + \cdots)(t + t^2 + \cdots)$

$=$ Coefficient of t^8 in $(1 + t^3 + t^6)(1 + t^2 + t^4 + t^6 + t^8)(1 + t + t^2 + \cdots + t^8)$ (neglecting higher powers)

$=$ Coefficient of t^8 in $(1 + t^2 + t^3 + t^4 + t^5 + 2t^6 + t^7 + 2t^8) \times (1 + t + t^2 + \cdots + t^8)$

$= 1 + 1 + 1 + 1 + 1 + 2 + 1 + 2 = 10$

Now, three distinct numbers can be assigned to three boys in $3!$ ways.

So, corresponding to each solution, we have six ways of distribution. So, total numbers of ways is $10 \times 6 = 60$.

Case V

When the required sum is not fixed.

To find the number of solutions of the equation,

$$x_1 + x_2 + \cdots + x_m \leq n \qquad (1)$$

We introduce a dummy variable x_{m+1} so that

$$x_1 + x_2 + \cdots + x_{m+1} = n, \; x_{m+1} \geq 0 \qquad (2)$$

Hence, the number of solutions of Eqs. (1) and (2) will be same.

To find the number of solution of $x_1 + x_2 + \cdots + x_n \geq n$ (when the values of $x_1, x_2, \ldots, x_n$ are restricted), first find the number of solutions of $x_1 + x_2 + \cdots + x_n \leq n - 1$ and then subtract it from the total number of solutions of $x_1 + x_2 + \cdots + x_n \leq n - 1$; and then subtract it from the total number of solutions.

Example 5.106 **In how many ways can we get a sum of at most 17 by throwing six distinct dice.**

Sol. Let $x_1, x_2, \ldots, x_6$ be the number that appears on the six dice. Let us find the number of ways to get the sum less than or equal to 17. This will be same as finding the number of solutions to the inequality $x_1 + x_2 + x_3 + \cdots + x_6 \leq 17$. Introducing a dummy variable x_7 ($x_7 \geq 0$), the inequality becomes an equation

$$x_1 + x_2 + \cdots + x_6 + x_7 = 17$$

Here, $1 \leq x_i \leq 6$ where $i = 1, 2, \ldots, 6$ and $x_7 \geq 0$. Therefore,

Number of solutions $=$ Coefficient of x^{17} in $(x + x^2 + \cdots + x^6)^6 \times (1 + x + x^2 + \cdots)$

$=$ Coefficient of x^{11} in $(1 - x^6)^6 (1 - x)^{-7}$

$=$ Coefficient of x^{11} in $(1 - 6x^6)(1 - x)^{-7} = {}^{17}C_6 - 6\,{}^{11}C_5$

Example 5.107 **In how many ways can we get a sum greater than 17 by throwing six distinct dice.**

Sol. Let $x_1, x_2, \ldots, x_6$ be the number that appears on the six dice. Here, $1 \leq x_i \leq 6$, $\forall i \in \{1, 2, 3, 4, 5, 6\} \Rightarrow$ total number of cases is 6^6. In the above example, we have calculated the number of ways to get the sum less than or equal to 17, which is

$${}^{17}C_{11} - 6 \times {}^{11}C_5.$$

Hence, the number of ways to get a sum greater than 17 is $6^6 - ({}^{17}C_{11} - 6\,{}^{11}C_5)$.

Concept Application Exercise 5.8

1. In how many ways can 30 marks be allotted to 8 questions if each question carries at least 2 marks?
2. Find the number of positive integral solutions of the inequality $3x + y + z \leq 30$.
3. Find the number of integers between 1 and 100000 having the sum of the digits 18.
4. Find the number of integral solutions of $x_1 + x_2 + x_3 = 24$ subjected to the condition that $1 \leq x_1 \leq 5$, $12 \leq x_2 \leq 18$ and $-1 \leq x_3$.

PRINCIPLE OF INCLUSION AND EXCLUSION

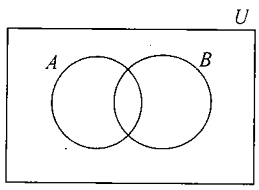


Fig. 5.26

In the above Venn's diagram, we get

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A_1' \cap A_2') = n(U) - n(A \cup B)$

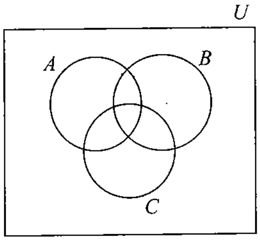


Fig. 5.27

In the above Venn's diagram, we get

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$n(A_1' \cap A_2' \cap A_3') = n(U) - n(A \cup B \cup C)$$

In general, we have

$$n(A_1 \cup A_2 \cup \cdots \cup A_n)$$
$$= \sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + \cdots + (-1)^n \sum n(A_1 \cap A_2 \cap \cdots \cap A_n)$$

Example 5.108 **Find the numbers of positive integers from 1 to 1000, which are divisible by at least 2, 3 or 5.**

Sol. Let A be the set of positive integers form 1 to 1000, which is divisible by k. Obviously, we have to find $n(A_2 \cup A_3 \cup A_5)$. If $[\cdot]$ denotes the greatest integer function, then

$$n(A_2) = \left[\frac{1000}{2}\right] = 500$$

$$n(A_3) = \left[\frac{1000}{3}\right] = 333$$

$$n(A_5) = \left[\frac{1000}{5}\right] = 200$$

Hence, $n(A_2 \cap A_3) = 166$, $n(A_3 \cap A_5) = 66$, $n(A_2 \cap A_5) = 100$, $n(A_2 \cap A_3 \cap A_5) = 33$.

Hence, $n(A_2 \cup A_3 \cup A_5) = 500 + 333 + 200 - 166 - 66 - 100 + 33 = 734$.

Note that the number of positive integers from 1 to 1000, which are not divisible by 2, 3 or 5 is $1000 - n(A_2 \cup A_3 \cup A_5) = 266$.

Example 5.109 **Find the number of ways in which two Americans, two British, one Chinese, one Dutch and one Egyptian can sit on a round table so that persons of the same nationality are separated.**

Sol. Total number of person is 6!. When A_1, A_2 are together,

$$n(A) = 5!\,2! = 240$$

When B_1, B_2 together,

$$n(B) = 5!\,2! = 240$$

$$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = 240 + 240 - 96 = 384$$

Hence,

$$n(\bar{A} \cap \bar{B}) = \text{Total persons} - n(A \cup B) = 6! - 384 = 720 - 384 = 336$$

Example 5.110 **Find the number of permutations of letters *a, b, c, d, e, f, g* taken all together if neither '*beg*' nor '*cad*' pattern appear.**

Sol. The total number of permutations without any restrictions is 7!.

beg *acdf*

Fig. 5.28

The number of permutations in which '*beg*' pattern always appears is 5!. Likewise, there are some cases in which '*cad*' pattern also appears.

cad *befg*

Fig. 5.29

The number of permutations in which '*cad*' pattern always appears is 5!. Likewise, there are some cases in which '*beg*' pattern also appears.

cad *beg* *f*

Fig. 5.30

The number of permutations in which '*beg*' and '*cad*' patterns appears is 3!.

Therefore, the total number of permutations in which '*beg*' and '*cad*' patterns do not appear is 7! − 5! −5! + 3!.

Example 5.111 **Find the number of *n* digit numbers, which contain the digits 2 and 7, but not the digits 0, 1, 8, 9.**

Sol. The total number without any restrictions containing digits 2, 3, 4, 5, 6, 7 is $n(S) = 6^n$.

The total number of numbers that contain 3, 4, 5, 6, 7 is $n(A) = 5^n$.

The total number of numbers that contain 2, 3, 4, 5, 6 is $n(B) = 5^n$.

The total number of numbers that contain 3, 4, 5, 6 is $n(A \cap B) = 4^n$.

The total number of numbers that do not contain digits 2 and 7 is $5^n + 5^n - 4^n$. The total number of numbers that contain 2 and 7 is $6^n - 5^n - 5^n + 4^n$.

Derangement

There are n letters and n corresponding envelopes. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is

$$n!\left[1-\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{(-1)^n}{n!}\right]$$

Proof: n letters are denoted by 1, 2, 3, ..., n. Let A_i denote the set of distribution of letters in envelopes (one letter in each envelop) so that the i^{th} letter is placed in the corresponding envelope. Then,

$n(A_i) = 1 \times (n-1)!$ [since the remaining $n-1$ letters can be placed in $n-1$ envelopes in $(n-1)!$ ways]

Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes. Then,

$$n(A_i \cap A_j) = 1 \times 1 \times (n-2)!$$

Also,

$$n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$$

Hence, the required number is

$$n(A_1' \cup A_2' \cup \cdots \cup A_n') = n! - n(A_1 \cup A_2 \cup \cdots \cup A_n)$$

$$= n! - \left[\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + \cdots + (-1)^n \sum n(A_1 \cap A_2 \cdots \cap A_n)\right]$$

$$= n! - [{}^nC_1(n-1)! - {}^nC_2(n-2)! + {}^nC_3(n-3)! + \cdots + (-1)^{n-1} \times {}^nC_n 1]$$

$$= n! - \left[\frac{n!}{1!(n-1)!}(n-1)! - \frac{n!}{2!(n-2)!}(n-2)! + \cdots + (-1)^{n-1}\right]$$

$$= n!\left[1-\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{(-1)^n}{n!}\right]$$

Example 5.112 **There are four balls of different colours and four boxes of colours same as those of the balls. Find the number of ways in which the balls, one in each box, could be placed in such a way that a ball does not go to box of its own colour.**

Sol. Number of derangements in such problems is given by

$$n!\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+(-1)^n\frac{1}{n!}\right\}$$

Hence, the required number of derangements is

$$4!\left\{\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right\} = 12-4+1 = 9$$

Distribution of n Distinct Objects into r Distinct Boxes if in Each Box at Least One Object is Placed

The number of ways in which n distinct objects can be distributed among r persons if each gets at least one object is

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)r^n - {}^rC_3(r-3)^n + \cdots + (-1)^{r-1}\,{}^rC_{r-1}1$$

Proof: Let A_i denote the set of distribution of objects if i^{th} person gets no object. Then,

$n(A_i) = (r-1)^n$ [as now, n objects can be distributed among $r-1$ persons in $(r-1)^n$ ways]

Then, $n(A_i \cap A_j)$ represents number of distribution ways in which persons i and j get no object. Then,

$$n(A_i \cap A_j) = (r-2)^n$$

Also,

$$n(A_i \cap A_j \cap A_k) = (r-3)^n$$

The required number is

$$n(A_1' \cup A_2' \cup \cdots \cup A_r') = n! - n(A_1 \cup A_2 \cup \cdots \cup A_r)$$

$$= r^n - \left[\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + \cdots (-1)^n \sum n(A_1 \cap A_2 \cdots \cap A_r)\right]$$

$$= r^n - [{}^rC_1(r-1)^n - {}^rC_2(r-2)r^n + {}^rC_3(r-3)^n - \cdots + {}^rC_{r-1}1]$$

$$= r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)r^n - {}^rC_3(r-3)^n + \cdots + (-1)^{r-1}\,{}^rC_{r-1}1$$

Example 5.113 **Find the number of ways in which 5 distinct balls can be distributed in three different boxes if no box remains empty.**

Sol. By above formula, the number of ways is $3^5 - {}^3C_1 \times (3-1)^5 + {}^3C_2(3-2)^5 = 243 - 96 + 3 = 150$.

Example 5.114 **If $n(A) = 5$ and $n(B) = 3$, then find the number of onto functions from A to B.**

Sol. We know that in onto function, each image must be assigned at least one pre-image.

This is equivalent to number of ways in which 5 different objects (pre-images) can be distributed in 3 different boxes (images) if no box remains empty. The total number is given by $3^5 - {}^3C_1(3-1)^5 + {}^3C_2(3-2)^5 = 243 - 96 + 3 = 150$.

EXERCISES

Subjective Type

Solutions on page 5.43

1. Prove by combinatorial argument that ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$.
2. Prove that $(n!)!$ is divisible by $(n!)^{(n-1)!}$.
3. If n_1 and n_2 are five-digit numbers, find the total number of ways of forming n_1 and n_2 so that these numbers can be added without carrying at any stage.
4. n_1 and n_2 are four-digit numbers. Find the total number of ways of forming n_1 and n_2 so that n_2 can be subtracted from n_1 without borrowing at any stage.
5. How many five-digit numbers can be made having exactly two identical digits?
6. An ordinary cubical dice having six faces marked with alphabets A, B, C, D, E and F is thrown n times and the list of n alphabets showing up are noted. Find the total number of ways in which among the alphabets A, B, C, D, E and F only three of them appear in the list.
7. Find the number of three-digit numbers from 100 to 999 including all numbers which have any one digit that is the average of the other two.
8. The members of a chess club took part in a round robin competition in which each player plays with other once. All members scored the same number of points, except four juniors whose total score were 17.5. How many members were there in the club? Assume that for each win a player scores 1 point: 1/2 for draw point and zero for losing.
9. There are $2n$ guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another and that there are two specified guests who must not be placed next to one another, show that the number of ways in which the company can be placed is $(2n-2)! \times (4n^2 - 6n + 4)$.
10. In how many ways can two distinct subsets of the set A of k $(k \geq 2)$ elements be selected so that they have exactly two common elements?
11. Prove that the number of ways to select n objects from $3n$ objects of which n are identical and the rest are different is

$$2^{2n-1} + \frac{1}{2}\frac{2n!}{(n!)^2}$$

12. There are n straight lines in a plane, in which no two are parallel and no three pass through the same point. Their points of intersection are joined. Show that the number of fresh lines thus introduced is

$$\frac{1}{8}n(n-1)(n-2)(n-3)$$

13. There are n points in a plane, in which no three are in a straight line except 'm' which are all in a straight line. Find the number of (a) different straight lines, (b) different triangles, (c) different quadrilaterals that can be formed with the given points as vertices.
14. Find the number of ways of disturbing n identical objects among n persons if at least $n-3$ persons get none of these objects.
15. The streets of a city are arranged like the lines of a chessboard. There are m streets running from north to south and n streets from east to west. Find the number of ways in which a man can travel from north-west to south-east corner, covering the shortest possible distance.
16. A batsman scores exactly a century by hitting fours and sixes in twenty consecutive balls. In how many different ways can he do it if some balls may not yield runs and the order of boundaries and overboundaries are taken into account?
17. In how many ways can $2t + 1$ identical balls be placed in three distinct boxes so that any two boxes together will contain more balls than the third?
18. Sohan has x children by his first wife. Geeta has $(x + 1)$ children by her first husband. They marry and have children of their own. The whole family has 24 children. Assuming that two children of the same parents do not fight, prove that the maximum possible number of fights that can take place is 191.
19. Let $S(n)$ denote the number of ordered pairs (x, y) satisfying $1/x + 1/y = 1/n$ where $n > 1$ and $x, y, n \in N$.
    (i) Find the value of $S(6)$.
    (ii) Show that if n is prime, then $S(n) = 3$ always.
20. Six apples and six mangoes are to be distributed among ten boys so that each boy receives at least one fruit. Find the number of ways in which the fruits can be distributed.

Objective Type

Solutions on page 5.47

Each question has four choices a, b, c and d, out of which *only one* is correct.

1. If $\alpha = {}^mC_2$, then ${}^{\alpha}C_2$ is equal to

   a. ${}^{m+1}C_4$ **b.** ${}^{m-1}C_4$
   c. $3\ {}^{m+2}C_4$ **d.** $3\ {}^{m+1}C_4$

2. If ${}^nC_3 + {}^nC_4 > {}^{n+1}C_3$, then

   a. $n > 6$ **b.** $n > 7$
   c. $n < 6$ **d.** none of these

3. The value of $\sum_{r=0}^{n-1} {}^nC_r / ({}^nC_r + {}^nC_{r+1})$ equals

   a. $n + 1$ **b.** $n/2$
   c. $n + 2$ **d.** none of these

4. In a city no two persons have identical set of teeth and there is no person without a tooth. Also no person has more than 32 teeth. If we disregard the shape and size of tooth and consider only the positioning of the teeth, the maximum population of the city is

   a. 2^{32} **b.** $(32)^2 - 1$
   c. $2^{32} - 1$ **d.** 2^{32-1}

5. In a room, there are 12 bulbs of the same wattage, each having a separate switch. The number of ways to light the room with different amount of illumination is

   a. $12^2 - 1$ **b.** 2^{12}
   c. $2^{12} - 1$ **d.** 12^2

6. The number of possible outcomes in a throw of n ordinary dice in which at least one of the dice shows an odd number is

   a. $6^n - 1$ **b.** $3^n - 1$
   c. $6^n - 3^n$ **d.** none of these

7. Let A be a set of n (≥ 3) distinct elements. The number of triplets (x, y, z) of the A elements in which at least two coordinates is equal to
a. ${}^nP_3$ b. $n^3 - {}^nP_3$
c. $3n^2 - 2n$ d. $3n^2(n - 1)$

8. The total number of flags with three horizontal strips in order, which can be formed using 2 identical red, 2 identical green and 2 identical white strips is equal to
a. 4! b. 3 × (4!)
c. 2 × (4!) d. none of these

9. The number of five-digit numbers that contain 7 exactly once is
a. (41) (9^3) b. (37) (9^3)
c. (7) (9^4) d. (41) (9^4)

10. A variable name in certain computer language must be either an alphabet or an alphabet followed by a decimal digit. The total number of different variable names that can exist in that language is equal to
a. 280 b. 290
c. 286 d. 296

11. The number less than 1000 that can be formed using the digits 0, 1, 2, 3, 4, 5 when repetition is not allowed is equal to
a. 130 b. 131
c. 156 d. 155

12. Total number of six-digit numbers that can be formed, having the property that every succeeding digit is greater than the preceding digit, is equal to
a. ${}^9C_3$ b. ${}^{10}C_3$
c. ${}^9p_3$ d. ${}^{10}p_3$

13. Numbers greater than 1000 but not greater than 4000, which can be formed with the digits 0, 1, 2, 3, 4 (repetition of digits is allowed), are
a. 350 b. 375
c. 450 d. 576

14. The total number of five-digit numbers of different digits in which the digit in the middle is the largest is
a. $\sum_{n=4}^{9} {}^nP_4$ b. 33(3!)
c. 30(3!) d. none of these

15. The number of four-digit numbers that can be made with the digits 1, 2, 3, 4 and 5 in which at least two digits are identical is
a. $4^5 - 5!$ b. 505
c. 600 d. none of these

16. The number of nine-non-zero digits such that all the digits in the first four places are less than the digit in the middle and all the digits in the last four places are greater than that in the middle is
a. 2(4!) b. 3(7!)/2
c. 2(7!) d. ${}^4P_4 \times {}^4P_4$

17. Total number of words that can be formed using all letters of the word 'BRIJESH' that neither begins with 'I' nor ends with 'B' is equal to
a. 3720 b. 4920
c. 3600 d. 4800

18. The total number of six-digit natural numbers that can be made with the digits 1, 2, 3, 4, if all digits are to appear in the same number at least once is
a. 1560 b. 840
c. 1080 d. 480

19. Total number of six-digit numbers in which all and only odd digits appear is
a. $\frac{5}{2}(6!)$ b. 6!
c. $\frac{1}{2}(6!)$ d. none of these

20. Total number less than 3×10^8 and can be formed using the digits 1, 2, 3 is equal to
a. $\frac{1}{2}(3^9 + 4 \times 3^8)$ b. $\frac{1}{2}(3^9 - 3)$
c. $\frac{1}{2}(7 \times 3^8 - 3)$ d. $\frac{1}{2}(3^9 - 3 + 3^8)$

21. If all the permutations of the letters in the word 'OBJECT' are arranged (and numbered serially) in alphabetical order as in a dictionary, then the 717th word is
a. TOJECB b. TOEJBC
c. TOCJEB d. TOJCBE

22. In a three-storey building, there are four rooms on the ground floor, two on the first and two on the second floor. If the rooms are to be allotted to six persons, one person occupying one room only, the number of ways in which this can be done so that no floor remains empty is
a. ${}^8P_6 - 2(6!)$ b. ${}^8P_6$
c. ${}^8P_5\,(6!)$ d. none of these

23. The total number not more than 20 digits that are formed by using the digits 0, 1, 2, 3 and 4 is
a. 5^{20} b. $5^{20} - 1$
c. $5^{20} + 1$ d. none of these

24. The number of ways in which we can select four numbers from 1 to 30 so as to exclude every selection of four consecutive numbers is
a. 27378 b. 27405
c. 27399 d. none of these

25. The number of words of four letters containing equal number of vowels and consonants, where repetition is allowed, is
a. 105^2 b. 210 × 243
c. 105 × 243 d. 150 × 21^2

26. The number of different seven digit numbers that can be written using only the three digits 1, 2 and 3 with the condition that the digit 2 occurs twice in each number is
a. ${}^7P_2\, 2^5$ b. ${}^7C_2\, 2^5$
c. ${}^7C_2\, 5^2$ d. none of these

27. The total number of three-letter words that can be formed from the letter of the word 'SAHARANPUR' is equal to
a. 210 b. 237
c. 247 d. 227

28. The number of ways to fill each of the four cells of the table with a distinct natural number such that the sum of the numbers is 10 and the sums of the numbers placed diagonally are equal is

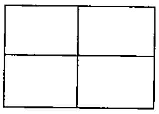


Fig. 5.31

a. 4 **b.** 8
c. 24 **d.** 6

29. The number of ways of arranging m positive and n $(< m + 1)$ negative signs in a row so that no two negative signs are together is

a. ${}^{m+1}P_n$ **b.** ${}^{n+1}P_m$
c. ${}^{m+1}C_n$ **d.** ${}^{n+1}C_m$

30. Three boys of class X, four boys of class XI and five boys of class XII sit in a row. The total number of ways in which these boys can sit so that all the boys of same class sit together is equal to

a. $(3!)^2 (4!) (5!)$ **b.** $(3!) (4!)^2 (5!)$
c. $(3!) (4!) (5!)$ **d.** $(3!) (4!) (5!)^2$

31. A library has 'a' copies of one book, 'b' copies each of two books, 'c' copies each of three books, an single copy of 'd' books. The total number of ways in which these books can be arranged in a shelf is equal to

a. $\dfrac{(a+2b+3c+d)!}{a!(b!)^2(c!)^3}$ **b.** $\dfrac{(a+2b+3c+d)!}{a!(2b!)(c!)^3}$
c. $\dfrac{(a+b+3c+d)!}{(c!)^3}$ **d.** $\dfrac{(a+2b+3c+d)!}{a!(2b)!(3c)!}$

32. The sum of the digits in the unit's place of all numbers formed with the help of 3, 4, 5, 6 taken all at a time is

a. 18 **b.** 432
c. 108 **d.** 144

33. The sum of all the numbers of four different digits that can be made by using the digits 0, 1, 2 and 3 is

a. 26664 **b.** 39996
c. 38664 **d.** none of these

34. The sum of all four-digit numbers that can be formed by using the digits 2, 4, 6, 8 (when repetition of digits is not allowed) is

a. 133320 **b.** 533280
c. 53328 **d.** none of these

35. The number of ordered pairs of integers (x, y) satisfying the equation $x^2 + 6x + y^2 = 4$ is

a. 2 **b.** 8
c. 6 **d.** none of these

36. The number of five-digit telephone numbers having at least one of their digits repeated is

a. 90000 **b.** 100000
c. 30240 **d.** 69760

37. How many numbers can be made with the digits 3, 4, 5, 6, 7, 8 lying between 3000 and 4000, which are divisible by 5 while repetition of any digit is not allowed in any number?

a. 60 **b.** 12
c. 120 **d.** 24

38. The number of ways in which ten candidates $A_1, A_2, \ldots, A_{10}$ can be ranked such that A_1 is always above A_{10} is

a. $5!$ **b.** $2(5!)$
c. $10!$ **d.** $\dfrac{1}{2}(10!)$

39. In the decimal system of numeration of six-digit numbers in which the sum of the digits is divisible by 5 is

a. 180000 **b.** 540000
c. 5×10^5 **d.** none of these

40. To fill 12 vacancies there are 25 candidates of which five are from scheduled caste. If three of the vacancies are reserved for scheduled caste candidates while the rest are open to all, the number of ways in which the selection can be made is

a. ${}^5C_3 \times {}^{22}C_9$ **b.** ${}^{22}C_9 - {}^5C_3$
c. ${}^{22}C_3 + {}^5C_3$ **d.** none of these

41. If the difference of the number of arrangements of three things from a certain number of dissimilar things and the number of selections of the same number of things from them exceeds 100, then the least number of dissimilar things is

a. 8 **b.** 6
c. 5 **d.** 7

42. Among 10 persons, A, B, C are to speak at a function. The number of ways in which it can be done if A wants to speak before B and B wants to speak before C is

a. $10!/24$ **b.** $9!/6$
c. $10!/6$ **d.** none of these

43. In how many ways can a team of 11 players be formed out of 25 players, if six out of them are always to be included and five always to be excluded

a. 2020 **b.** 2002
c. 2008 **d.** 8002

44. In how many ways can a team of six horses be selected out of a stud of 16, so that there shall always be three out of $A\ B\ C\ A'\ B'\ C'$, but never $A\ A'$, $B\ B'$ or $C\ C'$ together

a. 840 **b.** 1260
c. 960 **d.** 720

45. There are two bags each containing m balls. If a man has to select equals number of balls from both the bags the number of ways in which he can do so if he must choose at least one ball from each bag is

a. m^2 **b.** ${}^{2m}C_m$
c. ${}^{2m}C_m - 1$ **d.** none of these

46. The number of ways in which the letters of the word 'PERSON' can be placed in the squares of the given figure so that no row remains empty is

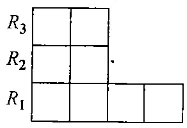


Fig. 5.32

a. $24 \times 6!$ **b.** $26 \times 6!$
c. $26 \times 7!$ **d.** $27 \times 6!$

47. The number of words of four letters that can be formed from the letters of the word 'EXAMINATION' is

a. 1464 **b.** 2454
c. 1678 **d.** none of these

48. The letters of word 'ZENITH' are written in all possible ways. If all these words are written in the order of a dictionary, then the rank of the word 'ZENITH' is

a. 716 **b.** 692
c. 698 **d.** 616

49. A class contains three girls and four boys. Every Saturday, five go on a picnic (a different group of students is sent every week). During the picnic, each girl in the group is given a doll by the accompanying teacher. If all possible groups of five have gone for picnic once, the total number of dolls that the girls have got is

a. 21 b. 45
c. 27 d. 24

50. A teacher takes three children from her class to the zoo at a time, but she does not take the same three children to the zoo more than once. She finds that she went to the zoo 84 times more than a particular child has gone to the zoo. The number of children in her class is

a. 12 b. 10
c. 60 d. none of these

51. Number of ways in which a lawn-tennis mixed double be made from seven married couples if no husband and wife play in the same set is

a. 240 b. 420
c. 720 d. none of these

52. In a class tournament, all participants were to play different games with one another, Two players fell ill after having played three games each. If the total number of games played in the tournament is equal to 84, the total number of participants in the beginning was equal to

a. 10 b. 15
c. 12 d. 14

53. A person always prefers to eat 'parantha' and 'vegetable dish' in his meal. How many ways can he make his platter in a marriage party if there are three types of paranthas, four types of 'vegetable dish', three types of 'salads' and two types of 'sauces'?

a. 3360 b. 4096
c. 3000 d. none of these

54. The number of even divisors of the number $N = 12600 = 2^3 3^2 5^2 7$ is

a. 72 b. 54
c. 18 d. none of these

55. A candidate is required to answer six out of 10 questions, which are divided into two groups, each containing five questions. He is not permitted to attempt more than four questions from either group. The number of different ways in which the candidate can choose six questions is

a. 50 b. 150
c. 200 d. 250

56. There were two women participating in a chess tournament. Every participant played two games with the other participates. The number of games that the men played among themselves proved to exceed by 66 number of games that the men played with the women. The number of participants is

a. 6 b. 11
c. 13 d. none of these

57. Two teams are to play a series of five matches between them. A match ends in a win, loss or draw for a team. A number of people forecast the result of each match and no two people make the same forecast for the series of matches. The smallest group of people in which one person forecasts correctly for all the matches will contain n people, where n is

a. 81 b. 243
c. 486 d. none of these

58. Ten IIT and 2 DCE students sit in a row. The number of ways in which exactly 3 IIT students sit between 2 DCE students is

a. ${}^{10}C_3 \times 2! \times 3! \times 8!$ b. $10! \times 2! \times 3! \times 8!$
c. $5! \times 2! \times 9! \times 8!$ d. none of these

59. A team of four students is to be selected from a total of 12 students. The total number of ways in which the team can be selected such that two particular students refuse to be together and other two particular students wish to be together only is equal to

a. 220 b. 182
c. 226 d. none of these

60. In an election, the number of candidates is one greater than the persons to be elected. If a voter can vote in 254 ways, the number of candidates is

a. 7 b. 10
c. 8 d. 6

61. Two players P_1 and P_2 play a series of '$2n$' games. Each game can result in either a win or a loss for P_1. The total number of ways in which P_1 can win the series of these games is equal to

a. $\frac{1}{2}(2^{2n} - {}^{2n}C_n)$ b. $\frac{1}{2}(2^{2n} - 2\times {}^{2n}C_n)$
c. $\frac{1}{2}(2^{n} - {}^{2n}C_n)$ d. $\frac{1}{2}(2^{n} - 2\times {}^{2n}C_n)$

62. In an examination of nine papers, a candidate has to pass in more papers than the number of papers in which he fails in order to be successful. The number of ways in which he can be unsuccessful is

a. 255 b. 256
c. 193 d. 319

63. A student is allowed to select at most n books from a collection of $(2n + 1)$ books. If the total number of ways in which he can select at least one book is 63, then the value of n is

a. 2 b. 3
c. 4 d. 5

64. In a group of 13 cricket players, four are bowlers. Find out in how many ways can they form a cricket team of 11 players in which at least 2 bowlers are included.

a. 55 b. 72
c. 78 d. None of these

65. A person predicts the outcome of 20 cricket matches of his home team. Each match can result in a either win, loss or tie for the home team. Total number of ways in which he can make the predictions so that exactly 10 predictions are correct is equal to

a. ${}^{20}C_{10} \times 2^{10}$ b. ${}^{20}C_{10} \times 3^{20}$
c. ${}^{20}C_{10} \times 3^{10}$ d. ${}^{20}C_{10} \times 2^{20}$

66. The number of different ways in which five 'alike dashes' and eight 'alike dots' can be arranged using only seven of these 'dashes' and 'dots' is

a. 350 b. 120
c. 1287 d. none of these

67. Let there be $n \geq 3$ circles in a plane. The value of n for which the number of radical centres is equal to the number of radical axes is (assume that all radical axes and radical centre exist and are different)

a. 7 b. 6
c. 5 d. none of these

68. The number of ways of choosing a committee of two women and three men from five women and six men, if Mr. A refuses to serve on the committee if Mr. B is a member and Mr. B can only serve, if Miss C is the member of the committee is

a. 60 **b.** 84
c. 124 **d.** none of these

69. The last digit of $(1! + 2! + \cdots + 2005!)^{500}$ is

a. 9 **b.** 2
c. 7 **d.** 1

70. $ABCD$ is a convex quadrilateral and 3, 4, 5 and 6 points are marked on the sides AB, BC, CD and DA, respectively. The number of triangles with vertices on different sides is

a. 270 **b.** 220
c. 282 **d.** 342

71. There are 10 points in a plane of which no three points are collinear and four points are concyclic. The number of different circles that can be drawn through at least three points of these points is

a. 116 **b.** 120
c. 117 **d.** none of these

72. n lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. The number of different points at which these lines will cut is

a. $\sum_{k=1}^{n-1} k$ **b.** $n(n-1)$
c. n^2 **d.** none of these

73. The number of triangles that can be formed with 10 points as vertices, n of them being collinear, is 110. Then n is

a. 3 **b.** 4
c. 5 **d.** 6

74. There are three coplanar parallel lines. If any p points are taken on each of the lines, the maximum number of triangles with vertices on these points is

a. $3p^2(p-1)+1$ **b.** $3p^2(p-1)$
c. $p^2(4p-3)$ **d.** none of these

75. The maximum number of points of intersection of five lines and four circles is

a. 60 **b.** 72
c. 62 **d.** none of these

76. If m parallel lines in a plane are intersected by a family of n parallel lines, the number of parallelograms that can be formed is

a. $\frac{1}{4}mn(m-1)(n-1)$ **b.** $\frac{1}{2}mn(m-1)(n-1)$
c. $\frac{1}{4}m^2n^2$ **d.** none of these

77. The number of integral solutions of $x + y + z = 0$ with $x \geq -5$, $y \geq -5$, $z \geq -5$ is

a. 134 **b.** 136
c. 138 **d.** 140

78. The number of ways in which we can get a score of 11 by throwing three dice is

a. 18 **b.** 27
c. 45 **d.** 56

79. In how many different ways can the first 12 natural numbers be divided into three different groups such that numbers in each group are in A.P.?

a. 1 **b.** 5
c. 6 **d.** 4

80. Fifteen identical balls have to be put in five different boxes. Each box can contain any number of balls. The total number of ways of putting the balls into the boxes so that each box contains at least two balls is equal to

a. ${}^9C_5$ **b.** ${}^{10}C_5$
c. ${}^6C_5$ **d.** ${}^{10}C_6$

81. If n objects are arranged in a row, then the number of ways of selecting three of these objects so that no two of them are next to each other is

a. ${}^{n-2}C_3$ **b.** ${}^{n-3}C_2$
c. ${}^{n-3}C_3$ **d.** none of these

82. The number of ways to give 16 different things to three persons A, B, C so that B gets one more than A and C gets two more than B, is

a. $\frac{16!}{4!5!7!}$ **b.** $4!\,5!\,7!$
c. $\frac{16!}{3!5!8!}$ **d.** none of these

83. The number of ways in which we can distribute mn students equally among m sections is given by

a. $\frac{(mn)!}{n!}$ **b.** $\frac{(mn)!}{(n!)^m}$
c. $\frac{(mn)!}{m!n!}$ **d.** $(mn)^m$

84. $2m$ white counters and $2n$ red counters are arranged in a straight line with $(m + n)$ counters on each side of a central mark. The number of ways of arranging the counters, so that the arrangements are symmetrical with respect to the central mark, is

a. ${}^{m+n}C_m$ **b.** ${}^{2m+2n}C_{2m}$
c. $\frac{1}{2}\frac{(m+n)!}{m!n!}$ **d.** none of these

85. A person buys eight packets of TIDE detergent. Each packet contains one coupon, which bears one of the letters of the word TIDE. If he shows all the letters of the word TIDE, he gets one free packet. If he gets exactly one free packet, then the number of different possible combinations of the coupons is

a. ${}^7C_3$ **b.** ${}^8C_4$
c. ${}^8C_3$ **d.** 4^4

86. There are three copies each of four different books. The number of ways in which they can be arranged in a shelf is

a. $\frac{12!}{(3!)^4}$ **b.** $\frac{12!}{(4!)^3}$
c. $\frac{21!}{(3!)^4 4!}$ **d.** $\frac{12!}{(4!)^3 3!}$

87. The number of ways in which 12 books can be put in three shelves with four on each shelf is

a. $\frac{12!}{(4!)^3}$ **b.** $\frac{12!}{(3!)(4!)^3}$
c. $\frac{12!}{(3!)^3 4!}$ **d.** none of these

88. The total number of ways in which $2n$ persons can be divided into n couples is

a. $\frac{2n!}{n!n!}$
b. $\frac{2n!}{(2!)^n}$
c. $\frac{2n!}{n!(2!)^n}$
d. none of these

89. Let $x_1, x_2, \ldots, x_k$ be the divisors of positive integer 'n' (including 1 and n). If $x_1 + x_2 + \cdots + x_k = 75$, then $\sum_{i=1}^{k} 1/x_i$ is equal to

a. $\frac{75}{n^2}$
b. $\frac{75}{n}$
c. $\frac{75}{k}$
d. none of these

90. Let $A = \{x_1, x_2, x_3, \ldots, x_7\}$, $B = \{y_1, y_2, y_3\}$. The total number of functions $f: A \to B$ that are on to and there are exactly three element x in A such that $f(x) = y_2$ is equal to

a. 490
b. 510
c. 630
d. none of these

91. The total number of ways in which n^2 number of identical balls can be put in n numbered boxes (1, 2, 3, ..., n) such that i^{th} box contains at least i number of balls is

a. ${}^{n^2}C_{n-1}$
b. ${}^{n^2-1}C_{n-1}$
c. ${}^{\frac{n^2+n-2}{2}}C_{n-1}$
d. none of these

92. The total number of ways in which 15 identical blankets can be distributed among four persons so that each of them gets at least two blankets is equal to

a. ${}^{10}C_3$
b. ${}^{9}C_3$
c. ${}^{11}C_3$
d. none of these

93. Number of ways in which 25 identical things be distributed among five persons if each gets odd number of things is

a. ${}^{25}C_4$
b. ${}^{12}C_8$
c. ${}^{14}C_{10}$
d. ${}^{13}C_3$

94. Number of ways in which Rs. 18 can be distributed amongst four persons such that nobody receives less than Rs. 3 is

a. 4^2
b. 2^4
c. 4!
d. none of these

95. In how many ways can 17 persons depart from railway station in 2 cars and 3 autos, given that 2 particular persons depart by same car (4 persons can sit in a car and 3 persons can sit in an auto)?

a. $\frac{15!}{2!4!(3!)^3}$
b. $\frac{16!}{(2!)^2 4!(3!)^3}$
c. $\frac{17!}{2!4!(3!)^3}$
d. $\frac{15!}{4!(3!)^3}$

96. The total number of ways of selecting six coins out of 20 one-rupee coins, 10 fifty-paise coins and 7 twenty-five paise coins is

a. 28
b. 56
c. ${}^{37}C_6$
d. none of these

97. Let $f(n, k)$ denote the number of ways in which k identical balls can be coloured with n colours so that there is at least one ball of each colour. Then $f(2n, n)$ must be equal to

a. ${}^{2n}C_n$
b. ${}^{2n-1}C_{n+1}$
c. ${}^{2n-1}C_n$
d. none of these

98. The total number of ways in which three distinct numbers in $A.P.$ can be selected from the set $\{1, 2, 3, \ldots, 24\}$ is equal to

a. 66
b. 132
c. 198
d. none of these

99. The total number of ways of selecting two number from the set $\{1, 2, 3, 4, \ldots, 3n\}$ so that their sum is divisible by 3 is equal to

a. $\frac{2n^2 - n}{2}$
b. $\frac{3n^2 - n}{2}$
c. $2n^2 - n$
d. $3n^2 - n$

100. Among the 8! permutations of the digits 1, 2, 3, ..., 8, consider those arrangements which have the following property. If we take any five consecutive positions, the product of the digits in these positions is divisible by 5. The number of such arrangements is equal to

a. 7!
b. 2. (7!)
c. ${}^{7}C_4$
d. none of these

101. The total number of divisors of 480, that are of the form $4n + 2$, $n \geq 0$, is equal to

a. 2
b. 3
c. 4
d. none of these

102. The total number of times, the digit '3' will be written, when the integers having less than 4 digits are listed is equal to

a. 300
b. 310
c. 302
d. 306

103. Straight lines are drawn by joining m points on a straight line to n points on another line. Then excluding the given points, the number of point of intersections of the lines drawn is (no two lines drawn are parallel and no three lines are concurrent)

a. $\frac{1}{4}mn(m-1)(n-1)$
b. $\frac{1}{2}mn(m-1)(n-1)$
c. $\frac{1}{2}m^2n^2$
d. $\frac{1}{4}m^2n^2$

104. In a polygon, no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon is 70, then the number of diagonals of the polygon is

a. 20
b. 28
c. 8
d. none of these

105. Two packs of 52 cards are shuffled together. The number of ways in which a man can be dealt 26 cards so that he does not get two cards of the same suit and same denomination is

a. ${}^{52}C_{26} \cdot 2^{26}$
b. ${}^{104}C_{26}$
c. $2 \cdot {}^{52}C_{26}$
d. none of these

106. There are $(n + 1)$ white and $(n + 1)$ black balls each set numbered 1 to $n + 1$. The number of ways in which the balls can be arranged in a row so that the adjacent balls are of different colours is

a. $(2n + 2)!$
b. $(2n + 2)! \times 2$
c. $(n + 1)! \times 2$
d. $2\{(n + 1)!\}^2$

107. The number of three-digit numbers of the form xyz such that $x < y$ and $z \leq y$ is

a. 276
b. 285
c. 240
d. 244

108. A is a set containing 'n' different elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q$ contains exactly two elements is

a. ${}^nC_3 \times 2^n$
b. ${}^nC_2 \times 3^{n-2}$
c. 3^{n-2}
d. none of these

109. Messages are conveyed by arranging four white, one blue and three red flags on a pole. Flags of the same colour are alike. If a message is transmitted by the order in which the colours are arranged, the total number of messages that can be transmitted if exactly six flags are used is

a. 45
b. 65
c. 125
d. 185

110. 20 persons are sitting in a particular arrangement around a circular table. Three persons are to be selected for leaders. The number of ways of selection of three persons such that no two were sitting adjacent to each other is

a. 600
b. 900
c. 800
d. none of these

111. A seven-digit number without repetition and divisible by 9 is to be formed by using seven digits out of 1, 2, 3, 4, 5, 6, 7, 8. 9. The number of ways in which this can be done is

a. 9!
b. 2(7!)
c. 4 (7!)
d. none of these

112. n is selected from the set {1, 2, 3, ..., 10} and the number $2^n + 3^n + 5^n$ is formed. Total number of ways of selecting n so that the formed number is divisible by 4 is equal to

a. 50
b. 49
c. 48
d. none of these

113. The number of distinct natural numbers up to a maximum of four digits and divisible by 5, which can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 each digit not occurring more than once in each number is

a. 1246
b. 952
c. 1106
d. none of these

114. A man has three friends. The number of ways he can invite one friend everyday for dinner on six successive nights so that no friend is invited more than three times is

a. 640
b. 320
c. 420
d. 510

115. There are four letters and four directed envelopes. The number of ways in which all the letters can be put in the wrong envelope is

a. 8
b. 9
c. 16
d. none of these

116. A bag contains four one-rupee coins, two twenty-five paisa coins and five ten-paisa coins. In how many ways can an amount, not less than Re 1 be taken out from the bag? (Consider coins of the same denominations to be identical.)

a. 71
b. 72
c. 73
d. 80

117. In a certain test, there are n questions. In the test 2^{n-i} students gave wrong answers to at least i questions, where $i = 1, 2, \ldots, n$. If the total number of wrong answers given is 2047, then n is equal to

a. 10
b. 11
c. 12
d. 13

118. Rajdhani Express going from Bombay to Delhi stops at five intermediate stations, 10 passengers enter the train during the journey with 10 different ticket of two classes. The number of different sets of tickets they may have is

a. ${}^{15}C_{10}$
b. ${}^{20}C_{10}$
c. ${}^{30}C_{10}$
d. none of these

119. A train timetable must be compiled for various days of the week so that two trains twice a day depart for three days, one train daily for two days and three trains once a day for two days. How many different timetables can be compiled?

a. 140
b. 210
c. 133
d. 72

120. The total number of positive integral solution of $15 < x_1 + x_2 + x_3 \le 20$ is equal to

a. 685
b. 785
c. 1125
d. none of these

Multiple Correct Answers Type

Solutions on page 5.55

Each question has 4 choices a, b, c and d, out of which *one or more* answers are correct.

1. Number of ways in which three numbers in A.P. can be selected from 1, 2, 3, ..., n is

a. $\left(\frac{n-1}{2}\right)^2$ if n is even
b. $\frac{n(n-2)}{4}$ if n is even
c. $\frac{(n-1)^2}{4}$ if n is odd
d. none of these

2. Kanchan has 10 friends among whom two are married to each other. She wishes to invite five of them for a party. If the married couples refuse to attend separately, then the number of different ways in which she can invite five friends is

a. ${}^8C_5$
b. $2 \times {}^8C_3$
c. ${}^{10}C_5 - 2 \times {}^8C_4$
d. none of these

3. A forecast is to be made of the results of five cricket matches, each of which can be a win or a draw or a loss for Indian team. Let,

p = number of forecasts with exactly 1 error
q = number of forecasts with exactly 3 errors and
r = number of forecasts with all five errors

Then the correct statement(s) is/are

a. $2q = 5r$
b. $8p = q$
c. $8p = 5r$
d. $2(p + r) > q$

4. Ten persons numbered 1, 2, ..., 10 play a chess tournament, each player playing against every other player exactly one game. It is known that no game ends in a draw. If $w_1, w_2, \ldots, w_{10}$ are the number of games won by players 1, 2, 3, ..., 10, respectively, and

$l_1, l_2, \ldots, l_{10}$ are the number of games lost by the players 1, 2, …, 10, respectively, then,

a. $\Sigma w_i = \Sigma l_i = 45$
b. $w_i + l_i = 9$
c. $\Sigma w l_i^2 = 81 + \Sigma l_i^2$
d. $\Sigma w_i^2 = \Sigma l_i^2$

5. The number of ways of choosing triplet (x, y, z) such that $z \geq \max\{x, y\}$ and $x, y, z \in \{1, 2, \ldots, n, n+1\}$ is

a. ${}^{n+1}C_3 + {}^{n+2}C_3$
b. $n(n+1)(2n+1)/6$
c. $1^2 + 2^2 + \cdots + n^2$
d. $2({}^{n+2}C_3) - {}^{n+1}C_2$

6. Number of ways in which 200 people can be divided in 100 couples is

a. $\dfrac{(200)!}{2^{100}(100)!}$
b. $1 \times 3 \times 5 \cdots 199$
c. $\left(\dfrac{101}{2}\right)\left(\dfrac{102}{2}\right)\cdots\left(\dfrac{200}{2}\right)$
d. $\dfrac{(200)!}{(100)!}$

7. If a seven-digit number made up of all distinct digits 8, 7, 6, 4, 3, x and y is divisible by 3, then

a. maximum value of $x - y$ is 9
b. maximum value of $x + y$ is 12
c. minimum value of xy is 0
d. minimum value of $x + y$ is 3

8. If n is number of necklaces which can be formed using 17 identical pearls and two identical diamonds and similarly m is number of necklaces which can be formed using 17 identical pearls and different diamonds, then

a. $n = 9$
b. $m = 18$
c. $n = 18$
d. $m = 9$

9. Let $f(n)$ be the number of regions in which n coplanar circles can divide the plane. If it is known that each pair of circles intersect in two different point and no three of them have common point of intersection, then

a. $f(20) = 382$
b. $f(n)$ is always an even number
c. $f^{-1}(92) = 10$
d. $f(n)$ can be odd

10. Given that the divisors of $n = 3^p \cdot 5^q \cdot 7^r$ are of the form $4\lambda + 1$, $\lambda \geq 0$. Then

a. $p + r$ is always even
b. $p + q + r$ is always odd
c. q can be any integer
d. if p is odd then r is even

11. Number of ways of selecting three integers from $\{1, 2, 3, \ldots, n\}$ if their sum is divisible by 3 is

a. $3({}^{n/3}C_3) + (n/3)^3$ if $n = 3k, k \in N$
b. $2({}^{(n-1)/3}C_3) + ({}^{(n+2)/3}C_3) + ((n-1)/3)^2(n+2)$, if $n = 3k + 1$, $k \in N$
c. $2({}^{(n-1)/3}C_3) + ({}^{(n+2)/3}C_3) + ((n-1)/3)^2(n+2)$, if $n = 3k + 2$, $k \in N$
d. independent of n

12. Number of points of intersection of n straight lines if n satisfies

$${}^{n+5}P_{n+1} = \frac{11(n-1)}{2} \times {}^{n+3}P_n \text{ is}$$

a. 15
b. 28
c. 21
d. 10

13. Number of shortest ways in which we can reach from the point (0, 0, 0) to point (3, 7, 11) in space where the movement is possible only along the x-axis, y-axis and z-axis or parallel to them and change of axes is permitted only at integral points (an integral point is one which has its coordinate as integer) is

a. equivalent to number of ways of dividing 21 different objects in three groups of size 3, 7, 11
b. equivalent to coefficient of y^3z^7 in the expansion of $(1 + y + z)^{21}$
c. equivalent to number of ways of distributing 21 different objects in three boxes of size 3, 7, 11
d. equivalent to number of ways of arranging 21 objects of which 3 are alike of one kind, 7 are alike of second type and 11 are alike of third type

14. Number of ways in which 30 identical things are distributed among six persons is

a. ${}^{17}C_5$ if each gets odd number of things
b. ${}^{16}C_{11}$ if each gets odd number of things
c. ${}^{14}C_5$ if each gets even number of things (excluding 0)
d. ${}^{15}C_{10}$ if each gets even number of things (excluding 0)

15. If N denotes the number of ways of selecting r objects out of n distinct objects $(r \geq n)$ with unlimited repetition but with each object included at least once in selection, then N is equal to

a. ${}^{r-1}C_{r-n}$
b. ${}^{r-1}C_n$
c. ${}^{r-1}C_{n-1}$
d. none of these

16. A is a set containing n elements. A subset P_1 of A is chosen. The set A is reconstructed by replacing the elements of P_1. Next, a subset P_2 of A is chosen and again the set is reconstructed by replacing the elements of P_2. In this way, m (>1) subsets $P_1, P_2, \ldots, P_m$ of A are chosen. The number of ways of choosing $P_1, P_2, \ldots, P_m$ is

a. $(2^m - 1)^n$ if $P_1 \cap P_2 \cap \cdots \cap P_m = \phi$
b. 2^{mn} if $P_1 \cup P_2 \cup \cdots \cup P_m = A$
c. 2^{mn} if $P_1 \cap P_2 \cap \cdots \cap P_m = \phi$
d. $(2^m - 1)^n$ if $P_1 \cup P_2 \cup \cdots \cup P_m = A$

17. If $10! = 2^p \cdot 3^q \cdot 5^r \cdot 7^s$, then

a. $2q = p$
b. $pqrs = 64$
c. number of divisors of 10! is 280
d. number of ways of putting 10! as a product of two natural numbers is 135

18. If $P = 21(21^2 - 1^2)(21^2 - 2^2)(21^2 - 3^2) \cdots (21^2 - 10^2)$, then P is divisible by

a. 22!
b. 21!
c. 19!
d. 20!

19. Let n is of four-digit integer in which all the digits are different. If x is number of odd integers and y is number of even integers, then

a. $x < y$
b. $x > y$
c. $x + y = 4500$
d. $|x - y| = 54$

Reasoning Type

Solutions on page 5.57

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.
b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.
c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** The number of positive integral solutions of $abc = 30$ is 27.

 Statement 2: Number of ways in which three prizes can be distributed among three persons is 3^3.

2. **Statement 1:** Number of ways in which 10 identical toys can be distributed among three students if each receives at least two toys is ${}^9C_2$.

 Statement 2: Number of positive integral solutions of $x + y + z + w = 7$ is ${}^6C_3$

3. **Statement 1:** $(n^2)!/(n!)^n$ is a natural number for all $n \in N$.

 Statement 2: Number of ways in which n^2 objects can be distributed among n persons equally is $(n^2)!/(n!)^n$.

4. **Statement 1:** The number of ways of writing 1400 as a product of two positive integers is 12.

 Statement 2: 1400 is divisible by exactly three prime factors.

5. **Statement 1:** Let $E = \{1, 2, 3, 4\}$ and $F = \{a, b\}$. Then the number of onto functions from E to F is 14.

 Statement 2: Number of ways in which four distinct objects can be distributed into two different boxes is 14 if no box remains empty.

6. **Statement 1:** Number of ways in which India can win the series of 11 matches is 2^{10}. (if no match is drawn).

 Statement 2: For each match there are two possibilities, either India wins or loses.

7. **Statement 1:** Number of ways in which Indian team (11 players) can bat, if Yuvraj wants to bat before Dhoni and Pathan wants to bat after Dhoni is 11!/3!.

 Statement 2: Yuvraj, Dhoni and Pathan can be arranged in batting order in 3! ways.

8. **Statement 1:** When number of ways of arranging 21 objects of which r objects are identical of one type and remaining are identical of second type is maximum, then maximum value of ${}^{13}C_r$ is 78.

 Statement 2: ${}^{2n+1}C_r$ is maximum when $r = n$.

9. **Statement 1:** Total number of five-digit numbers having all different digits and divisible by 4 can be formed using the digits {1, 3, 2, 6, 8, 9} is 192.

 Statement 2: A number is divisible by 4, if the last two digits of the number are divisible by 4.

10. **Statement 1:** Number of ways in which 30 can be partitioned into three unequal parts, each part being a natural number is 61.

 Statement 2: Number of ways of distributing 30 identical objects in three different boxes is ${}^{30}C_2$.

11. **Statement 1:** If $p, q < r$, the number of different selections of $p + q$ things taking r at a time, where p things are identical and q other things are identical, is $p + q - r + 1$.

 Statement 2: If $p, q > r$, the number of different selections of $p + q$ things taking r at a time, where p things are identical and q other things are identical, is $r - 1$.

12. **Statement 1:** The number of ways in which three distinct numbers can be selected from the set $\{3^1, 3^2, 3^3, ..., 3^{100}, 3^{101}\}$ so that they form a G.P. is 2500.

 Statement 2: If a, b, c are in A.P., then $3^a, 3^b, 3^c$ are in G.P.

13. **Statement 1:** Number of ways in which two persons A and B select objects from two different groups each having 20 different objects such that B selects always more objects than A (including the case when A selects no object) is $(2^{40} - {}^{40}C_{20})/2$.

 Statement 2: The sum $\sum_{0 \le i < j \le n} {}^nC_i\,{}^nC_j = (2^{2n} - {}^{2n}C_n)/2$.

14. **Statement 1:** Number of ways of selecting 10 objects from 42 objects of which, 21 objects are identical and remaining objects are distinct is 2^{20}.

 Statement 2: ${}^{42}C_0 + {}^{42}C_1 + {}^{42}C_2 + \cdots + {}^{42}C_{21} = 2^{41}$.

15. **Statement 1:** Number of terms in the expansion of $(x + y + z + w)^{50}$ is ${}^{53}C_3$.

 Statement 2: Number of non-negative solution of the equation $p + q + r + s = 50$ is ${}^{53}C_3$.

16. **Statement 1:** The number of ways in which n persons can be seated at a round table, so that all shall not have the same neighbours in any two arrangements is $(n - 1)!/2$.

 Statement 2: Number of ways of arranging n different beads in circles in which is $(n - 1)!/2$.

17. **Statement 1:** Number of zeros at the end of 50! is equal to 12.

 Statement 2: Exponent of 2 in 50! is 47.

Linked Comprehension Type

Solutions on page 5.59

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *is only one* is correct.

For Problems 1–3

We have to choose 11 players for cricket team from eight batsmen, six bowlers, four all rounders and two wicket keepers in the following conditions.

1. The number of selections when almost one all rounder and one wicket keeper will play

 a. ${}^4C_1 \times {}^{14}C_{10} + {}^2C_1 \times {}^{14}C_{10} + {}^4C_1 \times {}^2C_1 \times {}^{14}C_9 + {}^{14}C_{11}$
 b. ${}^4C_1 \times {}^{15}C_{11} + {}^{15}C_{11}$

c. ${}^{4}C_{1} \times {}^{15}C_{10} + {}^{15}C_{11}$
d. none of these

2. Number of selections when two particular batsmen do not want to play when a particular bowler will play
a. ${}^{17}C_{10} + {}^{19}C_{11}$
b. ${}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$
c. ${}^{17}C_{10} + {}^{20}C_{11}$
d. ${}^{19}C_{10} + {}^{19}C_{11}$

3. Number of selections when a particular batsman and a particular wicket keeper do not want to play together
a. $2\,{}^{18}C_{10}$
b. ${}^{19}C_{11} + {}^{18}C_{10}$
c. ${}^{19}C_{10} + {}^{19}C_{11}$
d. none of these

For Problems 4–6

Twelve 12 persons are to be arranged around two round tables such that one table can accommodate seven persons and another five persons only. Answer the following questions.

4. Number of ways in which these 12 persons can be arranged is
a. ${}^{12}C_{5}6!4!$
b. $6!4!$
c. ${}^{12}C_{5}6!4!$
d. none of these

5. Number of ways of arrangements if two particular persons A and B do not want to be on the same table is
a. ${}^{10}C_{4}\,6!\,4!$
b. $2\,{}^{10}C_{6}\,6!\,4!$
c. ${}^{11}C_{6}\,6!\,4!$
d. none of these

6. Number of ways of arrangement if two particular persons A and B want to be together and consecutive is
a. ${}^{10}C_{7}\,6!\,3!2! + {}^{10}C_{5}\,4!5!2!$
b. ${}^{10}C_{5}\,6!\,3! + {}^{10}C_{7}\,4!5!$
c. ${}^{10}C_{7}\,6!\,2! + {}^{10}C_{5}\,5!2!$
d. none of these

For Problems 7–9

Five balls are to be placed in three boxes. Each box should hold all the five balls so that no box remains empty.

7. Number of ways if balls are different but boxes are identical is
a. 30
b. 25
c. 21
d. 35

8. Number of ways if balls and boxes are identical is
a. 3
b. 1
c. 2
d. none of these

9. Number of ways if balls as well as boxes are identical but boxes are kept in a row is
a. 10
b. 15
c. 20
d. 6

For Problems 10 – 12

Let $f(n)$ denote the number of different ways in which the positive integer n can be expressed as the sum of 1s and 2s. For example, $f(4) = 5$, since $4 = 2 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 1 + 1 + 1 + 1$. Note that order of 1s and 2s is important.

10. The value of $f(6)$ is
a. 12
b. 13
c. 18
d. 21

11. The value of $f(f(6))$ is
a. 400
b. 350
c. 377
d. none of these

12. $f: N \to N$ is
a. one-one and onto
b. one-one and into
c. many-one and onto
d. many-one and into

For Problems 13–15

There are m seats in the first row of a theatre, of which n are to be occupied.

13. The number of ways of arranging n persons if no two persons sit side by side is
a. $\dfrac{(m-n+1)!}{(m-3n+1)!}$
b. $\dfrac{(m-n+1)!}{(m-2n)!}$
c. $\dfrac{(m-n+1)!}{(m-2n+1)!}$
d. $\dfrac{(m-n+2)!}{(m-2n-1)!}$

14. If n is even, the number of ways of arranging n persons if each person has exactly one neighbour is
a. $({}^{n}P_{n/2})\,({}^{m-n+1}P_{n/2})$
b. $({}^{n}P_{n})\,({}^{m-n+1}P_{n/2})$
c. $({}^{n}P_{n/2})\,({}^{m-n+1}P_{n})$
d. none of these

15. The number of ways of arranging n persons, if out of any two seats located symmetrically in the middle of the row at least one is empty is
a. $({}^{m/2}C_{n})(2^{n}) - 1$
b. ${}^{m/2}P_{n}$
c. $({}^{m/2}P_{n})(2^{n} - 1)$
d. $({}^{m/2}P_{n})(2^{n})$

For Problems 16–18

Consider the letters of the word 'MATHEMATICS'.

16. Possible number of words taking all letters at a time such that at least one repeating letter is at odd position in each word is
a. $\dfrac{11!}{2!2!2!} - \dfrac{9!}{2!2!}$
b. $\dfrac{9!}{2!2!2!}$
c. $\dfrac{9!}{2!\,2!}$
d. $\dfrac{11!}{2!\,2!\,2!}$

17. Possible number of words taking all letters at a time such that in each word both M's are together and both T's are together but both A's are not together is
a. $\dfrac{11!}{2!\,2!\,2!} - \dfrac{10!}{2!\,2!}$
b. $7!\,{}^{8}C_{2}$
c. $\dfrac{6!\,4!}{2!\,2!}$
d. $\dfrac{9!}{2!\,2!\,2!}$

18. Possible number of words in which no two vowels are together is
a. $7!\;{}^{8}C_{4}\,\dfrac{4!}{2!}$
b. $\dfrac{7!}{2!}\,{}^{8}C_{4}\,\dfrac{4!}{2!}$
c. $\dfrac{7!}{2!\,2!}\,{}^{8}C_{4}\,\dfrac{4!}{2!}$
d. $\dfrac{7!}{2!\,2!\,2!}\,{}^{8}C_{4}\,\dfrac{4!}{2!}$

Matrix–Match Type

Solutions on page 5.60

Each question contains statements given in two columns, which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct match are a→p, a→s, , b→q, b→r, c→p, c→q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1

Column I	Column II
a. Number of straight lines joining any two of 10 points of which four points are collinear	**p.** 30
b. Maximum number of points of intersection of 10 straight lines in the plane	**q.** 60
c. Maximum number of points of intersection of six circles in the plane	**r.** 40
d. Maximum number of points of intersection of six parabolas	**s.** 45

2. Consider a 6 × 6 chessboard. Then match the following columns.

Column I	Column II
a. Number of rectangles	**p.** $^{10}C_5$
b. Number of squares	**q.** 441
c. Number of ways three squares can be selected if they are not in same row or column	**r.** 91
d. In how many ways eleven '+' sign can be arranged in the squares if no row remains empty	**s.** 2400

3. Consider the convex polygon, which has 35 diagonals. Then match the following column.

Column I	Column II
a. Number of triangles joining the vertices of the polygon	**p.** 210
b. Number of points of intersections of diagonal which lies inside the polygon	**q.** 120
c. Number of triangles in which exactly one side is common with that of polygon	**r.** 10
d. Number of triangles in which exactly two sides are common with that of polygon	**s.** 60

4. A function is defined as $f: \{a_1, a_2, a_3, a_4, a_5, a_6\} \to \{b_1, b_2, b_3\}$.

Column I	Column II
a. Number of subjective functions	**p.** is divisible by 9
b. Number of functions in which $f(ai) \neq bi$	**q.** is divisible by 5
c. Number of invertible functions	**r.** is divisible by 4
d. Number of many one functions	**s.** is divisible by 3
	t. not possible

5.

Column I	Column II
a. Total number of function $f: \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ that are on to and $f(i) \neq i$ is equal to	**p.** divisible by 11
b. If $x_1x_2x_3 = 2 \times 5 \times 7^2$, then the number of solution set for (x_1, x_2, x_3) where $x_i \in N$, $x_i > 1$ is	**q.** divisible by 7
c. Number of factors of 3780 are divisible by either 3 or 2 or both is	**r.** divisible by 3
d. Total number of divisors of $n = 2^5 \times 3^4 \times 5^{10}$ that are of the form $4\lambda + 2$, $\lambda \geq 1$ is	**s.** divisible by 4

6.

Column I	Column II
a. Four dice (six faced) are rolled. The number of possible outcomes in which at least one dice shows 2 is	**p.** 210
b. Let A be the set of 4-digit number $a_1a_2a_3a_4$ where $a_1 > a_2 > a_3 > a_4$. Then $n(A)$ is equal to	**q.** 480
c. The total number of three-digit numbers, the sum of whose digits is even, is equal to	**r.** 671
d. The number of four-digit numbers that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7 so that each number contains digit 1 is	**s.** 450

7.

Column I	Column II
a. The number of five-digit numbers having the product of digits 20 is	(p) > 70
b. A closest has five pairs of shoes. The number of ways in which four shoes can be drawn from it such that there will be no complete pair is	(q) < 60
c. Three ladies have each brought their one child for admission to a school. The principal wants to interview the six persons one by one subject to the condition that no mother is interviewed before her child. The number of ways in which interviews can be arranged is	(r) ∈ (50, 110)
d. The figures 4, 5, 6, 7, 8 are written in every possible order. The number of numbers greater than 56000 is	(s) ∈ (40, 70)

8.

Column I	Column II
a. If a denotes the number of permutations of $x + 2$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $x - 11$ things taken all at a time such that $a = 182\,bc$, then the value of x is product of	**p.** 6
b. The number of six-digit numbers that can be made with the digits 0, 1, 2, 3, 4 and 5 so that even digits occupy odd places is product of	**q.** 5
c. The number of five-digit numbers that can be made using the digits 1 and 2 and in which at least one digit is different is product of	**r.** 4
d. In a polygon the number of diagonals is 54. The number of sides of the polygon is product of	**s.** 3

Integer Type

Solutions on page 5.62

1. If ${}^nP_r = {}^nP_{r+1}$ and ${}^nC_r = {}^nC_{r-1}$ then the value of $n + r$ is.

2. A person has 'n' friends. The minimum value of 'n' so that a person can invite a different pair of friends every day for four weeks in a row is.

3. There are n distinct white and n distinct black balls. If the number of ways of arranging them in a row so that neighboring balls are of different colors is 1152 then value of 'n' is.

4. Numbers from 1 to 1000 are divisible by 60 but not by 24 is.

5. Number of ways in which the letters of the word 'ABBCABBC can be arranged such that the word ABBC does not appear is any word, is N then the value of $(N^{1/2} - 10)$ is.

6. A class has three teachers, Mr. P, Ms. Q and Mrs. R and six students A,B,C,D,E,F. Number of ways in which they can be seated in a line of 9 chairs, if between any two teachers there are exactly two students, is $k!(18)$, then the value of k is.

7. Consider the five points comprising of the vertices of a square and the intersection point of its diagonals. How many triangles can be formed using these points?

8. If number of selections of 6 different letters that can be made from the words 'SUMAN' and 'DIVYA' so that each selection contains 3 letters from each word, is N^2 then the value of N is.

9. There are 20 books on Albegra and Calculus in one library. For the greatest number of selections each of which consists of 5 books on each topic possible number of Algebra books are N then the value of $N/2$ is.

10. Number of permutations of 1, 2, 3, 4, 5, 6, 7, 8 and 9 taken all at a time are such that the digit
 1 appearing somewhere to the left of 2
 3 appearing to the left of 4 and
 5 somewhere to the left of 6, is $k \times 7!$ Then the value of k is.

11. The number of n digit numbers which consists of the digits 1 and 2 only if each digit is to be used at least once, is equal to 510 then n is equal to.

12. Number of ways in which 7 people can occupy six seats, 3 seats on each side in a first class railway compartment if two specified persons are to be always included and occupy adjacent seats on the same side, is $(5!) \cdot k$ then k has the value equal to.

13. There are 2 women participating in a chess tournament. Every participant played 2 games with the other participants. The number of games that the men played between themselves exceeded by 66 as compared to the number of games that the men played with the women. If the number of participants is n then the value of $n - 6$ is.

14. The number of three digit numbers having only two consecutive digits identical is N, then the value of $(N/2)^{1/2}$ is.

15. Number of 4 digit numbers of the form $N = abcd$ which satisfy following three conditions
 (i) $4000 \le N < 6000$
 (ii) N is a multiple of 5
 (iii) $3 \le b < c \le 6$
 is equal to N then the value of $N/3$ is.

16. Number of ways in which 5 A's and 6 B's can be arranged in a row which reads the same backwards and forwards, is N then the value of $N/2$ is.

17. If N is the number of different paths of length-12 which leads from A to B in the grid which do not pass through M, then the value of $[N/10]$, where [.] represents the greatest integer function, is.

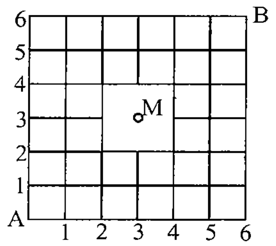


Fig. 5.33

18. There are 720 permutations of the digits 1, 2, 3, 4, 5, 6. Suppose these permutations are arranged from smallest to largest numerical values, beginning from 1 2 3 4 5 6 and ending with 6 5 4 3 2 1. Then the digit in unit place of number at 267th position is.

19. If N is the number of ways in which a person can walk up a stairway which has 7 steps if he can take 1 or 2 steps up the stairs at a time, then the value of $N/3$ is.

20. There are 3 men and 7 women taking a dance class. If N is number of different ways in which each man be paired with a woman partner, and the four remaining women be paired into two pairs each of two, then the value of $N/90$ is.

21. Let P_n denotes the number of ways in which three people can be selected out of 'n' people sitting in a row, if no two of them are consecutive. If, $P_{n+1} - P_n = 15$ then the value of 'n' is.

22. A man has 3 friends. If N is number of ways he can invite one friend everyday for dinner on 6 successive nights so that no friend is invited more than 3 times then the value of $N/170$ is.

23. If N is the number of ways in which 3 distinct numbers can be selected from the set $\{3^1, 3^2, 3^3, ...3^{10}\}$ so that they form a G.P. then the value of $N/5$ is.

24. Let $f(n) = \sum_{r=0}^{n} \sum_{k=r}^{n} \binom{k}{r}$. Find the total number of divisors of $f(9)$.

Archives

Solutions on page 5.64

Subjective Type

1. **(i)** In how many ways can a pack of 52 cards be divided equally among four players?

(ii) In how many ways can you divide these cards in four sets, three of them having 17 cards each and the fourth one just one card? **(IIT-JEE, 1979)**

2. Six Xs have to be placed in the squares of figure below in such a way that each row contains at least one X. In how many different ways can this be done?

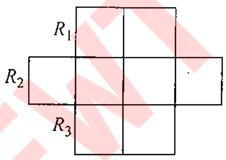


Fig. 5.34

(IIT-JEE, 1978)

3. Five balls of different colours are to be placed in the boxes of different size. Each box can hold all five. In how many different ways can we place the balls so that no box remains empty? **(IIT-JEE, 1981)**

4. m men and n women are to be seated in a row so that no two women sit together. If $m > n$, then show that the number of ways in which they can be seated is $m!(m + 1)!/(m - n + 1)!$. **(IIT-JEE, 1983)**

5. Seven relatives of a man comprises four ladies and three gentlemen; his wife has also seven relatives—three of them are ladies and four gentlemen. In how many ways can they invite 3 ladies and 3 gentlemen at a dinner party so that there are three man's relatives and three wife's relatives? **(IIT-JEE, 1985)**

6. A box contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw? **(IIT-JEE, 1986)**

7. A student is allowed to select almost n books from a collection of $2n + 1$ books. If the total number of ways in which he can select at least one book is 63, find the value of n. **(IIT-JEE, 1987)**

8. A number of 18 guests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on the other side. Determine the number of ways in which the sitting arrangements can be made. **(IIT-JEE, 1991)**

9. A committee of 12 is to be formed from nine women and eight men. In how many ways can this be done if at least five women have to be included in a committee? In how many of these committees

a. the women hold majority?

b. the men hold majority? **(IIT-JEE, 1994)**

10. Prove by permutation or otherwise that $(n^2)!/(n!)^n$ is an integer $(n \in I^+)$. **(IIT-JEE, 2004)**

Objective Type

Fill in the blanks

1. In a certain test, a_i students gave wrong answers to at least i questions, where $i = 1, 2, ..., k$. No student gave more than k wrong answers. The total number of wrong answers given is——. **(IIT-JEE, 1982)**

2. The sides AB, BC and CA of a triangle ABC have 3, 4 and 5 interior points, respectively on them. The number of triangles that can be constructed using these interior points as vertices is ——. **(IIT-JEE, 1984)**

3. The total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together is ——. **(IIT-JEE, 1988)**

4. There are four balls of different colours and four boxes of colours same as those of the balls. The number of ways in which the balls, one each in a box, could be placed such that a ball does not go to a box of its own colour is ——. **(IIT-JEE, 1992)**

True or false

1. The product of any r consecutive natural numbers is always divisible by $r!$. **(IIT-JEE, 1985)**

Multiple choice questions with one correct answer

1. If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, then r is

a. 1 b. 2 c. 3 d. none of these

(IIT-JEE, 1979)

2. Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. The number of words, which have at least one letter repeated is

a. 59720 b. 79260
c. 69760 d. none of these

3. The value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is equal to

a. ${}^{47}C_5$ b. ${}^{52}C_5$
c. ${}^{52}C_4$ d. none of these

4. Eight chairs are numbered 1 to 8. Two women and three men wish to occupy one chair each. First, the women choose the chairs from amongst the chairs marked 1 to 4, and then the men select the chairs from amongst the remaining. The number of possible arrangements is

a. ${}^6C_3 \times {}^4C_2$ b. ${}^4P_2 \times {}^4P_3$
c. ${}^4C_2 + {}^4P_3$ d. none of these

(IIT-JEE, 1982)

5. In a group of boys, two boys are brothers and six more boys are present in the group. In how many ways can they sit if the brothers are not to sit along with each other?

a. $2 \times 6!$ b. ${}^7P_2 \times 6!$
c. ${}^7C_2 \times 6!$ d. none of these

(IIT-JEE, 1982)

6. A five-digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5, without repetition. The total number of ways this can be done is

a. 216 b. 240
c. 600 d. 3125 **(IIT-JEE, 1989)**

7. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits 2, 5 and 7. The smallest value of n for which this is possible is

a. 6 b. 7
c. 8 d. 9 **(IIT-JEE, 1998)**

8. How many different nine-digit numbers can be formed from the number 223355888 by rearranging its digits so that the odd digits occupy even positions?

a. 16 b. 36
c. 60 d. 180 **(IIT-JEE, 2000)**

9. Let T_n denote the number of triangles, which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals

a. 5 b. 7
c. 6 d. 4 **(IIT-JEE, 2001)**

10. The number of arrangements of the letters of the word 'BANANA' in which the two Ns do not appear adjacently is

a. 40 b. 60
c. 80 d. 100 **(IIT-JEE, 2002)**

11. A rectangle with sides $2m - 1$ and $2n - 1$ is divided into squares of unit length by drawing parallel lines as shown in the diagram, then the number of rectangles possible with odd side lengths is

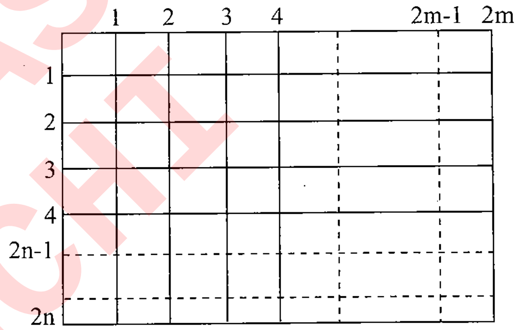


Fig. 5.35

a. $(m + n - 1)^2$ b. 4^{m+n-1}
c. m^2n^2 d. $m(m + 1)n(n + 1)$

(IIT-JEE, 2000)

12. If r, s, t are prime numbers and p, q are the positive integers such that the LCM of p, q is $r^2t^4s^2$, then the number of ordered pair (p, q) is

a. 252 b. 254
c. 225 d. 224 **(IIT-JEE, 2006)**

13. The letters of the word 'COCHIN' are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. The number of words that appear before the word 'COCHIN' is

a. 360 b. 192
c. 96 d. 48

(IIT-JEE, 2007)

14. Let $S = \{1, 2, 3, 4\}$. The total number of unordered pairs of disjoint subsets of S is equal to

a. 25 b. 34
c. 42 d. 41

(IIT-JEE, 2010)

ANSWERS AND SOLUTIONS

Subjective Type

1. Consider $n + 1$ different toys Then,

$\left(\begin{matrix}\text{no. of ways of}\\ \text{selecting } r \text{ toys out of}\\ (n+1) \text{ different toys}\end{matrix}\right)$

$= \left(\begin{matrix}\text{no. of ways of selecting}\\ (r-1) \text{ toys out of } n \text{ toys}\\ \text{when } T_0 \text{ is always included}\end{matrix}\right)$

$+ \left(\begin{matrix}\text{no. of ways of}\\ \text{selecting } r \text{ toys when } T_0\\ \text{is always excluded}\end{matrix}\right)$

$= {}^nC_{r-1} + {}^nC_r$

Hence, ${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$.

2. Now, $(n!)!$ is the product of the positive integers from 1 to $n!$
We write the integers from 1 to $n!$ in $(n-1)!$ rows as follows:

$$1\times 2\times 3\cdots n$$
$$(n+1)(n+2)(n+3)\cdots(2n)$$
$$(2n+1)(2n+2)\cdots(3n)$$
$$(3n+1)(3n+2)\cdots(4n)$$
$$\vdots$$
$$(n!-n+1)(n!-n+2)\cdots(n(n-1)!)$$

Each of these $(n-1)!$ rows contain n consecutive positive integers. The product of consecutive integers in each row is divisible by $n!$ Thus, the product of all the integers from 1 to n! is divisible by $(n!)^{(n-1)!}$.

3. $n_1 = x_1, x_2, x_3, x_4, x_5$

$n_2 = y_1, y_2, y_3, y_4, y_5$

and n_2 can be added without carrying at any stage if $x_i + y_i \le 9$.

Value of x_5	Value of y_5
0	0, 1, 2, ..., 9
1	0, 1, 2, ..., 8
2	0, 1, 2, ..., 7
3	0, 1, 2, 3, 4, 5, 6
4	0, 1, 2, 3, 4, 5
5	0, 1, 2, 3, 4
6	0, 1, 2, 3
7	0, 1, 2
8	0, 1
9	0

Thus, x_5 and y_5 can be selected collectively by $10 + 9 + 8 + \cdots + 1 = 55$ ways. Similarly, each pair (x_4, y_4), (x_3, y_3), (x_2, y_2) can be selected in 55 ways. But pair (x_1, y_1) can be selected in $1 + 2 + 3 + \cdots + 8 = 36$ ways as in this pair we cannot have 0 or 9. Thus, total number of ways is $36(55)^3$.

4. $n_1 = x_1, x_2, x_3, x_4, x_5$

$n_2 = y_1, y_2, y_3, y_4, y_5$

n_1 and n_2 can be subtracted without borrowing at any stage if $x_i \le y_i$.

Value of x_5	Value of y_5
9	0, 1, 2, ..., 9
8	0, 1, 2, ..., 8
7	0, 1, 2, ..., 7
6	0, 1, 2, 3, 4, 5, 6
5	0, 1, 2, 3, 4, 5
4	0, 1, 2, 3, 4
3	0, 1, 2, 3
2	0, 1, 2
1	0, 1
0	0

Thus, x_5 and y_5 can be selected collectively by $10 + 9 + 8 + \cdots + 1 = 55$ ways. Similarly, each pair (x_4, y_4), (x_3, y_3), (x_2, y_2) can be selected in 55 ways. But, pair (x_1, y_1) can be selected in $1 + 2 + 3 + \cdots + 9 = 36$ ways as in this pair we cannot have 0.

Thus, total number of ways is $45(55)^3$.

5. Case I: Two identical digits are 0, 0.

The number of ways to select three more digits is ${}^9C_3$. The number of arrangements of these five digits is $5!/2! - 4! = 36$.

Hence, the number of such numbers is

$${}^9C_3 \times 36 = 3024 \qquad (1)$$

Case II: Two identical digits are (1, 1) or (2, 2) or … or (9, 9).

If 0 is included, then number of ways of selection of two more digits is ${}^8C_2$. The number of ways of arrangements of these five digits is $5!/2! - 4!/2! = 48$. Therefore, Number of such numbers is ${}^8C_2 \times 48$. If 0 is not included, then selection of three more digits is ${}^8C_3$. Therefore, Number of such numbers is ${}^8C_3 \times 5!/2! = {}^8C_3 \times 60$. Hence, total number of five-digit numbers with identical digits (1, 1), …, (9, 9) is

$$9 \times ({}^8C_2 \times 48 + {}^8C_3 \times 60) = 42336 \qquad (2)$$

From Eqs. (1) and (2), the required number of numbers is $3024 + 42336 = 45360$.

6. Out of six faces, three can be selected in ${}^6C_3$ ways.

Consider one such selection, say ABC. Each of the 'n' places can be filled in three ways. So total number of ways is 3^n.

But this includes those ways also, which contain exactly one alphabet or exactly two alphabets which are to be subtracted.

Now, number of ways which contain only one letter is 3 and number of ways containing exactly two alphabets is ${}^3C_2\,(2^n - 2)$.

Hence, the number of ways is $3^n - {}^3C_2\,(2^n - 2) - 3$. So, required number of ways is ${}^6C_3\,[3^n - {}^3C_2\,(2^n - 2) - 3]$.

7.

One of the digits	Pattern	Number of numbers
1	102, 111	$4 + 1 = 5$
2	240, 231, 222	$4 + 6 + 1 = 11$
3	306, 315, 324, 333	$4 + 6 + 6 + 1 = 17$
4	408, 417, 426, 435, 444	$4 + 6 + 6 + 6 + 1 = 23$
5	519, 528, 537, 546, 555	$6 + 6 + 6 + 6 + 1 = 25$
6	639, 648, 657, 666	$6 + 6 + 6 + 1 = 19$
7	759, 768, 777	$6 + 6 + 1 = 13$
8	879, 888	$6 + 1 = 7$
9	999	1
	Total	121

Alternative solution:

Consider two sets.

(1) 1, 3, 5, 7, 9

(2) 0, 2, 4, 6, 8

The required number of ways = [any two from set (1) + any two from set (2) (excluding zero)] 3! + [0 along with any one from set (2)] $\times 4$ + all three alike

$$= ({}^5C_2 + {}^4C_2) \times 3! + {}^4C_1\, 4 + 9$$
$$= 121$$

8. Let the number of members be n. Total number of points is ${}^nC_2$.

Therefore, ${}^nC_2 - 17 \times 5 = (n - 4)x$ (where x is the number of point scored by each player)

$$\Rightarrow \quad n(n-1) - 35 = 2(n-4)x$$

$$\Rightarrow \quad 2x = \frac{n(n-1)-35}{n-4} \quad \text{(where } x \text{ takes the values } 0.5, 1, 1.5, \text{ etc.)}$$

$$= \frac{n^2 - n - 35}{n-4}$$

$$= \frac{n(n-4)+3(n-4)-23}{n-4}$$

$$= (n+3) - \frac{23}{n-4}$$

$$\Rightarrow \quad \frac{23}{n-4} \text{ must be an integer}$$

$$\Rightarrow \quad n = 27$$

9. Excluding the two specified guests, $2n$ persons can be divided into two groups one containing n and the other containing $n - 2$ in $(2n - 2)!/[n!(n-2)!]$ ways and can sit on either side of master and mistress in 2! ways and can arrange themselves in $n!\ (n-2)!$ Now, the two specified guests where $n - 2$ guests are seated will have $n - 1$ gaps and can arrange themselves in 2! ways. The number of ways when G_1G_2 will always be together is

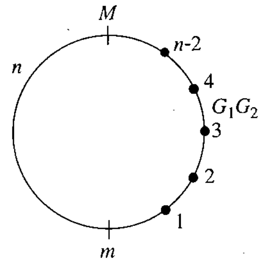


Fig. 5.36

$$\frac{(2n-2)!}{n!(n-2)!}\ 2!\ n!\ (n-2)!\ (n-1) \times 2!$$

$$= (2n-2)!\ 4(n-1)$$

Hence, the number of ways when G_1G_2 are never together is

$$(2n)! - 4(n-1)(2n-2)! = (2n-2)!\ [2n(2n-1) - 4(n-1)]$$
$$= (2n-2)!\ [4n^2 - 6n + 4]$$

10. Let the two subsets be called A and B. The elements for the two sets will be selected as follows.

First, two elements out of k elements for both the sets can be selected in ${}^kC_2$ ways. Now, remaining r elements for the subset A are selected from $k - 2$ elements and any number of elements for B from the remaining $k - 2 - r$ elements.

Here r can vary from 0 to $k - 2$. For a fixed r, the number of selections is ${}^{k-2}C_r \times 2^{k-2-r}$, because the number of selections of any number of things from n things is 2^n. Then, the total number of selections is $\sum_{r=0}^{k-2} {}^{k-2}C_r \times 2^{k-2-r} - 1$, excluding the case when both the subsets are equal having only the two common elements.

But, every pair of A, B is appearing twice like $\{a_1, a_2, a_3\}$, $\{a_1, a_2, a_4, a_5, a_6\}$ and $\{a_1, a_2, a_4, a_5, a_6\}$, $\{a_1, a_2, a_3\}$. Hence, the required number of ways is

$$\frac{1}{2} \times {}^kC_2 \left(\sum_{r=0}^{k-2} {}^{k-2}C_r \times 2^{k-2-r} - 1 \right)$$

$$= \frac{k(k-1)}{2} \times \frac{1}{2} \times [({}^{k-2}C_0 \times 2^{k-2} + {}^{k-2}C_1 \times 2^{k-3} + {}^{k-2}C_2 \times 2^{k-4} + \cdots + {}^{k-2}C_{k-2}) - 1]$$

$$= \frac{k(k-1)}{4}\ [(2+1)^{k-2} - 1]$$

$$= \frac{k(k-1)}{4}\ (3^{k-2} - 1)$$

11. The number of selections of r objects from n identical objects and $n - r$ objects from $2n$ different objects is $1 \times {}^{2n}C_{n-r} = {}^{2n}C_{n-r}$.

Here r varies form 0 to n. Therefore, the required number of selections is

$$\sum_{r=0}^{n} {}^{2n}C_{n-r} = {}^{2n}C_n + {}^{2n}C_{n-1} + {}^{2n}C_{n-2} + \cdots + {}^{2n}C_0$$

$$= {}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \cdots + {}^{2n}C_n \text{ (writing in reverse order)}$$

$$= \frac{1}{2}\{2 \cdot {}^{2n}C_0 + 2 \cdot {}^{2n}C_1 + 2 \cdot {}^{2n}C_2 + \cdots + 2 \cdot {}^{2n}C_n\}$$

$$= \frac{1}{2}\{({}^{2n}C_0 + {}^{2n}C_{2n}) + ({}^{2n}C_1 + {}^{2n}C_{2n-1}) + \cdots + ({}^{2n}C_{n-1} + {}^{2n}C_{n+1}) + 2 \cdot {}^{2n}C_n\}$$

$$= \frac{1}{2}\{{}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + {}^{2n}C_3 + \cdots + {}^{2n}C_{2n} + {}^{2n}C_n\}$$

$$= \frac{1}{2}\{2^{2n} + {}^{2n}C_n\}$$

$$= 2^{2n-1} + \frac{1}{2}\frac{2n!}{(n!)^2}$$

12. Since no two lines are parallel and no three this pass through the same point, their points of intersection, i.e., number of ways of selecting two lines from n lines is ${}^nC_2 = N$ (say). It should also be noted that on each line there will be $(n-1)$ points of intersection made by the remaining $(n-1)$ lines.

Now we have to find number of new lines formed by these points of intersections. Clearly, a straight line is formed by joining two points so the problem is equivalent to select two points from N points. But each old line repeats itself ${}^{(n-1)}C_2$ times [selection of two points from $(n-1)$ points on this line]. Hence, the required number of new lines is

$${}^NC_2 - n\cdot{}^{(n-1)}C_2 = \frac{1}{2}\times\frac{1}{2}n(n-1)\left[\frac{1}{2}n(n-1)-1\right] - \frac{1}{2}n(n-1)(n-2)$$

$$= \frac{1}{8}n(n-1)(n^2-n-2) - \frac{1}{2}n(n-1)(n-2)$$

$$= \frac{1}{8}n(n-1)[n^2-n-2-4n+8]$$

$$= \frac{1}{8}n(n-1)(n-2)(n-3)$$

13. (a) A straight line can be formed by joining any two points. so number of straight lines (i.e., selection of two points from n) is ${}^nC_2$. But, selection of two points from m collinear points gives no extra line. Hence, number of distinct straight lines is

$${}^nC_2 - ({}^mC_2 - 1) = \frac{1}{2}n(n-1) - \frac{1}{2}m(m-1) + 1 \qquad (1)$$

(b) Formation of triangles is equivalents to section of three points from n points. As m points are collinear, selection of three points from m collinear points gives no triangles. Hence, the number of triangles is

$${}^nC_3 - {}^mC_3 = \frac{1}{6}[n(n-1)(n-2) - m(m-1)(m-2)]$$

(c) Four points determine a quadrilateral. But of these four points, not more than two is to be selected from the four collinear points. Now, number of selections of four points from all n is ${}^nC_4$. The number of selections of three points from m collinear and one from rest is ${}^mC_3\,{}^{n-m}C_1$. The number of selections of four points from m collinear $= {}^mC_4$.

Hence, the number of quadrilaterals is ${}^nC_4 - {}^mC_3 \times {}^{(n-m)}C_1 - {}^mC_4$.

14. The number of ways of disturbing n identical things among n person when at least $n-3$ persons get none of these of the objects is $A + B + C$ (suppose), where A is the number of ways when exactly $n-3$ persons get none of these, i.e., ${}^{n-1}C_2\,{}^nC_3$; B is the number of ways when exactly $n-2$ persons get none of these, i.e., ${}^{n-1}C_1\,{}^nC_2$; C is the number of ways when exactly $n-1$ persons get none of these, i.e., ${}^nC_1$. Hence,

$$A + B + C = {}^{n-1}C_2\,{}^nC_3 + {}^{n-1}C_1\,{}^nC_2 + {}^nC_1$$

$$= \frac{(n-1)(n-2)}{2}\frac{n(n-1)(n-2)}{6} + \frac{(n-1)n(n-1)}{2} + \frac{n}{1}$$

$$= \frac{n(n-1)^2}{2}\left[\frac{(n-2)^2}{6}+1\right] + n$$

$$= \frac{n(n-1)^2}{2}\left[\frac{n^2-4n+10}{6}\right] + n$$

15. Let the city be represented by a rectangle whose sides are of length a and b north-south and west-east, respectively. Man has to go from P to Q. For this, he will have to travel a distance a vertically downward and a distance b horizontally from left to right.

Let $a_1, a_2, \ldots, a_{m-1}$ denote the distances between consecutive streets drawn horizontally beginning with the street passing through P and $b_1, b_2, \ldots, b_{n-1}$ denote the distances between consecutive streets drawn vertically beginning with the street passing through P.

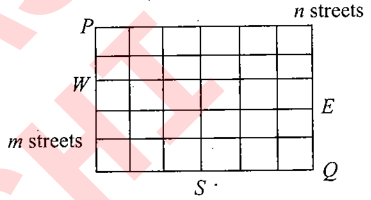


Fig. 5.37

Each arrangement of $(m+n-2)$ things $a_1, a_2, \ldots, a_{m-1}, b_1, b_2, \ldots, b_{n-1}$ in a row so that order of a_i's does not change and order b_i's does not change corresponds to one path to go from P to Q.

Therefore, the required number is equal to number of arrangements of $m+n-2$ things such that order of a_i's does not change and order of b_i's does not change, which is equal to $(m+n-2)!/(m-1)!(n-1)!$.

Alternative solution:

Let the city be represented by a rectangle whose sides are of length a and b.

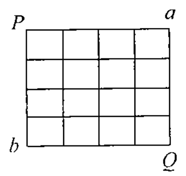


Fig. 5.38

For each path total distance covered in horizontal direction is a and that in vertical direction is b. a is the sum of lengths of $(m-1)$ horizontal line segments and b is the sum of lengths of $(n-1)$ vertical line segments.

Each path to go from P to Q will be a arrangement of $(m+n-2)$ line segments of which $(m-1)$ are horizontal and $(n-1)$ are vertical.

Therefore, the required number is equal to number of arrangements of $m+n-2$ different things of which $m-1$ are of one kind and $n-1$ are of another kind, which is given by $(m+n-2)!/(m-1)!(n-1)!$.

16. Let the batsman hit x fours, y sixes and let z balls may not yield runs. Then we have,

$$4x + 6y + 0z = 100 \qquad (1)$$

where

$$x + y + z = 20 \qquad (2)$$

From Eqs. (1) and (2),

$$4(20 - y - z) + 6y = 100$$

$\Rightarrow$ $2y - 4z = 20$

$\Rightarrow$ $y - 2z = 10$

$\Rightarrow$ $y = 10 + 2z$

Hence, z can be 0, 1, 2, 3 as any of x, y, z cannot exceed 20. So, we have following type of distribution of runs:

Number of fours, x	**Number of sixes, y**	**Number of zeros, z**	**Number of ways of arranging x, y and z**
10	10	0	$\frac{20!}{10!\,10!}$
7	12	1	$\frac{20!}{7!\,12!}$
4	14	2	$\frac{20!}{4!\,14!\,2!}$
1	16	3	$\frac{20!}{16!\,3!}$

Hence, the total number of ways is

$$20!\left(\frac{1}{16!\,3!} + \frac{1}{14!\,4!\,2!} + \frac{1}{12!\,7!} + \frac{1}{10!10!}\right)$$

17. The total number of ways to place the balls disregarding the constrains is ${}^{2t+1+3-1}C_{3-1} = {}^{2t+3}C_2$.

The total number of ways to place the balls so that the first box will have more balls than the other two is ${}^{t+3-1}C_{3-1} = {}^{t+2}C_2$.

[We place $t + 1$ balls in the first box and then divide the rest of t balls in the three boxes arbitrarily.]

The same result applies to the case of 2nd box holding more balls than 1st or 3rd combined and also for the 3rd box containing more balls than 1st and 2nd combined. Hence, required number of ways is

$${}^{2t+3}C_2 - 3\,{}^{t+2}C_2 = \frac{t}{2}(t+1)$$

18. The whole family has 24 children. Children of Geeta and Sohan are $24 - x - (x + 1) = 23 - 2x$ in number. When Sohan's child fights with a Geeta's child, there are $x(x + 1)$ fights. When Sohan's child fights with Sohan and Geeta's children, there are $x(23 - 2x)$ fights. Again, when Geeta's child fights with Sohan and Geeta's children, there are $(x + 1)(23 - 2x)$ fights. Therefore, total number of fights is

$$F(x) = x(x + 1) + x(23 - 2x) + (x + 1)(23 - 2x)$$
$$= 23 + 45x - 3x^2$$
$$= 3\left[\frac{23}{3} + \frac{225}{4} - \left(x - \frac{15}{2}\right)^2\right]$$

For $F(x)$ to be maximum,

$$x - \frac{15}{2} = 0,$$

i.e.,

$$x = 7.5$$

Since x is an integer, therefore, $x = 7$ or 8. For both $x = 7$ and $x = 8$. Total number of fights is $F(x) = 191$.

19. Given,

$$nx + ny = xy$$

$\Rightarrow$ $xy - nx - ny + n^2 = n^2$

$\Rightarrow$ $(x - n)(y - n) = n^2$

Hence, $(x - n)$ and $(y - n)$ are two integral factors of n^2. Obviously, if d is one divisor of n^2, then for each sub-divisor there will be an ordered pair (x, y). Let $S(n)$ be the number of divisors of n^2.

(i) For $n = 6$, we have $d = 1, 2, 3, 6, 9, 12, 18, 36$.

$\therefore$ $S(6) = 9$

(ii) If n is prime, then $d = 1$, n and n^2; hence $S(n) = 3$.

20. When 12 fruits are distributed subject to the given condition, either nine boys get one fruit each and the remaining one boy gets three fruits or eight boys get one fruit each and the remaining two boys get two fruit each. Three fruits to a boy can be given in the four ways.

Apples	3	2	1	0
Mangoes	0	1	2	3

After giving 3 apples to a single boy, the 3 remaining apples and 6 oranges can be distributed to the 9 boys in $({}^9C_3)\,({}^6C_6)$ ways. Thus, number of ways three fruits can be given to a particular boy is

$$({}^9C_3)({}^6C_6) + ({}^9C_4)({}^5C_5) + ({}^9C_5)({}^4C_4) + ({}^9C_6)({}^3C_3)$$
$$= 2[{}^9C_3 + {}^9C_4]$$

One particular boy can be chosen in ${}^{10}C_1 = 10$ ways. Therefore, three fruits can be given to a single boy in $({}^{10}C_1)\,(2)\,[{}^9C_3 + {}^9C_4] = 20\,({}^{10}C_4) = 4200$ ways.

We can give two fruits to two boys, say P and Q, in the following ways:

P	**Apples**	2	1	0	2	1	0	2	1	0
	Mangoes	0	1	2	0	1	2	0	1	2
Q	**Apples**	2	2	2	1	1	1	0	0	0
	Mangoes	0	0	0	1	1	1	2	2	2

The remaining eight fruits can be distributed among eight boys in the following ways:

$$({}^8C_2)({}^5C_6) + ({}^8C_3)({}^5C_5) + ({}^8C_4)({}^4C_4) + ({}^8C_3)({}^5C_5) + ({}^8C_4)$$
$$({}^4C_4) + ({}^8C_5)({}^3C_3) + ({}^8C_4)({}^4C_4) + ({}^8C_5)({}^5C_5) + ({}^8C_6)({}^2C_2)$$
$$= 2\,({}^8C_2) + 4\,({}^8C_3) + 4\,({}^8C_4)$$

Two boys out of 10 can be chosen in ${}^{10}C_2$ ways. Therefore, number of ways two boys can get two fruits each is

$({}^{10}C_2)\,[2({}^8C_2) + 4\,({}^8C_3) + 3\,({}^8C_4)]$

$$= \frac{10!}{2!\,8!}\left[2\frac{8!}{2!\,6!}+4\frac{8!}{3!\,5!}+3\times\frac{8!}{4!\,4!}\right]$$

$$= \frac{10!}{2!\,6!}\left[14+\frac{15}{4}\right] = 22050.$$

Hence, the required number of ways is 4200 + 22050 = 26250.

Objective Type

1. d. $\alpha = {}^mC_2 \Rightarrow \alpha = \dfrac{m(m-1)}{2}$

$$\therefore\ {}^{\alpha}C_2 = \frac{\alpha(\alpha-1)}{2} = \frac{1}{2}\frac{m(m-1)}{2}\left\{\frac{m(m-1)}{2}-1\right\}$$

$$= \frac{1}{8}m(m-1)(m-2)(m+1)$$

$$= \frac{1}{8}(m+1)m(m-1)(m-2) = 3\ {}^{m+1}C_4$$

2. a. ${}^nC_3 + {}^nC_4 > {}^{n+1}C_3$

$\Rightarrow\ {}^{n+1}C_4 > {}^{n+1}C_3$ $(\because\ {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$

$\Rightarrow\ \dfrac{{}^{n+1}C_4}{{}^{n+1}C_3} > 1$

$\Rightarrow\ \dfrac{n-2}{4} > 1$

$\Rightarrow\ n > 6$

3. b. $\displaystyle\sum_{r=0}^{n-1}\frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}}$

$$= \sum_{r=0}^{n-1}\frac{1}{1+\dfrac{{}^nC_{r+1}}{{}^nC_r}}$$

$$= \sum_{r=0}^{n-1}\frac{1}{1+\dfrac{n-r}{r+1}}$$

$$= \sum_{r=0}^{n-1}\frac{r+1}{n+1} = \frac{1}{n+1}\sum_{r=0}^{n-1}(r+1)$$

$$= \frac{1}{(n+1)}[1+2+\cdots+n] = \frac{n}{2}$$

4. c. We have 32 places for teeth. For each place, we have two choices either there is a tooth or there is no tooth. Therefore, the number of ways to fill up these places is 2^{32}. As there is no person without a tooth, the maximum population is $2^{32} - 1$.

5. c. For each bulb there are two possibilities. It will be switched either on or off. Hence, total number of ways in which the room can be illuminated is $2^{12} - 1$.

6. c. The total number of ways is $6 \times 6 \times \cdots$ to n times $= 6^n$. The total number of ways to show only even numbers is $3 \times 3 \times \cdots$ to n times $= 3^n$. Therefore, the required number of ways is $6^n - 3^n$.

7. b. Total number of triplets without restriction is $n \times n \times n$.
The number of triplets with all different coordinates is ${}^nP_3$.
Therefore, the required number of triplets is $n^3 - n(n-1)(n-2)$.

8. a. All strips are of different colours, then number of flags is $= 3! = 5$. When two strips are of same colour, then number of flags is ${}^3C_1 \times (3!/2) \times {}^2C_1 = 18$. Total number of flags is $6 + 18 = 24 = 4!$

9. a. If 7 is used at first place, the number of numbers is 9^4 and for any other four places it is 8×9^3.

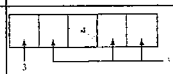
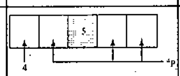
10. c. Total number of variables if only alphabet is used is 286.
Total number of variables if alphabets and digits both are used is 26×10. Hence, the total number of variables is $26(1 + 10) = 286$.

11. b. The number of one-digit numbers is 6.
The number of two-digit numbers is $5 \times 5 = 25$.
The number of thee-digit numbers is $5 \times 5 \times 4 = 100$.
Hence, the total number are is 131.

12. a. $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$, when the number is $x_1x_2x_3x_4x_5x_6$
Clearly no digit can be zero. Also, all the digits are distinct.
So, let us first select six digits from the list of digits 1, 2, 3, 4, 5, 6, 7, 8, 9 which can be done in ${}^9C_6$ ways.
After selecting these digits they can be put only in one order.
Thus, total number of such numbers is ${}^9C_6 \times 1 = {}^9C_3$.

13. b. Numbers greater than 1000 and less than or equal to 4000 will be of 4 digits and will have either 1 (except 1000) or 2 or 3 in the 1st place with 0 in each of remaining places. After fixing 1st place, the 2nd place can be filled by any of the 5 digits. Similarly the 3rd place can be filled up in 5 ways and 4th place can be filled up in 5 ways. Thus, there will be $5 \times 5 \times 5 = 125$ ways in which 1 will be in first place but this also includes 1000. Hence, there will be 124 numbers having 1 in the first place. Similarly, 125 for each 2 or 3. One number will be there in which 4 will be in the first place, i.e., 4000. Hence, the required number of ways is $124 + 125 + 125 + 1 = 375$.

14. d.

Middle digit	Digits available for remaining four places	Pattern	Number of ways filling remaining four places
4	0, 1, 2, 3		$3 \times {}^3P_3$
5	0, 1, ..., 4		$4 \times {}^4P_3$
6	0, 1, ..., 5	...	$5 \times {}^5P_3$
7	0, 1, ..., 6	...	$6 \times {}^6P_3$
8	0, 1, ..., 7	...	$7 \times {}^7P_3$
9	0, 1, ..., 8	...	$8 \times {}^8P_3$

15. **b.** The number of numbers when repetition is allowed is 5^4.
The number of numbers when digits cannot be repeated is ${}^5P_5$.
Therefore, the required number of numbers is $5^4 - 5!$.

16. **d.** According to given conditions, numbers can be formed by the following format:

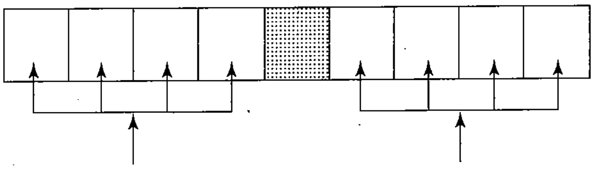


Fig. 5.39

The required number of numbers is ${}^4P_4 \times {}^4P_4$.

17. **a.** Total number of words without any restriction is 7!
Total number of words beginning with I is 6!
Total number of words ending with B is 6!
Total number of words beginning with I and ending with B is 5!
Thus the total number of required words is $7! - 6! - 6! + 5! = 7! - 2(6!) + 5!$.

18. **a.** There can be two types of numbers.
(i) Any one of the digits 1, 2, 3, 4 appears thrice and the remaining digits only once, i.e., of the type 1, 2, 3, 4, 4, 4, etc. Number of ways of selection of digit which appears thrice is ${}^4C_1$.

Then the number of numbers of this type is $(6!/3!) \times {}^4C_1 = 480$.

(ii) Any two of the digits 1, 2, 3, 4 appears twice and the remaining two only once, i.e., of the type 1, 2, 3, 3, 4, 4, etc. The number of ways of selection of two digits which appear twice is ${}^4C_2$. Then the number of numbers of this type is $[6!/(2!2!) \times {}^4C_2]$. Therefore, the required number of numbers is $480 + 1080 = 1560$.

19. **a.** Clearly, one of the odd digits 1, 3, 5, 7, 9 will be repeated. The number of selections of the sixth digit is ${}^5C_1 = 5$. Then the required number of numbers is $5 \times (6!/2!)$.

20. **c.** Formed number can be utmost of nine digits. Total number of such numbers is

$$3 + 3^2 + 3^3 + \cdots + 3^8 + 2 \times 3^8$$

$$= \frac{3(3^8-1)}{3-1} + 2\times 3^8 = \frac{3^9 - 3 + 4\times 3^8}{2} = \frac{7\times 3^8 - 3}{2}$$

21. **d.** The order of letters of the word 'OBJECT' is B C E J O T
Words starting with B can be formed in 5! ways.
Words starting with C can be formed in 5! ways.
Words starting with E can be formed in 5! ways.
Words starting with J can be formed in 5! ways.
Words starting with O can be formed in 5! ways.
Words starting with TB can be formed in 4! ways.
Words starting with TC can be formed in 4! ways.
Words starting with TE can be formed in 4! ways.
Words starting with TJ can be formed in 4! ways.
Words starting with TOB can be formed in 3! ways.
Words starting with TOC can be formed in 3! ways.
Words starting with TOE can be formed in 3! ways.
Words starting with TOJB can be formed in 2! ways.
Words starting with TOJC can be formed in 2! ways.
Therefore, the total number of words is 718 words.
Hence 717$^{\text{th}}$ word is TOJCBE.

22. **a.** The number of ways of allotment without any restriction is ${}^8P_6$. Now, it is possible that all rooms of 2$^{\text{nd}}$ floor or 3$^{\text{rd}}$ floor are not occupied. Thus, there are two ways in which one floor remains unoccupied. Hence, the number of ways of allotment in which a floor is unoccupied is $2 \times 6!$ Hence, number of ways in which none of the floor remains unoccupied is ${}^8P_6 - 2(6!)$.

23. **a.** Each position can be filled in 5 ways. Hence, the total number of numbers is 5^{20}.

24. **a.** The number of ways of selecting four numbers from 1 to 30 without any restriction is ${}^{30}C_4$. The number of ways of selecting four consecutive [i.e. (1, 2, 3, 4), (2, 3, 4, 5), …, (27, 28, 29, 30)] number is 27. Hence, the number of ways of selecting four integers which excludes consecutive four selections is

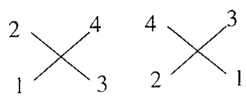
$${}^{30}C_4 - 27 = \frac{30\times 29\times 28\times 27}{24} - 27 = 27378$$

25. **d.** Let us first select two places for vowel, which can be selected from 4 places in ${}^4C_2$ ways. Now this places can be filled by vowels in $5\times 5 = 25$ ways as repetition is allowed. The remaining two places can be filled by consonants in 21×21 ways. Then the total number of words is ${}^4C_2 \times 25 \times 21^2 = 150 \times 21^2$.

26. **b.** Other than 2, remaining five places can be filled by 1 and 3 for each place. The number of ways for five places is $2 \times 2 \times 2 \times 2 \times 2 = 2^5$. For 2, selecting 2 places out of 7 is ${}^7C_2$. Hence, the required number of ways is ${}^7C_2 \times 2^5$.

27. **c.** 1 S, 3A, 1 H, 2 R, 1 N, 1 P, 1 U
When all letters are different corresponding ways is ${}^7C_3 \times 3!$ $= {}^7C_3 = 210$. When two letters are of one kind and other is different, corresponding number of ways is ${}^2C_1 \times {}^6C_1 \times (3!/2!) = 36$. When all letters are alike, corresponding number of ways is 1. Thus, total number of words that can be formed is $210 + 36 + 1 = 247$.

28. **b.** The natural numbers are 1, 2, 3, 4. Clearly, in one diagonal we have to place 1, 4 and in the other 2, 3.



(i) (ii)

Fig. 5.40

The number of ways in (i) is $2! \times 2! = 4$.
The number of ways in (ii) is $2! \times 2! = 4$.
Therefore, the total number of ways is 8.

29. **c.** First arrange m positive signs. The number of ways is just 1 (as all + signs are identical). Now, $m + 1$ gaps are created of which n are to be selected for placing '–' signs. Then the total number of ways of doing so is ${}^{m+1}C_n$. After selecting the gaps '–' signs can be arranged in one way only.

30. **a.** We can think of three packets. One consisting of three boys of class X, other consisting of 4 boys of class XI and last one consisting of 5 boys of class XII. These packets can be arranged in 3! ways and contents of these packets can be further arranged in 3! 4! and 5! ways, respectively. Hence, the total number of ways is 3!× 3! × 4! × 5!

31. a. The total number of books is $a + 2b + 3c + d$. The total number of ways in which these books can be arranged in a shelf (in same row) is

$$\frac{(a+2b+3c+d)!}{a!(b!)^2(c!)^3}$$

32. c. Required sum is $3!(3+4+5+6) = 6 \times 18 = 108$.

[If we fix 3 in the unit place, other three digits can be arranged in 3! ways. Similarly for 4, 5, 6.]

33. c. The number of numbers with 0 in the unit's place is $3! = 6$.

The number of numbers with 1 or 2 or 3 in the unit's place is $3! - 2! = 4$. Therefore the sum of the digits in the unit's place is $6 \times 0 + 4 \times 1 + 4 \times 2 + 4 \times 3 = 24$.

Similarly, for the ten's and hundred's places, the number of numbers with 1 or 2 in the thousand's place is 3! Therefore, the sum of the digits in the thousand's place is $6 \times 1 + 6 \times 2 + 6 \times 3 = 36$.

Hence, the required sum is $36 \times 1000 + 24 \times 100 + 24 \times 10 + 24$.

34. a. Total numbers ending with 2 is 3! as after fixing 2 in the unit's place other three places can be filled by 3! ways. Thus, 2 appears in the unit's place 3! times.

Similarly, all other digits 4, 6 and 8 also appear 3! times. Then sum of the digits in the unit's place is $6(2+4+6+8) = 120$ units. Similarly, sum of digits in ten's place is 120 tens and that in hundred's place is 120 hundreds, etc. Hence, sum of all the 24 numbers is $120(1 + 10 + 10^2 + 10^3) = 120 \times 1111 = 133320$.

35. b. $(x+3)^2 + y^2 = 13$

$\Rightarrow \quad x + 3 = \pm 2, y = \pm 3$ or $x + 3 = \pm 3, y = \pm 2$

36. d. Using the digits 0, 1, 2, …, 9 the number of five digit telephone numbers which can be formed is 10^5 (since repetition is allowed). The number of five digit telephone, numbers which have none of the digits repeated is ${}^{10}P_5 = 30240$. Therefore, the required number of telephone numbers is $10^5 - 30240 = 69760$.

37. b. 3 must be at thousand's place and since the number should be divisible by 5, or 5 must be at unit's place. Now, we have to fill two places (tens and hundreds), i.e., ${}^4P_2 = 12$.

38. d. Two positions for A_1 and A_{10} can be selected in ${}^{10}C_2$ ways. Rest 8 students can be ranked in 8! ways. Hence total number of ways is ${}^{10}C_2 \times 8! = (1/2)(10!)$.

39. a.

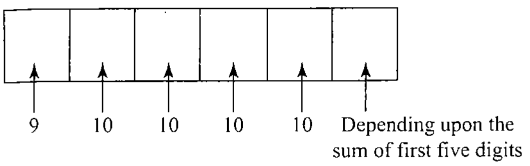


Fig. 5.41

First place from left cannot be filled with 0. Next four places can be filled with any of the 10 digits. After filling the first five places, the last place can be filled in following ways:

Sum of digits in first five places	Digit in the unit's place
$5k$	0 or 5
$5k+1$	4 or 9
$5k+2$	3 or 8
$5k+3$	2 or 7
$5k+4$	1 or 6

Thus, in any case the last place can be filled in two ways.

Hence, the required number of numbers is $9 \times 10^4 \times 2$.

40. a. The selection can be made in ${}^5C_3 \times {}^{22}C_9$ ways.

(Since 3 vacancies are filled from 5 candidates in ${}^5C_3$ ways and now remaining candidates are 22 and remaining seats are 9.)

41. d. Here,

${}^nP_3 - {}^nC_3 > 100$

$$\Rightarrow \quad \frac{n!}{(n-3)!} - \frac{n!}{3!(n-3)!} > 100$$

$$\Rightarrow \quad \frac{5}{6}n(n-1)(n-2) > 100$$

$$\Rightarrow \quad n(n-1)(n-2) > 120$$

$$\Rightarrow \quad n(n-1)(n-2) > 6 \times 5 \times 4$$

$$\Rightarrow \quad n = 7, 8, \ldots$$

42. c. Places for A, B, C can be chosen in ${}^{10}C_3$ ways. Remaining 7 persons can speak in 7! ways. Hence, the number of ways in which they can speak is ${}^{10}C_3 \times 7! = 10!/6$.

43. b. Since 5 players are always to be excluded and 6 players always to be included, therefore 5 players are to be chosen from 14. Hence required number of ways is ${}^{14}C_5 = 2002$.

44. c.

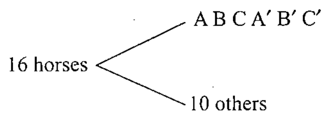


Fig. 5.42

The number of ways is ${}^{10}C_3 \times$ number of ways of choosing out of $ABC\,A'B'C'$, so that AA', BB' or CC' are not together

$= {}^{10}C_3$ (one from each of pairs AA', BB', CC')

$= {}^{10}C_3 \times 8$

$$= \frac{10 \times 9 \times 8}{1 \times 2 \times 3}\ 8 = 960$$

45. b. The number of ways of selecting r $(0 \le r \le m)$ balls out of m is ${}^mC_r$. Therefore, the number of ways if selecting r balls from each of the bag is $({}^mC_r)^2$. Further the number of ways of selecting equal number of balls from each of the two bags, choosing at least one from each bag, is

$({}^mC_1)^2 + ({}^mC_2)^2 + \cdots + ({}^mC_m)^2 = {}^{2m}C_m - 1$

$[\because ({}^mC_0)^2 + ({}^mC_1)^2 + \cdots + ({}^mC_m)^2 = {}^{2m}C_m]$

46. b. There are 6 different letters. We have to select 6 squares, taking at least one from each row and then arranging in each selection. Let us first select places in each row such that no row remains empty.

R_1	R_2	R_3	Number of selections
1	1	4	${}^2C_1\,{}^2C_1\,{}^4C_4 = 4$
1	2	3	${}^2C_1\,{}^2C_2\,{}^4C_3 = 8$
2	1	3	${}^2C_2\,{}^2C_1\,{}^4C_3 = 8$
2	2	2	${}^2C_2\,{}^2C_2\,{}^4C_2 = 6$

Therefore, the total number of selections of 6 squares is $4+8+8+6=26$. For each selection of 6 squares, the number of arrangements of 6 letters is $6! = 720$. Hence, the required number of ways is $26 \times 720 = 18720$.

47. b. There are 11 letters A, A; I, I; N, N; E, X, M, T, O. For the selection of 4 letters we have the following possibilities:

(A) 2 alike, 2 alike

(B) 2 alike, 2 different

(C) All four different

(A) There are 3 pairs of 2 letters. So, the number of ways of selection of 2 pairs is ${}^3C_2$ and permutation of these 4 letters is $4!\ /\ 2!\ 2!$ Therefore, the number of words in this case is ${}^3C_2 \times 4!/2!\,2! = 18$.

(B) We have to select one pair from 3 pairs and 2 distinct letters from remaining 7 distinct letters. For illustration, let us select both A, A; then we have I, N, E, X, M, T, O, i.e., 7 as remaining distinct letters. Hence, the number of selections is ${}^3C_1 \times {}^7C_2$ and these 4 (2 same, 2 distinct) can be permuted in 4!/2! ways. Therefore, number of words is ${}^3C_1 \times {}^7C_2 \times 4!\ /\ 2!$ $= 3 \times 21 \times 12 = 756$.

(C) There are 8 distinct letters so number of words of 4 letters is ${}^8C_4 \times 4! = 1680$. By sum rule, the total number of words is $18 + 756 + 1680 = 2454$.

48. d. The total number of words is $6! = 720$. Let us write the letters of word ZENITH alphabetically, i.e., EHINTZ.

For ZENITH word start with	Word starting with	Number of words
Z	E	5!
	H	5!
	I	5!
	N	5!
	T	5!
ZEN	ZEH	3!
	ZEI	3!
ZENI	ZENH	2!
ZENIT	ZENIH	1
	Total number of words before ZENITH	615

Hence, there are 615 words before ZENITH, so the rank of ZENITH is 616.

49. b.

Number of girls	Number of boys	Number of groups going to picnic	Total number of dolls
1	4	${}^3C_1\,{}^4C_4$	$1({}^3C_1\,{}^4C_4) = 3$
2	3	${}^3C_2\,{}^4C_3$	$2({}^3C_2\,{}^4C_3) = 24$
3	2	${}^3C_3\,{}^4C_2$	$3({}^3C_3\,{}^4C_2) = 18$
		Total	45

50. b. The number of times the teacher goes to the zoo is ${}^nC_3$. The number of times a particular child goes to the zoo is equal to number of ways two other children can be selected who accompany a particular child, i.e., ${}^{n-1}C_2$. From the question,

$${}^nC_3 - {}^{n-1}C_2 = 84$$

or

$$(n-1)(n-2)(n-3) = 6 \times 84 = 9 \times 8 \times 7 \Rightarrow n - 1 = 9$$

51. b. We first select 2 men out of 7 in ${}^7C_2$ ways. Now we exclude the wives of these two selected men and so select 2 ladies from remaining 5 ladies in ${}^5C_2$ ways. Let A, B be two men and X, Y be the ladies playing in one set. Then we can have

(i) A and X plying against B and Y.

(ii) A and Y playing against B and X.

Then the total number of ways is ${}^7C_2 \times {}^5C_2 \times 2 = 21 \times 10 \times 2$ $= 420$.

52. b. Suppose there 'n' players in the beginning. The total number of games to be played was equal to ${}^nC_2$ and each player would have played $n-1$ games.

Let us assume that A and B fell ill. Now the total number of games of A and B is $(n-1)+(n-1)-1 = 2n-3$. But they have played 3 games each. Then their remaining number of games is $2n - 3 - 6 = 2n - 9$. Given,

$${}^nC_2 - (2n - 9) = 84$$

$$\Rightarrow n^2 - 5n - 150 = 0$$

$$\Rightarrow n = 15$$

Alternative solution:

The number of games excluding A and B is ${}^{n-2}C_2$. But before leaving A and B played 3 games each. Then,

$${}^{n-2}C_2 + 6 = 84$$

Solving this equation, we get $n = 15$.

53. a. The number of ways he can select at least one parantha is $2^3 - 1 = 7$. The number of ways he can select at least one vegetable dish is $2^4 - 1 = 15$. The number of ways he can select zero or more items from salads and sauces is 2^5.

Hence, the total number of ways is $7 \times 15 \times 32 = 3360$.

54. b. Number of even divisors is equal to number of ways in which one or more '2', zero or more '3', zero or more '5' and zero or more '7' can be selected, and is given by $(3)(2+1)(2+1)(1+1)$ $= 54$.

55. c. The number of ways the candidate can choose questions under the given conditions is enumerated below.

Group 1	Group 2	Number of ways
4	2	$({}^5C_4)({}^5C_2) = 50$
3	3	$({}^5C_3)({}^5C_3) = 100$
2	4	$({}^5C_2)({}^5C_4) = 50$
	Total number of ways	200

56. c. Let there be n men participants. Then the number of games that the men play between themselves is $2 \times {}^nC_2$ and the number of games that the men played with the women is $2 \times (2n)$.

$\therefore$ $2 \times {}^nC_2 - 2 \times 2n = 66$ (by hypothesis)

$\Rightarrow$ $n^2 - 5n - 66 = 0 \Rightarrow n = 11$

Hence, the number of participants is 11 men + 2 women = 13.

57. b. The smallest number of people = total number of possible forecasts

= total number of possible results

$= 3 \times 3 \times 3 \times 3 \times 3$

58. a. Three IIT students who will be between the IIT students can be selected in ${}^{10}C_3$ ways. Now, two DCE students having three IIT students between them can be arranged in 2!×3! ways. Finally, a group of above five students and the remaining seven students together can be arranged in 8! ways. Hence, total number of ways is ${}^{10}C_3 \times 2! \times 3! \times 8!$

59. c. Let S_1 and S_2 refuse to be together and S_3 and S_4 want to be together only. The total number of ways when S_3 and S_4 are selected is$({}^8C_2 + {}^2C_1 \times {}^8C_1) = 44$. The total ways when S_3 and S_4 are not selected is $({}^8C_4 + {}^2C_1 \times {}^8C_3) = 182$. Thus, the total number of ways is 44 + 182 = 226.

60. c. Let there be n candidates. Then,

$${}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} = 254$$

$\Rightarrow \quad 2^n - 2 = 254$

$\Rightarrow \quad 2^n = 2^8 \Rightarrow n = 8$

61. b. 'P_1' must win at least $n + 1$ games. Let 'P_1' win $n + r$ games ($r = 1, 2, \ldots, n$). Therefore, corresponding number of ways is ${}^{2n}C_{n+r}$. The total number of ways is

$$\sum_{r=1}^{n} {}^{2n}C_{n+r} = {}^{2n}C_{n+1} + {}^{2n}C_{n+2} + \cdots + {}^{2n}C_{2n}$$

$$= \frac{2^{2n}}{2} - {}^{2n}C_n$$

$$= \frac{1}{2}(2^{2n} - 2 \times {}^{2n}C_n)$$

62. b. The candidate is unsuccessful if he fails in 9 or 8 or 7 or 6 or 5 papers. Therefore, the number of ways to be unsuccessful is

$${}^9C_9 + {}^9C_8 + {}^9C_7 + {}^9C_6 + {}^9C_5 = {}^9C_0 + {}^9C_1 + {}^9C_2 + {}^9C_3 + {}^9C_4$$

(recall the concept of half series)

$$= \frac{1}{2}({}^9C_0 + {}^9C_1 + \cdots + {}^9C_9)$$

$$= \frac{1}{2} \times 2^9 = 2^8$$

63. b. Since the student is allowed to select at most n books out of $(2n + 1)$ books, therefore in order to select one book he has the choice to select one, two, three, …, n books. Thus, if T is the total number of ways of selecting one book, then

$$T = {}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n = 63 \qquad \text{(i)}$$

Again the sum of binomial coefficients is

$${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \cdots + {}^{2n+1}C_{2n+1} = (1+1)^{2n+1} = 2^{2n+1}$$

or

$${}^{2n+1}C_0 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n) + {}^{2n+1}C_{2n+1} = 2^{2n+1}$$

$\Rightarrow \quad 1 + 2(T) + 1 = 2^{2n+1}$

$\Rightarrow \quad 1 + T = \dfrac{2^{2n+1}}{2} = 2^{2n}$

$\Rightarrow \quad 1 + 63 = 2^{2n}$

$\Rightarrow \quad 2^6 = 2^{2n} \Rightarrow n = 3$

64. c. The number of ways can be given as follows:

2 bowlers and 9 other players: ${}^4C_2 \times {}^9C_9$

3 bowlers and 8 other players: ${}^4C_3 \times {}^9C_8$

4 bowlers and 7 other players: ${}^4C_4 \times {}^9C_7$

Hence, required number of ways is $6 \times 1 + 4 \times 9 + 1 \times 36 = 78$.

65. a. Matches whose predictions are correct can be selected in ${}^{20}C_{10}$ ways. Now each wrong prediction can be made in 2 ways. Thus, the total number of ways is ${}^{20}C_{10} \times 2^{10}$.

66. b.

Dashes	Dots	Arrangements
5	2	${}^7C_2$
4	3	${}^7C_3$
3	4	${}^7C_4$
2	5	${}^7C_5$
1	6	${}^7C_6$
0	7	${}^7C_7$

The total number of ways is ${}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5 + {}^7C_6 = 2^7 - 8 = 120$.

67. c. For a radical centre, 3 circles are required. The total number of radical centres is ${}^nC_3$. The total number of radical axis is ${}^nC_2$. Now,

$${}^nC_2 = {}^nC_3 \Rightarrow n = 5$$

68. c.

(i) Miss C is taken

(A) B included ⇒ A excluded ⇒ ${}^4C_1 \times {}^4C_2 = 24$

(B) B excluded ⇒ ${}^4C_1 \times {}^5C_3 = 40$

(ii) Miss C is not taken

⇒ B does not come; ⇒ ${}^4C_2 \times {}^5C_3 = 60$

⇒ Total = 124

Alternative method:

Case I:

Mr. 'B' is present

⇒ 'A' is excluded and 'C' included

Hence, the number of ways is ${}^4C_2\,{}^4C_1 = 24$.

Case II:

Mr. 'B' is absent

⇒ No constraint

Hence, the number of ways is ${}^5C_3\,{}^5C_2 = 100$.

∴ Total = 124.

69. d. $N = 1! + 2! + \cdots + 2005!$

$= (1! + 2! + 3! + 4!) + (5! + \cdots + 2005!)$

= 33 + an integer having 0 in its unit's place

= an integer having 3 in its unit's place

Hence, N^{500} is an integer having 1 in its unit's place.

70. d.

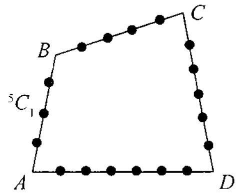


Fig. 5.43

The number of triangles with vertices on sides AB, BC, CD is ${}^3C_1 \times {}^4C_1 \times {}^5C_1$.

Similarly, for other cases, the total number of triangles is

$${}^3C_1 \times {}^4C_1 \times {}^5C_1 + {}^3C_1 \times {}^4C_1 \times {}^6C_1 + {}^3C_1 \times {}^5C_1 \times {}^6C_1 + {}^4C_1 \times {}^5C_1 \times {}^6C_1 = 342.$$

71. **c.** Number of points required for the fixed circle is 3. So, first select any three points from the 10 points in ${}^{10}C_3$ ways.

In these ways, circle with four concyclic points is selected in ${}^4C_3$ ways. But it should be taken once then total number of circles is $({}^{10}C_3 - {}^4C_3) + 1$.

72. **a.** The number of points of intersection is equal to the number of ways two lines are selected, which is given by

$${}^nC_2 = \frac{n(n-1)}{2} = \sum_{k=1}^{n-1} K$$

73. **c.** Out of 10 points let n points are collinear. Then the number of triangles is

${}^{10}C_3 - {}^nC_3 = 110$

$\Rightarrow \frac{10\times9\times8}{6} - \frac{n(n-1)(n-2)}{6} = 110$

$\Rightarrow n(n-1)(n-2) = 60$

$\Rightarrow n = 5$

74. **c.** Select any three points from total $3p$ points, which can be done ${}^{3p}C_3$ ways. But this also includes selection of three collinear points. Now three collinear points from each straight line can be selected in ${}^pC_3$ ways. Then the number of triangles is ${}^{3p}C_3 - 3{}^pC_3 = p^2(4p-3)$.

75. **c.** Two circles intersect at two distinct points.Two straight lines intersect at one point. One circle and one straight line intersect at two distinct points. Then the total numbers of points of intersections are as follows:

Number of ways of selection	Points of intersection
Two straight lines: ${}^5C_2$	${}^5C_2 \times 1 = 10$
Two circles: ${}^4C_2$	${}^4C_2 \times 2 = 12$
One line and one circle: ${}^5C_1 \times {}^4C_1$	${}^5C_1 \times {}^4C_1 \times 2 = 40$
Total	62

76. **a.** The number of selection of two parallel lines from m lines is ${}^mC_2$.

The number of selection of two parallel lines from n lines is ${}^nC_2$.

Hence, the number of parallelograms lines is

$${}^mC_2 \times {}^nC_2 = \frac{1}{4} mn(m-1)(n-1)$$

77. **b.** Let $x = p - 5$, $y = q - 5$ and $z = r - 5$, where $p, q, r \geq 0$.

Then the given equation reduces to

$$p + q + r = 15 \qquad (1)$$

Now, we have to find non-negative integral solution of Eq. (1). The total number of such solutions is ${}^{15+3-1}C_{3-1} = {}^{17}C_2 = 136$.

78. **b.** Dice is marked with numbers 1, 2, 3, 4, 5, 6. If the sum of dice in three throws is 11, then observations must be 1, 4, 6; … 1, 5, 5; … 2, 3, 6; … 2, 4, 5; … 3, 3, 5; … 3, 4, 4.

We can get this observation in 3! + 3!/2! + 3! + 3! + 3!/2! +3!/2! = 27 ways.

79. **d.** No group of four numbers from the first 12 natural numbers can have the common difference 4.

If one group including 1 is selected with the common difference 1, then the other two group can have the common difference 1 or 2.

If one group including 1 is selected with the common difference 2, then one of the other two groups can have the common difference 2 and the remaining group will have common difference 1.

If one group including 1 is selected with the common difference 3, then the other two groups can have the common difference 3.

Therefore, the required number of ways is 2 + 1 + 1 = 4.

80. **a.** Let the balls put in the box are x_1, x_2, x_3, x_4 and x_5. We have,

$x_1 + x_2 + x_3 + x_4 + x_5 = 15, x_i \geq 2$

$\Rightarrow (x_1 - 2) + (x_2 - 2) + (x_3 - 2) + (x_4 - 2) + (x_5 - 2) = 5$

$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 = 5, y_i = x_i - 2 \geq 0$

The total number of ways is equal to number of non-negative integral solutions of the last equation, which is equal to ${}^{5+5-1}C_5 = {}^9C_5$.

81. **a.** Let x be the number of objects to the left of the first object chosen, y the number of objects between the first and the second, z the number of objects between the second and the third and u the number of objects to the right of the third objects. Then, $x, u \geq 0; y, z \geq 1$ and $x + y + z + u = n - 3$. Let $y_1 = y - 1$ and $z_1 = z - 1$. Then, $y_1 \geq 0, z_1 \geq 0$ such that $x + y_1 + z_1 + u = n - 5$.

The total number of non-negative integral solutions of this equation is ${}^{n-5+4-1}C_{4-1} = {}^{n-2}C_3$.

82. **a.** Obviously, A, B and C get 4, 5 and 7 objects, respectively. Then, number of distribution ways is equal to number of division of ways, which is given by 16!/(4!5!7!).

83. **b.** The number of ways is

$$\frac{(mn)!}{(n!)^m m!} m! = \frac{(mn)!}{(n!)^m}$$

84. **a.** $m + n$ counters on one side can be arranged in $\frac{(m+n)!}{m!n!}$ ways.

For each arrangement on one side, corresponding arrangement on the other side is just one as arrangements are symmetrical. Hence, the total number of arrangements is

$$\frac{(m+n)!}{m!n!} = {}^{m+n}C_m$$

85. **a.** Let x_1, x_2, x_3, x_4 be the number of times T, I, D, E appears on the coupon. Then we must have $x_1 + x_2 + x_3 + x_4 = 8$, where $1 \leq x_1, x_2, x_3, x_4 \leq 8$ (as each letter must appear once).Then the required number of combinations of coupons is equivalent to number of positive integral solutions of the above equation, which is further equivalent to number of ways of 8 identical objects distributed among 4 persons when each gets at least one objects, and is given by ${}^{8-1}C_{4-1} = {}^7C_3$.

86. **a.** Here, we have to divide 12 books into 4 sets of 3 books each. Therefore, the required number of ways is

$$\frac{12!}{(3!)^4 4!} \times 4!$$

87. **a.** Since the shelves which are to receive the books are different, therefore the required number of ways is 12!/(4!)^3.

88. **c.** Here, we are dividing $2n$ people in n groups of 2 each, and we are concerned with mere grouping. Hence, the required number of ways is $\frac{2n!}{n!(2!)^n}$.

89. b. $\sum_{i=1}^{k} \frac{1}{x_i} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} = \frac{\Sigma x_i}{n} = \frac{75}{n}$

(as L.C.M. of $x_1, x_2, \ldots, x_n$ is n)

90. a. Three elements from set 'A' can be selected in ${}^7C_3$ ways. Their image has to be y_2. Remaining 2 images can be assigned to remaining 4 pre-images in 2^4 ways. But the function is onto, hence the number of ways is $2^4 - 2$. Then the total number of functions is ${}^7C_3 \times 14 = 490$.

91. c. If we put minimum number of balls required in each box, balls left are $n(n-1)/2$ which can be put in ${}^{(n^2+n-1)/2}C_{n-1}$ ways without restriction.

92. a. Let the blankets received by the persons are x_1, x_2, x_3 and x_4. We have,

$x_1 + x_2 + x_3 + x_4 = 15$ and $x_i \geq 2$

$\Rightarrow$ $(x_1 - 2) + (x_2 - 2) + (x_3 - 2) + (x_4 - 2) = 7$

$\Rightarrow$ $y_1 + y_2 + y_3 + y_4 = 7$ (where $y_i = x_i - 2 \geq 0$)

The required number is equal to the number of non-negative integral solutions of this equation which is equal to ${}^{4+7-1}C_7$, i.e., ${}^{10}C_7 = {}^{10}C_3$.

93. c. Let person P_i gets x_i number of things such that

$x_1 + x_2 + x_3 + x_4 + x_5 = 25$

Let $x_i = 2\lambda_i + 1$, where $\lambda_i \geq 0$. Then,

$2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) + 5 = 25$

$\Rightarrow$ $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 10$

We have to simply obtain the number of non-negative integral solutions of this equation, which is equal to ${}^{14}C_4$.

94. d. Suppose i^{th} person receives Rs x_i; $i = 1, 2, 3, 4$

Then, $x_1 + x_2 + x_3 + x_4 = 18$, where $x_i \geq 4$

Let $y_i = x_i - 3$, $i = 1, 2, 3, 4$. Then,

$y_1 + y_2 + y_3 + y_4 = 6$

The total number of ways is equal to number of solutions of the above equation, which is given by ${}^{6+4-1}C_{4-1} = {}^9C_3 = 84$.

95. d. Make 1 group of 2 persons, 1 group of 4 persons and 3 groups of 3 persons among 15 persons (except 2 particular persons). Hence the number of ways by grouping method is

$$\frac{15!}{2!4!(3!)^3 3!}$$

Now we add that 2 persons in the group of 2 persons and thus number of arrangements of these groups into cars and autos is

$$\frac{15!}{2!4!(3!)^3 3!} \times 2! \times 3! = \frac{15!}{4!(3!)^3}$$

96. a. Since the total number of selections of r things from n things where each thing can be repeated as many times as one can is ${}^{n+r-1}C_r$. Therefore the required number is ${}^{3+6-1}C_6 = 28$.

97. c. $f(2n, n)$ must be equal to number of positive integer solutions of $x_1 + x_2 + \cdots + x_n = 2n$, which must be equal to ${}^{2n-1}C_{n-1} = {}^{2n-1}C_n$.

98. b. Let the numbers selected be x_1, x_2, x_3. We must have

$2x_2 = x_1 + x_3$

$\Rightarrow$ $x_1 + x_3 =$ even.

Therefore, x_1, x_3 both are odd or both are even.

If x_1 and x_3 both are even, we can select them in ${}^{12}C_2$ ways.

Similarly, if x_1 and x_3 both are odd, we can again select them in ${}^{12}C_2$ ways. Thus, the total number of ways is $2 \times {}^{12}C_2 = 132$.

99. b. Given number can be rearranged as

$1, 4, 7, \ldots, 3n - 2 \rightarrow 3\lambda - 2$

$2, 5, 8, \ldots, 3n - 1 \rightarrow 3\lambda - 1$

$3, 6, 9, \ldots, 3n \rightarrow 3\lambda$

That means, we must take two numbers from last row or one number each from first and second rows. Therefore, the total number of ways is

$${}^nC_2 + {}^nC_1 \times {}^nC_1 = \frac{n(n-1)}{2} + n^2 = \frac{3n^2 - n}{2}$$

100. b. Let the arrangement be $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8$ clearly 5 should occupy the position x_4 or x_5. Thus required number is 2(7!).

101. c. $480 = 2^5 \times 3 \times 5$

Now, $4n + 2 = 2(2n + 1) =$ odd multiple of 2. Thus, the total number of such divisors is $1 \times 2 \times 2 = 4$.

102. a.

Number of times 3 used	Pattern	Numbers of type	Number of times 3 appears
1	– – 3 – 3 – 3 – –	$3 \times 9 \times 9$	$1(3 \times 9 \times 9)$
2	– 3 3 3 3 – 3 – 3	3×9	$2(3 \times 9)$
3	333	1	3
		Total	300

Any place other than 3 is filled by 9 ways as '0' can appear anywhere which gives all types of numbers like single digit, two digits, etc.

Alternative solution:

A three-digit block from 000 to 999 means 1000 numbers, each number constituting 3 digits. Hence, the total numbers of digits which we have to write is 3000.

Since the total number of digits is 10 (0 to 9) no digit is filled preferentially. Therefore, number of times we write 3 is 3000/10 = 300.

103. b.

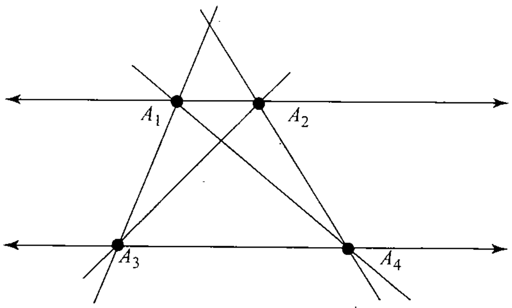


Fig. 5.44

For intersection point we must have two straight lines, for which 2 points from each straight line must be selected. Now selection of these points can be done in ${}^mC_2 \times {}^nC_2$ ways. Now as shown in diagram these four points can give two different sets of straight lines, which generate two distinct points of intersection.

Then total number of points of intersection is ${}^mC_2 \times {}^nC_2 \times 2$.

104. a.

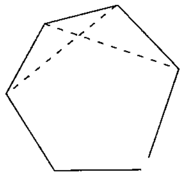


Fig. 5.45

Let the number of sides be n. A selection of four vertices of the polygon gives an interior intersection.

$\therefore$ $^nC_4 = 70$

$\Rightarrow$ $n(n-1)(n-2)(n-3) = 24 \times 70 = 8 \times 7 \times 6 \times 5$

$\Rightarrow$ $n = 8$

105. a. 26 cards can be chosen out of 52 cards in $^{52}C_{26}$ ways. There are two ways in which each card can be dealt, because a card can be either from the first pack or from the second. Hence the total number of ways is $^{52}C_{26} \times 2^{26}$.

106. d. Since the balls are to be arranged in a row so that the adjacent balls are of different colours, we can therefore begin with a white ball or a black ball. If we begin with a white ball, we find that $n+1$ white balls numbered 1 to $n+1$ can be arranged in a row in $(n+1)!$ ways. Now $n+2$ places are created among $n+1$ white balls which can be filled by $n+1$ black balls in $(n+1)!$ ways.

So, the total number of arrangements in which adjacent balls are of different colours and first ball is a white ball is $(n+1)! \times (n+1)! = [(n+1)!]^2$. But we can begin with a black ball also. Hence, the required number of arrangements is $2[(n+1)!]^2$.

107. a. If zero is included it will be at $z \Rightarrow {}^9C_2$

$$\text{If zero is excluded } \begin{cases} x, y, z \text{ all diff.} & \Rightarrow\ {}^9C_3 \times 2! \\ x = z < y & \Rightarrow\ {}^9C_2 \\ x < y = z & \Rightarrow\ {}^9C_2 \end{cases}$$

The total number of ways is 276.

Alternative method:

y can be from 2 to 9; so total number of ways is

$$\sum_{r=2}^{9}(r^2-1) = 276$$

108. b. The two common elements can be selected in $^nC_2$ ways. Remaining $n-2$ elements, each can be chosen in three ways, i.e., $a \in P$ and $a \notin Q$ or $a \in Q$ and $a \notin P$ or a is neither in P nor in Q. Therefore, the total number of ways is $^nC_2 \times 3^{n-2}$.

109. d. We will consider the following cases

Case	Flags	No. of signals
4 alike and 2 others alike	4 white and 2 red	$\frac{6!}{4!\,2!} = 15$
4 alike and 2 others different	4 white, 1 red and 1 blue	$\frac{6!}{4!} = 30$
3 alike and 3 others alike	3 white, 3 red	$\frac{6!}{3!\,3!} = 20$
3 alike and 2 other alike and 1 different	3 white, 1 blue, 2 red or 3 red, 1 blue, 2 white	${}^2C_1 \times \frac{6!}{3!2!} = 120$
	Total	185

110. c. The total number of ways of selection without restriction is $^{20}C_3$. The number of ways of selection when two are adjacent is $20 \times {}^{16}C_1$. The number of ways of selection when all the three are adjacent is 20. The required number of ways is

$$^{20}C_3 - 20 \times 16 - 20 = \frac{20 \times 19 \times 18}{6} - 20 \times 16 - 20$$

$$= 20[57 - 16 - 1]$$

$$= 20 \times 40 = 800$$

111. c. Sum of 7 digits is a multiple of 9. Sum of numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 is 45; so two left digits should also have sum of 9. The pairs of left numbers are (1, 8), (2, 7), (3, 6), (4, 5). With each pair left number of 7-digit number is 7! So with all 4 pairs, total number is $4 \times 7!$.

112. b. If n is odd,

$3^n = 4\lambda_1 - 1,\ 5^n = 4\lambda_2 + 1$

$\Rightarrow$ $2^n + 3^n + 5^n$ is divisible by 4 if $n \geq 2$

Thus, $n = 3, 5, 7, 9, \ldots, 99$, i.e., n can take 49 different values. If n is even, $3^n = 4\lambda_1 + 1,\ 5^n = 4\lambda_2 + 1$

$\Rightarrow$ $2^n + 3^n + 5^n$ is not divisible by 4

as $2^n + 3^n + 5^n$ will be in the form of $4\lambda + 2$.

Thus, the total number of ways of selecting 'n' is equal to 49.

113. c.

Number of digits	Numbers ending with 0	Numbers ending with 5	Total
×	0	1	1
××	8	9	17
×××	$9 \cdot 8 = 72$	$8 \cdot 8 = 64$	136
××××	$9 \cdot 8 \cdot 7 = 504$	$8 \cdot 8 \cdot 7 = 448$	952
		Total	1106

114. d. Let x, y, z be the friends and a, b, c denote the case when x is invited a times, y is invited b times and z is invited c times. Now, we have the following possibilities:

$(a, b, c) = (1, 2, 3)$ or $(3\ 3\ 0)$ or $(2\ 2\ 2)$

[grouping of 6 days of week]

Hence, the total number of ways is

$$\frac{6!}{1!2!3!}3! + \frac{6!}{3!3!2!}3! + \frac{6!}{(2!2!2!)3!}3!$$

$$= 360 + 60 + 90 = 510$$

115. b. There is concept of derangement. The required number is

$$4!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = 9$$

116. c. When at least one one-rupee coin is selected we can select any number of twenty five coins and ten paisa coins. Then number of ways of such selection is $4(2+1)(5+1) = 72$ as we can select zero or more twenty-five paisa and ten paisa coins to ensure that amount selected is Re. 1 or more.

But when none of one-rupee coins is selected we have to select all twenty-five paisa coins and ten paisa coins to ensure sum of Re. 1, which can be done only in one way. Then the total number of ways is 73.

117. b. Since the number of students giving wrong answers to at least i questions ($i = 1, 2, \ldots, n$) is 2^{n-i}.

The number of students answering exactly i $(1 \le i \le -1)$ questions wrongly = {the number of students answering at least i questions wrongly, $i = 1, 2, \ldots$)} − {the number of students answering at least $(i + 1)$ questions wrongly $(2 \le i + 1 \le n)$} $= 2^{n-i} - 2^{n-(i+1)}$ $(1 \le i \le n - 1)$.

Now, the number of students answering all the n questions wrongly is $2^{n-2} = 2^0$. Thus, the total number of wrong answers is

$$1(2^{n-1} - 2^{n-2}) + 2(2^{n-2} - 2^{n-3}) + 3(2^{n-3} - 2^{n-4}) + \cdots + (n-1)(2^1 - 2^0) + n(2^0)$$

$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2^0 = 2^n - 1 \qquad (\because \text{ it is a G.P.})$$

Therefore, as given,

$$2^n - 1 = 2047 \Rightarrow 2^n = 2048 = 2^{11} \Rightarrow n = 11$$

118. c. For a particular class the total number of different tickets from first intermediate station is 5. Similarly, number of different tickets from second intermediate station is 4. So the total number of different tickets is $5 + 4 + 3 + 2 + 1 = 15$. And same number of tickets for another class is equal to total number of different tickets, which is equal to 30 and number of selection is ${}^{30}C_{10}$.

119. b. The number of trains a day (the digits 1, 2, 3) are three groups of like elements from which a sample must be formed. In the time-table for a week, the number 1 is repeated twice, the number 2 is repeated 3 times and the number 3 is repeated twice.

The number of different time-tables is given by

$$p(2, 3, 2) = \frac{7!}{2!3!2!} = 210$$

120. a. $15 < x_1 + x_2 + x_3 \le 20$

$\Rightarrow$ $x_1 + x_2 + x_3 = 16 + r$, $r = 0, 1, 2, 3, 4$

Now, the number of positive integral solution of $x_1 + x_2 + x_3 = 16 + r$ is ${}^{13+r+3-1}C_{13+r}$, i.e., ${}^{15+r}C_{13+r} = {}^{15+r}C_2$

Thus, the total number of solutions is

$$\sum_{r=0}^{4} {}^{15+r}C_2 = {}^{15}C_2 + {}^{16}C_2 + {}^{17}C_2 + {}^{18}C_2 + {}^{19}C_2$$

$$= \frac{1}{2}(15\times14 + 16\times15 + 17\times16 + 18\times17 + 19\times18)$$

$$= 685$$

Multiple Correct Answers Type

1. b, c.

If a, b, c are in A.P., then a and c both are odd or both are even.

Case I: n is even.

The number of ways of selection of two even numbers a and c is ${}^{n/2}C_2$. Number of ways of selection of two odd numbers is ${}^{n/2}C_2$. Hence the number of A.P.'s is

$$2\,{}^{n/2}C_2 = 2\,\frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{2} = \frac{n(n-2)}{4}$$

Case II: n is odd.

The number of ways of selection of two odd numbers a and c is ${}^{(n+1)/2}C_2$. The number of ways of selection of two even numbers a and c is ${}^{(n-1)/2}C_2$. Hence the number of A.P.'s is ${}^{(n+1)/2}C_2 + {}^{(n-1)/2}C_2$

$$= \frac{\left(\frac{n+1}{2}\right)\left(\frac{n+1}{2}-1\right)}{2} + \frac{\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}-1\right)}{2}$$

$$= \frac{1}{8}(n-1)\big((n+1)+(n-3)\big)$$

$$= \frac{(n-1)^2}{4}$$

2. b, c.

The number of ways of inviting, with the couple not included, is ${}^8C_5$. The number of ways of inviting with the couple included, is ${}^8C_3$. Therefore the required number of ways is

$${}^8C_5 + {}^8C_3 = {}^8C_3 + {}^8C_3 \quad (\because {}^8C_5 = {}^8C_3)$$

Also,

$${}^{10}C_5 - 2 \times {}^8C_4 = \frac{10!}{5!5!} - 2\times\frac{8!}{4!4!}$$

$$= \frac{10\times9\times8\times7\times6}{120} - 2\times\frac{8\times7\times6\times5}{24}$$

$$= 9 \times 4 \times 7 - 140$$

$$= 112 = 2 \times \frac{8!}{3!5!}$$

3. a, b, d.

$p = {}^5C_4 \times {}^2C_1 = 10$

$q = {}^5C_2({}^2C_1)^3 = 80$

$r = {}^5C_0({}^2C_1)^5 = 32$

$\Rightarrow$ $2q = 5r$, $8p = q$ and $2(p + r) > q$

4. a, b, d.

Clearly, each player will play 9 games. And total number of games is ${}^{10}C_2 = 45$. Clearly,

$w_i + l_i = 9$ and $\Sigma w_i = \Sigma l_i = 45$

$\Rightarrow$ $w_i = 9 - l_i \Rightarrow w_i^2 = 81 + l_i^2 - 18l_i$

$\Rightarrow$ $\Sigma w_i^2 = 81 \times 10 + \Sigma l_i^2 - 180\,\Sigma l_i = 810 + \Sigma l_i^2 - 18 \times 45 = \Sigma l_i^2$

5. b, c, d.

When $z = n + 1$, we can choose x, y from $\{1, 2, \ldots, n\}$.

When $z = n + 1$, x, y can be chosen in n^2 ways and when $z = n$, x, y can be chosen in $(n - 1)^2$ ways and so on. Therefore, the number of ways of choosing triplets is

$$n^2 + (n-1)^2 + \cdots + 1^2 = \frac{1}{6}n(n+1)(2n+1)$$

Alternatively triplets with $x = y < z$, $x < y < z$, $y < z < x$ can be chosen in ${}^{n-1}C_2$, ${}^{n+1}C_3$, ${}^{n+1}C_3$ ways. Therefore,

$${}^{n+1}C_2 + 2({}^{n+1}C_3) = {}^{n+2}C_2 + {}^{n+1}C_3 = 2({}^{n+2}C_3) - {}^{n+1}C_2$$

6. a, b, c.

$$\frac{(200)!}{\underbrace{2!\,2!\ldots2!}_{100\text{ times}}(100)!}$$

$$= \frac{(200)!}{100!\,2^{100}}$$

$$= 1 \times 3 \times 5 \cdots 199$$

Also,

$$\frac{(200)!}{100!\,2^{100}} = \binom{101}{2}\binom{102}{2}\cdots\binom{200}{2}$$

7. a, b, c, d.

8, 7, 6, 4, 2, x and y

Any number is divisible by 3 if sum of digits is divisible by 3, i.e., $x + y + 27$ is divisible by 3. x and y can take values from 0, 1, 3, 5, 9. Possible pairs are (5, 1) (3, 0) (9, 0) (9, 3) and (1, 5), (0, 3) (0, 9) (3, 9).

8. a, d.

Problem is same as dividing 17 identical things in two groups.

$$\therefore \quad n = \frac{17+1}{2} = 9$$

There is no effect if two diamonds are different as necklace can be flipped over. Hence, $n = m = 9$.

9. a, b, c.

The number of regions for 'n' circles be $f(n)$. Clearly, $f(1) = 2$. Now,

$f(n) = f(n-1) + 2(n-1),\ \forall\ n \geq 2$

$\Rightarrow\ f(n) - f(n-1) = 2(n-1)$

Putting $n = 2, 3, \ldots, n$, we get

$f(n) - f(1) = 2(1 + 2 + 3 + \cdots n - 1) = (n-1)n$

$\Rightarrow\ f(n) = n(n-1) + 2 = (n^2 - n + 2)$ (which is always even)

$\Rightarrow\ f(20) = 20^2 - 20 + 2 = 382$

Also,

$n^2 - n + 2 = 92$

$\Rightarrow\ n^2 - n - 90 = 0 \Rightarrow n = 10$

10. a, c.

$3^p = (4-1)^p = 4\lambda_1 + (-1)^p$

$5^q = (4+1)^q = 4\lambda_2 + 1$

$7^r = (8-1)^r = 4\lambda_3 + (-1)^r$

Hence, any positive integer power of 5 will be in the form of $4\lambda_2 + 1$. Even power of 3 and 7 will be in the form of $4\lambda + 1$ and odd power of 3 and 7 will be in the form of $4\lambda - 1$. Hence, both p and r must be odd or both must be even. Thus $p + r$ is always even. Also, $p + q + r$ can be odd or even.

11. a, b, c.

When $n = 3k$, there are exactly $n/3$ integers of each type $3p$, $3p + 1$, $3p + 2$.

Now, sum of three selected integers is divisible by 3. Then either all the integers of the same type $3p$, $3p + 1$ or $3p + 2$ or one-one integer from each type. Then number of selection ways is ${}^{n/3}C_3 + {}^{n/3}C_3 + {}^{n/3}C_3 + ({}^{n/3}C_1)\,({}^{n/3}C_1)\,({}^{n/3}C_1) = 3({}^{n/3}C_3) + (n/3)^3$.

If $n = 3k + 1$, then there are $(n-1)/3$ integers of the type $3p$, $3p + 2$ and $(n+2)/3$ integers of the type $3p + 1$. Then number of selection ways is $2({}^{(n-1)/3}C_3) + ({}^{(n+2)/3}C_3) + ((n-1)/3)^2(n + 2)$.

When $n = 3k + 2$, the number of selection ways are same as in the case of $n = 3k + 1$.

12. a, c.

$${}^{n+5}P_{n+1} = \frac{11(n-1)}{2} \times {}^{n+3}P_n$$

$$\Rightarrow\ {}^{n+5}P_{n+1} = \frac{(n+5)!}{4!} = \frac{11(n-1)}{2}\,\frac{(n+3)!}{3!}$$

$$\Rightarrow\ (n+5)(n+4) = 22(n-1)$$

After solving, we get $n = 6$ or $n = 7$.

The number of points of intersection of lines is ${}^6C_2$ or ${}^7C_2$ $\equiv 15$ or 21.

13. a, b, d.

Total number of units to be covered is $3 + 7 + 11 = 21$. A person can choose 3 units in ${}^{21}C_3$ ways. A person can choose 7 units in ${}^{18}C_7$ ways. The rest 11 units can be chosen in 1 way. Therefore, total number of ways is ${}^{21}C_3 \times {}^{18}C_7 \times 1 = 21!/(3!7!11!)$.

For correct answers (a) and (d), see the respective theory.

For correct answer (b) see the theory of multinomial expansion in the binomial theorem.

14. a, c.

Let person P_i get x_i number of things such that

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 30$

If x_i is odd or $x_i = 2\lambda_i + 1$, where $\lambda_i \geq 0$, then

$2(\lambda_i + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) + 6 = 30$

$\Rightarrow\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 12$

Then number of solutions is ${}^{12+6-1}C_{6-1} = {}^{17}C_5$. If x_i is even or $x_i = 2\lambda_p$, where $\lambda_i \geq 1$, then

$2(\lambda_i + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) = 30$

$\Rightarrow\ \lambda_i + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 15$

Therefore, required number of ways is ${}^{15-1}C_{6-1} = {}^{14}C_5$.

15. a, c.

Let x_i $(1 \leq i \leq n)$ be the number of objects selected of the i^{th} type. Since each object is to be selected at least once, we must have $x_i \geq 1$ and $x_1 + x_2 + \cdots + x_n = r$. We have to find number of positive integral solutions of the above equation. Total number of such solutions is ${}^{r-1}C_{n-1} = {}^{r-1}C_{r-n}$.

16. a, d.

Let $A = \{a_1, a_2, \ldots, a_n\}$. For each $a_i (1 \leq i \leq n)$, we have either $a_i \in P_j$ or $a_i \notin P_j$ $(1 \leq j \leq m)$. That is, there are 2^m choices in which a_i $(1 \leq i \leq n)$ may belong to the P_j's. One of these, there is only one choice, in which $a_i \in P_j$ for all $j = 1, 2, \ldots, m$ which is not favourable for $P_1 \cap P_2 \cap \cdots \cap P_m$ to be ϕ. Thus, $a_i \notin P_1 \cap P_2 \cap \cdots \cap P_m$ in $2^m - 1$ ways.

Since there are n elements in set A, the total number of choices is $(2^m - 1)^n$.

Also, there is exactly one choice, in which, $a_i \notin P_j$ for all $j = 1, 2, \ldots, m$ which is not favourable for $P_1 \cup P_2 \cup \cdots \cup P_m$ to be equal to A.

Thus, a_i can belong to $P_1 \cup P_2 \cup \cdots \cup P_m$ in $(2^m - 1)$ ways.

Since there are n elements in set A, the number of ways in which $P_1 \cup P_2 \cup \cdots \cup P_m$ can be equal to A is $(2^m - 1)^n$.

17. b, c, d.

Exponent of 2 is

$$\left[\frac{10}{2}\right]+\left[\frac{10}{2^2}\right]+\left[\frac{10}{2^3}\right]=5+2+1=8$$

Exponent of 3 is

$$\left[\frac{10}{3}\right]+\left[\frac{10}{3^2}\right]=3+1=4$$

Exponent of 5 is

$$\left[\frac{10}{5}\right]=2$$

Exponent of 7 is

$$\left[\frac{10}{7}\right]=1$$

The number of divisors of 10! is $(8+1)(4+1)(2+1)(1+1) = 270$. The number of ways of putting N as a product of two natural numbers is $270/2 = 135$.

18. b, c, d.

$$P = 21(21+1)(21-1)(21+2)(21-2)\cdots(21+10)(21-10)$$
$$= (21-10)(21-9)\cdots(21-1)\,21\,(21+1)(21+10)\cdots(21+10)$$
$$= 41 \times 40 \cdots 11$$

which is divisible by 21!, and hence by 20! and 19!

19. a, d.

When n is odd:

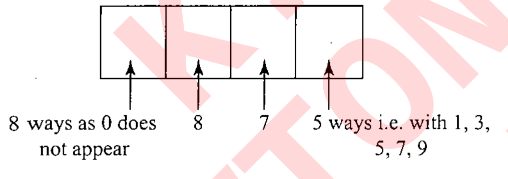


Fig. 5.46

The number of such numbers is $8 \times 8 \times 7 \times 5 = 2240$.

When n is even:

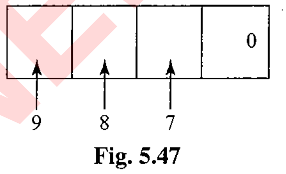


Fig. 5.47

If unit's place is filled with 0, then the total number is $9 \times 8 \times 7 = 504$.

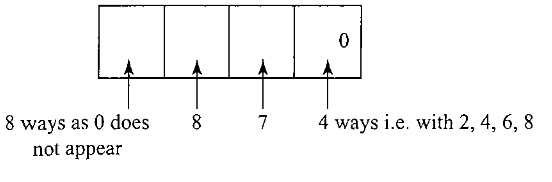


Fig. 5.48

If unit's place is not filled with 0, then the total number is $8 \times 8 \times 7 \times 4 = 1792$.

Hence, the total number of even numbers is $y = 504 + 1792 = 2296$.

Reasoning Type

1. **a.** We have, $30 = 2 \times 3 \times 5$. So, 2 can be assigned to either a or b or c, i.e. 2 can be assigned in 3 ways. Similarly, each of 3 and 5 can be assigned in 3 ways. Thus, the number of solutions is $3 \times 3 \times 3 = 27$.

2. **d.** Each student receives at least two toys. Let us first give each student one toy. Now, we are left with 7 toys, which can be distributed among three students such that each receives at least one toy, which is equivalent to number of positive integral solutions of the equation $x + y + z + w = 7$, which is given by ${}^{7-1}C_{4-1} = {}^6C_3$.

   Hence, statement 1 is false and statement 2 is correct.

3. **a.** Number of ways of dividing n^2 objects into n groups of same size is $\dfrac{(n^2)!}{(n!)^n n!}$.

Now number of ways of distributing these n groups among n persons is

$$\left[\frac{(n^2)!}{(n!)^n n!}\right] n! = \frac{(n^2)!}{(n!)^n} \text{ which is always an integer.}$$

Also we know that product of r is divisible by $r!$ Now, $(n^2)! = 1 \times 2 \times 3 \times 4 \cdots n^2$

$$= 1 \times 2 \times 3 \cdots n$$
$$\times (n+1)(n+2)\cdots 2n$$
$$\times (2n+1)(2n+2)\cdots 3n$$
$$\times (n^2-(n^2-1))(n^2-(n^2-1)) \cdots n^2$$

Thus, in n^2! there are n rows each consisting product of n integers. Each row is divisible by $n!$

Hence $(n^2)!$ is divisible by $(n!)^n$ or $\dfrac{(n^2)!}{(n!)^n}$ is a natural number.

Hence, both statements are correct and statement 2 is correct explanation of statement 1.

4. **b.** $1400 = 2^3 5^2 7$

The number of ways in which 1400 can be expressed as a product of two positive integers is

$$\frac{(3+1)(2+1)(1+1)}{2} = 12$$

Statement 2 is correct but does not explain statement 1 as it just gives the information about the prime factor about which 1400 is divisible.

5. **a.** In onto functions each image must be assigned at least one pre-image. Now if we consider the images a and b as two different boxes, then four distinct objects 1, 2, 3 and 4 (pre-images) can be distributed in $2^4 - {}^2C_1(2-1)^4 = 14$ ways.

   Hence, both statements are correct and statement 2 is correct explanation of statement 1.

6. **b.** India must win at least 6 matches of 11 matches. Then number of ways in which India can win the series is ${}^{11}C_6 + {}^{11}C_7 + \cdots + {}^{11}C_{11} = 2^{10}$

   Thus, both the statements are true, but statement 2 is not correct explanation of statement 1.

7. **a.** The batting order of 11 players can be decided in 11! ways. Now Yuvraj, Dhoni and Pathan can be arranged in 3! ways.

But the order of these three players is fixed, i.e., Yuvraj–Dhoni–Pathan. Now, 11! answer is 3! times more, hence the required answer is 11!/3!.

8. d. Number of ways of arranging 21 identical objects when r is identical of one type and remaining are identical of second type is $\frac{21!}{r!(21-r)!} = {}^{21}C_r$ which is maximum when $r = 10$ or 11.

Therefore, ${}^{13}C_r = {}^{13}C_{10}$ or ${}^{13}C_{11}$, hence maximum value of ${}^{13}C_r$ is ${}^{13}C_{10} = 286$.

Hence, statement 1 is false. Obviously statement 2 is true.

9. a. A number is divisible by 4, if the last two digits are divisible by 4. Last two digits can be 12, 16, 28, 32, 36, 68, 92, 96. Thus, last two places can be filled in 8 ways. The remaining three places can be filled with remaining 4 digits in ${}^4C_3\,3!$ ways. Total number of such numbers is $8 \times ({}^4C_3\,3!) = 192$.

10. b. We have $a + b + c = 30$, and $a \neq b \neq c$. Let $a < b < c$.

Now, relative values of a, b, c are tabulated as follows.

a	b	c	Number of triplets (a, b, c)
1	2	27	
	3	26	
	4	25	
	⋮	⋮	
	14	15	13
2	3	25	
	4	24	
	⋮	⋮	
	13	15	11
3	4	23	
	5	22	
	⋮	⋮	
	13	14	10
4	5	21	
	6	20	
	⋮	⋮	
	12	14	8
5	6	19	
	7	18	
	⋮	⋮	
	12	13	7
6	7	17	
	⋮	⋮	
	11	13	5
7	8	15	
	⋮	⋮	
	11	12	4
8	9	13	
	10	12	2
9	10	11	1
		Total	61

Statement 2 is correct but it does not explain statement 1.

11. c. When $p, q < r$, we have selection procedure as follows :

From p identical things	From q identical things
p	$r - p$
$p - 1$	$r - (p - 1)$
$p - 2$	$r - (p - 2)$
⋮	⋮
⋮	⋮
$r - q$	q
	Total: $p + q - r + 1$

When $p, q > r$, we have selection procedure as follows:

From p identical things	From q identical things
r	0
$r - 1$	1
$r - 2$	2
⋮	⋮
⋮	⋮
0	r
	Total: $r + 1$

Thus, statement 1 is correct, but statement 2 is false.

12. a. Statement 2 is correct as when $3^a, 3^b, 3^c$ are in G.P., we have $(3^b)^2 = (3^a)(3^c) \Rightarrow 2b = a + c \Rightarrow a, b, c$ are in A.P. Thus, selecting three numbers in G.P. from $\{3^1, 3^2, 3^3, \ldots, 3^{100}, 3^{101}\}$ is equivalent to selecting 3 numbers from $\{1, 2, 3, \ldots, 101\}$ which are in A.P. Now, a, b, c are in A.P. if either a and c are odd or a and c are even.

Number of selection ways of 2 odd numbers is ${}^{51}C_2$. Number of selection ways of 2 even numbers is ${}^{50}C_2$. Hence, total number of ways is ${}^{51}C_2 + {}^{50}C_2 = 1275 + 1225 = 2500$.

13. a. Statement 2 is true, see the proof in binomial theorem. Also in statement 1, if A selects i objects and B selects j objects then $i < j$. Hence number of ways is $\sum_{0 \le i < j \le 20} {}^{20}C_i\,{}^{20}C_j$.

14. c.

Number of objects from 21 different objects	Number of objects from 21 identical objects	Number of ways of selections
10	0	${}^{21}C_{10} \times 1$
9	1	${}^{21}C_9 \times 1$
⋮	⋮	⋮
0	10	${}^{21}C_0 \times 1$

Thus, total number of ways of selection is ${}^{21}C_0 + {}^{21}C_1 + {}^{21}C_2 + \cdots + {}^{21}C_{10} = 2^{20}$.

Statement 2 is false, as given series is not exact half series. (For details, see the theory in binomial theorem.)

15. a. General in the expansion of $(x + y + z + w)^{50}$ is

$$\frac{50!}{p!q!r!s!}x^p y^q z^r w^s$$

where $p + q + r + s = 50$, $0 \le p, q, r, s \le 50$. (see theory in binomial theorem).

Now number of terms is equal to number of ways in which we can adjust powers of x, y, z and w such that their sum is

50, i.e., equal to the non-negative solutions of $p + q + r + s = 50$, which is given by ${}^{50+4-1}C_{4-1}$.

16. a. When n persons be seated at a round table, so that all shall not have the same neighbours in any two arrangements, clockwise and anticlockwise arrangements are considered to be the same, which is the case when n different beads are arranged in the circle. Hence, number of ways is $(n-1)!/2$.

17. b. Exponent of 2 in 50! is

$$\left[\frac{50}{2}\right]+\left[\frac{50}{4}\right]+\left[\frac{50}{8}\right]+\left[\frac{50}{16}\right]+\left[\frac{50}{32}\right]=25+12+6+3+1=47.$$

And exponent of 5 in 50! is

$$\left[\frac{50}{5}\right]+\left[\frac{50}{25}\right]=12$$

Now number of zeros in the end of 50! is equal to exponent of 10 in 50! which is equal to exponent of 5 in 50! Therefore, number of zeros in the end depends on exponent of 5, but not on the exponent of 2.

Hence both statements 1 and 2 are true; but statement 2 is not a correct explanation for statement 1.

Linked Comprehension Type

For Problems 1–3

1. a, 2. b, 3. b.

Sol. 1. When one all rounder and ten players from bowlers and batsmen play, number of ways is ${}^4C_1\ {}^{14}C_{10}$.

When one wicketkeeper and 10 players from bowlers and batsmen play, number of ways is ${}^2C_1\ {}^{14}C_{10}$.

When one all rounder, one wicketkeeper and nine from batsmen and bowlers play, number of ways is ${}^4C_1\ {}^2C_1\ {}^{14}C_9$.

When all elven players play from bowlers and batsmen then, number of ways is ${}^{14}C_{11}$.

Total number of selections is ${}^4C_1\ {}^{14}C_{10} + {}^2C_1\ {}^{14}C_{10} + {}^4C_1\ {}^2C_1\ {}^{14}C_9 + {}^{14}C_{11}$.

2. If the particular bowler plays then two batsmen will not play. So, rest of 10 players can be selected from 17 other players. Number of such selections is ${}^{17}C_{10}$.

If the particular bowler does not play, then number of selections is ${}^{19}C_{11}$.

If all the three players do not play, then number of selections is ${}^{17}C_{11}$.

Total number of selections is ${}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$.

3. If the particular batsman is selected, then rest of 10 players can be selected in ${}^{18}C_{10}$ ways.

If particular wicketkeeper is selected, then rest of 10 players can be selected in ${}^{18}C_{10}$ ways.

If both are not selected, then number of ways is ${}^{18}C_{11}$.

Hence, total number of ways is $2\ {}^{18}C_{10} + {}^{18}C_{11} = {}^{19}C_{11} + {}^{18}C_{10}$.

For Problems 4–6

4. c, 5. b, 6. a.

Sol. 4. Seven persons can be selected for first table in ${}^{12}C_7$ ways. Now these seven persons can be arranged in 6! ways. The remaining five persons can be arranged on the second table in 4! ways.Hence, total number of ways is ${}^{12}C_5 6!4!$.

5. Here, A can sit on first table and B on the second or A on second table and B on the second table.

If A is on the first table, then remaining six for first table can be selected in ${}^{10}C_6$ ways. Now these seven persons can be arranged in 6! ways. Remaining five can be arranged on the other table in 4! ways.

Hence, total number of ways is $2\ {}^{10}C_6\ 6!4!$.

6. If A, B are on the first table, then remaining five can be selected in ${}^{10}C_5$ ways.

Now seven persons including A and B can be arranged on the first table in which A and B are together in 2!5! ways. Remaining five can be arranged on the second table in 4! ways. Total number of ways is ${}^{10}C_5\ 4!5!2!$

If A, B are on the second table, then remaining three can be selected in ${}^{10}C_3$ ways.

Now five persons including A and B can be arranged on the second table in which A and B are together 2!3! ways. Remaining seven can be arranged on the first table in 6! ways. Hence, number of ways for first table is ${}^{10}C_7\ 6!3!2!$.

For Problems 7–9

7. b, 8. c, 9. d.

Sol. If no box remains empty, then we can have (1, 1, 3) or (1, 2, 2) distribution pattern.

7. When balls are different and boxes are identical, number of distributions is equal to number of divisions in (1, 2, 3) or (1, 2, 2) ways. Hence, total number of ways is

$$\frac{5!}{1!\cdot 2!\cdot 3!}+\frac{5!}{(2!)^2\,1!\cdot 2!}=25$$

8. When balls as well as boxes are identical, we have only two ways (1, 1, 3) and (1, 2, 2). Hence, number of ways is 2.

9. When boxes are kept in a row, they will be treated as different. In this case the number of ways will be ${}^{5-1}C_{3-1} = {}^4C_2 = 6$.

For Problems 10–12

10. b, 11. c, 12. b.

Sol. $6 = 0(2) + 6(1) = 1(2) + 4(1) = 2(2) + 2(1) = 3(2) + 0(1)$

Number of 2s	Number of 1s	Number of permutations
0	6	1
1	4	$\frac{5!}{4!}=5$
2	2	$\frac{4!}{2!\,2!}=6$
3	0	$\frac{3!}{3!}=1$
		Total = 13

$\therefore\ f(6) = 13$

Now, $f(f(6)) = f(13)$

Number of 1s	Number of 2s	Number of permutations
13	0	1
11	1	$\frac{12!}{11!}=12$

9	2	$\frac{11!}{9!2!}=55$
7	3	$\frac{10!}{7!3!}=120$
5	4	$\frac{9!}{5!4!}=126$
3	5	$\frac{8!}{3!5!}=56$
1	6	$\frac{7!}{6!}=7$
		Total = 377

$\therefore$ $f(f(6)) = f(13) = 377$

$f(1) = 1$ (1)

$f(2) = 2$ (1, 1 or 2)

$f(3) = 3$ (1, 1, 1 or 2, 1 or 1, 2)

$f(4) = 5$ (explained in the paragraph)

By taking higher value of n in $f(n)$, we always get more value of $f(n)$. Hence, $f(x)$ is one-one. Clearly, $f(x)$ is into.

For Problems 13–15

13. c, 14. a, 15. d.

Sol.

13. Let x_0 denote the number of empty seats to the left of the first person, x_i $(1 \le i \le n-1)$ the number of empty seats between the i^{th} and $(i+1)^{\text{th}}$ persons of the n^{th} person. Then $x_0, x_n \ge 0$ and $x_i \ge 1$ for $1 \le i \le n-1$.

$$x_0 + x_1 + \cdots + x_n = (m-n) \quad (1)$$

Putting $x_i = y_i + 1$, where $y_i \ge 0$, we have

$$x_0 + y_1 + \cdots + y_{n-1} + x_n + (1 + 1 + 1 + \cdots + (n-1) \text{ times}) = (m-n)$$

$$\Rightarrow \quad x_0 + y_1 + \cdots + y_{n-1} + x_n = m - n - (n-1)$$

$$\Rightarrow \quad x_0 + y_1 + \cdots + y_{n-1} + x_n = m - 2n + 1$$

Now number of non-negative integral solutions is ${}^{n+1+(m-2n+1)-1}C_{n+1-1} = {}^{m-n+1}C_n$. Since we can permute n persons in $n!$ ways, the required number of ways is

$$\left({}^{m-n+1}C_n\right)(n!) = \frac{(m-n+1)!}{n!(m-2n+1)!}\, n! = \frac{(m-n+1)!}{(m-2n+1)!}$$

14. Let $n = 2k$, where k is some positive integer. Let x_0 denote the number of empty seats to the left of the first pair, x_i $(1 \le i \le k-1)$ the number of empty seats between i^{th} and $(i+1)^{\text{th}}$ pair and x_k the number of empty seats to the right of the k^{th} pair. Note that $x_0, x_k \ge 0$, $x_i \ge 1$ $(1 \le i \le k-1)$ and

$$x_0 + x_1 + \cdots + x_k = (m-2k) \quad (2)$$

The number of integral solutions of Eq. (2) is ${}^{m-2k+1}C_k$ (see the solution to problem 13).

Since we can permute n persons in $n!$ ways, the required number of ways is

$$\left({}^{m-2k+1}C_k\right)(2k)! = \frac{(m-2k+1)!}{k!(m-3k+1)!}(2k)!$$

$$= \frac{(2k)!}{(k)}\,\frac{(m-2k+1)!}{(m-3k+1)!}$$

$$= ({}^{2k}P_k)\,({}^{m-2k+1}P_k)$$

$$= ({}^{n}P_{n/2})\,({}^{m-n+1}P_{n/2})$$

15. m is even. Let $m = 2k$, where k is some positive integer. We can choose n seats out of the k seats to the left of the middle seat in ${}^kC_n$ ways. Each chosen seat can be either empty or occupied. Thus, the number of ways of choosing seats for n persons is equal to $({}^kC_n)(2^n)$. We can arrange n persons at these seats in ${}^nP_n$ ways. Hence, the required number of arrangements is given by

$$(n!)\,({}^kC_n)\,(2^n) = ({}^kP_n)\,(2^n) = ({}^{m/2}P_n)(2^n)$$

For Problems 16–18

16. d, 17. b, 18. c.

Sol.

16. Since there are 5 even places and 3 pairs of repeated letters, therefore at least one of these must be at an odd place. Therefore, the number of ways is $11!/(2!2!2!)$.

17. Make a group of both M's and another group of T's. Then except A's we have 5 letters remaining. So M's, T's and the letters except A's can be arranged in 7! ways. Therefore, total number of arrangements is $7! \times {}^8C_2$.

18. Consonants can be placed in $7!/(2!2!)$ ways. Then there are 8 places and 4 vowels. Therefore, number of ways is

$$\frac{7!}{2!2!}\,{}^8C_4\,\frac{4!}{2!}$$

Matrix-Match Type

1. a → r; b → s; c → p; d → q.

a. ${}^{10}C_2 - {}^4C_2 + 1 = 45 - 6 + 1 = 40$

b. $1 \times {}^{10}C_2 = 45$

c. $2 \times {}^6C_2 = 30$

d. ${}^6C_2 \times 4 = 60$

2. a → q; b → r; c → s; d → p.

a. Number of rectangles is equal to number of ways we can select two vertical lines and two horizontal lines. Total number of ways is ${}^7C_2 \times {}^7C_2 = 441$.

b.

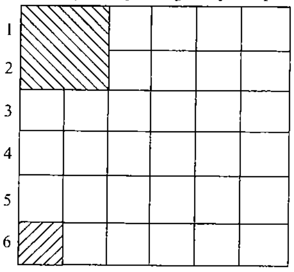


Fig. 5.49

If the square is of 1 sq. units like $a6$, then we have such $6 \times 6 = 36$ squares.

If the square is of 4 sq. units like shaded region of the squares a1, a2, b1, b2, then we have such 5 squares in the belt formed by rows 1 and 2. Similarly we have 4 more belts 23, 34, 45 and 56. Hence, there are $5 \times 5 = 25$ such squares.

Similarly we have 4×4, 3×3, 2×2, 1×1 squares of increasing sizes.

Hence, total number of squares is $1 + 4 + 9 + 16 + 25 + 36 = 91$.

c.

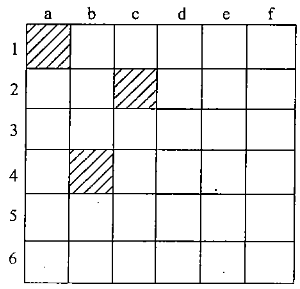


Fig. 5.50

The first square can be selected in 36 ways. If one such square $a1$ is selected, we are left with 25 squares; second square cannot be selected from row 1 and column a. If second square is $c2$, we are left with 16 squares, from which third square can be selected, e.g., $b4$.

Hence, number of ways of selections is $36 \times 25 \times 16$. But in this one-by-one type of selection order of selection is also considered. Hence, actual number of ways is $(36 \times 25 \times 16)/3! = 2400$.

d. Given number of ways is equivalent to selecting 11 squares from 36 squares if no row remains empty.

Suppose $x_1, x_2, x_3, x_4, x_5, x_6$ be the number of squares selected from the 1st, 2nd, 3rd, 4th, 5th and 6th row.

Then we must have $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 11$ (where $1 \le x_i \le 6$).

The number of positive integral solutions of the above equation is ${}^{11-1}C_{6-1} = {}^{10}C_5$.

3. a → q; b → p; c → s; d → r.

a. If polygon has n sides, then number of diagonals is ${}^nC_2 - n = 35$ (given). Solving we get $n = 10$. Thus, there are 10 vertices, from which ${}^{10}C_3$ (= 120) triangles can be formed.

b. Four vertices can be selected in ${}^{10}C_4$ (= 210) ways. Using these four vertices two diagonals can be formed, which has exactly one point of intersection lying inside the polygon.

Hence, number of points of intersections of diagonal which lies inside the polygon is ${}^{10}C_4 \times 1 = 210$.

c.

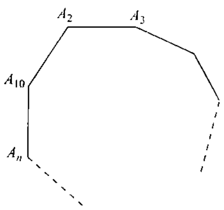


Fig. 5.51

Suppose one of the sides of the triangle is A_1A_2. Then third vertex cannot be A_3 or A_{10}. Thus, for the third vertex six vertices are left. There are six triangles in which side A_1A_2 is common with that of polygon. Similarly, for each of the sides A_2A_3, A_3A_4, $\cdots$, A_9A_{10} there are six triangles. Then total number of triangles is 60.

d. Triangles $A_1A_2A_3$, $A_2A_3A_4$, …, $A_8A_9A_{10}$ have two sides common with that of polygon. Hence, there are 10 such triangles.

4. a → p, q, r, s; b. → p, r, s; c. → q, r, s; d. → s.

a. Number of subjective functions is
$3^6 - {}^3C_1(3-1)^6 + {}^3C_2(3-2)^6 = 729 - 192 + 3 = 540$

b. If $f(a_i) \ne b_i$, then pre-image a_1, a_2, a_3 cannot be assigned images b_1, b_2, b_3, respectively.

Hence each of a_1, a_2, a_3 can be assigned images in 2 ways.

a_4, a_5, a_6 can be assigned images in 3 ways each.

Hence number of functions is $2^3 3^3 = 216$.

c. One-one functions are not possible ar pre-images are more than images.

d. Number of many-one functions is

Total number of functions – number of one-one functions
$= 3^6 - {}^6P_3 = 729 - 120 = 609$

5. a → p, s; b → q, r; c → p, s; d → r.

a. Total number of required functions is equal to number of derangement of 5 objects, which is given by

$$5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44$$

b. $x_1x_2x_3 = 2 \times 35 \times 7 = 2 \times 49 \times 5 = 10 \times 7 \times 7 = 14 \times 7 \times 5$

So total number of solution set is $3 \times 3! + 3!/2! = 21$.

c. $3780 = 2^2 \times 3^3 \times 5 \times 7$

Number of divisors which are divisible by 2 but not by 3 is $2 \times 2 \times 2 = 8$.

Number of divisors which are divisible by 3 but not by 2 is $3 \times 2 \times 2 = 12$.

Number of divisors which are divisible by 2 as well as 3 is $2 \times 3 \times 2 \times 2 = 24$.

Hence total number of divisors is 44.

d. $4\lambda + 2 = 2(2\lambda + 1)$ = odd multiple of 2

Thus, total number of divisors is $1 \times 5 \times 11 - 1 = 54$. (1 is subtracted and powers of three and five are zero each and this will make $\lambda = 0$.)

6. a → r; b → p; c → s; d → q.

a. The number of possible outcomes with 2 on at least one dice
= The total number of outcomes with 2 on at least one dice
= (The total number of outcomes) – (The number of outcomes in which 2 does not appear on any dice) $= 6^4 - 5^4$
$= 1296 - 625 = 671$

b. Any selection of four digits from the 10 digits 0, 1, 2, 3, …, 9 gives one number. So, the required number of numbers is ${}^{10}C_4$.

c. Let the number be $n = pqr$. Since $p + q + r$ is even, p can be filled in 9 ways and q can be filled in 10 ways.

r can be filled in number of ways depending upon what is the sum of p and q.

If $p + q$ is odd, then r can be filled with any one of five odd digits.

If $p + q$ is even, then r can be filled with any one of five even digits.

In any case, r can be filled in five ways.

Hence, total number of numbers is $9 \times 10 \times 5 = 450$.

d. After fixing 1 at one position out of 4 places 3 places can be filled by $^7P_3$ ways. But for some numbers whose fourth digit is zero, such type of ways is $^6P_2$. Therefore, total number ways is $^7P_3 - {}^6P_2 = 480$.

7. a → q, s; b. → p, r; c → p, r; d → p, r.

a. There are two cases.

(i) 5, 4, 1, 1, 1

Number of ways of selection is 5!/3! = 20.

(ii) 5, 2, 2, 1, 1

Number of ways of selection is 5!/2!2! = 30.

Hence, total number is 20 + 30 = 50.

b. Select 4 pairs in $^5C_4 = 5$ ways. Now select exactly one shoe from each of the pairs selected in $(^2C_1)^4$ ways. This will fulfill the condition. Hence, required answer is 5 × 16 = 80.

c. The first child C_1 can be chosen in 3 ways; his/her mother can be interviewed in 5 ways; the second child C_2 can be chosen in 2 ways, and his/her mother can be interviewed in 3 ways.

Hence total number of ways is 3 × 5 × 2 × 3 = 90.

d. Required number of ways is 5! – 4! – 3! = 120 – 64 – 6 = 90. (Number will be less than 56000 only if either 4 occurs on the first place or 5, 4 occurs on the first two places.)

8. a → r, s; b → p, r; c → p, q; d → r, s

a. We have,

$a = {}^{x+2}P_{x+2} = (x+2)!$,

$b = {}^xP_{11} = \dfrac{x!}{(x-11)!}$

$c = {}^{x-11}P_{x-11} = (x-11)!$

Now,

$a = 182\, bc \Rightarrow (x+2)! = 182 \times \dfrac{x!}{(x-11)!}(x-11)!$

$\Rightarrow (x+2)! = 182x! \Rightarrow (x+2)(x+1) = 182 \Rightarrow x = 12$

b. ×I×I×I

Even digits occupy odd places shown by crosses. Crosses can be filled in 2 × 2 × 1 ways (∵ 0 cannot go in the first place from the left). The remaining places can be filled in 3! ways. Therefore, the required number of numbers is 2 × 2 × 1 × 3! = 24.

c. Total number of numbers without restriction is 2^5. Two numbers have all the digits equal. So, the required number of numbers is $2^5 - 2$.

d. Let number of sides of polygon be n. Number of sides of polygon is equal to number of vertices of polygon. Now number of diagonals of polygon is

$^nC_2 - n = 54$

$\Rightarrow \dfrac{n(n-1)}{2} - n = 54$

$\Rightarrow n^2 - 3n - 162 = 0$

$\Rightarrow (n-12)(n+9) = 0$

$\Rightarrow n = 12$

Integer Type

1.(5) $^nP_r = {}^nP_{r+1}$

$\Rightarrow \dfrac{n!}{(n-r)!} = \dfrac{n!}{(n-r-1)!} \Rightarrow n - r = 1$ (1)

Again $^nC_r = {}^nC_{r-1} = \dfrac{n!}{(n-r)!r!} = \dfrac{n!}{(n-r+1)!(r-1)!}$

$\Rightarrow \dfrac{1}{r} = \dfrac{1}{n-r+1} \Rightarrow n - 2r = -1$ (2)

Solving (1) and (2), $n = 3$, $r = 2$

2.(8) We have $^nC_2 = 28$

$\Rightarrow n = 8$ (as there are 7 days in week)

3.(4) Number of arrangements are $2\, n!\, n!$

Giventhat $2\, n!\, n! = 1152$

$\Rightarrow (n!)^2 = 576$

$\Rightarrow n! = 24$

$\Rightarrow n = 4$

4.(8) Let $n(A)$ = number divisible by 60 = (60, 120, …, 960) = 16

$n(B)$ = number divisible by 24 = (24, 48, …, 984) = 41

$n(A \cap B)$ = number divisible by both

= 120 + 240 + ⋯ + 960 = 8

Hence $n(A \cap B) = n(A) - n(A \cap B) = 16 - 8 = 8$

5.(9) We have A's = 2; B's = 4; C's = 2

Total words formed = $\dfrac{8!}{4!\,2!\,2!} = 420$ (1)

Let ABBC = '×'

Number of ways in which × ABBC can be arranged = $\dfrac{5!}{2!} = 60$ but this includes × ABBC and ABBC ×.

But the word ABBCABBC is counted twice in 60 hence it should be 59

Hence required number of ways = 420 – 59 = 361

6.(6) Let T and S denotes teacher and student respectively

Then we have following possible patterns according to question

(i) T S S T S S T S S

(ii) S T S S T S S T S

(iii) S S T S S T S S T

Hence total number of arrangements are 3 · (3!)6! = 18 × 6!

$\Rightarrow k = 6$

7.(8) To form a triangle, 3 points out of 5 can be chosen in $^5C_3 = 10$ ways.

But of these, the three points lying on the 2 diagonals will be collinear.

So 10 – 2 = 8 triangles can be formed

8.(8) Here A is common letter in words 'SUMAN' and 'DIVYA'

Now for selecting six different letters we must select A either from word 'SUMAN' or from word 'DIVYA'.

Hence for possible selections, we have

A excluded from SUMAN +A included in SUMAN

$= {}^4C_3 \cdot {}^5C_3 + {}^4C_2 \cdot {}^4C_3 = 40 + 24 = 64$

Hence $N^2 = 64 \quad \Rightarrow N = 8$

9.(5) Let r no. of books of algebra and $20 - r$ of calc. no. of selections $= {}^{r}C_5 \times {}^{20-r}C_5$

Which has maximum value when $r = 10$

10.(9) Number of digits are 9

select 2 places for the digit 1 and 2 in ${}^{9}C_2$ ways

from the remaining 7 places select any two places for 3 and 4 in ${}^{7}C_2$ ways

and from the remaining 5 places select any two for 5 and 6 in ${}^{5}C_2$ ways

now, the remaining 3 digits can be filled in 3! ways

$\therefore$ Total ways $= {}^{9}C_2 \cdot {}^{7}C_2 \cdot {}^{5}C_2 \cdot 3!$

$$= \frac{9!}{2!\cdot 7!}\cdot\frac{7!}{2!\cdot 5!}\cdot\frac{5!}{2!\cdot 3!}\cdot 3!$$

$$= \frac{9!}{8} = \frac{9\cdot 8\cdot 7!}{8} = 9\cdot 7!$$

11.(9) We have $2^n - 2 = 510$;

$\Rightarrow \quad 2^n = 512$

$\Rightarrow \quad n = 9$

12.(8) Including the two specified people, 4 others can be selected in ${}^{5}C_4$ ways.

The two adjacent seats can be taken in 4 ways and the two specified people can be arranged in 2!ways, remaining 4 people can be arranged in 4! ways.

$\Rightarrow \quad 5C_4 .4 .2!4! = 5!\ 8 = 8\ .5!$

13.(7) There are 2 women and let number of men are n

According to question

$2 \times {}^{n}C_2 = 66 + 2 \times {}^{n}C_1 \times {}^{2}C_1$

$$\Rightarrow \quad 2.\frac{n(n-1)}{2} = 2\,[33 + 2n]$$

$$\Rightarrow \quad \frac{n(n-1)}{1.2} = 33 + n(2)$$

$\Rightarrow \quad n^2 - 5n - 66 = 0 \Rightarrow (n-11)(n+6) = 0$

$\therefore \quad n = 11 \; (\because sn > 0)$

total participants = 2 + 11 = 13

14.(9) $\boxed{x}\boxed{x}\boxed{}$ when two consecutive digits are 11, 22, etc = 9 . 9 = 81

$\boxed{}\boxed{0}\boxed{0}$ when two consecutive digits are 0 0 = 9

$\boxed{}\boxed{x}\boxed{x}$ when two consecutive digits are 11, 22, 33, ...

= 9 . 8 = 72

Total number of numbers are $N = 162$.

15.(8) We have $N = \boxed{a}\boxed{b}\boxed{c}\boxed{d}$

First place a can be filled in 2 ways i.e. 4, 5 $(4000 \le N < 6000)$

For b and c, total possibilities are '6' $(3 \le b < c \le 6)$

i.e. 34, 35, 36, 45, 46, 56

Last place d can be filled in 2 ways i.e. 0, 5 (N is a multiple of 5)

Hence, total numbers $= 2 \times 6 \times 2 = 24 = N$ then $N/3 = 8$.

16.(5) A AAAA | B BBBBB

Since word reads the same backwards and forwards, the middle digit must be A.

M

× × × × × ↓× × × × ×

so that even number of A's and B's are available for arrangement about middle position M in the above figure.

Take AABBB on one side of M (6th place) and then their image about M in a unique way

$$\therefore \quad \text{Number of ways N} = \frac{5!}{2!\cdot 3!} = 10$$

17.(5)

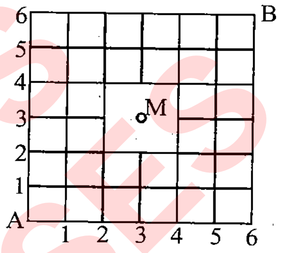


Fig. 5.52

Here the path which leads from A to B is of length-12.

Now without considering the constrain of passing through the point M, number of ways in which we can reach B from A is equal to number of ways we can select 6 steps from left to right and 6 from bottom to top which is equal to ${}^{12}C_6$.

Now we can reach from A to M in 6 steps in ${}^{6}C_3$ ways and can reach from M to B in ${}^{6}C_3$ ways.

Hence we can reach from A to B through M in ${}^{6}C_3 \times {}^{6}C_3$ ways.

Hence required number of ways $= {}^{12}C_6 - [{}^{6}C_3 \times {}^{6}C_3] = 924 - 400 = 524$

18.(6)

number of numbers beginning with 1 = 120 $\boxed{1}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}$

number of numbers beginning with 2 = 120 $\boxed{2}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}$

starting with 31 ....= 24 $\boxed{3}\boxed{1}\boxed{}\boxed{}\boxed{}\boxed{}$

starting with 3214 ....= 2 $\boxed{3}\boxed{2}\boxed{1}\boxed{4}\boxed{}\boxed{}$

finally = 1 $\boxed{3}\boxed{2}\boxed{1}\boxed{5}\boxed{4}\boxed{6}$

Hence unit place digit of 267th number is 6

19.(7) x denotes the number of times he can take unit step and y denotes the number of times he can take 2 steps, then $x + 2y = 7$,

Then we must have $x = 1, 3, 5,$

If $x = 1$, the steps will be 1 2 2 2

$$\Rightarrow \quad \text{number of ways} = \frac{4!}{3!} = 4$$

If $x = 3$, the steps will be 1 1 1 2 2

$$\Rightarrow \quad \text{number of ways} = \frac{5!}{2!\,.\,3!} = 10$$

If $x = 5$, the steps will be 1 1 1 1 1 2

$\Rightarrow \quad$ number of ways $= {}^{6}C_1 = 6$

If $x = 7$, the steps will be 1 1 1 1 1 1 1

$\Rightarrow \quad {}^{7}C_0 = 1$

Hence total number of ways $= N = 21$

$\Rightarrow \quad N/3 = 7$

20.(7) 3 women can be selected in ${}^{7}C_3$ ways and can be paired with 3 men in 3! ways.

Remaining 4 women can be grouped into two couples in

$$\frac{4!}{2!\cdot 2!\cdot 2!} = 3$$

$\therefore$ Total = ${}^7C_3 \cdot 3! \cdot 3 = 630 = N$

Then the value of N/90 is 7

21.(8) Here $P_n = {}^{n-2}C_3$ and $P_{n+1} = {}^{n-1}C_3$

Hence ${}^{n-1}C_3 - {}^{n-2}C_3 = 15$

$\Rightarrow \quad {}^{n-2}C_3 + {}^{n-2}C_2 - {}^{n-2}C_3 = 15$

$\Rightarrow \quad {}^{n-2}C_2 = 15 \Rightarrow n = 8$

22.(3)

(i) He can invite 2 friends three times each

Lets select first those 2 friends in ${}^3C_2$ ways

Now these two friends each three time can be invited on 6 days in $\frac{6!}{3!3!}$

Thus total number of ways 2 friends can be invited three times

$= {}^3C_2 \frac{6!}{3!3!}$

(ii) Another possibility is that he invites all three friends 2 times each

Then number of ways $= \frac{6!}{2!2!2!}$

(ii) One more possibility is that he invites one friend three times, one two times and one three times.

Then number of ways $= \frac{6! \times 6}{3!2!}$

Hence total number of ways

$$= {}^3C_2 \times \frac{6!}{3!3!} + \frac{6! \times 6}{3!2!} + \frac{6!}{2!2!2!} = 510$$

23.(4) If three numbers are in G.P., then their exponent must be in A.P.

If a, b, c are selected number in G.P., then the exponents of a and c both are either odd or both even, or otherwise exponent b will not be integer.

Now two odd exponent (from 1, 2, 3, …, 10) can be selected in ${}^5C_2$ ways and two even exponent can be selected in ${}^5C_2$ ways.

Hence number of G.P.'s are $2\,{}^5C_2 = 20$

24.(8) $\sum_{k=r}^{n} {}^kC_r = {}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \cdots + {}^nC_r$

$= 1 + {}^{r+1}C_1 + {}^{r+2}C_2 + {}^{r+3}C_3 + \cdots + {}^nC_{n-r}$

$= {}^{r+1}C_0 + {}^{r+1}C_1 + {}^{r+2}C_2 + \cdots + {}^nC_{n-r}$

$= {}^{r+2}C_1$

${}^{r+3}C_2$ and so on finally ${}^{n+1}C_{n-r}$

now, ${}^{n+1}C_{n-r} = {}^{n+1}C_{r+1}$

$\therefore\ f(n) = \sum_{r=0}^{n} {}^{n+1}C_{r+1} = {}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \cdots + {}^{n+1}C_{n+1}$

$= {}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1} - 1$

$f(n) = (2^{n+1}) - 1$

$f(9) = 2^{10} - 1 = 1023 = 3 \cdot 11 \cdot 31$

hence number of divisors are $(1 + 1)(1 + 1)(1 + 1) = 8$

Archives

Subjective Type

1. (i) Distribution of 52 cards can be equally divided amongst four players.

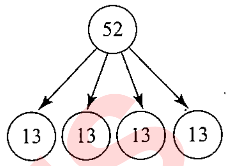


Fig. 5.53

Hence, number of ways is

$$\frac{52!}{(13!)4!}4! = \frac{52!}{(13!)}.$$

(ii)

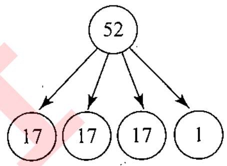


Fig. 5.54

Number of divisions is

$$\frac{52!}{(17!)^3 1!3!}.$$

Note: *There is division by* 3! *since* 3 *groups can be arranged in* 3! *ways and here* 3 *groups are of equal number of cards.*

2. As all the X's are identical, the question is of selection of 6 squares from 8 squares, so that no row remains empty. Here R_1 has 2 squares, R_2 has 4 squares and R_3 has 2 squares. The selection scheme is as follows:

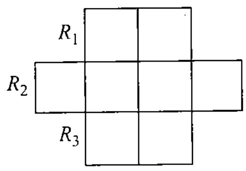


Fig. 5.55

R_1	R_2	R_3
1	4	1
1	3	2
2	3	1
2	2	2

Therefore, number of selection is

$${}^2C_1 \times {}^4C_4 \times {}^2C_1 + {}^2C_1 \times {}^4C_3 \times {}^2C_2 + {}^2C_2 \times {}^4C_3 \times {}^2C_1 + {}^2C_2 \times {}^4C_2 \times {}^2C_2 = 4 + 8 + 8 + 6 = 26$$

3. As no box should remain empty, boxes can have balls in the following numbers:

Possibilities 1, 2, 3 or 1, 2, 2

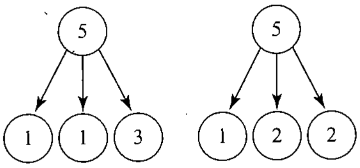


Fig. 5.56

Division ways for tree (i) is $\dfrac{5!}{(1!)^2\,3!\,2!}$

Division ways for tree (ii) is $\dfrac{5!}{(2!)^2\,1!\,2!}$

Now, total number of ways of distribution of these groups into three boxes is

$$\left[\frac{5!}{(1!)^2\,3!\,2!}+\frac{5!}{(2!)^2\,1!\,2!}\right]\times 3! = 150$$

4. m men can be seated in $m!$ ways creating $(m + 1)$ for ladies to sit n ladies out of $(m + 1)$ places (as $n < m$) can be seated in ${}^{m+1}P_n$ ways.

Therefore, total number of ways is

$$m!\times {}^{m+1}P_n = m!\times\frac{(m+1)!}{(m+1-n)!}=\frac{(m+1)!\,m!}{(m-n+1)!}$$

5. The scheme is as follows:

Husband's relatives		**Wife's relatives**		**Number of selections**
Male (3)	Female (4)	Male (4)	Female (3)	
3	0	0	3	${}^3C_3\times{}^3C_3=1$
2	1	1	2	${}^3C_2\times{}^4C_1\times{}^4C_1$ $x\,{}^3C_2=144$
1	2	2	1	${}^3C_1\times{}^4C_2\times{}^4C_2$ $x\,{}^3C_1=324$
0	3	3	0	${}^4C_3\times{}^4C_3=16$
			Total	485

6. Out of 2 white, 3 black and 4 red balls, three balls have to be drawn.

If at least one black ball is selected, then we have following cases:

Black balls (3)	**White + red balls (6)**	**Number of ways of selection**
1	2	${}^3C_1\times{}^6C_2=45$
2	1	${}^3C_2\times{}^6C_1=18$
3	0	${}^3C_3\times{}^6C_0=1$
	Total	64

7. Number of ways in which a student can select at least one and almost n books out of $2n + 1$ books is

$${}^{2n+1}C_1+{}^{2n+1}C_2+{}^{2n+1}C_3+\cdots+{}^{2n+1}C_n$$

$$=\frac{1}{2}\,[2\times{}^{2n+1}C_1+2\times{}^{2n+1}C_2+2\times{}^{2n+1}C_3+\cdots+2\times{}^{2n+1}C_n]$$

$$=\frac{1}{2}\Big[({}^{2n+1}C_1+{}^{2n+1}C_{2n})+({}^{2n+1}C_2+{}^{2n+1}C_{2n-1})$$
$$+({}^{2n+1}C_3+{}^{2n+1}C_{2n-2})+\cdots+({}^{2n+1}C_n+{}^{2n+1}C_{n+1})\Big]$$

[Using ${}^nC_r={}^nC_{n-r}$]

$$=\frac{1}{2}\Big[{}^{2n+1}C_1+{}^{2n+1}C_2+{}^{2n+1}C_3+\cdots+{}^{2n+1}C_n+{}^{2n+1}C_{n+1}$$
$$+{}^{2n+1}C_{n+2}+\cdots+{}^{2n+1}C_{2n}\Big]$$

$$=\frac{1}{2}\Big[{}^{2n+1}C_0+{}^{2n+1}C_1+{}^{2n+1}C_2+\cdots+{}^{2n+1}C_{2n+1}-1-1\Big]$$

$$=\frac{1}{2}\Big[2^{2n+1}-2\Big]=2^{2n}-1$$

Now given,

$2^{2n}-1=63$

$\Rightarrow$ $2^{2n}=64=2^6$

$\Rightarrow$ $2n=6$

$\Rightarrow$ $n=3$

8. Out of 18 guests, 9 are to be seated on side A and rest 9 on side B.

Now out of 18 guests, 4 particular guests desire to sit on one particular side say side A and other 3 on other side B. Out of rest $18 - 4 - 3 = 11$ guests we can select 5 more for side A and rest 6 can be seated on side B. Selection of 5 out of 11 can be done in ${}^{11}C_5$ ways. Nine guests on each side of table can be seated in $9!\times 9!$ ways.

Thus, there are total ${}^{11}C_5\times 9!\times 9!$ arrangements.

9. A committee of 12 is to be formed from 9 women and 8 men with minimum 5 women. Then we have following selection ways.

Women (9)	**Men (8)**	**Number of ways of selecting**
5	7	${}^9C_5\times{}^8C_7=1008=s_1$
6	6	${}^9C_6\times{}^8C_6=2352=s_2$
7	5	${}^9C_7\times{}^8C_5=2016=s_3$
8	4	${}^9C_8\times{}^8C_4=630=s_4$
9	3	${}^9C_9\times{}^8C_3=56=s_3$
	Total	6062

a. Number of committee when women are in majority is $s_3 + s_4 + s_5 = 2702$.

b. Number of committee when men are in majority is $s_1 = 1008$.

10. Let there be n sets of different objects each set containing n identical objects, e.g., ((1, 1, 1, … , 1(n times)), (2, 2, 2, …, 2 (n times)), …, (n, n, n, …, n (n times)). Then the number of ways in which these $n\times n=n^2$ objects can be arranged in a row is

$$\frac{(n^2)!}{n!\,n!\cdots n!}=\frac{(n^2)!}{(n!)^n}$$

But, these number of ways should be a natural number. Hence, $(n^2)!/(n!)^n$ is an integer ($n\in I^+$).

Objective Type

Fill in the blanks

1. Number of students who gave wrong answers to exactly one question is $a_1 - a_2$.

Number of students who gave wrong answers to exactly two questions is $a_2 - a_3$.

Number of students who gave wrong answers to exactly three questions is $a_3 - a_4$.

Number of students who gave wrong answers to exactly k question is $a_{k-1} - a_k$.

Therefore, total number of wrong answers is

$$1(a_1 - a_2) + 2(a_2 - a_3) + 3(a_3 - a_4) + \cdots + k(a_{k-1} - a_k)$$
$$= a_1 + a_2 + a_3 + \cdots + a_k$$

2. We have total 3 + 4 + 5 = 12 points. So, number of Δs that can be formed using 12 such points is given by

Total number of ways of selecting three point – Number of ways three collinear points are selected

$$= {}^{12}C_3 - {}^3C_3 - {}^4C_3 - {}^5C_3$$
$$= \frac{12\times11\times10}{6} - 1 - 4 - \frac{5\times4}{2\times1}$$
$$= 220 - 15 = 205$$

3. '+' signs can be put in a row in one way creating seven gaps shown as arrows:

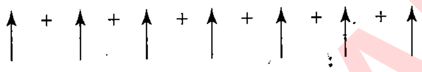


Fig. 5.57

Now 4 '–' signs must be kept in these gaps. So, no two '–' signs should be together.

Out of these 7 gaps 4 can be chosen in ${}^7C_4$ ways. Hence, required number of arrangements is

$${}^7C_4 = {}^7C_3 = \frac{7\times6\times5}{3\times2\times1} = 35$$

4. We know that number of derangements of n objects is

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{1}{n!}\right]$$

Therefore, number of ways of putting all the 4 balls into boxes of different colour is

$$4!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = 4!\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right)$$
$$= 24\left(\frac{12-4+1}{24}\right)$$
$$= 9$$

True or false

1. True. See the proof in theory section in properties of ${}^nC_r$.

Multiple choice questions with one correct answer

1. c. ${}^nC_{r-1} = 36$, ${}^nC_r = 84$, ${}^nC_{r+1} = 126$

We know that

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{r}{n-r+1}$$

$$\Rightarrow \quad \frac{36}{84} = \frac{r}{n-r+1}$$

$$\Rightarrow \quad \frac{r}{n-r+1} = \frac{3}{7}$$

$$\Rightarrow \quad 3n - 10r + 3 = 0 \qquad (1)$$

Also,

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r} = \frac{84}{126} = \frac{2}{3}$$

$$\Rightarrow \quad 2n - 5r - 3 = 0 \qquad (2)$$

Solving (1) and (2), we get $n = 9$ and $r = 3$.

2. Number of words when repetition is allowed is $10 \times 10 \times 10 \times 10 \times 10 = 10^5$.

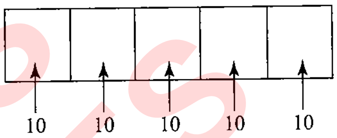


Fig. 5.58

Number of words when repetition is not allowed is

$10 \times 9 \times 8 \times 7 \times 6 = 30240$.

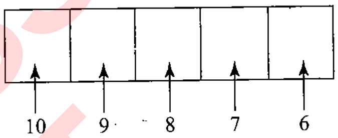


Fig. 5.59

Hence, required number of words in which at least one letter is repeated is 100000 – 30240 = 69760.

3. c. ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$

$$= {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$$
$$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 + {}^{47}C_4$$
[Using ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$]
$$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + ({}^{48}C_3 + {}^{48}C_4)$$
$$= {}^{51}C_3 + {}^{50}C_3 + ({}^{49}C_3 + {}^{49}C_4)$$
$$= {}^{51}C_3 + ({}^{50}C_3 + {}^{50}C_4)$$
$$= {}^{51}C_3 + {}^{51}C_4$$
$$= {}^{52}C_4$$

4. d. $\overline{1}\,\overline{2}\,\overline{3}\,\overline{4}\,\overline{5}\,\overline{6}\,\overline{7}\,\overline{8}$

Two women can choose two chairs out of 1, 2, 3, 4 in ${}^4C_2$ ways, and can arrange among themselves in 2! ways. Three men can choose 3 chairs out of 6 remaining chairs in ${}^6C_3$ ways and can arrange themselves in 3! ways.

Therefore, total number of possible arrangements is ${}^4C_2 \times 2! \times {}^6C_3 \times 3! = {}^4P_2 \times {}^6P_3$.

5. b. $\times B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6 \times$

Let first six boys sit, which can be done in 6! ways. Once they have been seated, the two brothers can be made to occupy seats in between or in extreme (i.e. on crosses) in ${}^7P_2$ ways.

Hence, required number of ways is ${}^7P_2 \times 6!$.

6. a. We know that a number is divisible by 3 if the sum of its digits is divisible by 3. Now out of 0, 1, 2, 3, 4, 5 if we take 1, 2, 3, 4, 5 or 0, 1, 2, 4, 5, then the 5-digit numbers will be divisible by 3.

Case I:

Total number of five-digit numbers formed using the digits 1, 2, 3, 4, 5 is 5! = 120.

Case II:
Taking 0, 1, 2, 4, 5, total number is $4 \times 4! = 96$.

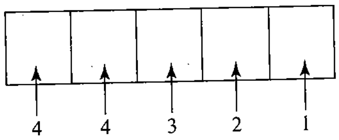


Fig. 5.60

From case I and case II, total number divisible by 3 is $120 + 96 = 216$.

7. b. Distinct n-digit numbers which can be formed using digits 2, 5 and 8 are 3^n. We have to find n so that

$3^n \geq 900$

$\Rightarrow 3^{n-2} \geq 100$

$\Rightarrow n - 2 \geq 5$

$\Rightarrow n \geq 7$

So the least value of n is 7.

8. c. X – X – X – X – X

The four digits 3, 3, 5, 5 can be arranged at (–) places in $\dfrac{4!}{2!2!} = 6$ ways. The five digits 2, 2, 8, 8, 8 can be arranged at (X) place in $\dfrac{5!}{2!3!} = 10$ ways.

Total number of arrangements is $6 \times 10 = 60$.

9. b. A regular polygon of n sides has n vertices, no two of which are collinear. Out of these n points, ${}^nC_3$ triangles can be formed.

$\therefore \quad T_n = {}^nC_3;\ T_{n+1} = {}^{n+1}C_3$

Given,

$T_{n+1} - T_n = 21$

$\Rightarrow {}^{n+1}C_3 - {}^nC_3 = 21$

$\Rightarrow \dfrac{(n+1)n(n-1)}{3 \times 2 \times 1} - \dfrac{n(n-1)(n-2)}{3 \times 2 \times 1} = 21$

$\Rightarrow n(n-1)(n+1-n+2) = 126$

$\Rightarrow n(n-1) = 42$

$\Rightarrow n(n-1) = 7 \times 6$

$\Rightarrow n = 7$

10. a. Total number of ways of arranging the letters of the word BANANA is $\dfrac{6!}{2!3!} = 60$.

Number of words in which 2N's come together is $5!/3! = 20$.

Hence, the required number is $60 - 20 = 40$.

11. c.

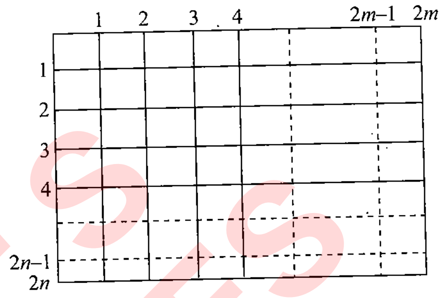


Fig. 5.61

If we see the blocks in terms of lines, then there are $2m$ vertical lines and $2n$ horizontal lines.

To form the required rectangle we must select two horizontal lines, one even numbered (out of 2, 4, ..., $2n$) and one odd numbered (out of 1, 3, ..., $2n - 1$) and similarly two vertical lines.

The number of rectangles is ${}^mC_1 \times {}^mC_1 \times {}^nC_1 \times {}^nC_1 = m^2n^2$

12. c. If L.C.M. of p and q is $r^2t^4s^2$, then distribution of factors r is as follows:

p	q
r^0	r^2
r^1	r^2
r^2	r^2
r^2	r^0
r^2	r^1

Thus, factor r can be distributed in $2 \times 3 - 1$ ways. Similarly, factors t and s can be distributed in $2 \times 5 - 1$ and $2 \times 3 - 1$ ways, respectively.

Hence, number of ordered pairs are $(2 \times 3 - 1) \times (2 \times 5 - 1) \times (2 \times 3 - 1) = 225$.

13. c. The letters of word COCHIN in alphabetic order are C, C, H, I, N, O. Fixing first letter C and keeping C at second place, rest 4 can be arranged in 4! ways.

Similarly, the total number of words starting with CH, CI, CN is 4! in each case.

Then fixing first two letters as CO, next four places when filled in alphabetic order gives the word COCHIN.

Therefore, number of words coming before COCHIN is $4 \times 4! = 4 \times 24 = 96$.

14. d. Total number of unordered pairs of disjoint subsets = $\dfrac{3^4 + 1}{2} = 41$.

CHAPTER
6

Binomial Theorem

INTRODUCTION

Consider

$$(x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2$$
$$= x^2 + 2xy + y^2$$

Here, xy can be written in two ways (xy and yx). Hence the coefficient of xy is equal to the number of ways x, y can be arranged, which is 2! Consider

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Coefficient of x^2y is 3 as for x^2y, we can arrange x, x, y in $3!/2!$ ways.

With similar arguments, we have

$$(x+y)^4 = x^4 + \frac{4!}{3!}x^3y + \frac{4!}{2!2!}x^2y^2 + \frac{4!}{3!}xy^3 + y^4$$
$$= {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$$

Now by this development, we can find the coefficient of any term in the binomial expansion of any positive integral power.

Thus in the expansion of $(x+y)^7$, the coefficient of x^3y^4 is equivalent to number of ways x, x, x, y, y, y, y can be arranged which is $7!/(3!4!) = {}^7C_3$.

Hence, in general

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n,$$
where $n \in N$

Note:

- *This expansion has $(n+1)$ terms.*
- *Its general term is given by $T_{r+1} = {}^nC_r\, x^{n-r}y^r$, where* $r = 0, 1, 2, 3, \ldots, n$.
- *In each term, the degree is n and the coefficient of $x^{n-r}y^r$ is equal to the number of ways $(n-r)$ x's and r y's can be arranged, which is given by*

$$\frac{n!}{(n-r)!\,r!} = {}^nC_r$$

- *$(p+1)^{th}$ term from the end is $(n-p+1)^{th}$ term from the beginning, i.e. T_{n-p+1}.*
- *$S = (x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n$, where $n \in N$*

$$= \sum_{r=0}^{n} {}^nC_r x^{n-r}y^r$$

Replacing r by $n-r$, we have

$$S = \sum_{r=0}^{n} {}^nC_{n-r}x^{n-(n-r)}y^{n-r} \qquad (\because {}^nC_r = {}^nC_{n-r})$$

$$= \sum_{r=0}^{n} {}^nC_r x^r y^{n-r}$$

$$= {}^nC_0y^n + {}^nC_1y^{n-1}x + {}^nC_2y^{n-2}x^2 + \cdots + {}^nC_nx^n$$

$$= {}^nC_ny^n + {}^nC_{n-1}y^{n-1}x + {}^nC_{n-2}y^{n-2}x^2 + \cdots + {}^nC_0x^n$$

Thus replacing r by $n-r$, we are in fact writing the binomial expansion in the reverse order.

Properties of Binomial Coefficient

- Sum of two consecutive binomial coefficients, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$
- Ratio of two consecutive binomial coefficients,

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

- If ${}^nC_x = {}^nC_y$, then either $x = y$ or $x + y = n$. So,

So, ${}^nC_r = {}^nC_{n-r} = \dfrac{n!}{r!(n-r)!}$

Pascal's Triangle

Coefficient of binomial expansion can also be easily determined by Pascal's triangle.

$n=0$						1						
$n=1$					1		1					
$n=2$				1		2		1				
$n=3$			1		3		3		1			
$n=4$		1		4		6		4		1		
$n=5$	1		5		10		10		5		1	

Construction of this triangle also justifies ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ as

$n=3$	1	$3 = {}^3C_1$	$3 = {}^3C_2$		1
$n=4$	1	4	$6 = {}^4C_2$	4	1

Some Standard Expansions

We know that

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n, \text{ where } n \in N$$

Putting $y = 1$, we have

$$(1+x)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + \cdots + {}^nC_n$$
$$= {}^nC_nx^n + {}^nC_{n-1}x^{n-1} + {}^nC_{n-1}x^{n-2} + \cdots + {}^nC_0$$
$$= {}^nC_0 + {}^nC_1x^1 + {}^nC_2x^2 + \cdots + {}^nC_nx^n$$
$$= \sum_{r=0}^{n} {}^nC_rx^r$$

In the above expansion, replacing x by $-x$, we have

$$(1-x)^n = {}^nC_0 - {}^nC_1x^1 + {}^nC_2x^2 - \cdots + (-1)^r\,{}^nC_rx^r + \cdots + (-1)^n\,{}^nC_nx^n$$

$$= \sum_{r=0}^{n} {}^nC_r(-x)^r$$

$$= \sum_{r=0}^{n} (-1)^r\,{}^nC_rx^r$$

Tips for finding coefficients of required term

- Coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$
  e.g. coefficient of x^6 in the expansion of $(1+x)^{10}$ is ${}^{10}C_6$

- Coefficient of x^r in the expansion of $(1 + ax)^n$ is ${}^nC_r a^r$ as term containing x^r is ${}^nC_r a^r x^r$
  e.g. coefficient of x^5 in the expansion of $(1 + 2x)^{12}$ is ${}^{12}C_5 2^5$
  coefficient of x^6 in the expansion of $(1 - 3x)^{15}$ is ${}^{15}C_6 3^6$
  coefficient of x^7 in the expansion of $(1 - 4x)^{13}$ is $-{}^{13}C_7 4^7$
- Coefficient of x^r in the expansion of $(a + bx)^n$ is ${}^nC_r b^r a^{n-r}$ as term containing x^r is ${}^nC_r a^{n-r}(bx)^r$
  e.g. coefficient of x^4 in the expansion of $(3 + 2x)^9$ is ${}^9C_4 2^4 3^5$ (understand the distribution of exponent '9' when exponent 4 is taken by $2x$ the remaining exponent 5 will be taken by 3)
  Coefficient of x^3 in the expansion of $(4 - 5x)^{10}$ is $-{}^{10}C_3 5^3 4^7$
- Coefficient of x^r in the expansion of $(1 + x^p)^n$ is ${}^nC_{r/p}$ if r is multiple of p
  e.g. coefficient of x^{10} in the expansion of $(1 + x^2)^{15}$ is ${}^{15}C_5$
  coefficients of x^{12} in the expansion of $(1 + x^3)^{10}$ is ${}^{10}C_4$
  coefficient of x^{10} in the expansion of $(1 + 3x^2)^{12}$ is ${}^{12}C_5 3^5$
  coefficient of x^{15} in the expansion of $(2 + 5x^3)^{20}$ is ${}^{20}C_5 5^5 2^{15}$

Example 6.1 **Simplify: $x^5 + 10x^4a + 40x^3a^2 + 80x^2a^3 + 80xa^4 + 32a^5$.**

Sol. We have,

$$(x + a)^n = {}^nC_0 + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \cdots$$

$$(x + 2a)^5 = {}^5C_0 x^5 + {}^5C_1 x^4(2a) + {}^5C_2 x^3 (2a)^2 + {}^5C_3 x^2 (2a)^3 + {}^5C_4 x(2a)^4 + {}^5C_5 (2a)^5$$

$$= x^5 + 10x^4a + 40x^3a^2 + 80x^2a^3 + 80xa^4 + 32a^5$$

Example 6.2 **Find the value of**

$$\frac{18^3 + 7^3 + 3\times18\times7\times25}{3^6 + 6\times243\times2 + 15\times81\times4 + 20\times27\times8 + 15\times9\times16 + 6\times3\times32 + 64}$$

Sol. The form of the numerator is

$$a^3 + b^3 + 3ab(a + b) = (a + b)^3$$

$$\therefore\ N' = (18 + 7)^3 = 25^3$$

$$\therefore\ D' = 3^6 + {}^6C_1 3^5 \times 2^1 + {}^6C_2 3^4 \times 2^2 + {}^6C_3 3^3 \times 2^3 + {}^6C_4 3^2 \times 2^4 + {}^6C_5 3 \times 2^5 + {}^6C_6 2^6$$

This is clearly the expansion of $(3 + 2)^6 = 5^6 = (25)^3$.

$$\therefore\ \frac{N'}{D'} = \frac{(25)^3}{(25)^3} = 1$$

Example 6.3 **Evaluate $(1.0025)^{10}$, correct to six decimal places.**

Sol. $(1.0025)^{10} = (1 + 0.0025)^{10}$

$$= 1 + {}^{10}C_1 (0.0025) + {}^{10}C_2 (0.0025)^2 + {}^{10}C_3 (0.0025)^3 + \cdots + (0.0025)^{10}$$

$$= 1 + 10 \times (0.0025) + 45(0.00000625)$$

Leaving other terms, as we require the value up to five places of decimals, we have

$$(1.0025)^{10} = 1 + 0.025 + 0.000281 = 1.025281$$

Example 6.4 **Find the 6th term in the expansion of $(2x^2 - 1/3x^2)^{10}$.**

Sol. $T_{r+1} = {}^nC_r x^{n-r} y^r$ for $(x + y)^n$

Hence for $(2x^2 - 1/3x^2)^{10}$,

$$T_6 = {}^{10}C_5 (2x^2)^5 \left(-\frac{1}{3x^2}\right)^5$$

$$= -\frac{10!}{5!5!} 32 \times \frac{1}{243}$$

$$= -\frac{896}{27}$$

Example 6.5 ***If* the 21st and 22nd terms in the expansion of $(1 - x)^{44}$ are equal, then find the value of x**

Sol. $T_{22} = T_{21} \Rightarrow {}^{44}C_{21}(-x)^{21} = {}^{44}C_{20}(-x)^{20}$

$$\therefore\ \frac{{}^{44}C_{21}}{{}^{44}C_{20}} = \frac{1}{-x}\ or\ \frac{{}^nC_r}{{}^nC_{r-1}} = -\frac{1}{x} = \frac{n - r + 1}{r}$$

Put $n = 44$, $r = 21$

$$\therefore\ -\frac{1}{x} = \frac{44 - 21 + 1}{21} = \frac{24}{21} = \frac{8}{7}$$

$$\therefore\ x = -7/8$$

Example 6.6 **Find the coefficient of x^4 in the expansion of $(x/2 - 3/x^2)^{10}$.**

Sol. In the expansion of $(x/2 - 3/x^2)^{10}$, the general term is

$$T_{r+1} = {}^{10}C_r \left(\frac{x}{2}\right)^{10-r} \left(-\frac{3}{x^2}\right)^r$$

$$= {}^{10}C_r (-1)^r \frac{3^r}{2^{10-r}} x^{10-3r}$$

Here, the exponent of x is

$$10 - 3r = 4 \Rightarrow r = 2$$

$$\therefore\ T_{2+1} = {}^{10}C_2 \left(\frac{x}{2}\right)^8 \left(-\frac{3}{x^2}\right)^2$$

$$= \frac{10\times9}{1\times2} \times \frac{1}{2^8} \times 3^2 \times x^4$$

$$= \frac{405}{256} x^4$$

Therefore, the required coefficient is 405/256.

Example 6.7 **Find the term in $\left(\sqrt[3]{\frac{a}{\sqrt{b}}}+\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^{21}$ which has the same power of a and b.**

Sol. We have,

$$T_{r+1}={}^{21}C_r\left(\sqrt[3]{\frac{a}{\sqrt{b}}}\right)^{21-r}\left(\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^r$$

$$={}^{21}C_r a^{7-(r/2)}b^{(2/3)r-(7/2)}$$

Since the powers of a and b are the same, therefore

$$7-\frac{r}{2}=\frac{2}{3}r-\frac{7}{2}\Rightarrow r=9$$

Example 6.8 **Find the term independent of x in the expansion of $(1+x+2x^3)\,[(3x^2/2)-(1/3x)]^9$.**

Sol. We have,

$$(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$$

$$=(1+x+2x^3)\left[\left(\frac{3}{2}x^2\right)^9-{}^9C_1\left(\frac{3}{2}x^2\right)^8(1/3x)+\cdots+(-1)^9(1/3x)^9\right] \quad (1)$$

Therefore, the term independent of x in the expansion is

$$1a_0+1a_1+2a_3 \quad (2)$$

where a_m is the coefficient of x^m in the second bracket [ ] of (1).
Now, $(r+1)^{\text{th}}$ term in [ ] of (1) is

$${}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}(-1/3x)^r$$

$$=(-1)^r\,{}^9C_r\left(\frac{3}{2}\right)^{9-r}(1/3^r)(x^{18-3r})$$

$\therefore$ a_{18-3r} = coefficient of x^{18-3r}

$$=(-1)^r\,{}^9C_r\left(\frac{3}{2}\right)^{9-r}(1/3^r)$$

Now for a_0,

$18-3r=0$

$$\Rightarrow r=6\Rightarrow a_0=(-1)^6\,{}^9C_6\left(\frac{3}{2}\right)^{9-6}\times\frac{1}{3^6}=\frac{7}{18}$$

For a_1,

$18-3r=-1$

$\Rightarrow$ $r=19/3$, which is fractional

$\therefore$ $a_1=0$

For a_3,

$18-3r=-3$

$$\Rightarrow r=7\Rightarrow a_3=(-1)^7\,{}^9C_7\left(\frac{3}{2}\right)^{9-7}\frac{1}{3^7}=-\frac{1}{27}$$

Hence from (2), the required term is

$$1\times\frac{7}{18}+0+2\times\left(-\frac{1}{27}\right)=\frac{17}{52}$$

Example 6.9 **Find the coefficient of x^k in $1+(1+x)+(1+x)^2+\cdots+(1+x)^n\ (0\le k\le n)$.**

Sol. The expression being in G.P., we have

$$E=1+(1+x)+(1+x)^2+\cdots+(1+x)^n$$

$$=\frac{(1+x)^{n+1}-1}{(1+x)-1}=x^{-1}[(1+x)^{n+1}-1]$$

Therefore, the coefficient of x^k in E is equal to the coefficient of x^{k+1} in $[(1+x)^{n+1}-1]$, which is given by ${}^{n+1}C_{k+1}$.

Example 6.10 **Find the coefficient of x^4 in the expansion of $(2-x+3x^2)^6$.**

Sol. $(2-x+3x^2)^6=[2-x(1-3x)]^6$

$$=2^6-{}^6C_1\,2^5x(1-3x)+{}^6C_2\,2^4x^2\times(1-3x)^2-{}^6C_3\,2^3x^3(1-3x)^3+{}^6C_4\,2^2x^4(1-3x)^4-{}^6C_5\,2x^5(1-3x)^5+{}^6C_6\,2^6$$

Obviously, x^4 occurs in 3rd, 4th and 5th terms. Now, 3rd term is $15\times16x^2(1-6x+9x^2)$. Here, the coefficient of x^4 is $15\times16\times9=2160$. The 4th term is $-20\times8x^3(1-9x+27x^2-27x^3)$. Here, the coefficient of x^4 is $20\times8\times9=1440$. The 5th term is $15\times4x^4\,[1-4\times3x+\cdots+(3x)^4]$. Here, the coefficient of x^4 is $15\times4=60$. Hence, the required coefficient of x^4 is $2160+1440+60=3660$.

Example 6.11 **Find the coefficient of x^{50} in $(1+x)^{101}\times(1-x+x^2)^{100}$.**

Sol. $(1+x)^{101}(1-x+x^2)^{100}$

$$=(1+x)[(1+x)^{100}(1-x+x^2)^{100}]$$

$$=(1+x)(1-x^3)^{100}$$

$$=(1-x^3)^{100}+x(1-x^3)^{100}$$

Now, coefficient of x^{50} in $[(1-x^3)^{100}+x(1-x^3)^{100}]$ is
Coefficient of x^{50} in $(1-x^3)^{100}$ + coefficient of x^{49} in $(1-x^3)^{100}$

= 0 (as 49 and 50 are not a multiple of 3)

Example 6.12 **If sum of the coefficients of the first, second and third terms of the expansion of $\left(x^2+\frac{1}{x}\right)^m$ is 46, then find the coefficient of the term that does not contain x**

Sol. We are given

$${}^mC_0+{}^mC_1+{}^mC_2=46$$

$\Rightarrow$ $2m+m(m-1)=90$

$\Rightarrow$ $m^2+m-90=0$

$\Rightarrow$ $m=9$ as $m>0$

Now, $(r+1)$th term of $\left(x^2+\frac{1}{x}\right)^m$ is ${}^mC_r(x^2)^{m-r}\left(\frac{1}{x}\right)^r$

$$= {}^mC_r\, x^{2m-3r}$$

For this to be independent of x, $2m - 3r = 0 \Rightarrow r = 6$

$\therefore$ term independent of x is ${}^9C_6 = 84$.

Example 6.13 **Find the coefficient of x^{20} in**

$$\left(x^2 + 2 + \frac{1}{x^2}\right)^{-5} (1 + x^2)^{40}.$$

Sol. $E = (1 + x^2)^{40} \left(x + \dfrac{1}{x}\right)^{-10} = x^{10} (1 + x^2)^{30}$

Hence we have to choose the term of x^{20} in $x^{10}(1 + x^2)^{30}$ or the term of $(x^2)^5$ in $(1 + x^2)^{30}$ which will be ${}^{30}C_5$.

Example 6.14 **Find the coefficient of x^5 in the expansion of $(1 + x^2)^5 . (1 + x)^4$ is 60.**

Sol. Coefficient of x^5 in $(1 + x^2)^5 . (1 + x)^4$

$= $ Coefficient of x^5 in $({}^5C_0 + {}^5C_1x^2 + {}^5C_2x^4 + {}^5C_3x^6 + \cdots)$
$({}^4C_0 + {}^4C_1x + {}^4C_2x^2 + \cdots)$

$= {}^5C_1{}^4C_3 + {}^5C_2{}^4C_1$

$= 20 + 40$

$= 60$

Example 6.15 **Find the coefficient of x^{13} in the $(1 - x)^5 \times (1 + x + x^2 + x^3)^4$.**

Sol. $E = (1 - x)^5 (1 + x)^4 (1 + x^2)^4$

$= (1 - x)(1 - x^2)^4 (1 + x^2)^4$

$= (1 - x)(1 - x^4)^4$

$= (1 - x)[1 - 4(x^4) + 6(x^4)^2 - 4(x^4)^3 + (x^4)^4]$

Coefficient of x^{13} is $(-1)(-4) = 4$

Example 6.16 Find the coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$

Sol. $(1 + x)^{11} (1 + x^2)^{11}$

$= (1 + {}^{11}C_1\, x + {}^{11}C_2\, x^2 + {}^{11}C_3x^3 + {}^{11}C_4x^4 + \cdots)$

$\times (1 + {}^{11}C_1\, (x^2) + {}^{11}C_2(x^2)^2 + \cdots)$

Coefficient of x^4 is ${}^{11}C_2 .1 + {}^{11}C_1 . {}^{11}C_2 + {}^{11}C_4 = 990$

Example 6.17 **Find the number of terms which are free from radical signs in the expansion of $(y^{1/5} + x^{1/10})^{55}$.**

Sol. In the expansion of $(y^{1/5} + x^{1/10})^{55}$,

$$T_{r+1} = {}^{55}C_r (y^{1/5})^{55-r} (x^{1/10})^r = {}^{55}C_r\, y^{11-r/5} x^{r/10}$$

Thus T_{r+1} will be independent of radicals if the exponents $r/5$ and $r/10$ are integers for $0 \le r \le 55$, which is possible only when $r = 0, 10, 20, 30, 40, 50$.

Therefore, there are six terms, i.e., T_1, T_{11}, T_{21}, T_{31}, T_{41}, T_{51} which are independent of radicals.

Concept Application Exercise 6.1

1. Find the constant term in the expansion of $(x - 1/x)^6$.
2. Find the coefficient of x^{-10} in the expansion of $\left(\dfrac{a}{x} + bx\right)^{12}$.
3. If x^4 occurs in the r^{th} term in the expansion of $\left(x^4 + \dfrac{1}{x^3}\right)^{15}$, then find the value of r.
4. The first three terms in the expansion of $(1 + ax)^n$ $(n \ne 0)$ are 1, $6x$ and $16x^2$. Then find the value of a and n.
5. If p and q be positive, then prove that the coefficients of x^p and x^q in the expansion of $(1 + x)^{p+q}$ will be equal.
6. If the coefficient of 4^{th} term in the expansion of $(a + b)^n$ is 56, then find the value of n.
7. In $\left(\sqrt[3]{2} + \dfrac{1}{\sqrt[3]{3}}\right)^n$ if the ratio of 7^{th} term from the beginning to the 7^{th} term from the end is 1/6, then find the value of n.
8. If the coefficient of $(2r + 3)^{\text{th}}$ and $(r - 1)^{\text{th}}$ terms in the expansion of $(1 + x)^{15}$ are equal, then find the value of r.
9. If x^p occurs in the expansion of $(x^2 + 1/x)^{2n}$, prove that its coefficient is
$$\frac{(2n)!}{\left[\frac{1}{3}(4n-p)\right]!\left[\frac{1}{3}(2n+p)\right]!}.$$
10. Find the number of irrational terms in the expansion of $(5^{1/6} + 2^{1/8})^{100}$.

Multinomial Expansions

Consider the expansion of $(x + y + z)^{10}$. In the expansion, each term has different powers of x, y and z and sum of these powers is always 10.

One of the terms is $\lambda x^2y^3z^5$. Now, the coefficient of this term λ is equal to the number of ways 2 x's, 3 y's and 5 z's are arranged, i.e., $10!/(2!3!5!)$. Thus,

$$(x + y + z)^{10} = \sum \frac{10!}{P_1!P_2!P_3!} x^{P_1} y^{P_2} z^{P_3}$$

where $P_1 + P_2 + P_3 = 10$ and $0 \le P_1, P_2, P_3 \le 10$. In general,

$$(x_1 + x_2 + \cdots + x_r)^n = \sum \frac{n!}{P_1!P_2!\cdots P_r!} x_1^{P_1} x_2^{P_2} \cdots x_r^{P_r}$$

where $P_1 + P_2 + P_3 + \cdots + P_r = n$ and $0 \le P_1, P_2, \ldots, P_r \le n$.

Number of Terms in the Expansion of $(x_1 + x_2 + \cdots + x_r)^n$

From the general term of the above expansion, we can conclude that number of terms is equal to the number of ways different powers can be distributed to $x_1, x_2, x_3, \ldots, x_n$ such that sum of powers is always n.

Number of non-negative integral solutions of $x_1 + x_2 + \cdots + x_r = n$ is ${}^{n+r-1}C_{r-1}$.

For example, number of terms in the expansion of $(x + y + z)^3$ is ${}^{3+3-1}C_{3-1} = {}^5C_2 = 10$.

As in the expansion, we have terms like $x^0y^0z^3$, $x^0y^1z^2$, $x^0y^2z^1$, $x^0y^3z^0$, $x^1y^0z^2$, $x^1y^1z^1$, $x^1y^2z^0$, $x^2y^0z^1$, $x^2y^1z^0$, $x^3y^0z^0$.

Number of terms in $(x + y + z)^n$ is ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$.

Number of terms in $(x + y + z + w)^n$ is ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ and so on.

Example 6.18 **If the number of terms in the expansion of $(x + y + z)^n$ are 36, then find the value of n.**

Sol. Number of terms in the expansion of $(x + y + z)^n$ is $(n + 1)(n + 2)/2 = 36$

$\Rightarrow \quad (n + 1)(n + 2) = 72$

$\Rightarrow \quad n = 7$

Example 6.19 **Find the coefficient of a^3b^4c in the expansion of $(1 + a - b + c)^9$.**

Sol. $(1 + a - b + c)^9 = \Sigma \dfrac{9!}{x_1!x_2!x_3!x_4!}(1)^{x_1}(a)^{x_2}(-b)^{x_3}(c)^{x_4}$

(for $x_1 = 1, x_2 = 3, x_3 = 4, x_4 = 1$)

Hence, the coefficient of a^3b^4c is $\dfrac{9!}{1!3!4!1!} = \dfrac{9!}{3!4!}$.

Example 6.20 **Find the coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$.**

Sol. In this case, write $a^3b^4c^5 = (ab)^x(bc)^y(ca)^z$ (say). Then,

$a^3b^4c^5 = a^{z+x}\, b^{x+y}\, c^{y+z}$

$\Rightarrow \quad z + x = 3, x + y = 4, y + z = 5$

Adding all,

$2(x + y + z) = 12$

$\Rightarrow \quad x + y + z = 6$

Then, $x = 1, y = 3, z = 2$.

Therefore, the coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$ or the coefficient of $(ab)^1(bc)^3(ca)^2$ in the expansion of $(bc + ca + ab)^6$ is $6!/(1!3!2!) = 60$.

Example 6.21 **Find the coefficient of x^7 in the expansion of $(1 + 3x - 2x^3)^{10}$.**

Sol. Coefficient of x^7 in the expansion of $(1 + 3x - 2x^3)^{10}$ is

$$\sum \frac{10!}{n!\, n_2!\, n_3!}(1)^{n_1}(3x)^{n_2}(-2x^3)^{n_3}$$

where $n_1 + n_2 + n_3 = 10$ and $n_2 + 3n_3 = 7$, the possible values of n_1, n_2 and n_3 are shown in the margin.

n_1	n_2	n_3
3	7	0
5	4	1
7	1	2

Therefore the coefficient of x^7 is

$$\frac{10!}{3!7!0!}(1)^3(3)^7(-2)^0 + \frac{10!}{5!4!1!}(1)^5(3)^4(-2)^1 + \frac{10}{7!1!2!}(1)^7(3)^1(-2)^2$$

$= 262440 - 204120 + 4320$

$= 62640$

ANALYSIS OF BINOMIAL EXPANSION

Sum of Binomial Coefficients

For the sake of convenience, the coefficients ${}^nC_0, {}^nC_1, \ldots, {}^nC_r, \ldots, {}^nC_n$ are usually denoted by $C_0, C_1, \ldots, C_r, \ldots, C_n$, respectively.

- $C_0 + C_1 + C_2 + \cdots + C_n = 2^n$

Proof:

$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n$

Putting $x = y = 1$, we have

$(1+1)^n = 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$

Similarly,

${}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1} = (1+1)^{n+1} = 2^{n+1}$

- $C_0 - C_1 + C_2 - \cdots + C_n = 0$

Proof:

$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n$

Putting $x = 1$ and $y = -1$, we have

$(1 - 1)^n = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - \cdots = 0$

$\Rightarrow 0 = ({}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots) - ({}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots)$

$\Rightarrow ({}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots) = ({}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots)$

$\Rightarrow C_0 + C_1 + C_2 + \cdots + C_n = 2^n = ({}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots) + ({}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots) = 2({}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots)$

$\Rightarrow ({}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots) = 2^{n-1}$

Similarly,

$({}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots) = 2^{n-1}$

- $C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots = 2^{n-1}$

See above proof.

e.g. ${}^{n+1}C_0 + {}^{n+1}C_2 + {}^{n+1}C_4 + {}^{n+1}C_6 + \cdots = 2^n$

${}^{n-1}C_1 + {}^{n-1}C_3 + {}^{n-1}C_5 + {}^{n-1}C_7 + \cdots = 2^{n-2}$

${}^{20}C_0 + {}^{20}C_2 + {}^{20}C_4 + {}^{20}C_6 + \cdots = 2^{19}$

${}^{15}C_1 + {}^{15}C_3 + {}^{15}C_5 + \cdots = 2^{14}$

etc.

Example 6.22 **Find the sum ${}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9$.**

Sol. We know that

$$2^{n-1} = {}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots$$

So,

$${}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + \cdots + {}^{10}C_9 = 2^{10-1} = 2^9$$

Example 6.23 Find the sum

$$\frac{1}{1!(n-1)!}+\frac{1}{3!(n-3)}+\frac{1}{5!(n-5)}+\cdots.$$

Sol. $S=\dfrac{1}{1!(n-1)!}+\dfrac{1}{3!(n-3)!}+\dfrac{1}{5!(n-5)!}+\cdots$

Multiplying each term by $n!/n!$, S reduces to

$$S=\frac{1}{n!}\left[\frac{n!}{1!(n-1)!}+\frac{1}{3!}\frac{n!}{(n-3)!}+\frac{1}{5!}\frac{n!}{(n-5)!}+\cdots\right]$$

$$=\frac{1}{n!}\left[{}^nC_1+{}^nC_3+{}^nC_5+\cdots\right]$$

$$=\frac{2^{n-1}}{n!}$$

Example 6.24 Find the sum $\sum_{k=0}^{10}{}^{20}C_k$.

Sol. $S=\sum_{k=0}^{10}{}^{20}C_k={}^{20}C_0+{}^{20}C_1+\cdots+{}^{20}C_{10}$

Now,

$${}^{20}C_0+{}^{20}C_1+\cdots+{}^{20}C_9+{}^{20}C_{10}+{}^{20}C_{11}+\cdots+{}^{20}C_{20}=2^{20}$$

$$\Rightarrow ({}^{20}C_0+{}^{20}C_{20})+({}^{20}C_1+{}^{20}C_{19})+\cdots+({}^{20}C_9+{}^{20}C_{11})+{}^{20}C_{10}=2^{20}$$

$$\Rightarrow 2[{}^{20}C_0+{}^{20}C_1+\cdots+{}^{20}C_{10}]+{}^{20}C_{10}=2^{20}\ (\because {}^nC_r={}^nC_{n-r})$$

$$\therefore\ S={}^{20}C_0+{}^{20}C_1+\cdots+{}^{20}C_{10}=2^{19}-\frac{1}{2}\,{}^{20}C_{10}$$

Example 6.25 Find the sum of the series ${}^{15}C_0+{}^{15}C_1+{}^{15}C_2+\ldots\ldots+{}^{15}C_7$

Sol. Let $S={}^{15}C_0+{}^{15}C_1+{}^{15}C_2+\ldots+{}^{15}C_7$

Here the series is exactly half series(8 terms)

As the full series (16 terms) will be ${}^{15}C_0+{}^{15}C_1+{}^{15}C_2+\ldots+{}^{15}C_7+{}^{15}C_8+\ldots+{}^{15}C_{15}$

So the sum of S is exactly half of the full series that is half of 2^{15} which is 2^{14}

Hence, ${}^{15}C_0+{}^{15}C_1+{}^{15}C_2+\ldots+{}^{15}C_7=2^{14}$.

Example 6.26 If the sum of coefficient of first half terms in the expansion of $(x+y)^n$ is 256 then find the greatest coefficient in the expansion

Sol. Sum of coefficient of first half of the terms $=2^{n-1}$ (half series) $=256=2^8$

$\Rightarrow\ n=9$

$\Rightarrow$ greatest coefficient $={}^9C_4=126$

Sum of Coefficients in Binomial Expansion

For $(x+y)^n={}^nC_0x^n+{}^nC_1x^{n-1}y+{}^nC_2x^{n-2}y^2+\cdots+{}^nC_ny^n$, we get the sum of coefficients by putting $x=y=1$, which is 2^n.

Similarly, in the expansion of $(x+y+z)^n$, we get the sum of coefficients by putting $x=y=z=1$.

For expansion of the type $(x^2+x+1)^n=a_0+a_1x+a_2x^2+\cdots+a_{2n}x^{2n}$, we get the sum of coefficients by putting $x=1$ or $3^n=a_0+a_1+a_2+\cdots+a_{2n}$ which is the required sum of coefficients.

Infact to find sum of coefficients we put the value of all variables as 1. (where variables are in x^n form)

In the above expansion, to get the sum of coefficients of even powers of x and odd powers of x, put $x=1$ and $x=-1$ alternatively and then add or subtract the two results.

Example 6.27 **Find the sum of all the coefficients in the binomial expansion of $(x^2+x-3)^{319}$.**

Sol. Putting $x=1$ in $(x^2+x-3)^{319}$, we get the sum of coefficients equal to $(1+1-3)^{319}=-1$.

Example 6.28 **If the sum of the coefficients in the expansion of $(\alpha^2x^2-2\alpha x+1)^{51}$ vanishes, then find the value of α.**

Sol. The sum of the coefficients of the polynomial $(\alpha^2x^2-2\alpha x+1)^{51}$ is obtained by putting $x=1$ in $(\alpha^2x^2-2\alpha x+1)^{51}$. Hence, from the given condition,

$$(\alpha^2-2\alpha+1)^{51}=0\Rightarrow\alpha=1$$

Example 6.29 **If $(1+x-2x^2)^{20}=a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_{40}x^{40}$, then find the value of $a_1+a_3+a_5+\cdots+a_{39}$.**

Sol. $(1+x-2x^2)^{20}=a_0+a_1x+a_2x^2+\cdots+a_{40}x^{40}$

Putting $x=1$, we get

$$a_0+a_1+a_2+a_3+\cdots+a_{40}=0 \quad (1)$$

Putting $x=-1$, we get

$$a_0-a_1+a_2-a_3+\cdots-a_{39}+a_{40}=2^{20} \quad (2)$$

Subtracting (1) from (2), we get

$$2[a_1+a_3+\cdots+a_{39}]=-2^{20}$$

$$\Rightarrow a_1+a_3+\cdots+a_{39}=-2^{19}$$

Middle Term in Binomial Expansion

Consider

$$(x+y)^n={}^nC_0x^n+{}^nC_1x^{n-1}y+{}^nC_2x^{n-2}y^2+\cdots+{}^nC_ny^n$$

- The middle term depends upon the value of n.
  (a) If n is even, then total number of terms in the expansion is odd. So, there is only one middle term, i.e., $(n/2+1)^{\text{th}}$ term is the middle term.
  (b) If n is odd, then total number of terms in the expansion is even. So, there are two middle terms, i.e., $[(n+1)/2]^{\text{th}}$ and $[(n+3)/2]^{\text{th}}$ terms are the two middle terms.
- Middle term always carries the greatest binomial coefficient. As when n is an even middle term, $T_{n/2+1}$ has the greatest binomial coefficient ${}^nC_{n/2}$.

  And when n is an odd middle term, $T_{(n+1)/2}$ and $T_{(n+3)/2}$ or $T_{\left(\frac{n-1}{2}\right)+1}$ and $T_{\left(\frac{n+1}{2}\right)+1}$ have the greatest binomial coefficients ${}^nC_{\left(\frac{n-1}{2}\right)}$ and ${}^nC_{\left(\frac{n+1}{2}\right)}$.

Example 6.30 **If the middle term in the expansion of $(x^2+1/x)^n$ is $924x^6$, then find the value of n.**

Sol. Since there is only one middle term here, n is even, therefore $(n/2+1)^{th}$ term is the middle term. Hence,

$$^nC_{n/2}\,(x^2)^{n/2}\left(\frac{1}{x}\right)^{n/2}=924x^6$$

$$\Rightarrow\quad x^{n/2}=x^6$$

$$\Rightarrow\quad n=12\ (\text{also } ^{12}C_6=924)$$

Example 6.31 **If the coefficient of the middle term in the expansion of $(1+x)^{2n+2}$ is α and the coefficients of middle terms in the expansion of $(1+x)^{2n+1}$ are β and γ, then relate α, β and γ.**

Sol. Since $(n+2)^{th}$ term is the middle term in the expansion of $(1+x)^{2n+2}$, therefore $\alpha={}^{2n+2}C_{n+1}$. Since $(n+1)^{th}$ and $(n+2)^{th}$ terms are middle terms in the expansion of $(1+x)^{2n+1}$, therefore,

$$\beta={}^{2n+1}C_n \text{ and } \gamma={}^{2n+1}C_{n+1}$$

But

$$^{2n+1}C_n+{}^{2n+1}C_{n+1}={}^{2n+2}C_{n+1}\Rightarrow\beta+\gamma=\alpha$$

Concept Application Exercise 6.2

1. Find the sum of coefficients in $(1+x-3x^2)^{4163}$.
2. If the sum of coefficients in the expansion of $(x-2y+3z)^n$ is 128, then find the greatest coefficient in the expansion of $(1+x)^n$.
3. If $(1+x-2x^2)^6=1+a_1x+a_2x^2+\cdots+a_{12}x^{12}$, then find the value of $a_2+a_4+a_6+\cdots+a_{12}$.
4. Find the middle term in the expansion of $\left(x^2+\frac{1}{x^2}+2\right)^n$.
5. In the expansion of $(1+x)^{50}$, find the sum of coefficients of odd powers of x.
6. Find the following sum:
$$\frac{1}{n!}+\frac{1}{2!\,(n-2)!}+\frac{1}{4!\,(n-4)!}+\cdots.$$
7. Find the sum of the last 30 coefficients in the expansion of $(1+x)^{59}$, when expanded in ascending powers of x.
8. Find the sum $\sum_{j=0}^{n}({}^{4n+1}C_j+{}^{4n+1}C_{2n-j})$.

RATIO OF CONSECUTIVE TERMS/COEFFICIENTS

Coefficients of x^r and x^{r+1} are $^nC_{r-1}$ and ^nC_r, respectively. Also, we know that

$$\frac{^nC_r}{^nC_{r-1}}=\frac{n-r+1}{r}$$

Similarly,

$$\frac{^nC_{r+1}}{^nC_r}=\frac{n-r}{r+1}\quad(\text{replacing } r \text{ by } r+1)$$

$$\frac{^{n+1}C_{r+1}}{^{n+1}C_r}=\frac{n-r+1}{r+1}\quad(\text{replacing } r \text{ by } r+1 \text{ and } n \text{ by } n+1) \text{ and so on.}$$

Example 6.32 **If the coefficients of three consecutive terms in the expansion of $(1+x)^n$ are 165, 330 and 462, respectively, then find the value of n.**

Sol. The coefficients of three consecutive terms, i.e. $(r+1)^{th}$, $(r+2)^{th}$, $(r+3)^{th}$ in expansion of $(1+x)^n$ are 165, 330 and 462 respectively, Then, coefficient of $(r+1)^{th}$ term is $^nC_r=165$, coefficient of $(r+2)^{th}$ term is $^nC_{r+1}=330$ and coefficient of $(r+3)^{th}$ term is $^nC_{r+2}=462$.

$$\therefore\quad\frac{^nC_{r+1}}{^nC_r}=\frac{n-r}{r+1}=2\Rightarrow n-r=2(r+1)\Rightarrow r=\frac{1}{3}(n-2)$$

and

$$\frac{^nC_{r+2}}{^nC_{r+1}}=\frac{n-r-1}{r+2}=\frac{231}{165}$$

$$\Rightarrow\quad 165(n-r-1)=231(r+2) \text{ or } 165n-627=396r$$

$$\Rightarrow\quad 165n-627=396\times\frac{1}{3}\times(n-2)$$

$$\Rightarrow\quad 165n-627=132(n-2) \text{ or } n=11$$

Example 6.33 **If $C_r={}^nC_r$, then prove that $(C_0+C_1)(C_1+C_2)\cdots(C_{n-1}+C_n)=(C_1C_2\cdots C_{n-1}C_n)(n+1)^n/n!$**

Sol. We have,

$$(C_0+C_1)(C_1+C_2)(C_2+C_3)\cdots(C_{n-1}+C_n)$$

$$=C_1C_2\cdots C_{n-1}C_n\left(1+\frac{C_0}{C_1}\right)\left(1+\frac{C_1}{C_2}\right)\left(1+\frac{C_2}{C_3}\right)\cdots\left(1+\frac{C_{n-1}}{C_n}\right)$$

$$=C_1C_2\cdots C_{n-1}C_n\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n-1}\right)\left(1+\frac{3}{n-2}\right)\cdots\left(1+\frac{n}{1}\right)$$

$$=C_1C_2\cdots C_{n-1}C_n\frac{(n+1)^n}{n!}$$

Example 6.34 **If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, then prove that $\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}$.**

Sol. Let the coefficients of $T_r, T_{r+1}, T_{r+2}, T_{r+3}$ be a_1, a_2, a_3, a_4, respectively, in the expansion of $(1+x)^n$. Then,

$$\frac{a_2}{a_1}=\frac{^nC_r}{^nC_{r-1}}=\frac{n-r+1}{r}$$

$$\Rightarrow\quad 1+\frac{a_2}{a_1}=\frac{n+1}{r}$$

Similarly,

$$1+\frac{a_3}{a_2}=\frac{n+1}{r+1},\quad 1+\frac{a_4}{a_3}=\frac{n+1}{r+2}$$

Now,

$$\text{L.H.S.}=\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}$$

$$=\frac{1}{1+\frac{a_2}{a_1}}+\frac{1}{1+\frac{a_4}{a_3}}$$

$$=\frac{r}{n+1}+\frac{r+2}{n+1}=\frac{2(r+1)}{n+1}=2\frac{1}{1+\frac{a_3}{a_2}}=\frac{2a_2}{a_2+a_3}$$

= R.H.S.

Example 6.35 **Find the sum $\sum_{r=1}^{n}\frac{r\,{}^nC_r}{{}^nC_{r-1}}$.**

Sol. $$\sum_{r=1}^{n}\frac{r\,{}^nC_r}{{}^nC_{r-1}}=\sum_{r=1}^{n}r\cdot\frac{n-r+1}{r}$$

$$=(n+1)\sum_{r=1}^{n}-\sum_{r=1}^{n}r$$

$$=n(n+1)-\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}$$

Concept Application Exercise 6.3

1. In the expansion of $(1+x)^n$, 7^{th} and 8^{th} terms are equal. Find the value of $(7/x+6)^2$.
2. Find the sum $\sum_{r=1}^{n}r^2\frac{{}^nC_r}{{}^nC_{r-1}}$.
3. Show that no three consecutive binomial coefficients can be in (i) G.P., (ii) H.P.
4. If the 3^{rd}, 4^{th}, 5^{th} and 6^{th} terms in the expansion of $(x+a)^n$ be, respectively, a, b, c and d, prove that $\frac{b^2-ac}{c^2-bd}=\frac{5a}{3c}$.

APPLICATIONS OF BINOMIAL EXPANSION

Important Result

$$2\le\left(1+\frac{1}{n}\right)^n<3,\ n\ge 1,\ n\in N.$$

Proof:

By the use of binomial theorem, we have

$$\left(1+\frac{1}{n}\right)^n=1+n\frac{1}{n}+\frac{n(n-1)}{2!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}$$

$$\times\frac{1}{n^3}+\cdots+\frac{n(n-1)(n-2)\cdots[n-(n-1)]}{n!}\frac{1}{n^n}$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$

$$+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$<1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$$

$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}\qquad(1)$$

$$=1+1\frac{\left\{1-\left(\frac{1}{2}\right)^n\right\}}{1-\frac{1}{2}}$$

$$=1+2\left\{1-\left(\frac{1}{2}\right)^n\right\}$$

$$=3-\frac{1}{2^{n-1}}\qquad(2)$$

Hence, from (1) and (2),

$$2\le\left(1+\frac{1}{n}\right)^n<3,\ n\ge1$$

Example 6.36 **Find the positive integer just greater than $(1+0.0001)^{10000}$.**

Sol. $$(1+0.0001)^{10000}=\left(1+\frac{1}{10000}\right)^{10000}$$

Now we know that

$$2\le\left(1+\frac{1}{n}\right)^n<3,\ n\ge1,\ n\in N$$

Hence positive integer just greater than $(1+0.0001)^{10000}$ is 3.

Example 6.37 **Find (i) the last digit, (ii) the last two digits and (iii) the last three digits of 17^{256}.**

Sol. We have,

$$17^{256}=(17)^{128}=(289)^{128}=(290-1)^{128}$$

$$\therefore\ 17^{256}={}^{128}C_0(290)^{128}-{}^{128}C_1(290)^{127}+{}^{128}C_2(290)^{126}-\cdots$$
$$-{}^{128}C_{125}(290)^3+{}^{128}C_{126}(290)^2-{}^{128}C_{127}(290)+1$$
$$=[{}^{128}C_0(290)^{128}-{}^{128}C_1(290)^{127}+{}^{128}C_2(290)^{126}-\cdots-$$
$${}^{128}C_{125}(290)^3]+{}^{128}C_{126}(290)^2-{}^{128}C_{127}(290)+1$$
$$=1000m+{}^{128}C_2(290)^2-{}^{128}C_1(290)+1\qquad(m\in I)$$
$$=1000m+\frac{(128)(127)}{2}(290)^2-128\times290+1$$
$$=1000m+(128)(127)(290)(145)-128\times290+1$$
$$=1000m+(128)(290)(127\times145-1)+1$$
$$=1000m+(128)(290)(18414)+1$$
$$=1000m+683527680+1$$
$$=1000m+683527000+680+1$$
$$=1000(m+683527)+681$$

Hence, the last three digits of 17^{256} must be 681. As a result, the last two digits of 17^{256} are 81 and the last digit of 17^{256} is 1.

Example 6.38 **If 10^m divides the number $101^{100}-1$, then find the greatest value of m.**

Sol. $$(1+100)^{100}=1+100\times100+\frac{100\times99}{1\times2}\times(100)^2$$
$$+\frac{100\times99\times98}{1\times2\times3}(100)^3+\cdots$$

$$\Rightarrow \quad (101)^{100} - 1 = 100$$

$$\times 100\left[1 + \frac{100 \times 99}{1 \times 2} + \frac{100 \times 9 \times 98}{1 \times 2 \times 3} \times 100 + \cdots\right]$$

From above, it is clear that $(101)^{100} - 1$ is divisible by $(100)^2 = 10000$. So greatest value of m is 4.

Important Results

$(1 + x)^n - 1 = {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$ is always divsible by x.

Also, $(1 + x)^n - 1 - nx = {}^nC_2 x^2 + \cdots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$ is always divisible by x^2.

Example 6.39 **Prove that for each $n \in N$, $2^{3n} - 1$ is divisible by 7.**

Sol. $2^{3n} - 1 = (2^3)^n - 1 = (1 + 7)^n - 1$

$= [1 + {}^nC_1 (7) + {}^nC_2 (7)^2 + \cdots + {}^nC_n (7)^n] - 1$

$= 7 [{}^nC_1 + {}^nC_2 (7) + \cdots + {}^nC_n (7)^{n-1}]$

$\Rightarrow$ $2^{3n} - 1$ is divisible by 7 for all $n \in N$

Example 6.40 **Find the remainder when $6^n - 5n$ is divided by 25**

Sol. $6^n - 5n = (1 + 5)^n - 5n$

$= (1 + 5n + {}^nC_2 . 5^2 + {}^nC_3 5^3 + \cdots) - 5n$

$= 25 ({}^nC_2 + {}^nC_3 + \cdots) + 1$

Hence $6^n - 5n$ when divided by 25 leaves 1 as remainder.

Example 6.41 **Using binomial theorem, show that $2^{3n} - 7n - 1$ is divisible by 49. Hence, show that $2^{3n+3} - 7n - 8$ is divisible by 49, $n \in N$.**

Sol. $2^{3n} - 7n - 1 = (2^3)^n - 7n - 1$

$= (1 + 7)^n - 7n - 1$

$= 1 + 7n + {}^nC_2 7^2 + {}^nC_3 7^3 + \cdots + {}^nC_n 7^n - 7n - 1$

$= 7^2[{}^nC_2 + {}^nC_3 7 + \cdots + {}^nC_n 7^{n-2}] = 49K \quad (1)$

where K is an integer.

Therefore, $2^{3n} - 7n - 1$ is divisible by 49. Now,

$2^{3n+3} - 7n - 8 = 2^3\, 2^{3n} - 7n - 8$

$= 8(2^{3n} - 7n - 1) + 49n$

$= 8 \times 49K + 49n \quad$ [From (1)]

$= 49(8K + n)$

Therefore, $2^{3n+3} - 7n - 8$ is divisible by 49.

Finding Remainder Using Binomial Theorem

To find the remainder when a^n is divided by b, we adjust the power of a to a^m which is very close to b say with difference 1. Also, the remainder is always positive. When number of the type $3k - 1$ is divided by 3, we have

$$\frac{3k-1}{3} = \frac{3k-3+2}{3} = k - 1 + \frac{2}{3}$$

Hence, the remainder is 2.

Following illustrations will explain the exact procedure.

Example 6.42 **Find the remainder when 5^{99} is divided by 8 is**

Sol. $5^{99} = 5(5^2)^{49}$

$= 5(24 + 1)^{49}$

$= 5({}^{49}C_0 24^{49} + {}^{49}C_1 24^{48} + \ldots + {}^{49}C_{48} 24 + 1)$

Hence remainder when 5^{99} is divided by 8 is 5

Example 6.43 **Find the remainder when 5^{99} is divided by 13.**

Sol. Here $5^2 = 25$ which is close to $26 = 13 \times 2$. Hence,

$E = 5^{99} = 5 \times 5^{98} = 5 \times (5^2)^{49} = 5(26 - 1)^{49}$

$\Rightarrow$ $E = 5[{}^{49}C_0\, 26^{49} - {}^{49}C_1\, 26^{48} + {}^{49}C_2\, 26^{47} - \cdots + {}^{49}C_{48}\, 26 - {}^{49}C_{49}]$

$= 5 \times 26k - 5$

Now,

$$\frac{E}{13} = 10k - \frac{5}{13} = 10k - 1 + \frac{8}{13}$$

Hence, the remainder is 8.

Example 6.44 **Find the value of $\{3^{2003}/28\}$, where $\{\cdot\}$ denotes the fractional part.**

Sol. $E = 3^{2003} = 3^{2001} \times 3^2 = 9(27)^{667} = 9(28 - 1)^{667}$

$\Rightarrow$ $E = 9[{}^{667}C_0\, 28^{667} - {}^{667}C_1\, (28)^{666} + \cdots - {}^{667}C_{667}]$

$\Rightarrow$ $E = 9 \times 28k - 9$

$$\Rightarrow \frac{E}{28} = 9k - \frac{9}{28} = 9k - 1 + \frac{19}{28}$$

That means if we divide 3^{2003} by 28, the remainder is 19. Thus,

$$\left\{\frac{3^{2003}}{28}\right\} = \frac{19}{28}$$

Example 6.45 **Find the remainder when $1690^{2608} + 2608^{1690}$ is divided by 7.**

Sol. Here base (1690 and 2608) is too big, so first let us reduce it.

$1690 = 7 \times 241 + 3$ and $2608 = 7 \times 372 + 4$

Let,

$S = 1690^{2608} + 2608^{1690}$

$= (7 \times 241 + 3)^{2608} + (7 \times 372 + 4)^{1690}$

$= 7k + 3^{2608} + 4^{1690}$ (where k is some positive integer)

Let,

$S' = 3^{2608} + 4^{1690}$

Clearly, the remainder in S and S' will be the same when divided by 7.

$S' = 3 \times 3^{3 \times 867} + 4 \times 4^{3 \times 563}$

$= 3 \times 27^{867} + 4 \times 64^{563}$

$= 3(28 - 1)^{867} + 4(63 + 1)^{563}$

$= 3[7n - 1] + 4[7m + 1]$ $(m, n \in I)$

$= 7p + 1$ (where p is some positive integer)

Hence, the remainder is 1.

Example 6.46 **Find the remainder when $x = 5^{5^{5^{\cdot^{\cdot^{5}}}}}$ (24 times 5) is divided by 24.**

Sol. Here exponent is $5^{5^{5^{\cdot^{\cdot^{5}}}}}$ (23 times 5) is an odd natural number. Therefore, $x = 5^{2m+1} = 5 \times (25^m)$, where m is a natural number. Thus,

$$x = 5 \times (24 + 1)^m$$
$$= 5 + \text{a multiple of } 24$$

Hence, the remainder is 5.

Expansion: $(x + y)^n \pm (x - y)^n$

We know that

$$(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \cdots + {}^nC_n y^n \quad (1)$$

and

$$(x - y)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \cdots + (-1)^n\, {}^nC_n y^n \quad (2)$$

Adding (1) and (2), we have

$$(x + y)^n + (x - y)^n = 2[{}^nC_0 x^n + {}^nC_2 x^{n-2} y^2 + {}^nC_4 x^{n-4} y^4 + \cdots]$$

Subtracting (2) from (1), we have

$$(x + y)^n - (x - y)^n = 2[{}^nC_1 x^{n-1} y + {}^nC_3 x^{n-3} y^3 + {}^nC_5 x^{n-5} y^5 + \cdots]$$

Above results are best used in the following illustrations.

Example 6.47 **Prove that $\sqrt{10}[(\sqrt{10}+1)^{100} - (\sqrt{10}-1)^{100}]$ is an even integer.**

Sol. $\sqrt{10}[(\sqrt{10}+1)^{100} - (\sqrt{10}-1)^{100}]$

$$= 2\sqrt{10}[{}^{100}C_1(\sqrt{10})^{99} + {}^{100}C_3(\sqrt{10})^{97} + {}^{100}C_5(\sqrt{10})^{95} + \cdots]$$
$$= 2[{}^{100}C_1(\sqrt{10})^{100} + {}^{100}C_3(\sqrt{10})^{98} + {}^{100}C_5(\sqrt{10})^{96} + \cdots]$$
$$= 2[{}^{100}C_1(10)^{50} + {}^{100}C_3(10)^{49} + {}^{100}C_5(10)^{48} + \cdots]$$

which is an even number.

Example 6.48 **If $9^7 - 7^9$ is divisible by 2^n, then find the greatest value of n, where $n \in N$.**

Sol. We have,

$$9^7 - 7^9 = (1 + 8)^7 - (1 - 8)^9$$
$$= (1 + {}^7C_1\, 8^1 + {}^7C_2\, 8^2 + \cdots + {}^7C_7\, 8^7) - (1 - {}^9C_1\, 8^1 + {}^9C_2\, 8^2 - \cdots - {}^9C_9\, 8^9)$$
$$= 16 \times 8 + 64[({}^7C_2 + \cdots + {}^7C_7\, 8^5) - ({}^9C_2 - \cdots - {}^9C_9\, 8^7)]$$
$$= 64k \text{ (where } k \text{ is some integer)}$$

Therefore, $9^7 - 7^9$ is divisible by 64.

Example 6.49 **Find the degree of the polynomial**

$$\frac{1}{\sqrt{4x+1}}\left\{\left(\frac{1+\sqrt{4x+1}}{2}\right)^7 - \left(\frac{1-\sqrt{4x+1}}{2}\right)^7\right\}$$

Sol. $\dfrac{1}{\sqrt{4x+1}}\left\{\left(\dfrac{1+\sqrt{4x+1}}{2}\right)^7 - \left(\dfrac{1-\sqrt{4x+1}}{2}\right)^7\right\}$

$$= \frac{2}{2^7\sqrt{4x+1}}[{}^7C_1\sqrt{4x+1} + {}^7C_3(\sqrt{4x+1})^3 + {}^7C_5(\sqrt{4x+1})^5 + {}^7C_7(\sqrt{4x+1})^7]$$

$$= \frac{1}{2^6}[{}^7C_1 + {}^7C_3(4x+1) + {}^7C_5(4x+1)^2 + {}^7C_7(4x+1)^3]$$

Clearly, the degree of the polynomial is 3.

Example 6.50 **If $(2+\sqrt{3})^n = I + f$ where I and n are +ive integers and $0 < f < 1$, show that I is an odd integer and $(1 - f)(I + f) = 1$.**

Sol. $(2 + \sqrt{3})^n = I + f$

or

$$I + f = 2^n + {}^nC_1\, 2^{n-1}\sqrt{3} + {}^nC_2\, 2^{n-2}(\sqrt{3})^2 + {}^nC_3\, 2^{n-3}(\sqrt{3})^3 + \cdots \quad (1)$$

Now,

$$0 < 2 - \sqrt{3} < 1 \Rightarrow 0 < (2 - \sqrt{3})^n < 1$$

Let $(2 - \sqrt{3})^n = f'$ where $0 < f' < 1$.

$$\therefore\ f' = 2^n - {}^nC_1\, 2^{n-1}\sqrt{3} + {}^nC_2\, 2^{n-2}(\sqrt{3})^2 - {}^nC_3 2^{n-3}(\sqrt{3})^3 + \quad (2)$$

Adding (1) and (2), we get

$$I + f + f' = 2[2^n + {}^nC_2\, 2^{n-2} \times 3 + \ldots]$$

or

$$I + f + f' = \text{even integer} \quad (3)$$

Now, $0 < f < 1$ and $0 < f' < 1$.

$$\therefore\ 0 < f + f' < 2$$

Hence from (3), we conclude that $f + f'$ is an integer between 0 and 2.

$$\therefore\ f + f' = 1 \Rightarrow f' = 1 - f \quad (4)$$

From (3) and (4), we get $I + 1$ is an even integer. Therefore, I is an odd integer. Now,

$$I + f = (2 + \sqrt{3})^n,\ f' = 1 - f = (2 - \sqrt{3})^n$$
$$\therefore\ (I + f)(1 - f) = [(2 + \sqrt{3})(2 - \sqrt{3})]^n = (4 - 3)^n = 1$$
$$\therefore\ (I + f)(1 - f) = 1$$

Concept Application Exercise 6.4

1. Using binomial theorem, show that $3^{2n+2} - 8n - 9$ is divisible by 64, $\forall\, n \in N$.
2. For each $n \in N$, prove that $49^n + 16n - 1$ is divisible by 64.
3. Find the remainder when 27^{40} is divided by 12.
4. Let n be an odd natural number greater than 1. Then, find the number of zeros at the end of the sum $99^n + 1$.
5. Find the last two digits of the number $(23)^{14}$.
6. The number of non-zero terms in the expansion of $(1+3\sqrt{2}x)^9 + (1-3\sqrt{2}x)^9$ is______.
7. Find the value of $(\sqrt{2}+1)^6 - (\sqrt{2}-1)^6$.
8. If
$$\frac{1}{\sqrt{4x+1}}\left\{\left(\frac{1+\sqrt{4x+1}}{2}\right)^n - \left(\frac{1-\sqrt{4x+1}}{2}\right)^n\right\} = a_0 + a_1 x + \cdots + a_5 x^5$$
then find the possible values of n.
9. Show that the integer next above $(\sqrt{3}+1)^{2m}$ contains 2^{m+1} as a factor.

USE OF COMPLEX NUMBERS IN BINOMIAL THEOREM

- Writing the binomial expression of $(\cos\theta + i\sin\theta)^n$ and equating the real part to $\cos n\theta$ and the imaginary part to $\sin n\theta$, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \cdots$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta + \cdots$$

$$\Rightarrow \tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - {}^nC_7\tan^7\theta + \cdots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - {}^nC_6\tan^6\theta + \cdots}$$

- We get a very interesting results if in binomial expansion, any variable is replaced with $i = \sqrt{-1}$ or ω (cube roots of unity).

Example 6.51 **Find the sum $C_0 - C_2 + C_4 - C_6 + \cdots$ where $C_r = {}^nC_r$.**

Sol. Consider

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$$

Put $x = i$ where $i = \sqrt{-1}$. Then,

$$(1+i)^n = C_0 + C_1 i + C_2 i^2 + C_3 i^3 + C_4 i^4 + \cdots$$
$$= (C_0 - C_2 + C_4 - C_6 + \cdots) + i(C_1 - C_3 + C_5 + \cdots)$$

$$\Rightarrow C_0 - C_2 + C_4 - C_6 + \cdots = \text{Real part of } (1+i)^n$$

$$= \text{Re}\left[(\sqrt{2})^n\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^n\right]$$

$$= \text{Re}\left[2^{n/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n\right]$$

$$= \text{Re}\left[2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)\right] \text{ [De' Moivre's theorem]}$$

$$= 2^{n/2}\cos\frac{n\pi}{4}$$

Also,

$$C_1 - C_3 + C_5 - \cdots = \text{Im}[(1+i)^n] = 2^{n/2}\sin\left(\frac{n\pi}{4}\right)$$

Example 6.52 **Prove that**

$${}^nC_0 + {}^nC_3 + {}^nC_6 + \cdots = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$$

Sol. We have $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots + C_nx^n$

Putting $x = 1$, ω and ω^2, we have $2^n = C_0 + C_1 + C_2 + \cdots + C_n$

$$(1+\omega)^n = C_0 + C_1\omega + C_2\omega^2 + \cdots + C_n\omega^n$$
$$(1+\omega^2)^n = C_0 + C_1\omega^2 + C_2\omega^4 + \cdots + C_n\omega^{2n}$$

Adding the above three equations,

$$2^n + (1+\omega)^n + (1+\omega^2)^n = 3(C_0 + C_3 + C_6 + \cdots)$$

because

$$1 + \omega^k + \omega^{2k} = 0 \text{ if } k \neq 3m \qquad m \in N$$
$$= 3 \text{ if } k = 3m$$

Now,

$$1 + \omega = -\omega^2 = -\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$1 + \omega^2 = -\omega = -\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$$

Hence by De' Moivre's theorem,

$$2^n + (1+\omega)^n + (1+\omega^2)^n$$
$$= 2^n + \left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + \left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$$
$$= 2^n + 2\cos\frac{n\pi}{3}$$

Example 6.53 **If $T_0, T_1, T_2, \ldots, T_n$ represent the terms in the expansion of $(x+a)^n$, then find the value of $(T_0 - T_2 + T_4 - \cdots)^2 + (T_1 - T_3 + T_5 - \cdots)^2$, $n \in N$.**

Sol. $(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + {}^nC_3x^{n-3}a^3 + \cdots$

$$= T_0 + T_1 + T_2 + T_3 + \cdots$$

Replacing a by ai, we have

$$(x+ai)^n = {}^nC_0x^n + {}^nC_1x^{n-1}ai + {}^nC_2x^{n-2}(ai)^2 + {}^nC_3x^{n-3}(ai)^3 + \cdots$$
$$= ({}^nC_0x^n - {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 - \cdots) + i({}^nC_1x^na - {}^nC_3x^{n-3}a^3 + {}^nC_5x^{n-5}a^5 - \cdots)$$
$$= (T_0 - T_2 + T_4 - \cdots) + i(T_1 - T_3 + T_5 - \cdots)$$

Taking modulus of both sides and squaring

$$|x + ai|^{2n} = |(T_0 - T_2 + T_4 - \cdots) + i(T_1 - T_3 + T_5 - \cdots)|^2$$

$$\Rightarrow (x^2 + a^2)^n = (T_0 - T_2 + T_4 - \cdots)^2 + (T_1 - T_3 + T_5 - \cdots)^2$$

Example 6.54 **If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, find the value of $a_0 + a_3 + a_6 + \cdots$, $n \in N$.**

Sol. $(1 + x + x^2) = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$ (1)

In (1), putting $x = 1$, we get

$$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots \quad (2)$$

In (1), putting $x = \omega$, we get

$$(1 + \omega + \omega^2)^n = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + a_5\omega^5 + a_6\omega^6 + \cdots$$

$$\Rightarrow 0 = a_0 + a_1\omega + a_2\omega^2 + a_3 + a_4\omega + a_5\omega^2 + a_6 + \cdots \quad (3)$$

In (1), putting $x = \omega^2$, we get

$$(1 + \omega^2 + \omega^4)^n = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + a_5\omega^{10} + a_6\omega^{12} + \cdots$$

$$\Rightarrow 0 = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega^2 + a_5\omega + a_6 + \cdots \quad (4)$$

Adding (2), (3) and (4), we have

$$3^n = 3(a_0 + a_3 + a_6 + \cdots)$$

$$\Rightarrow a_0 + a_3 + a_6 + \cdots = 3^{n-1}$$

Concept Application Exercise 6.5

1. Find the sum ${}^nC_0 + {}^nC_4 + {}^nC_8 + \cdots$.
2. Find the value of ${}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \cdots + {}^{4n}C_{4n}$.

GREATEST TERM IN BINOMIAL EXPANSION

Consider the expansion of $(1 + x)^7$ when $x = 1/2$. Terms of the expansion are given by the following table.

Term	Value
T_1	$^7C_0 = 1$
T_2	$^7C_1\ (1/2) = 7/2$
T_3	$^7C_2\ (1/2)^2 = 21/4$
T_4	$^7C_3\ (1/2)^3 = 35/8$
T_5	$^7C_4\ (1/2)^4 = 35/16$
T_6	$^7C_5\ (1/2)^5 = 21/32$
T_7	$^7C_6\ (1/2)^6 = 7/64$
T_8	$^7C_7\ (1/2)^7 = 1/128$

Here, we can observe that value of the term increases till the 3rd term and then the value of the term decreases. Then, here T_3 is the greatest term.

T_3 is the greatest term, hence $T_3/T_2 > 1$ and also $T_4/T_3 < 1$.

In any binomial expansion, the values of the terms increase, reach a maximum and then decreases.

So in general to locate the maximum term in the expansion $(1 + x)^n$ we find the value of r till the ratio T_{r+1}/T_r is greater than 1 as for this value of r, any term is always greater than its previous term. The value of r till this occurs gives the greatest term.

So for the greatest term, let

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}x \geq 1 \Rightarrow r \leq \frac{(n+1)x}{1+x}$$

Then the greatest term occurs for $r = [(n + 1)x/(1 + x)]$, which is the integral part of $[(n + 1)x/(1 + x)]$.

If $[(n + 1)x/(1 + x)]$ is an exact integer, then T_r and T_{r+1} both are the greatest terms.

When we have an expansion in which positive and negative sign occurs alternatively, we find the numerically greatest term (ignoring the negative value), for which we find the value of r considering ratio $|T_{r+1}/T_r|$.

Note: *The greatest coefficient in the binomial expansion is equivalent to the greatest term when $x = 1$.*

Example 6.55 **Find the greatest term in the expansion of $\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)^{20}$.**

Sol. Let $(r + 1)^{th}$ term be the greatest term in the expansion of

$$\left(1+\frac{1}{\sqrt{3}}\right)^{20}. \text{ Now,}$$

$$\frac{T_{r+1}}{T_r} = \frac{20-r+1}{r}\left(\frac{1}{\sqrt{3}}\right)$$

Let

$$T_{r+1} \geq T_r$$

$$\Rightarrow\ 20 - r + 1 \geq \sqrt{3}r$$

$$\Rightarrow\ 21 \geq r(\sqrt{3}+1)$$

$$\Rightarrow\ r \leq \frac{21}{\sqrt{3}+1}$$

$$\Rightarrow\ r \leq 7.686$$

$$\Rightarrow\ r = 7$$

for which the greatest term occurs.

Hence, the greatest term in $\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)^{20}$ is

$$T_8 = \sqrt{3}\ {}^{20}C_7\left(\frac{1}{\sqrt{3}}\right)^7 = \frac{25840}{9}$$

Example 6.56 **Find the numerically greatest term in the expansion of $(3 - 5x)^{15}$ when $x = 1/5$.**

Sol. $(3 - 5x)^{15} = 3^{15}(1 - 5x/3)^{15} = 3^{15}(1 - 1/3)^{15}$ (for $x = 1/5$)

Now consider $(1 - 1/3)^{15}$.

$$\left|\frac{T_{r+1}}{T_r}\right| = \frac{15-r+1}{r}\left|-\frac{1}{3}\right| \geq 1$$

$$\Rightarrow\ 16 - r \geq 3r$$

$$\Rightarrow\ r \leq 4$$

Hence, T_4 and T_5 are the numerically greatest terms.

$$T_4 = {}^{15}C_3\ 3^{15-3}(-5x)^3 = -455 \times (3^{12})$$

and

$$T_5 = {}^{15}C_4\ 3^{15-4}(-5x)^4 = 455 \times (3^{12})$$

Also, $|T_4| = |T_5| = 455 \times (3^{12})$.

Example 6.57 **Given that the 4th term in the expansion of $[2 + (3/8)x]^{10}$ has the maximum numerical value. Then find the range of value of x.**

Sol. Let T_4 be numerically the greatest term in the expansion of $2^{10}\left[1+\left(\frac{3}{16}\right)x\right]^{10}$. Therefore,

$$\left|\frac{T_4}{T_3}\right| \geq 1 \text{ and } \left|\frac{T_4}{T_5}\right| \geq 1$$

$$\Rightarrow\ \left|\frac{T_4}{T_3}\right| \geq 1 \text{ and } \left|\frac{T_5}{T_4}\right| \leq 1$$

Now,

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}x$$

Taking $r = 3$ and $r = 4$ and replacing x by $3x/16$ and putting $n = 10$ in the above two relations, we get

$$\left|\frac{11-3}{3}\frac{3x}{16}\right| \geq 1 \text{ and } \left|\frac{11-4}{4}\frac{3x}{16}\right| \leq 1$$

$$\Rightarrow\ |x| \geq 2 \text{ and } |x| \leq \frac{64}{21} \quad (1)$$

$\Rightarrow \quad x^2 \ge 4$ and $x^2 \le \left(\dfrac{64}{21}\right)^2$

$\Rightarrow \quad 4 \le x^2 \le \left(\dfrac{64}{21}\right)^2$

$\Rightarrow \quad 2 \le x \le \dfrac{64}{21}$ or $-\dfrac{64}{21} \le x \le -2$

Example 6.58 **Find the greatest coefficient in the expansion of $(1 + 2x/3)^{15}$.**

Sol. The greatest coefficient is equal to the greatest term when $x = 1$.

For $x = 1$, $\dfrac{T_{r+1}}{T_r} = \dfrac{15-r+1}{r}\,\dfrac{2}{3}$

Let $\dfrac{T_{r+1}}{T_r} \ge 1$

$\Rightarrow \quad \dfrac{15-r+1}{r}\,\dfrac{2}{3} \ge 1$

$\Rightarrow \quad 32 - 2r \ge 3r$

$\Rightarrow \quad r \le 32/5$

$\Rightarrow \quad r = 6$

Hence, 7th term has the greatest coefficient and its value is $T_{6+1} = {}^{15}C_6(2/3)^6$.

Concept Application Exercise 6.6

1. Find the largest term in the expansion of $(3 + 2x)^{50}$ where $x = 1/5$.
2. If $x = 1/3$, find the greatest term in the expansion of $(1 + 4x)^8$.
3. If n is an even positive integer, then find the values of x if the greatest term in the expansion of $(1 + x)^n$ may have the greatest coefficient also.
4. If in the expansion of $(2x + 5)^{10}$, the numerically greatest term is equal to the middle term, then find the values of x.

SUM OF SERIES

Important Facts and Formulas for Finding Sum of Series

- $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$ (1)

$$r\,{}^nC_r = r\,\frac{n!}{(n-r)!\,r!}$$
$$= \frac{n!}{(n-r)!\,(r-1)!}$$
$$= n\,\frac{(n-1)!}{(n-r)!\,(r-1)!}$$
$$= n\,{}^{n-1}C_{r-1}$$

Similarly,

$$r\,{}^{n+1}C_r = (n+1)\,{}^nC_{r-1}$$
$$r\,{}^nC_{r-1} = [(r-1)+1]\,{}^nC_{r-1} = (r-1)\,{}^nC_{r-1} + {}^nC_{r-1} = n\,{}^{n-1}C_{r-2} + {}^nC_{r-1}$$
$$r^2\,{}^nC_r = r\,n\,{}^{n-1}C_{r-1} = n\,[(r-1)+1]\,{}^{n-1}C_{r-1} = n\,[(r-1)\,{}^{n-1}C_{r-1} + {}^{n-1}C_{r-1}] = n\,[(n-1)\,{}^{n-2}C_{r-2} + {}^{n-1}C_{r-1}]$$

and so on.

In the problems, we must remove any factor which is in terms of r which is multiplied by binomial coefficient.

- $\dfrac{{}^nC_r}{r+1} = \dfrac{{}^{n+1}C_{r+1}}{n+1}$

In $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$ replace n by $n+1$ and r by $r+1$.

$\therefore \quad (r+1)\,{}^{n+1}C_{r+1} = (n+1)\,{}^nC_r$

$\Rightarrow \quad \dfrac{{}^nC_r}{r+1} = \dfrac{{}^{n+1}C_{r+1}}{n+1}$ (2)

Similarly,

$\dfrac{{}^{n-1}C_{r-1}}{r} = \dfrac{{}^nC_r}{n}$ [replacing r by $r-1$ and n by $n-1$ in (2)]

Also,

$$\frac{{}^nC_r}{(r+1)(r+2)} = \frac{{}^{n+2}C_{r+2}}{(n+1)(n+2)},$$
$$\frac{{}^nC_r}{(r+1)(r+2)(r+3)} = \frac{{}^{n+3}C_{r+3}}{(n+1)(n+2)(n+3)},$$

and so on.

- Always adjust the power of variable to suffix r of binomial coefficient ${}^nC_r$.

Consider the following example:

$(-1)^r\,r\,{}^nC_r = (-1)^r\,n\,{}^{n-1}C_{r-1} = -n\,{}^{n-1}C_{r-1}(-1)^{r-1}$. Here ${}^{n-1}C_{r-1}(-1)^{r-1}$ is standard general term of binomial series $(1-x)^{n-1}$. Similaraly,

$$\frac{{}^nC_r\,2^r}{(r+1)} = \frac{{}^{n+1}C_{r+1}\,2^r}{(n+1)} = \frac{{}^{n+1}C_{r+1}\,2^{r+1}}{2(n+1)}$$

Example 6.59 **Find the sum $C_0 + 3\,C_1 + 3^2\,C_2 + \cdots + 3^nC_n$.**

Sol. $S = C_0 + 3\,C_1 + 3^2\,C_2 + \cdots + 3^n\,C_4$
$= (1 + 3)^n = 4^n$

Example 6.60 **If $(1 + x)^n = \sum_{r=0}^{n} C_r x^r$, then prove that $C_1 + 2C_2 + 3C_3 + \cdots + nC_n = n2^{n-1}$.**

Sol. Method (i): By summation

r^{th} term of the given series, is

$t_r = r\,{}^nC_r \Rightarrow t_r = n\,{}^{n-1}C_{r-1}$

Sum of the series is

$$\sum_{r=1}^{n} t_r = n\sum_{r=1}^{n} {}^{n-1}C_{r-1}$$

$= n\,({}^{n-1}C_0 + {}^{n-1}C_1 + \cdots + {}^{n-1}C_{n-1})$
$= n\,2^{n-1}$

Method (ii): By calculus

We have,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n \quad (1)$$

Differentiating (1) with respect to x, we get

$$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \cdots + nC_nx^{n-1} \quad (2)$$

Putting $x = 1$ in (2),

$$n\,2^{n-1} = C_1 + 2C_2 + \cdots + {}^nC_n$$

Example 6.61 **Find the sum $1\,C_0 + 2\,C_1 + 3\,C_2 + \cdots + (n+1)\,C_n$ where $C_r = {}^nC_r$.**

Sol. $1C_0 + 2C_1 + 3C_2 + \cdots + (n+1)C_n$

$$= \sum_{r=0}^{n}(r+1)\,{}^nC_r$$

$$= \sum_{r=0}^{n}\left[r\,{}^nC_r + {}^nC_r\right]$$

$$= n\sum_{r=1}^{n}{}^{n-1}C_{r-1} + \sum_{r=0}^{n}{}^nC_r$$

$$= ({}^{n-1}C_0 + {}^{n-1}C_1 + \cdots + {}^{n-1}C_{n-1}) + ({}^nC_0 + {}^nC_1 + \cdots + {}^nC_n)$$

$$= n\,2^{n-1} + 2^n = 2^{n-1}(n+2)$$

Example 6.62 **Find the sum $1\times 2\times C_1 + 2\times 3\,C_2 + \cdots + n(n+1)C_n$, where $C_r = {}^nC_r$.**

Sol. $S = 1\times 2\times C_1 + 2\times 3\times C_2 + \cdots + n\,(n+1)\,C_n$

$$= \sum_{r=1}^{n} r(r+1)\,{}^nC_r$$

$$= \sum_{r=0}^{n}(r+1)[r\,{}^nC_r]$$

$$= \sum_{r=1}^{n}(r+1)[n\,{}^{n-1}C_{r-1}]$$

$$= n\sum_{r=1}^{n}[(r-1)+2]\,{}^{n-1}C_{r-1}$$

$$= n\sum_{r=1}^{n}\left[(n-1)\,{}^{n-2}C_{r-2} + 2\,{}^{n-1}C_{r-1}\right]$$

$$= n(n-1)\sum_{r=2}^{n}{}^{n-2}C_{r-2} + 2n\sum_{r=1}^{n}{}^{n-1}C_{r-1}$$

$$= n\,(n-1)\,2^{n-2} + 2n\,2^{n-1}$$
$$= 2^{n-2}\,n\,[n-1+4]$$
$$= n(n+3)\,2^{n-2}$$

Example 6.63 **If $n > 2$, then prove that $C_1\,(a-1) - C_2\times(a-2) + \cdots + (-1)^{n-1}\,C_n(a-n) = a$, where $C_r = {}^nC_r$.**

Sol. $C_1\,(a-1) - C_2\,(a-2) + \cdots + (-1)^{n-1}\,C_n(a-n)$

$$T_r = (-1)^{r-1}\,(a-r)\,{}^nC_r$$
$$= (-1)^{r-1}\,(a\,{}^nC_r - r\,{}^nC_r)$$
$$= (-1)^{r-1}\,(a\,{}^nC_r - n\,{}^{n-1}C_{r-1})$$
$$= -a\,(-1)^r\,{}^nC_r - n\,(-1)^{r-1}\times{}^{n-1}C_{r-1}$$

Now,

$$S = \sum_{r=1}^{n} T_r$$
$$= -a\,[(1-1)^n - {}^nC_0] - n\,(1-1)^{n-1}$$
$$= an$$

Example 6.64 **Find the sum $3\,{}^nC_0 - 8\,{}^nC_1 + 13\,{}^nC_2 - 18\times{}^nC_3 + \cdots$.**

Sol. The general term of the series is $T_r = (-1)^r\,(3+5r)\,{}^nC_r$ where $r = 0, 1, 2, \ldots, n$. Therefore, sum of the series is given by

$$S = \sum_{r=0}^{n}(-1)^r(3+5r)\,{}^nC_r$$

$$= 3\left(\sum_{r=0}^{n}(-1)^r\,{}^nC_r\right) + 5\left(\sum_{r=1}^{n}(-1)^r\,n\,{}^{n-1}C_{r-1}\right)$$

$$= 3\left(\sum_{r=0}^{n}(-1)^r\,{}^nC_r\right) - 5n\left(\sum_{r=1}^{n}(-1)^{r-1}\,{}^{n-1}C_{r-1}\right)$$

$$= 3(1-1)^n - 5n(1-1)^{n-1}$$
$$= 0$$

Example 6.65 **If $x + y = 1$, prove that $\sum_{r=0}^{n} r\,{}^nC_r\,x^r\,y^{n-r} = nx$.**

Sol. We have,

$$\sum_{r=0}^{n} r\,{}^nC_rx^ry^{n-r} = \sum_{r=1}^{n} n\,{}^{n-1}C_{r-1}x^{r-1}x^1y^{n-r}$$

$$= nx\sum_{r=1}^{n}{}^{n-1}C_{r-1}x^{r-1}y^{(n-1)-(r-1)}$$

$$= nx(x+y)^{n-1}$$

$$= nx \quad [\because x+y=1]$$

Example 6.66 **If $(1+x)^n = \sum_{r=0}^{n} C_rx^r$, show that**

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

Sol. Method (i): By summation

r^{th} term of the given series is

$$t_r = \frac{{}^nC_{r-1}}{r} = \frac{{}^{n+1}C_r}{n+1}$$

Required sum is

$$\sum_{r=1}^{n+1} t_r = \sum_{r=1}^{n+1}\frac{{}^{n+1}C_r}{n+1}$$

$$= \frac{1}{n+1}\,({}^{n+1}C_1 + {}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1})$$

$$= \frac{2^{n+1}-1}{n+1}$$

Method (ii): By calculus

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n \quad (1)$$

Integrating both the sides of (1) with respect to x between the limits 0 and x, we have

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^x = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \cdots + \frac{C_nx^{n+1}}{n+1}\right]_0^x$$

$$\Rightarrow \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \cdots + \frac{C_nx^{n+1}}{n+1} \quad (2)$$

Substituting $x = 1$ in (2), we get

$$\frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}$$

Example 6.67 **Prove that** $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots = \frac{2^n-1}{n+1}$ **where** $C_r = {}^nC_r$.

Sol. $S = \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots$

General term of the series is

$$\frac{{}^nC_{2r-1}}{2r} = \frac{{}^{n+1}C_{2r}}{n+1}$$

where $r = 1, 2, 3, \ldots$

$$\therefore \quad S = \frac{1}{n+1}\left[{}^{n+1}C_2 + {}^{n+1}C_4 + {}^{n+1}C_6 + \cdots\right]$$

$$= \frac{1}{n+1}\left[({}^{n+1}C_0 + {}^{n+1}C_4 + {}^{n+1}C_6 + \cdots) - {}^{n+1}C_0\right]$$

$$= \frac{1}{n+1}[2^n - 1]$$

Example 6.68 **Find the sum** $2C_0 + \frac{2^2}{2}C_1 + \frac{2^3}{3}C_2 + \frac{2^4}{4}C_3 + \cdots + \frac{2^{11}}{11}C_{10}$.

Sol. We have,

$$2C_0 + \frac{2^2}{2}C_1 + \frac{2^3}{3}C_2 + \cdots + \frac{2^{11}}{11}C_{10}$$

$$= \sum_{r=0}^{10} {}^{10}C_r \frac{2^{r+1}}{r+1}$$

$$= \frac{1}{11}\sum_{r=0}^{10} \frac{11}{r+1}\,{}^{10}C_r\,2^{r+1}$$

$$= \frac{1}{11}\sum_{r=0}^{10} {}^{11}C_{r+1}\,2^{r+1}$$

$$= \frac{1}{11}({}^{11}C_1 2^1 + \cdots + {}^{11}C_{11}2^{11})$$

$$= \frac{1}{11}({}^{11}C_0 2^0 + {}^{11}C_1 2^1 + \cdots + {}^{11}C_{11}\,2^{11} - {}^{11}C_0\,2^0)$$

$$= \frac{1}{11}[(1+2)^{11} - 1] = \frac{3^{11}-1}{11}$$

Example 6.69 **Prove that**

$$\frac{1}{n+1} = \frac{{}^nC_1}{2} - \frac{2({}^nC_2)}{3} + \frac{3({}^nC_3)}{4} - \cdots + (-1)^{n+1}\frac{n({}^nC_n)}{n+1}.$$

Sol. $S = \frac{{}^nC_1}{2} - \frac{2({}^nC_2)}{3} + \frac{3({}^nC_3)}{4} \cdots + (-1)^{n+1}\frac{n({}^nC_n)}{n+1}$

$$= \sum_{r=1}^{n} \frac{r\,{}^nC_r}{(r+1)}(-1)^{r+1}$$

$$= \sum_{r=1}^{n} r\frac{{}^{n+1}C_{r+1}}{n+1}(-1)^{r+1}$$

$$= \frac{1}{n+1}\sum_{r=1}^{n}[(r+1)-1]\,{}^{n+1}C_{r+1}(-1)^{r+1}$$

$$= \frac{1}{n+1}\sum_{r=1}^{n}[(r+1)\,{}^{n+1}C_{r+1}(-1)^{r+1} - {}^{n+1}C_{r+1}(-1)^{r+1}]$$

$$= \frac{1}{n+1}\sum_{r=1}^{n}[-(n+1)\,{}^nC_r(-1)^r - {}^{n+1}C_{r+1}(-1)^{r+1}]$$

$$= -[-{}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots + (-1)^n\,{}^nC_n]$$

$$-\frac{1}{n+1}[{}^{n+1}C_2 - {}^{n+1}C_3 + \cdots (-1)^{n+1}\;{}^{n+1}C_{n+1}]$$

$$= -[(1-1)^n - 1] - \frac{1}{n+1}[(1-1)^{n+1} - {}^{n+1}C_0 + {}^{n+1}C_1]$$

$$= 1 - \frac{1}{n+1}[(n+1)-1]$$

$$= 1 - \frac{n}{n+1} = \frac{1}{n+1}$$

Example 6.70 **If k and n be +ive integers and $s_k = 1^k + 2^k + 3^k + \cdots + n^k$, then prove that**

$$\sum_{r=1}^{m} {}^{m+1}C_r s_r = (n+1)^{m+1} - (n+1)$$

Sol. $S = \sum_{r=1}^{m} {}^{m+1}C_r\, s_r = [{}^{m+1}C_1\,s_1 + {}^{m+1}C_2\,s_2 + \cdots + {}^{m+1}C_m s_m]$

$$= {}^{m+1}C_1\,(1 + 2 + 3 + \cdots + n)$$
$$+ {}^{m+1}C_2\,(1^2 + 2^2 + 3^2 + \cdots + n^2)$$
$$+ {}^{m+1}C_3\,(1^3 + 2^3 + 3^3 + \cdots + n^3)$$
$$+ {}^{m+1}C_m\,(1^m + 2^m + 3^m + \cdots + n^m)$$

$$= ({}^{m+1}C_1\,1 + {}^{m+1}C_2\,1^2 + {}^{m+1}C_3\,1^3 + \cdots + {}^{m+1}C_m\,1^m)$$
$$+ ({}^{m+1}C_1\,2 + {}^{m+1}C_2\,2^2 + {}^{m+1}C_3\,2^3 + \cdots + {}^{m+1}C_m 2^m)$$
$$+ \cdots + ({}^{m+1}C_1\,n + {}^{m+1}C_2\,n^2 + \cdots + {}^{m+1}C_m\,n^m)$$

$$= [(1+1)^{m+1} - 1 - {}^{m+1}C_{m+1}\,1^{m+1}]$$
$$+ [(1+2)^{m+1} - 1 - {}^{m+1}C_{m+1}\,2^{m+1}]$$
$$+ [(1+3)^{m+1} - 1 - {}^{m+1}C_{m+1}\,3^{m+1}] + \cdots$$

$$= (2^{m+1} - 1^{m+1}) + (3^{m+1} - 2^{m+1}) + (4^{m+1} - 3^{m+1})$$
$$+ \cdots + [(1+n)^{m+1} - n^{m+1}] - n$$

$$= (1+n)^{m+1} - 1 - n = (1+n)^{m+1} - (n+1)$$

Example 6.71 Prove that $\frac{C_1}{1}-\frac{C_2}{2}+\frac{C_3}{3}-\frac{C_4}{4}+\cdots+\frac{(-1)^{n-1}}{n}C_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$

Sol. $S=\frac{C_1}{1}-\frac{C_2}{2}+\frac{C_3}{3}-\frac{C_4}{4}+\cdots+\frac{(-1)^{n-1}}{n}C_n$

Now,

$$(1+x)^n=C_0+C_1x+C_2x^2+\cdots+C_nx^n$$

Dividing by x,

$$\frac{(1+x)^n}{x}-\frac{C_0}{x}=C_1+C_2x+\cdots+C_nx^{n-1}$$

or

$$C_1+C_2x+C_2x^2+\cdots+C_nx^{n-1}$$
$$=\frac{(1+x)^n-1}{(1+x)-1}$$
$$=1+(1+x)+(1+x)^2+\cdots(1+x)^{n-1}$$

Integrating both sides between limits -1 and 0, we have

$$\int_{-1}^{0}(C_1+C_2x+C_3x^2+\cdots+C_nx^{n-1})dx$$
$$=\int_{-1}^{0}[1+(1+x)+\cdots+(1+x)^{n-1}]dx$$
$$\Rightarrow\left[C_1x+\frac{C_2x^2}{2}+\frac{C_3x^3}{3}+\cdots+\frac{C_nx^n}{n}\right]_{-1}^{0}$$
$$=\left[x+\frac{(1+x)^2}{2}+\frac{(1+x)^3}{3}+\cdots+\frac{(1+x)^n}{n}\right]_{-1}^{0}$$
$$\Rightarrow C_1-\frac{C_2}{2}+\frac{C_3}{3}-\cdots+(-1)^n\frac{C_n}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

Replacing r by n – r is equivalent to writing the series in the reverse order.

Example 6.72 Prove that $\sum_{r=0}^{n}{}^nC_r\sin rx\cos(n-r)x=2^{n-1}\sin(nx)$.

Sol. Here sum is given by

$$S=\sum_{r=0}^{n}{}^nC_r\sin rx\cos(n-r)x$$
$$\Rightarrow S=\sum_{r=0}^{n}{}^nC_{n-r}\sin(n-r)x\cos rx \text{ (replacing } r \text{ by } n-r)$$
$$\Rightarrow 2S=\sum_{r=0}^{n}{}^nC_r\sin nx=\sin nx\times 2^n$$
$$\Rightarrow S=2^{n-1}\sin nx$$

Example 6.73 If for $n\in N$, $\sum_{k=0}^{2n}(-1)^k\left({}^{2n}C_k\right)^2=A$, then find the value of $\sum_{k=0}^{2n}(-1)^k(k-2n)\left({}^{2n}C_k\right)^2$.

Sol. Let,

$$S=\sum_{k=0}^{2n}(-1)^k(k-2n)\left({}^{2n}C_k\right)^2 \quad (1)$$
$$\Rightarrow S=-\sum_{k=0}^{2n}(-1)^{2n-k}(2n-k)\left({}^{2n}C_{2n-k}\right)^2$$

Writing the terms in S in the reverse order, we get

$$S=-\sum_{k=0}^{2n}(-1)^k\left({}^{2n}C_k\right)^2 \quad (2)$$

Adding (1) and (2), we get

$$2S=-2n\sum_{k=0}^{2n}(-1)^k\left({}^{2n}C_k\right)^2=-2nA$$
$$\Rightarrow S=-nA$$

Concept Application Exercise 6.7

1. Prove that
$$\frac{1^2}{3}{}^nC_1+\frac{1^2+2^2}{5}{}^nC_2+\frac{1^2+2^2+3^2}{7}{}^nC_3+\cdots+\frac{1^2+2^2+\cdots+n^2}{2n+1}{}^nC_n=\frac{n(n+3)}{6}2^{n-2}.$$
2. If $p+q=1$, then show that $\sum_{r=0}^{n}r^2\,{}^nC_rp^rq^{n-r}=npq+n^2p^2$.
3. Prove that $1-{}^nC_1\frac{1+x}{1+nx}+{}^nC_2\frac{1+2x}{(1+nx)^2}-{}^nC_3\frac{1+3x}{(1+nx)^3}+\cdots$ up to $(n+1)$ terms $=0$.
4. Prove that $\frac{{}^nC_0}{1}+\frac{{}^nC_2}{3}+\frac{{}^nC_4}{5}+\frac{{}^nC_6}{7}+\cdots=\frac{2^n}{n+1}$.
5. If $(1+x)^{15}=C_0+C_1x+C_2x^2+\cdots+C_{15}x^{15}$, then find the sum of $C_2+2C_3+3C_4+\cdots+14C_{15}$.
6. Find the coefficient of x^n in the polynomial $(x+{}^nC_0)(x+3\,{}^nC_1)\times(x+5\,{}^nC_2)\cdots[x+(2n+1)\,{}^nC_n]$.
7. Find the value of ${}^{20}C_0-\frac{{}^{20}C_1}{2}+\frac{{}^{20}C_2}{3}-\frac{{}^{20}C_3}{4}+\cdots$.
8. Find the value of
$$\frac{1}{81^n}-\frac{10}{81^n}{}^{2n}C_1+\frac{10^2}{81^n}{}^{2n}C_2-\frac{10^3}{81^n}{}^{2n}C_3+\cdots+\frac{10^{2n}}{81^n}$$

MISCELLANEOUS SERIES

Series from Multiplication of Two Series

Standard Results

1. ${}^mC_r+{}^mC_{r-1}\,{}^nC_1+{}^mC_{r-2}\,{}^nC_2+\cdots+{}^nC_r={}^{m+n}C_r$, where $r<m$, $r<n$ and m, n, r are +ve integers.

Proof:

$$\text{L.H.S.}={}^mC_r+{}^mC_{r-1}\,{}^nC_1+{}^mC_{r-2}\,{}^nC_2+\cdots+{}^nC_r$$
$$={}^mC_r\,{}^nC_0+{}^mC_{r-1}\,{}^nC_1+{}^mC_{r-2}\,{}^nC_2+\cdots+{}^mC_0\,{}^nC_r$$
$$=\text{coefficient of } x^r \text{ in } [(1+x)^m(1+x)^n]$$
$$=\text{coefficient of } x^r \text{ in } (1+x)^{m+n}$$
$$={}^{m+n}C_r={}^{(\text{sum of prefixes } m \text{ and } n)}C_{\text{constant sum of suffixes}}$$

2. ${}^nC_0^2+{}^nC_1^2+\cdots+{}^nC_n^2={}^{2n}C_n$

Proof:

$${}^nC_0^2+{}^nC_1^2+{}^nC_2^2+\cdots+{}^nC_n^2$$

$= ({}^nC_0\,{}^nC_n + {}^nC_1\,{}^nC_{n-1} + {}^nC_2\,{}^nC_{n-2} + \cdots + {}^nC_n\,{}^nC_0)$
$=$ Coefficient of x^n in $(1+x)^n\,(1+x)^n$
$=$ Coefficient of x^n in $(1+x)^{2n}$
$= {}^{2n}C_n$

3. ${}^nC_0^{\,2} - {}^nC_1^{\,2} + {}^nC_2^{\,2} + \cdots + (-1)^n\,{}^nC_n^{\,2}$

Proof:

${}^nC_0^{\,2} - {}^nC_1^{\,2} + {}^nC_2^{\,2} - \cdots + (-1)^n\,{}^nC_n^{\,2}$
$= {}^nC_0\,{}^nC_n - {}^nC_1\,{}^nC_{n-1} + {}^nC_2\,{}^nC_{n-2} + \cdots + (-1)^n\,{}^nC_n\,{}^nC_0$
$=$ Coefficient of x^n in $(1+x)^n\,(1-x)^n$
$=$ Coefficient of x^n in $(1-x^2)^n$

Now, general term in $(1-x^2)^n$ is $T_{r+1} = {}^nC_r(-x^2)^r$.

For x^n, $2r = n \Rightarrow r = n/2 \Rightarrow n$ must be even.

If n is odd, r is not an integer or we can say, when n is odd, x^n term does not occur in $(1-x^2)^n$ which is obvious. When r is even, $r = n/2$. Hence,

$$T_{n/2+1} = {}^nC_{n/2}\,(-x^2)^{n/2} = (-1)^n\,{}^nC_{n/2}x^n$$

$$\Rightarrow C_0^2 - C_1^2 + C_2^2 + \cdots + (-1)^n C_n^2$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^n\,{}^nC_{n/2}, & \text{if } n \text{ is even} \end{cases}$$

Example 6.74 **Find the sum $\sum_{i=0}^{r} {}^{n_1}C_{(r-i)}\ {}^{n_2}C_i$.**

Sol. $S = \sum_{i=0}^{r} ({}^{n_1}C_{r-i})({}^{n_2}C_i)$
$=$ Coefficient of x^r in $(1+x)^{n_1}\ (1+x)^{n_2}$
$= {}^{n_1+n_2}C_r$

Example 6.75 **Prove that $\sum_{i=0}^{2n} r({}^{2n}C_r)^2 = {}^{4n-1}C_{2n+1}$.**

Sol. $S = \sum_{r=0}^{2n} r({}^{2n}C_r)^2$

$$= \sum_{r=0}^{2n} (r\,{}^{2n}C_r)({}^{2n}C_r)$$

$$= \sum_{r=0}^{2n} (2n)\,{}^{2n-1}C_{r-1}\ {}^{2n}C_r$$

$$= 2n\sum_{r=0}^{2n} {}^{2n-1}C_{r-1}\ {}^{2n}C_{2n-r}$$

Here, the sum of suffixes is $r-1+(2n-r) = 2n-1$ which is constant.

$\therefore$ $S =$ Coefficient of x^{2n-1} in the expansion of $(1+x)^{2n-1}\,(1+x)^{2n}$
$=$ Coefficient of x^{2n-1} in the expansion of $(1+x)^{4n-1}$
$= {}^{4n-1}C_{2n-1}$

Example 6.76 **Using binomial theorem (without using the formula for ${}^nC_r$), prove that**

$$ {}^nC_4 + {}^mC_2 - {}^mC_1\,{}^nC_2 = {}^mC_4 - {}^{m+n}C_1\,{}^mC_3 + {}^{m+n}C_2\,{}^mC_2 - {}^{m+n}C_3\,{}^mC_1 + {}^{m+n}C_4$$

Sol. ${}^mC_4 - {}^{m+n}C_1\,{}^mC_3 + {}^{m+n}C_2\,{}^mC_2 - {}^{m+n}C_3\,{}^mC_1 + {}^{m+n}C_4$
$= {}^{m+n}C_0\,{}^mC_4 - {}^{m+n}C_1\,{}^mC_3 + {}^{m+n}C_2\,{}^mC_2 - {}^{m+n}C_3\,{}^mC_1 + {}^{m+n}C_4\,{}^mC_0$
$=$ Coefficient of x^4 in $(1+x)^{m+n}\,(1-x)^m$
$=$ Coefficient of x^4 in $(1-x^2)^m\,(1+x)^n$
$=$ Coefficient of x^4 in $[1 - {}^mC_1\,x^2 + {}^mC_2\,x^4 - \cdots]\,[1 + {}^nC_1x + {}^nC_2\,x^2 + \cdots + {}^nC_n\,x^n]$
$= {}^nC_4 - {}^mC_1 \times {}^nC_2 + {}^mC_2$

Binomial Inside Binomial

Example 6.77 **Find the sum $\sum_{r=0}^{n} {}^{n+r}C_r$.**

Sol. $\sum_{r=0}^{n} {}^{n+r}C_r = {}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \cdots + {}^{n+n}C_n$
$= {}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \cdots + {}^{2n}C_n$
$=$ Coefficient of x^n in $\{(1+x)^n + (1+x)^{n+1} + \cdots + (1+x)^{2n}\}$
$=$ Coefficient of x^n in $\dfrac{(1+x)^n\left\{1-(1+x)^{n+1}\right\}}{1-(1+x)}$
$=$ Coefficient of x^{n+1} in $\{(1+x)^{2n+1} - (1+x)^n\}$
$= {}^{2n+1}C_{n+1}$

Example 6.78 **Prove that ${}^nC_0\ {}^{2n}C_n - {}^nC_1\ {}^{2n-1}C_n + {}^nC_2 \times {}^{2n-2}C_n + \cdots + (-1)^n\,{}^nC_n\,{}^nC_n = 1$.**

Sol. We know that

${}^{2n}C_n =$ Coefficient of x^n in $(1+x)^{2n}$
${}^{2n-1}C_n =$ Coefficient of x^n in $(1+x)^{2n-1}$
${}^{2n-2}C_n =$ Coefficient of x^n in $(1+x)^{2n-2}$
${}^{2n-3}C_n =$ Coefficient of x^n in $(1+x)^{2n-3}$
$\vdots$
${}^nC_n =$ Coefficient of x^n in $(1+x)^n$

Thus we have,

${}^nC_0\,{}^{2n}C_n - {}^nC_1\,{}^{2n-1}C_n + {}^nC_2\,{}^{2n-2}C_n + \cdots + (-1)^n\,{}^nC_n\,{}^nC_n$
$=$ Coefficient of x^n in $[C_0\,(1+x)^{2n} - C_1\,(1+x)^{2n-1} + C_2\,(1+x)^{2n-2} - C_3(1+x)^{2n-3} + \cdots + (-1)^n\,C_n\,(1+x)^n]$
$=$ Coefficient of x^n in $(1+x)^n\,[C_0\,(1+x)^n - C_1\,(1+x)^{n-1} + C_2(1+x)^{n-2} - C_3(1+x)^{n-3} + \cdots + (-1)^n C_n]$
$=$ Coefficient of x^n in $(1+x)^n\,[(1+x)-1]^n$
$=$ Coefficient of x^n in $(1+x)^n\,x^n$
$= 1$

Example 6.79 **Prove that ${}^mC_1\,{}^nC_m - {}^mC_2\,{}^{2n}C_m + {}^mC_3\,{}^{3n}C_m - \cdots = (-1)^{m-1}\,n^m$.**

Sol. ${}^mC_1\,{}^nC_m - {}^mC_2\,{}^{2n}C_m + {}^mC_3\,{}^{3n}C_m - \cdots + (-1)^{m-1}\,{}^mC_m\,{}^{mn}C_m$
$=$ Coefficient of x^m in

$${}^mC_1\,(1+x)^n - {}^mC_2\,(1+x)^{2n} + {}^mC_3(1+x)^{3n} - \cdots + (-1)^{m-1}\,{}^mC_m\,(1+x)^{mn}$$

$=$ Coefficient of x^m in

$${}^mC_0 - [{}^mC_0 - {}^mC_1\,(1+x)^n + {}^mC_2\,(1+x)^{2n} - \cdots + (-1)^m\,{}^mC_m(1+x)^{mn}]$$

$=$ Coefficient of x^m in $[1-\{1-(1+x)^n\}^m]$

$=$ Coefficient of x^m in $[1-\{-nx - {}^nC_2\,x^2 - {}^nC_3x^3 - \cdots - {}^nC_n\,x^n\}^m]$

$= -(-n)^m = -(-1)^m n^m = -(-1)^{m-1}\,(-1)^2 n^m$

$= -(-1)^{m-1} n^m$

Example 6.80 **If $(18x^2+12x+4)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, prove that**

$$a_r = 2^n\,3^r\,({}^{2n}C_r + {}^nC_1\,{}^{2n-2}C_r + {}^nC_2\,{}^{2n-4}C_r + \cdots)$$

Sol. $(18x^2+12x+4)^n = 2^n\,[2+9x^2+6x]^n$

Now, a_r is coefficient of x^r in $2^n\,[(3x+1)^2+1]^n$. Hence,

$$a_r = \text{Coefficient of } x^r \text{ in } 2^n\,[{}^nC_0(3x+1)^{2n} + {}^nC_1(3x+1)^{2n-2} + {}^nC_2(3x+1)^{2n-4} + \cdots + {}^nC_r(3x+1)^{2n-2r} + \cdots]$$

$$\Rightarrow\ a_r = 2^n\,[{}^nC_0\,3^r\,{}^{2n}C_r + {}^nC_1\,3^r\,{}^{2n-2}C_r + {}^nC_2\,3^r\,{}^{2n-4}C_r + \cdots]$$

$$= 2^n\,3^r\,[{}^nC_0\,{}^{2n}C_r + {}^nC_1\,{}^{2n-2}C_r + {}^nC_2\,{}^{2n-4}C_r + \cdots]$$

Concept Application Exercise 6.8

1. Prove that $\sum_{r=0}^{n} r(n-r)C_r^2 = n^2({}^{2n-2}C_n)$.
2. Prove that $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \cdots + ({}^{2n}C_{2n})^2 = (-1)^n\,{}^{2n}C_n$.
3. Prove that ${}^nC_0\;{}^nC_0 - {}^{n+1}C_1\;{}^nC_1 + {}^{n+2}C_2\;{}^nC_2 - \cdots = (-1)^n$.
4. Prove that ${}^nC_0\,{}^{2n}C_n - {}^nC_1\,{}^{2n-2}C_n + {}^nC_2\,{}^{2n-4}C_n - \cdots = 2^n$.
5. Find the value of $\sum_{p=1}^{n}\left(\sum_{m=p}^{n}{}^nC_m\;{}^mC_p\right)$. And hence, find the value of $\lim_{n\to\infty}\frac{1}{3^n}\sum_{p=1}^{n}\left(\sum_{m=p}^{n}{}^nC_m\;{}^mC_p\right)$.

Sum of the Series when *i* and *j* are Dependent

Consider, sum of the series $\sum\sum_{0\le i<j\le n} f(i)\,f(j)$

In the given summation, i and j are not independent.

In the sum of series $\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)\,f(j) = \sum_{i=1}^{n}\left(f(i)\left(\sum_{j=1}^{n} f(j)\right)\right)$, i and j are independent. In this summation, three types of terms occur, those when $i<j$, $i>j$ and $i=j$.

Also, sum of terms when $i<j$ is equal to the sum of the terms when $i>j$ if $f(i)$ and $f(j)$ are symmetrical. So, in that case

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = \sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{0\le j<i\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$= 2\sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$\Rightarrow\ \sum\sum_{0\le i<j\le n} f(i)f(j) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) - \sum\sum_{i=j} f(i)f(j)}{2}$$

When $f(i)$ and $f(j)$ are not symmetrical, we find the sum by listing all the terms.

Example 6.81 **Find the sum**

a. $\sum\sum_{0\le i<j\le n} {}^nC_i\,{}^nC_j$

b. $\sum\sum_{0\le i\le j\le n} {}^nC_i\,{}^nC_j$

c. $\sum\sum_{i\ne j} {}^nC_i\,{}^nC_j$

Sol. a. $\sum\sum_{0\le i<j\le n} {}^nC_i\,{}^nC_j$

$$= \frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n}{}^nC_i\,{}^nC_j\right) - \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{\left(\sum_{i=0}^{n}{}^nC_i\,2^n\right) - \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{2^n2^n - \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{2^{2n} - {}^{2n}C_n}{2}$$

b. $\sum\sum_{0\le i\le j\le n} {}^nC_i\,{}^nC_j$

$$= \frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n}{}^nC_i\,{}^nC_j\right) + \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{\left(\sum_{i=0}^{n}{}^nC_i\,2^n\right) + \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{2^n2^n + \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{2^{2n} + {}^{2n}C_n}{2}$$

c. $\sum\sum_{i\ne j} {}^nC_i\,{}^nC_j$

$$= \left(\sum_{i=0}^{n}\sum_{j=0}^{n}{}^nC_i\,{}^nC_j\right) - \sum_{i=0}^{n}({}^nC_i)^2$$

$$= \left(\sum_{i=0}^{n}{}^nC_i\,2^n\right) - \sum_{i=0}^{n}({}^nC_i)^2$$

$$= 2^{2n} - {}^{2n}C_n$$

Example 6.82 **Find the value of $\sum\sum_{0\le i<j\le n} ({}^nC_i + {}^nC_j)$.**

Sol. $$\sum\sum_{0\le i<j\le n}({}^nC_i+{}^nC_j)=\frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n}({}^nC_i+{}^nC_j)\right)-\sum_{i=0}^{n}2\ {}^nC_i}{2}$$

$$=\frac{\left(\sum_{i=0}^{n}\left(\sum_{j=0}^{n}{}^nC_i+\sum_{j=0}^{n}{}^nC_j\right)\right)-2\times 2^n}{2}$$

$$=\frac{\left(\sum_{i=0}^{n}\left({}^nC_i\sum_{j=0}^{n}1+2^n\right)\right)-2^{n+1}}{2}$$

$$=\frac{\left(\sum_{i=0}^{n}({}^nC_i(n+1)+2^n)\right)-2^{n+1}}{2}$$

$$=\frac{(n+1)\sum_{i=1}^{n}{}^nC_i+2^n\sum_{i=0}^{n}1-2^{n+1}}{2}$$

$$=\frac{(n+1)2^n+2^n(n+1)-2^{n+1}}{2}$$

$$=(n+1)2^n-2^n=n2^n$$

Example 6.83 **Find the value of** $\sum\sum_{1\le i<j\le n-1}(i\,j)\,{}^nC_i\,{}^nC_j$.

Sol. $$S=\sum\sum_{1\le i<j\le n-1}(i\ {}^nC_i)(j\ {}^nC_j)$$

$$=n^2\sum\sum_{1\le i<j\le n-1}{}^{n-1}C_{i-1}\ {}^{n-1}C_{j-1}$$

$$=n^2\left(\frac{2^{2(n-1)}-{}^{2(n-1)}C_{n-1}}{2}\right)$$

Example 6.84 **Find the value of** $\sum_{0\le i<j\le n}(i+j)\,({}^nC_i+{}^nC_j)$.

Sol. Here sum does not change if we replace i by $n-1$ and j by $n-j$

By doing so, in fact we are writing the series in the reverse order.

$$\therefore\quad S=\sum\sum_{0\le i<j\le n}(i+j)({}^nC_i+{}^nC_j)\qquad(1)$$

$$=\sum\sum_{0\le i<j\le n}(n-i+n-j)({}^nC_{n-i}+{}^nC_{n-j})$$

$$\therefore\quad S=\sum\sum_{0\le i<j\le n}(2n-(i+j))\ ({}^nC_i+{}^nC_n)\qquad(2)$$

Adding (1) and (2), we have

$$2S=2n\sum\sum_{0\le i<j\le n}({}^nC_i+{}^nC_j)$$

$$\Rightarrow\quad S=n\times n2^n=n^2\,2^n$$

Example 6.85 **Find the sum** $\sum\sum_{0\le i<j\le n}{}^nC_i$.

Sol. $$\sum\sum_{0\le i<j\le n}{}^nC_i=\frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n}{}^nC_i\right)-\sum_{i=0}^{n}{}^nC_i}{2}$$

$$=\frac{\left(\sum_{i=0}^{n}(n+1)\,{}^nC_i\right)-\sum_{i=0}^{n}{}^nC_i}{2}$$

$$=\frac{(n+1)2^n-2^n}{2}$$

$$=n\times 2^{n-1}$$

Example 6.86 **Find the sum** $\sum\sum_{0\le i<j\le n}j\,{}^nC_i$.

Sol. $$\sum\sum_{0\le i<j\le n}j\ {}^nC_i$$

$$=\sum_{r=0}^{n-1}{}^nC_r[(r+1)+(r+2)+\cdots+(n)]$$

$$=\sum_{r=0}^{n-1}{}^nC_r[(r+1)+(r+2)+\cdots+(n)]$$

$$=\sum_{r=0}^{n}{}^nC_r\left(\frac{n+1}{2}(n-r)-\frac{r(n-r)}{2}\right)$$

$$=\frac{n+1}{2}\sum_{r=0}^{n}(n-r)\ {}^nC_r\quad-\frac{n}{2}\sum_{r=0}^{n}r\ {}^nC_r+\frac{1}{2}\sum_{r=0}^{n}r^2\ {}^nC_r$$

$$=\frac{n+1}{2}\sum_{r=0}^{n}r\ {}^nC_r-\frac{n}{2}\sum_{r=0}^{n}r\ {}^nC_r+\frac{1}{2}\sum_{r=0}^{n}r^2\ {}^nC_r$$

$$=\frac{1}{2}\left(\sum_{r=0}^{n}r\ {}^nC_r+\sum_{r=0}^{n}r^2\ {}^nC_r\right)$$

$$=\frac{1}{2}(n\ 2^{n-1}+n(n-1)\ 2^{n-2}+n\ 2^{n-1})=n(n+3)\ 2^{n-3}$$

BINOMIAL THEOREM FOR ANY INDEX

Let n be a rational number and x be a real number such that $|x|<1$. Then

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$
$$+\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r+\cdots\infty$$

Note:

- *The condition $|x|<1$ is unnecessary if n is a whole number, while the same condition is essential if n is a rational number other than a whole number.*
- *Note that there are infinite number of terms in the expansion of $(1+x)^n$, when n is a negative integer or a fraction.*
- *In the above expansion, the first term is unity. If the first term is not unity and the index of the binomial is either a negative integer or a fraction, then we expand as follows:*

$$(x+a)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n = a^n\left(1+\frac{x}{a}\right)^n$$

$$= a^n\left\{1+n\frac{x}{a}+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+\cdots\right\}$$

$$= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2+\cdots$$

The expansion is valid when $|x/a| < 1$ *or equivalently* $|x| < |a|$.

- *Expansion of* $(x + a)^n$ *for any rational index:*

Case I: When $|x| > |a|$, *i.e.,* $|a/x| < 1$

$$(x+a)^n = \left\{x\left(1+\frac{a}{x}\right)\right\}^n$$

$$= x^n\left\{1+n\frac{a}{x}+\frac{n(n-1)}{2!}\left(\frac{a}{x}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{a}{x}\right)^3+\cdots\right\}$$

Case II: When $|x| < |a|$, *i.e.,* $|x/a| < 1$

$$(x+a)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n$$

$$= a^n\left\{1+n\frac{x}{a}+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3+\cdots\right\}$$

- *If n is a positive integer, the above expansion contains* $(n + 1)$ *terms and coincides with* $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_nx^n$, *because*

$$ {}^nC_0 = 1,\ {}^nC_1 = n,\ {}^nC_2 = \frac{n(n-1)}{2!},\ {}^nC_3 = \frac{n(n-1)(n-2)}{3!}$$

- *The general term in the expansion of* $(1 + x)^n$ *is given by*

$$T_{r+1} = \frac{n(n-1)(n-2)\cdots[n-(r-1)]}{r!}x^r$$

- *Let n be a positive integer, then by replacing n by* $-n$ *in the expansion for* $(1 + x)^n$, *we get*

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3+\cdots$$

$$+(-1)^r\frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}x^r+\cdots$$

$$= 1 - {}^nC_1x + {}^{n+1}C_2x^2 - {}^{n+2}C_3x^3 + \cdots + {}^{n+r-1}C_r(-x)^r + \cdots$$

Now replacing x by $-x$ *and n by* $-n$ *in the expression of* $(1 + x)^n$, *we get*

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3+\cdots$$

$$+\frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}x^r+\cdots$$

If it is –ve integer then $(1-x)^{-n} = 1 + {}^nC_1x + {}^{n+1}C_2x^2 + {}^{n+2}C_3x^3 + \cdots + {}^{n+r-1}C_rx^r + \cdots$

Important Expansions

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^r x^r + \cdots$

(ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots$

(iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

(iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Example 6.87 **Find the condition for which the formula** $(a+b)^m = a^m + ma^{m-1}b + \dfrac{m(m-1)}{1\times 2}a^{m-2}b^2 + \cdots$ **holds.**

Sol. The expression can be written as $a^m\left\{\left(1+\dfrac{b}{a}\right)^m\right\}$. Hence, it is valid only when

$$|b/a| < 1$$

$$\Rightarrow |b| < |a|$$

Example 6.88 **Find the values of** x, **for which** $1/(\sqrt{5+4x})$ **can be expanded as infinite series.**

Sol. The given expression can be written as $5^{1/2}\left(1+\dfrac{4x}{5}\right)^{-1/2}$ and is valid only when

$$\left|\frac{4}{5}x\right| < 1 \Rightarrow |x| < \frac{5}{4}$$

Example 6.89 **Find the fourth term in the expansion of** $(1-2x)^{3/2}$.

Sol. We have,

$$(1-2x)^{3/2} = 1 + \frac{3}{2}(-2x) + \frac{\frac{3}{2}\times\frac{1}{2}}{2!}(-2x)^2 + \frac{\frac{3}{2}\times\frac{1}{2}\left(-\frac{1}{2}\right)}{3!}(-2x)^3+\cdots$$

Hence, the 4th term is $x^3/2$.

Example 6.90 **Prove that the coefficient of** x^r **in the expansion of** $(1-2x)^{-1/2}$ **is** $(2r)!/[2^r(r!)^2]$.

Sol. Coefficient of x^r is

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-r+1\right)}{r!}(-2)^r$$

$$= \frac{1\times3\times5\times\cdots\times(2r-1)}{2^r}\ \frac{(-1)^r\ (-1)^r\ 2^r}{r!}$$

$$= \frac{1\times3\times5\times\cdots\times(2r-1)}{r!}$$

$$= \frac{1\times2\times3\times4\times5\times\cdots\times(2r-1)(2r)}{(2\times4\times6\times8\times\cdots\times(2r))r!}$$

$$= \frac{(2r)!}{2^r(r!)^2}$$

Example 6.91 **Find the sum**

$$1-\frac{1}{8}+\frac{1}{8}\times\frac{3}{16}-\frac{1\times3\times5}{8\times16\times24}+\ldots$$

Sol. Comparing the given series to

$$1+nx+\frac{n(n-1)}{2!}x^2+\cdots=(1+x)^n$$

we get

$$nx=-\frac{1}{8} \text{ and } \frac{n(n-1)}{2!}x^2=\frac{3}{128}$$

$$\Rightarrow \quad x=\frac{1}{4},\ n=-\frac{1}{2}$$

Hence,

$$1-\frac{1}{8}+\frac{1}{8}\frac{3}{16}-\ldots=\left(1+\frac{1}{4}\right)^{-1/2}=\frac{2}{\sqrt{5}}$$

Example 6.92 **Assuming x to be so small that x^2 and higher powers of x can be neglected, prove that**

$$\frac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}}=1-\frac{305}{96}x$$

Sol. We have,

$$\frac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}}=\frac{\left(1+\frac{3}{4}x\right)^{-4}(16)^{1/2}\left(1-\frac{3x}{16}\right)^{1/2}}{8^{2/3}\left(1+\frac{x}{8}\right)^{2/3}}$$

$$=\left(1+\frac{3}{4}x\right)^{-4}\left(1-\frac{3x}{16}\right)^{1/2}\left(1+\frac{x}{8}\right)^{-2/3}$$

$$=\left\{1+(-4)\left(\frac{3}{4}x\right)\right\}\left\{1+\frac{1}{2}\left(\frac{-3x}{16}\right)\right\}\left\{1+\left(-\frac{2}{3}\right)\left(\frac{x}{8}\right)\right\}$$

$$=(1-3x)\left(1-\frac{3}{32}x\right)\left(1-\frac{x}{12}\right)$$

$$=\left(1-3x-\frac{3}{32}x\right)\left(1-\frac{x}{12}\right) \quad [\text{neglecting } x^2]$$

$$=\left(1-\frac{99}{32}x\right)\left(1-\frac{x}{12}\right)=1-\frac{99}{32}x-\frac{x}{12} \quad [\text{neglecting } x^2]$$

$$=1-\frac{305}{96}x$$

Example 6.93 **Find the coefficient of x^n in the expansion of $(1-9x+20x^2)^{-1}$.**

Sol. We have,

$$(1-9x+20x^2)^{-1}=[(1-5x)(1-4x)]^{-1}$$

$$=\frac{1}{(1-5x)(1-4x)}=\frac{5}{1-5x}-\frac{4}{1-4x}$$

$$=5(1-5x)^{-1}-4(1-4x)^{-1}$$

$$=5[1+5x+(5x)^2+\cdots+(5x)^n+\cdots]-4[1+4x+(4x)^2+\cdots+(4x)^n+\cdots]$$

Therefore the coefficient of x^n is $5^{n+1}-4^{n+1}$.

Concept Application Exercise 6.9

1. If the third term in the expansion of $(1+x)^m$ is $-\frac{1}{8}x^2$, then find the value of m.
2. Find the cube root of 217, correct to two decimal places.
3. Find the coefficient of x^2 in $\left(\frac{a}{a+x}\right)^{1/2}+\left(\frac{a}{a-x}\right)^{1/2}$.
4. If $|x|<1$, then find the coefficient of x^n in the expansion of $(1+x+x^2+\cdots)^2$.
5. If $|x|>1$, then expand $(1+x)^{-2}$.
6. If $|x|<1$, then find the coefficient of x^n in the expansion of $(1+2x+3x^2+4x^3+\cdots)^{1/2}$.
7. If $(r+1)$th term is the first negative term in the expansion of $(1+x)^{7/2}$, then find the value of r.

EXERCISES

Subjective Type

Solutions on page 6.34

1. Find the coefficients of x^n in $\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)^2$.

2. If $\sum_{r=0}^{n}\{a_r(x-\alpha+2)^r - b_r(\alpha-x-1)^r\}=0$, then prove that $b_n-(-1)^n a_n=0$.

3. Evaluate ${}^nC_0\ {}^nC_2+2\,{}^nC_1\ {}^nC_3+3\,{}^nC_2\ {}^nC_4+\cdots+(n-1)\,{}^nC_{n-2}\ {}^nC_n$.

4. Prove that $\sum_{r=0}^{n}{}^nC_r(-1)^r\left[i^r+i^{2r}+i^{3r}+i^{4r}\right]$
$= 2^n + 2^{n/2+1}\cos(n\pi/4)$ where $i=\sqrt{-1}$.

5. Let $a = (4^{1/401}-1)$ and for each $n\ge 2$, let $b_n = {}^nC_1 + {}^nC_2 .\,a + {}^nC_3 .\,a^2+\ldots+{}^nC_n.a^{n-1}$. Find the value of $(b_{2006}-b_{2005})$.

6. If n be a positive integer, prove that

$$1-2n+\frac{2n(2n-1)}{2!}-\frac{2n(2n-1)(2n-2)}{3!}+\cdots+(-1)^{n-1}\frac{2n(2n-1)\cdots(n+2)}{(n-1)!}$$
$$=(-1)^{n+1}(2n)!/2(n!)^2$$

7. Prove that $\frac{{}^nC_0}{x}-\frac{{}^nC_1}{x+1}+\frac{{}^nC_2}{x+2}-\cdots+(-1)^n\frac{{}^nC_n}{x+n}$
$$=\frac{n!}{x(x+1)\cdots(x+n)},$$
where n is any positive integer and x is not a negative integer.

8. Prove that
$${}^{100}C_0\ {}^{100}C_2+{}^{100}C_2\ {}^{100}C_4+{}^{100}C_4\ {}^{100}C_6+\cdots+{}^{100}C_{98}\ {}^{100}C_{100}$$
$$=\frac{1}{2}\left[{}^{200}C_{98}-{}^{100}C_{49}\right].$$

9. Prove that $\sum_{r=1}^{m-1}\frac{2r^2-r(m-2)+1}{(m-r)\,{}^mC_r}=m-\frac{1}{m}$.

10. Find the coefficient of x^{50} in the expression $(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}+\cdots+1001x^{1000}$

11. Prove that
$${}^nC_1-\left(1+\frac{1}{2}\right){}^nC_2+\left(1+\frac{1}{2}+\frac{1}{3}\right){}^nC_3+\cdots+(-1)^{n-1}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right){}^nC_n=\frac{1}{n}$$

12. Prove that ${}^nC_1\,({}^nC_2)^2\,({}^nC_3)^3\cdots({}^nC_n)^n\le\left(\frac{2^n}{n+1}\right)^{{}^{n+1}C_2}$, $\forall\, n\in N$.

13. Prove that
$$\frac{1}{m!}{}^nC_0+\frac{n}{(m+1)!}{}^nC_1+\frac{n(n-1)}{(m+2)!}{}^nC_2+\cdots+\frac{n(n-1)\cdots2\times1}{(m+n)!}{}^nC_n$$
$$=\frac{(m+n+1)(m+n+2)\cdots(m+2n)}{(m+n)!}$$

14. If $n = 12\,m$ $(m\in N)$, prove that
$${}^nC_0-\frac{{}^nC_2}{(2+\sqrt{3})^2}+\frac{{}^nC_4}{(2+\sqrt{3})^4}-\frac{{}^nC_6}{(2+\sqrt{3})^6}+\cdots=(-1)^m\left(\frac{2\sqrt{2}}{1+\sqrt{3}}\right)^n$$

15. Prove that in the expansion of $(1+x)^n(1+y)^n(1+z)^n$, the sum of the coefficients of the terms of degree r is ${}^{3n}C_r$.

Objective Type

Solutions on page 6.36

Each question has four choices a, b, c and d, out of which *only one* is correct.

1. If the coefficients of 5th, 6th and 7th terms in the expansion of $(1+x)^n$ be in A.P., then $n=$
   a. 7 only b. 14 only
   c. 7 or 14 d. none of these

2. If x^m occurs in the expansion of $(x+1/x^2)^{2n}$, then the coefficient of x^m is
   a. $\frac{(2n)!}{(m)!(2n-m)!}$ b. $\frac{(2n)!3!3!}{(2n-m)!}$
   c. $\frac{(2n)!}{\left(\frac{2n-m}{3}\right)!\left(\frac{4n+m}{3}\right)!}$ d. none of these

3. The coefficient of x^5 in the expansion of $(1+x)^{21}+(1+x)^{22}+\cdots+(1+x)^{30}$ is
   a. ${}^{51}C_5$ b. ${}^{9}C_5$
   c. ${}^{31}C_6-{}^{21}C_6$ d. ${}^{30}C_5+{}^{20}C_5$

4. The coefficient of $1/x$ in the expansion of $(1+x)^n(1+1/x)^n$ is
   a. $\frac{n!}{(n-1)!(n+1)!}$ b. $\frac{(2n)!}{(n-1)!(n+1)!}$
   c. $\frac{(2n)!}{(2n-1)!(2n+1)!}$ d. none of these

5. In the expansion of $(1+3x+2x^2)^6$, the coefficient of x^{11} is
   a. 144 b. 288
   c. 216 d. 576

6. If the coefficients of r^{th} and $(r+1)^{th}$ terms in the expansion of $(3+7x)^{29}$ are equal, then r equals
   a. 15 b. 21
   c. 14 d. none of these

7. If the coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio 1:7:42, then the value of n is
   a. 60 b. 70
   c. 55 d. none of these

8. If $(1+2x+x^2)^n=\sum_{r=0}^{2n}a_rx^r$, then $a_r=$
   a. $({}^nC_r)^2$ b. ${}^nC_r\cdot{}^nC_{r+1}$
   c. ${}^{2n}C_r$ d. ${}^{2n}C_{r+1}$

9. If in the expansion of $(1 + x)^n$, a, b, c are three consecutive coefficients, then $n =$
 a. $\dfrac{ac+ab+bc}{b^2+ac}$ b. $\dfrac{2ac+ab+bc}{b^2-ac}$
 c. $\dfrac{ab+ac}{b^2-ac}$ d. none of these

10. The coefficient of the middle term in the binomial expansion in powers of x of $(1 + \alpha x)^4$ and of $(1 - \alpha x)^6$ is the same, if α equals
 a. $-\dfrac{5}{3}$ b. $\dfrac{10}{3}$
 c. $-\dfrac{3}{10}$ d. $\dfrac{3}{5}$

11. The coefficient of x^n in the expansion of $(1 - x)(1 - x)^n$ is
 a. $n - 1$ b. $(-1)^n(1 + n)$
 c. $(-1)^{n-1}(n - 1)^2$ d. $(-1)^{n-1}n$

12. If the coefficients of r^{th}, $(r + 1)^{th}$ and $(r + 2)^{th}$ terms in the binomial expansion of $(1 + y)^m$ are in A.P., then m and r satisfy the equation
 a. $m^2 - m(4r + 1) + 4r^2 + 2 = 0$
 b. $m^2 - m(4r - 1) + 4r^2 - 2 = 0$
 c. $m^2 - m(4r - 1) + 4r^2 + 2 = 0$
 d. $m^2 - m(4r + 1) + 4r^2 - 2 = 0$

13. If the coefficient of x^7 in $\left[ax^2 + \left(\dfrac{1}{bx}\right)\right]^{11}$ equals the coefficient of x^{-7} in $\left[ax - \left(\dfrac{1}{bx^2}\right)\right]^{11}$, then a and b satisfy the relation
 a. $a + b = 1$ b. $a - b = 1$
 c. $ab = 1$ d. $\dfrac{a}{b} = 1$

14. The coefficient of $a^8b^4c^9d^9$ in $(abc + abd + acd + bcd)^{10}$ is
 a. $10!$ b. $\dfrac{10!}{8!4!9!9!}$
 c. 2520 d. none of these

15. The coefficient of x^5 in the expansion of $(x^2 - x - 2)^5$ is
 a. –83 b. –82
 c. –86 d. –81

16. The coefficient of x^2y^3 in the expansion of $(1 - x + y)^{20}$ is
 a. $\dfrac{20!}{2!3!}$ b. $-\dfrac{20!}{2!3!}$
 c. $\dfrac{20!}{5!2!3!}$ d. none of these

17. If the term independent of x in the $\left(\sqrt{x} - \dfrac{k}{x^2}\right)^{10}$ is 405, then k equals
 a. 2, –2 b. 3, –3
 c. 4, –4 d. 1, –1

18. The coefficient of x^{10} in the expansion of $(1 + x^2 - x^3)^8$ is
 a. 476 b. 496
 c. 506 d. 528

19. If the coefficient of x^n in $(1 + x)^{101}(1 - x + x^2)^{100}$ is non-zero, then n cannot be of the form
 a. $3r + 1$ b. $3r$
 c. $3r + 2$ d. none of these

20. The coefficient of x^{28} in the expansion of $(1 + x^3 - x^6)^{30}$ is
 a. 1 b. 0
 c. ${}^{30}C_6$ d. ${}^{30}C_3$

21. The term independent of a in the expansion of $\left(1 + \sqrt{a} + \dfrac{1}{\sqrt{a} - 1}\right)^{-30}$ is
 a. ${}^{30}C_{20}$ b. 0
 c. ${}^{30}C_{10}$ d. none of these

22. The coefficient of x^{53} in the expansion $\sum_{m=0}^{100} {}^{100}C_m (x-3)^{100-m} 2^m$ is
 a. ${}^{100}C_{47}$ b. ${}^{100}C_{53}$
 c. $-{}^{100}C_{53}$ d. $-{}^{100}C_{100}$

23. The coefficient of the term independent of x in the expansion of $\left(\dfrac{x+1}{x^{2/3} - x^{1/3} + 1} - \dfrac{x-1}{x - x^{1/2}}\right)^{10}$ is
 a. 210 b. 105
 c. 70 d. 112

24. In the expansion of $(1 + x + x^3 + x^4)^{10}$, the coefficient of x^4 is
 a. ${}^{40}C_4$ b. ${}^{10}C_4$
 c. 210 d. 310

25. The approximate value of $(1.0002)^{3000}$ is
 a. 1.6 b. 1.4
 c. 1.8 d. 1.2

26. The last two digits of the number 3^{400} are
 a. 81 b. 43
 c. 29 d. 01

27. The expression
$$\left(\sqrt{2x^2+1} + \sqrt{2x^2-1}\right)^6 + \left(\frac{2}{\sqrt{2x^2+1} + \sqrt{2x^2-1}}\right)^6$$
is a polynomial of degree
 a. 6 b. 8
 c. 10 d. 12

28. The coefficient of $x^r [0 \le r \le (n - 1)]$ in the expansion of $(x + 3)^{n-1} + (x + 3)^{n-2}(x + 2) + (x + 3)^{n-3}(x + 2)^2 + \cdots + (x + 2)^{n-1}$ are
 a. ${}^nC_r(3^r - 2^n)$ b. ${}^nC_r(3^{n-r} - 2^{n-r})$
 c. ${}^nC_r(3^r + 2^{n-r})$ d. none of these

29. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \cdots + a_{20}x^{20}$, then a_1 equals
 a. 10 b. 20
 c. 210 d. none of these

30. In the expansion of $(5^{1/2} + 7^{1/8})^{1024}$, the number of integral terms is
 a. 128 b. 129
 c. 130 d. 131

31. The total number of terms which are dependent on the value of x, in the expansion of $\left(x^2-2+\frac{1}{x^2}\right)^n$ is equal to

a. $2n+1$ b. $2n$
c. n d. $n+1$

32. If the 6th term in the expansion of $\left(\frac{1}{x^{8/3}}+x^2\log_{10}x\right)^8$ is 5600, then x equals

a. 1 b. $\log_e 10$
c. 10 d. x does not exist

33. If n is an integer between 0 and 21, then the minimum value of $n!(21-n)!$ is attained for $n=$

a. 1 b. 10
c. 12 d. 20

34. In the expansion of $(3^{-x/4}+3^{5x/4})^n$ the sum of binomial coefficient is 64 and term with the greatest binomial coefficient exceeds the third by $(n-1)$, the value of x must be

a. 0 b. 1
c. 2 d. 3

35. If the last term in the binomial expansion of $\left(2^{1/3}-\frac{1}{\sqrt{2}}\right)^n$ is $\left(\frac{1}{3^{5/3}}\right)^{\log_3 8}$, then the 5th term from the beginning is

a. 210 b. 420
c. 105 d. none of these

36. If ${}^{n+1}C_{r+1}:{}^{n}C_{r}:{}^{n-1}C_{r-1}=11:6:3$, then $nr=$

a. 20 b. 30
c. 40 d. 50

37. The value of x for which the sixth term in the expansion of

$$\left[2^{\log_2\sqrt{9^{x-1}+7}}+\frac{1}{2^{\frac{1}{5}\log_2(3^{x-1}+1)}}\right]^7$$

is 84 is

a. 4 b. 1 or 2
c. 0 or 1 d. 3

38. The number of integral terms in the expansion of $(\sqrt{3}+\sqrt[8]{5})^{256}$ is

a. 33 b. 34
c. 35 d. none of these

39. The number of real negative terms in the binomial expansion of $(1+ix)^{4n-2}$, $n\in N$, $x>0$ is

a. n b. $n+1$
c. $n-1$ d. $2n$

40. If in the expansion of $(a-2b)^n$, the sum of 5th and 6th terms is 0, then the values of $a/b=$

a. $\frac{n-4}{5}$ b. $\frac{2(n-4)}{5}$
c. $\frac{5}{n-4}$ d. $\frac{5}{2(n-4)}$

41. The number of distinct terms in the expansion of $\left(x+\frac{1}{x}+x^2+\frac{1}{x^2}\right)^{15}$ is/are (with respect to different power of x)

a. 255 b. 61
c. 127 d. none of these

42. The sum of the coefficients of even power of x in the expansion of $(1+x+x^2+x^3)^5$ is

a. 256 b. 128
c. 512 d. 64

43. Maximum sum of coefficient in the expansion of $(1-x\sin\theta+x^2)^n$ is

a. 1 b. 2^n
c. 3^n d. 0

44. If the sum of the coefficients in the expansion of $(a+b)^n$ is 4096, then the greatest coefficient in the expansion is

a. 924 b. 792
c. 1594 d. none of these

45. If the sum of the coefficients in the expansion of $(1-3x+10x^2)^n$ is a and if the sum of the coefficients in the expansion of $(1+x^2)^n$ is b, then

a. $a=3b$ b. $a=b^3$
c. $b=a^3$ d. none of these

46. If $(1+x-2x^2)^6=1+a_1x+a_2x^2+a_3x^3+\cdots$, then the value of $a_2+a_4+a_6+\cdots+a_{12}$ will be

a. 32 b. 31
c. 64 d. 1024

47. The fractional part of $2^{4n}/15$ is $(n\in N)$

a. $\frac{1}{15}$ b. $\frac{2}{15}$
c. $\frac{4}{15}$ d. none of these

48. The value of ${}^{15}C_0^2-{}^{15}C_1^2+{}^{15}C_2^2-\cdots-{}^{15}C_{15}^2$ is

a. 15 b. -15
c. 0 d. 51

49. The value of

$$\binom{30}{0}\binom{30}{10}-\binom{30}{1}\binom{30}{11}+\binom{30}{2}\binom{30}{12}+\cdots+\binom{30}{20}\binom{30}{30}=$$

a. ${}^{60}C_{20}$ b. ${}^{30}C_{10}$
c. ${}^{60}C_{30}$ d. ${}^{40}C_{30}$

50. If $f(x)=x^n$, then the value of $f(1)+\frac{f^1(1)}{1}+\frac{f^2(1)}{2!}+\cdots+\frac{f^n(1)}{n!}$, where $f^r(x)$ denotes the r^{th} order derivative of $f(x)$ with respect to x, is

a. n b. 2^n
c. 2^{n-1} d. none of these

51. The value of ${}^{20}C_0+{}^{20}C_1+{}^{20}C_2+{}^{20}C_3+{}^{20}C_4+{}^{20}C_{12}+{}^{20}C_{13}+{}^{20}C_{14}+{}^{20}C_{15}$ is

a. $2^{19}-\frac{({}^{20}C_{10}+{}^{20}C_9)}{2}$ b. $2^{19}-\frac{({}^{20}C_{10}+2\times{}^{20}C_9)}{2}$
c. $2^{19}-\frac{{}^{20}C_{10}}{2}$ d. none of these

52. The value of $\frac{^nC_0}{n}+\frac{^nC_1}{n+1}+\frac{^nC_2}{n+2}+\cdots+\frac{^nC_n}{2n}$ is equal to

a. $\int_0^1 x^{n-1}(1-x)^n dx$
b. $\int_1^2 x^n(x-1)^{n-1}dx$
c. $\int_1^2 x^{n-1}(1+x)^n dx$
d. $\int_0^1 (1-x)^n x^{n-1}dx$

53. If $C_0, C_1, C_2, \ldots, C_n$ are the binomial coefficients, then $2\times C_1+2^3\times C^3+2^5\times C_5+\cdots$ equals

a. $\frac{3^n+(-1)^n}{2}$
b. $\frac{3^n-(-1)^n}{2}$
c. $\frac{3^n+1}{2}$
d. $\frac{3^n-1}{2}$

54. The value of $\sum_{r=1}^{n}(-1)^{r+1}\frac{^nC_r}{r+1}$ is equal to

a. $-\frac{1}{n+1}$
b. $-\frac{1}{n}$
c. $\frac{1}{n+1}$
d. $\frac{n}{n+1}$

55. $(1+x)^n = C_0+C_1x+C_2x^2+\cdots+C_nx^n$ then $C_0C_2+C_1C_3+C_2C_4+\cdots+C_{n-2}C_n=$

a. $\frac{(2n)!}{(n!)^2}$
b. $\frac{(2n)!}{(n-1)!(n+1)!}$
c. $\frac{(2n)!}{(n-2)!(n+2)!}$
d. none of these

56. The sum of series $^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+\cdots+{}^{20}C_{10}$ is

a. $\frac{1}{2}\,{}^{20}C_{10}$
b. 0
c. $^{20}C_{10}$
d. $-{}^{20}C_{10}$

57. $^{404}C_4-{}^4C_1{}^{303}C_4+{}^4C_2{}^{202}C_4-{}^4C_3{}^{101}C_4$ is equal to

a. $(401)^4$
b. $(101)^4$
c. 0
d. $(201)^4$

58. If $(3+x^{2008}+x^{2009})^{2010}=a_0+a_1x+a_2x^2+\cdots+a_nx^n$, then the value of $a_0-\frac{1}{2}a_1-\frac{1}{2}a_2+a_3-\frac{1}{2}a_4-\frac{1}{2}a_5+a_6-\cdots$ is

a. 3^{2010}
b. 1
c. 2^{2010}
d. none of these

59. The value of $\sum_{r=0}^{10} r\,{}^{10}C_r\,3^r(-2)^{10-r}$ is

a. 20
b. 10
c. 300
d. 30

60. The value of $\sum_{r=0}^{40} r\,{}^{40}C_r\,{}^{30}C_r$ is

a. $40\,{}^{69}C_{29}$
b. $40\,{}^{70}C_{30}$
c. $^{69}C_{29}$
d. $^{70}C_{30}$

61. The value of $\sum_{r=1}^{15}\frac{r2^r}{(r+2)!}$ is equal to

a. $\frac{(17)!-12^{16}}{(17)!}$
b. $\frac{(18)!-2^{17}}{(18)!}$
c. $\frac{(16)!-2^{15}}{(16)!}$
d. $\frac{(15)!-2^{14}}{(15)!}$

62. $(n+2)\,{}^nC_0\,2^{n+1}-(n+1)\,{}^nC_1\,2^n+n\,{}^nC_2\,2^{n-1}-\cdots$ is equal to

a. 4
b. $4n$
c. $4(n+1)$
d. $2(n+2)$

63. The value of $\sum_{r=0}^{50}(-1)^r\frac{^{50}C_r}{r+2}$ is equal to

a. $\frac{1}{50\times 51}$
b. $\frac{1}{52\times 50}$
c. $\frac{1}{52\times 51}$
d. none of these

64. In the expansion of $[(1+x)/(1-x)]^2$, the coefficient of x^n will be

a. $4n$
b. $4n-3$
c. $4n+1$
d. none of these

65. The sum of $1+n\left(1-\frac{1}{x}\right)+\frac{n(n+1)}{2!}\left(1-\frac{1}{x}\right)^2+\cdots\infty$ will be

a. x^n
b. x^{-n}
c. $\left(1-\frac{1}{x}\right)^n$
d. none of these

66. $\sum_{k=1}^{\infty}k\left(1-\frac{1}{n}\right)^{k-1}=$

a. $n(n-1)$
b. $n(n+1)$
c. n^2
d. $(n+1)^2$

67. The coefficient of x^4 in the expansion of $\{\sqrt{1+x^2}-x\}^{-1}$ in ascending powers of x, when $|x|<1$, is

a. 0
b. $\frac{1}{2}$
c. $-\frac{1}{2}$
d. $-\frac{1}{8}$

68. $1+\frac{1}{3}x+\frac{1\times 4}{3\times 6}x^2+\frac{1\times 4\times 7}{3\times 6\times 9}x^3+\cdots$ is equal to

a. x
b. $(1+x)^{1/3}$
c. $(1-x)^{1/3}$
d. $(1-x)^{-1/3}$

69. $1+\frac{1}{4}+\frac{1\times 3}{4\times 8}+\frac{1\times 3\times 5}{4\times 8\times 12}+\cdots=$

a. $\sqrt{2}$
b. $\frac{1}{\sqrt{2}}$
c. $\sqrt{3}$
d. $\frac{1}{\sqrt{3}}$

70. If $|x|<1$, then

$$1+n\left(\frac{2x}{1+x}\right)+\frac{n(n+1)}{2!}\left(\frac{2x}{1+x}\right)^2+\cdots$$

is equal to

a. $\left(\frac{2x}{1+x}\right)^n$ b. $\left(\frac{1+x}{2x}\right)^n$

c. $\left(\frac{1-x}{1+x}\right)^n$ d. $\left(\frac{1+x}{1-x}\right)^n$

71. The coefficient of x^5 in $(1+2x+3x^2+\cdots)^{-3/2}$ is $(|x|<1)$

a. 21 b. 25

c. 26 d. none of these

72. If $|x|<1$, then the coefficient of x^n in expansion of $(1+x+x^2+x^3+\cdots)^2$ is

a. n b. $n-1$

c. $n+2$ d. $n+1$

73. If x is positive, the first negative term in the expansion of $(1+x)^{27/5}$ is $(|x|<1)$

a. 5th term b. 8th term

c. 6th term d. 7th term

74. If x is so small that x^3 and higher powers of x may be neglected, then

$$\frac{(1+x)^{3/2}-\left(1+\frac{1}{2}x\right)^3}{(1-x)^{1/2}}$$

may be approximated as

a. $3x+\frac{3}{8}x^2$ b. $1-\frac{3}{8}x^2$

c. $\frac{x}{2}-\frac{3}{x}x^2$ d. $-\frac{3}{8}x^2$

75. If the expansion in powers of x of the function $1/[(1-ax)(1-bx)]$ is $a_0+a_1x+a_2x^2+a_3x^3+\cdots$, then a_n is

a. $\frac{b^n-a^n}{b-a}$ b. $\frac{a^n-b^n}{b-a}$

c. $\frac{a^{n+1}-b^{n+1}}{b-a}$ d. $\frac{b^{n+1}-a^{n+1}}{b-a}$

76. Value of $\sum_{k=1}^{\infty}\sum_{r=0}^{k}\frac{1}{3^k}({}^kC_r)$ is

a. $\frac{2}{3}$ b. $\frac{4}{3}$

c. 2 d. 1

77. The sum of rational term in $(\sqrt{2}+\sqrt[3]{3}+\sqrt[6]{5})^{10}$ is equal to

a. 12632 b. 1260

c. 126 d. none of these

78. If $f(x)=1-x+x^2-x^3+\cdots-x^{15}+x^{16}-x^{17}$, then the coefficient of x^2 in $f(x-1)$ is

a. 826 b. 816

c. 822 d. none of these

79. If $p=(8+3\sqrt{7})^n$ and $f=p-[p]$, where $[\cdot]$ denotes the greatest integer function, then the value of $p(1-f)$ is equal to

a. 1 b. 2

c. 2^n d. 2^{2n}

80. The value of $\sum_{r=0}^{10}(r)\,{}^{20}C_r$ is equal to

a. $20(2^{18}+{}^{19}C_{10})$ b. $10(2^{18}+{}^{19}C_{10})$

c. $20(2^{18}+{}^{19}C_{11})$ d. $10(2^{18}+{}^{19}C_{11})$

81. The last two digits of the number $(23)^{14}$ are

a. 01 b. 03

c. 09 d. none of these

82. Let $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$ and $\frac{f(x)}{1-x}=b_0+b_1x+b_2x^2+\cdots+b_nx^n+\cdots$, then

a. $b_n+b_{n-1}=a_n$ b. $b_n-b_{n-1}=a_n$

c. $b_n/b_{n-1}=a_n$ d. none of these

83. If $(1-x^2)^n=\sum_{r=0}^{n}a_rx^r(1-x)^{2n-r}$, then a_r is equal to

a. ${}^nC_r$ b. ${}^nC_r3^r$

c. ${}^{2n}C_r$ d. ${}^nC_r2^r$

84. $[({}^nC_0+{}^nC_3+\cdots)-1/2({}^nC_1+{}^nC_2+{}^nC_4+{}^nC_5+\cdots)]^2+3/4({}^nC_1-{}^nC_2+{}^nC_4-{}^nC_5+\cdots)^2=$

a. 3 b. 4

c. 2 d. 1

85. If $\frac{x^2+x+1}{1-x}=a_0+a_1x+a_2x^2+\cdots$, then $\sum_{r=1}^{50}a_r$ is equal to

a. 148 b. 146

c. 149 d. none of these

86. 'p' is a prime number and $n<p<2n$. If $N={}^{2n}C_n$, then

a. p divides N b. p^2 divides N

c. p cannot divide N d. none of these

87. $\sum_{r=0}^{300}a_rx^r=(1+x+x^2+x^3)^{100}$. If $a=\sum_{r=0}^{300}a_r$, then $\sum_{r=0}^{300}ra_r$ is equal to

a. $300a$ b. $100a$

c. $150a$ d. $75a$

88. The value of $\sum_{r=1}^{n+1}\left(\sum_{k=1}^{n}{}^kC_{r-1}\right)$ (where $r,k,n\in N$) is equal to

a. $2^{n+1}-2$ b. $2^{n+1}-1$

c. 2^{n+1} d. none of these

89. If $(1-x)^{-n}=a_0+a_1x+a_2x^2+\cdots+a_rx^r+\cdots$, then $a_0+a_1+a_2+\cdots+a_r$ is equal to

a. $\frac{n(n+1)(n+2)\cdots(n+r)}{r!}$ b. $\frac{(n+1)(n+2)\cdots(n+r)}{r!}$

c. $\frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}$ d. none of these

90. The value $\sum_{r=0}^{20}r(20-r)({}^{20}C_r)^2$ is equal to

a. $400\,{}^{39}C_{20}$ b. $400\,{}^{40}C_{19}$

c. $400\,{}^{39}C_{19}$ d. $400\,{}^{38}C_{20}$

Multiple Correct Answers Type

Solutions on page 6.44

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. The value/values of x in the expression $\left(x + x^{\log_{10} x}\right)^5$ if the third term in the expansion is 10,00,000 is/are
 a. 10 b. 100
 c. $10^{-5/2}$ d. $10^{-3/2}$

2. If the coefficients of r^{th}, $(r+1)^{th}$ and $(r+2)^{th}$ terms in the expansion of $(1+x)^{14}$ are in A.P., then r is/are
 a. 5 b. 12
 c. 10 d. 9

3. In the expansion of $(x+a)^n$ if the sum of odd terms be P and sum of even terms be Q, then
 a. $P^2 - Q^2 = (x^2 - a^2)^n$
 b. $4PQ = (x+a)^{2n} - (x-a)^{2n}$
 c. $2(P^2 + Q^2) = (x+a)^{2n} + (x-a)^{2n}$
 d. none of these

4. If $\left(4+\sqrt{15}\right)^n = I + f$, where n is an odd natural number, I is an integer and $0 < f < 1$, then
 a. I is an odd integer b. I is an even integer
 c. $(I+f)(1-f) = 1$ d. $1-f = \left(4-\sqrt{5}\right)^n$

5. The number of values of r satisfying the equation ${}^{69}C_{3r-1} - {}^{69}C_{r^2}$ $= {}^{69}C_{r^2-1} - {}^{69}C_{3r}$ is
 a. 1 b. 2
 c. 3 d. 7

6. If the 4^{th} term in the expansion of $(ax + 1/x)^n$ is 5/2, then
 a. $a = \dfrac{1}{2}$ b. $n = 8$
 c. $a = \dfrac{2}{3}$ d. $n = 6$

7. The sum of the coefficient in the expansion of $(1 + ax - 2x^2)^n$ is
 a. positive, when $a < 1$ and $n = 2k$, $k \in N$
 b. negative, when $a < 1$ and $n = 2k+1$, $k \in N$
 c. positive, when $a > 1$ and $n \in N$
 d. zero, when $a = 1$

8. If
$$f(m) = \sum_{i=0}^{m} \binom{30}{30-i}\binom{20}{m-i}$$
where
$\binom{p}{q} = {}^pC_q$, then
 a. maximum value of $f(m)$ is ${}^{50}C_{25}$
 b. $f(0) + f(1) + \cdots + f(50) = 2^{50}$
 c. $f(m)$ is always divisible by 50 $(1 \le m \le 49)$
 d. The value of $\displaystyle\sum_{m=0}^{50} (f(m))^2 = {}^{100}C_{50}$

9. If for z as real or complex, $(1 + z^2 + z^4)^8 = C_0 + C_1 z^2 + C_2 z^4 + \cdots + C_{16} z^{32}$, then
 a. $C_0 - C_1 + C_2 - C_3 + \cdots + C_{16} = 1$
 b. $C_0 + C_3 + C_6 + C_9 + C_{12} + C_{15} = 3^7$
 c. $C_2 + C_5 + C_8 + C_{11} + C_{14} = 3^6$
 d. $C_1 + C_4 + C_7 + C_{10} + C_{13} + C_{16} = 3^7$

10. In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$,
 a. there are exactly 730 rational terms
 b. there are exactly 5831 irrational terms
 c. the term which involves greatest binomial coefficients is irrational
 d. the term which involves greatest binomial coefficients is rational

11. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$, then $C_0 - (C_0 + C_1) + (C_0 + C_1 + C_2) - (C_0 + C_1 + C_2 + C_3)$ $+ \cdots (-1)^{n-1}(C_0 + C_1 + \cdots + C_{n-1})$, where n is even integer is
 a. a positive value
 b. a negative value
 c. divisible by 2^{n-1}
 d. divisible by 2^n

12. In the expansion of $\left(x^2 + 1 + \dfrac{1}{x^2}\right)^n$, $n \in N$,
 a. number of terms is $2n+1$
 b. coefficient of constant term is 2^{n-1}
 c. coefficient of x^{2n-2} is n
 d. coefficient of x^2 in n

13. The value of ${}^nC_1 + {}^{n+1}C_2 + {}^{n+2}C_3 + \cdots + {}^{n+m-1}C_m$ is equal to
 a. ${}^{m+n}C_{n-1}$
 b. ${}^{m+n}C_{n-1}$
 c. ${}^mC_1 + {}^{m+1}C_2 + {}^{m+2}C_3 + \cdots + {}^{m+n-1}C_n$
 d. ${}^{m+n}C_{m-1}$

14. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$, $n \in N$, then $C_0 - C_1 + C_2 - \cdots + (-1)^{m-1} C_{m-1}$ is equal to $(m < n)$
 a. $\dfrac{(n-1)(n-2)\cdots(n-m+1)}{(m-1)!}(-1)^{m-1}$
 b. ${}^{n-1}C_{m-1}(-1)^{m-1}$
 c. $\dfrac{(n-1)(n-2)\cdots(n-m)}{(m-1)!}(-1)^{m-1}$
 d. ${}^{n-1}C_{n-m}(-1)^{m-1}$

15. 10^{th} term of $\left(3 - \sqrt{\dfrac{17}{4} + 3\sqrt{2}}\right)^{20}$
 a. an irrational number
 b. a rational number
 c. a positive integer
 d. a negative integer

16. For the expansion $(x \sin p + x^{-1} \cos p)^{10}$, $(p \in R)$,

a. the greatest value of the term independent of x is $10!/2^5(5!)^2$

b. the least value of sum of coefficient is zero

c. the greatest value of sum of coefficient is 32

d. the least value of the term independent of x occurs when $p = (2n+1)\frac{\pi}{4}, n \in Z$

17. Let $(1 + x^2)^2 (1 + x)^n = \sum_{k=0}^{n+4} a_k x^k$. If a_1, a_2 and a_3 are in arithmetic progression, then the possible value/values of n is/are

a. 5
b. 4
c. 3
d. 2

18. The middle term in the expansion of $(x/2 + 2)^8$ is 1120; then $x \in R$ is equal to

a. -2
b. 3
c. -3
d. 2

19. For which of the following values of x, 5^{th} term is the numerically greatest term in the expansion of $(1 + x/3)^{10}$:

a. -2
b. 1.8
c. 2
d. -1.9

20. For natural numbers m, n, if $(1 - y)^m (1 + y)^n = 1 + a_1 y + a_2 y^2 + \cdots$, and $a_1 = a_2 = 10$, then

a. $m < n$
b. $m > n$
c. $m + n = 80$
d. $m - n = 20$

Reasoning Type

Solutions on page 6.47

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.

b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.

c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** The value of $({}^{10}C_0) + ({}^{10}C_0 + {}^{10}C_1) + ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2) + \cdots + ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + \cdots + {}^{10}C_9)$ is $10 \cdots 2^9$.
Statement 2: ${}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \cdots + n\,{}^nC_n = n2^{n-1}$.

2. **Statement 1:** Greatest term in the expansion of $(1 + x)^{12}$, when $x = 11/10$ is 7^{th}.
Statement 2: 7^{th} term in the expansion of $(1 + x)^{12}$ has the factor ${}^{12}C_6$ which is greatest value of ${}^{12}C_r$.

3. **Statement 1:** Remainder when 3456^{2222} is divided by 7 is 4.
Statement 2: Remainder when 5^{2222} is divided 7 is 4.

4. **Statement 1:** Three consecutive binomial coefficients are always in A.P.
Statement 2: Three consecutive binomial coefficients are not in H.P. or G.P.

5. **Statement 1:** If $n \in N$ and 'n' is not a multiple of 3 and $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then the value of $\sum_{r=0}^{n} (-1)^r a_r\, {}^nC_r$ is zero.
Statement 2: The coefficient of x^n in the expansion of $(1 - x^3)$ is zero, if $n = 3k + 1$ or $n = 3k + 2$.

6. **Statement 1:** $3^{2n+2} - 8n - 9$ is divisible by 64, $\forall\, n \in N$.
Statement 2: $(1 + x)^n - nx - 1$ is divisible by x^2, $\forall\, n \in N$.

7. **Statement 1:** The number of distinct terms in $(1 + x + x^2 + x^3 + x^4)^{1000}$ is 4001.
Statement 2: The number of distinct terms in the expansion $(a_1 + a_2 + \ldots + a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

8. **Statement 1:** The coefficient of x^n in $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)^3$ is $\frac{3^n}{n!}$.
Statement 2: The coefficient of x^n in e^{3x} is $\frac{3^n}{n!}$.

9. **Statement 1:** In the expansion of $(1 + x)^{41}(1 - x + x^2)^{40}$, the coefficient of x^{85} is zero.
Statement 2: In the expansion of $(1 + x)^{41}$ and $(1 - x + x^2)^{40}$, x^{85} term does not occur.

10. **Statement 1:** The total number of dissimilar terms in the expansion of $(x_1 + x_2 + \cdots + x_n)^3$ is $\frac{n(n+1)(n+2)}{6}$.
Statement 2: The total number of dissimilar terms in the expansion of $(x_1 + x_2 + x_3)^n$ is $\frac{n(n+1)(n+2)}{6}$.

11. **Statement 1:** If p is a prime number $(p \neq 2)$, then $\left[\left(2 + \sqrt{5}\right)^p\right] - 2^{p+1}$ is always divisible by p (where $[\cdot]$ denotes the greatest integer function).
Statement 2: If n is prime, then ${}^nC_1$, ${}^nC_2$, ${}^nC_3$, …, ${}^nC_{n-1}$ must be divisible by n.

12. Let n be a positive integer and k be a whole number, $k \le 2n$.
Statement 1: The maximum value of ${}^{2n}C_k$ is ${}^{2n}C_n$.
Statement 2: $\frac{{}^{2n}C_{k+1}}{{}^{2n}C_k} < 1$, for $k = 0, 1, 2, \ldots, n-1$ and $\frac{{}^{2n}C_k}{{}^{2n}C_{k-1}} > 1$ for $k = n + 1, n + 2, \ldots, 2n$

13. **Statement 1:** The sum of coefficients in the expansion of $(3^{-x/4} + 3^{5x/4})^n$ is 2^n.
Statement 2: The sum of coefficients in the expansion of $(x + y)^n$ is 2^n when we put $x = y = 1$.

14. **Statement 1:** ${}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \cdots + {}^nC_r = 0$, if $m + n < r$.
Statement 2: ${}^nC_r = 0$ if $n < r$.

15. **Statement 1:** $\sum\sum_{0 \le i < j \le n} \left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j}\right)$ is equal to $\frac{n^2}{2}\, a$, where $a = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$.
Statement 2: $\sum_{r=0}^{n} \frac{r}{{}^nC_r} = \sum_{r=0}^{n} \frac{n-r}{{}^nC_r}$.

Linked Comprehension Type

Solutions on page 6.48

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

The sixth term in the expansion of $\left[\sqrt{\left\{2^{\log\left(10-3^x\right)}\right\}}+\sqrt[5]{\left\{2^{(x-2)\log 3}\right\}}\right]^m$ is equal to 21, if it is known that the binomial coefficient of the 2nd, 3rd and 4th terms in the expansion represents, respectively, the first, third and fifth terms of an A.P. (the symbol log stands for logarithm to the base 10).

1. The value of m is

a. 6 **b.** 7
c. 8 **d.** 9

2. The sum of possible values of x is

a. 1 **b.** 3
c. 4 **d.** none of these

3. The minimum value of expression is

a. 64 **b.** 32
c. 128 **d.** none of these

For Problems 4–6

The 2nd, 3rd and 4th terms in the expansion of $(x + a)^n$ are 240, 720 and 1080, respectively.

4. The value of $(x - a)^n$ can be

a. 64 **b.** –1
c. –32 **d.** none of these

5. The value of least term in the expansion is

a. 16 **b.** 160
c. 32 **d.** 81

6. The sum of odd-numbered terms is

a. 1664 **b.** 2376
c. 1562 **d.** 1486

For Problems 7–9

If $(1 + x + x^2)^{20} = a_0 + a_1x + a_2x^2 \cdots + a_{40}x^{40}$, then answer the following questions.

7. The value of $a_0 + a_1 + a_2 + \cdots + a_{19}$ is

a. $\frac{1}{2}(9^{10}+a_{20})$ **b.** $\frac{1}{2}(9^{10}-a_{20})$
c. $\frac{9^{10}}{2}$ **d.** none of these

8. The value of $a_0^2 - a_1^2 + a_2^2 - \cdots - a_{19}^2$ is

a. $\frac{1}{2}a_{20}(1-a_{20})$ **b.** $\frac{1}{2}a_{20}(1+a_{20})$
c. $\frac{1}{2}a_{20}^2$ **d.** none of these

9. The value of $a_0 + 3a_1 + 5a_2 + \cdots + 81a_{40}$ is

a. 161×3^{20} **b.** 41×3^{40}
c. 41×3^{20} **d.** none of these

For Problems 10–12

An equation $a_0 + a_1x + a_2x^2 + \cdots + a_{99}x^{99} + x^{100} = 0$ has roots ${}^{99}C_0$, ${}^{99}C_1$, ${}^{99}C_2$, …, ${}^{99}C_{99}$.

10. The value of a_{99} is equal to

a. 2^{98} **b.** 2^{99}
c. -2^{99} **d.** none of these

11. The value of a_{98} is

a. $\frac{2^{198}-{}^{198}C_{99}}{2}$ **b.** $\frac{2^{198}+{}^{198}C_{99}}{2}$
c. $2^{99}-{}^{99}C_{49}$ **d.** none of these

12. The value of $({}^{99}C_0)^2 + ({}^{99}C_1)^2 + \cdots + ({}^{99}C_{99})^2$ is equal to

a. $2a_{98} - a_{99}^2$ **b.** $a_{99}^2 - a_{98}$
c. $a_{99}^2 - 2a_{98}$ **d.** none of these

For Problems 13–15

Any complex number in polar form can be an expression in Euler's form as $\cos\theta + i\sin\theta = e^{i\theta}$. This form of the complex number is useful in finding the sum of series $\sum_{r=0}^{n} {}^nC_r(\cos\theta + i\sin\theta)^r$.

$$\sum_{r=0}^{n} {}^nC_r(\cos r\theta + i\sin r\theta) = \sum_{r=0}^{n} {}^nC_r e^{ir\theta}$$
$$= \sum_{r=0}^{n} {}^nC_r\left(e^{i\theta}\right)^r$$
$$= \left(1+e^{i\theta}\right)^n$$

Also, we know that the sum of binomial series does not change if r is replaced by $n - r$.

Using these facts, answer the following questions.

13. The value of $\sum_{r=0}^{100} {}^{100}C_r\,(\sin rx)$ is equal to

a. $2^{100}\cos^{100}\frac{x}{2}\sin 50x$ **b.** $2^{100}\sin(50x)\cos\frac{x}{2}$
c. $2^{101}\cos^{100}(50x)\sin\frac{x}{2}$ **d.** $2^{101}\sin^{100}(50x)\cos(50x)$

14. In triangle ABC, the value of $\sum_{r=0}^{50} {}^{50}C_r a^r b^{n-r}\cos(rB - (50 - r)A)$ is equal to (where a, b, c are sides of triangle opposite to angle A, B, C, respectively, and s is semi-perimeter)

a. c^{49} **b.** $(a + b)^{50}$
c. $(2s - a - b)^{50}$ **d.** none of these

15. If

$$f(x)=\frac{\sum_{r=0}^{50} {}^{50}C_r\sin 2rx}{\sum_{r=0}^{50} {}^{50}C_r\cos 2rx}$$

then the value of $f(\pi/8)$ is equal to

a. 1 **b.** –1
c. irrational value **d.** none of these

For Problems 16–18

Let

$$P=\sum_{r=1}^{50}\frac{{}^{50+r}C_r(2r-1)}{{}^{50}C_r(50+r)},\ Q=\sum_{r=0}^{50}\left({}^{50}C_r\right)^2,\ R=\sum_{r=0}^{100}(-1)^r\left({}^{100}C_r\right)^2$$

16. The value of $P-Q$ is equal to

a. 1 **b.** -1
c. 2^{50} **d.** 2^{100}

17. The value of $P-R$ is equal to

a. 1 **b.** -1
c. 2^{50} **d.** 2^{100}

18. $Q+R$ is equal to

a. $2P+1$ **b.** $2P-1$
c. $2P+2$ **d.** $2P-2$

For Problems 19–21

P is a set containing n elements. A subset A of P is chosen and the set P is reconstructed by replacing the elements of A. A subset B of P is chosen again.

19. The number of ways of choosing A and B such that A and B have no common elements is

a. 3^n **b.** 2^n
c. 4^n **d.** none of these

20. The number of ways of choosing A and B such that B contains just one element more than A is

a. 2^n **b.** ${}^{2n}C_{n-1}$
c. ${}^{2n}C_n$ **d.** $(3^n)^2$

21. The number of ways of choosing A and B such that B is a subset of A is

a. ${}^{2n}C_n$ **b.** 4^n
c. 3^n **d.** none of these

Matrix-Match Type

Solutions on page 6.51

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a→p, a→s, b→r, c→p, c→q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1.

Column I	Column II
a. If ${}^{(n+1)}C_4+{}^{(n+1)}C_3+{}^{(n+2)}C_3>{}^{(n+3)}C_3$, then possible value/values of n is/are	**p.** 4
b. The remainder when $(3053)^{456}-(2417)^{333}$ is divided by 9 is less than	**q.** 5
c. The digit in the unit place of the number $183!+3^{183}$ is greater than	**r.** 6
d. If sum of the coefficients of the first, second and third terms of the expansion of $\left(x^2+\dfrac{1}{x}\right)^m$ is 46, then the index of the term that does not contain x is greater than	**s.** 7

2.

Column I	Column II
a. ${}^{32}C_0^2-{}^{32}C_1^2+{}^{32}C_2^2-\cdots+{}^{32}C_{32}^2=$	**p.** ${}^{63}C_{32}$
b. ${}^{32}C_0^2+{}^{32}C_1^2+{}^{32}C_2^2-\cdots+{}^{32}C_{32}^2=$	**q.** ${}^{32}C_{16}$
c. $\dfrac{1}{32}(1\times{}^{32}C_1{}^2+2\times{}^{32}C_2{}^2+\cdots+32\times{}^{32}C_{32})=$	**r.** 0
d. ${}^{31}C_0^2-{}^{31}C_1^2+{}^{31}C_2^2-\cdots-{}^{31}C_{31}^2=$	**s.** ${}^{64}C_{32}$

3.

Column I	Column II
a. $\sum\sum_{i\neq j}{}^{10}C_i\,{}^{10}C_j$	**p.** $\dfrac{2^{20}-{}^{20}C_{10}}{2}$
b. $\sum\sum_{0\le i\le j\le n}{}^{10}C_i\,{}^{10}C_j$	**q.** $2^{20}-{}^{20}C_{10}$
c. $\sum\sum_{0\le i<j\le n}{}^{10}C_i\,{}^{10}C_j$	**r.** 2^{20}
d. $\sum_{i=0}^{10}\sum_{j=0}^{10}{}^{10}C_i\,{}^{10}C_j$	**s.** $\dfrac{2^{20}+{}^{20}C_{10}}{2}$

4

Column I	Column II
a. The sum of binomial coefficients of terms containing power of x more than x^{20} in $(1+x)^{41}$ is divisible by	**p.** 2^{39}
b. The sum of binomial coefficients of rational terms in the expansion of $(1+\sqrt{2})^{42}$ is divisible by	**q.** 2^{40}
c. If $\left(x+\dfrac{1}{x}+x^2+\dfrac{1}{x^2}\right)^{21}=a_0x^{-42}+a_1x^{-41}+a_2x^{-40}+\cdots+a_{84}x^{42}$, then $a_0+a_2+\cdots+a_{84}$ is divisible by	**r.** 2^{41}
d. The sum of binomial coefficients of positive real terms in the expansion of $(1+ix)^{42}$ $(x>0)$ is divisible by	**s.** 2^{38}

5

Column I	Column II
a. The coefficient of two consecutive terms in the expansion of $(1+x)^n$ will be equal, then n can be	**p.** 9
b. If 15^n+23^n is divided by 19, then n can be	**q.** 10
c. ${}^{10}C_0\,{}^{20}C_{10}-{}^{10}C_1\,{}^{18}C_{10}+{}^{10}C_2\,{}^{16}C_{10}-\cdots$ is divisible by 2^n, then n can be	**r.** 11
d. If the coefficients of T_r, T_{r+1}, T_{r+2} terms of $(1+x)^{14}$ are in A.P., then r is less than	**s.** 12

Integer Type

Solutions on page 6.52

1. If the coefficients x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ and coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ are equal then the value of ab is.
2. If the coefficients of the $(2r+4)^{th}$, $(r+2)^{th}$ terms in the expansion of $(1+x)^{18}$ are equal, then the value of r is.
3. If the coefficients of the r^{th}, $(r+1)^{th}$, $(r-2)^{th}$ terms in the expansion of $(1+x)^{14}$ are in AP, then the largest value of r is.
4. If the three consecutive coefficient in the expansion of $(1+x)^n$ are 28, 56 and 70, then the value of n is.
5. Degree of the polynomial
$$[\sqrt{x^2+1}+\sqrt{x^2-1}]^8+\left[\frac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}}\right]^8 \text{ is.}$$
6. Least positive integer just greater than $(1+0.00002)^{50000}$ is.
7. If the middle term in the expansion of $\left(\frac{x}{2}+2\right)^8$ is 1120; then the sum of possible real values of x is.
8. If the second term of the expansion $\left[a^{1/13}+\frac{a}{\sqrt{a^{-1}}}\right]^n$ is $14a^{5/2}$, then the value of $\frac{^nC_3}{^nC_2}$ is.
9. Number of values in set of values of 'r' for which $^{23}C_r+2.\,^{23}C_{r+1}+{}^{23}C_{r+2}\geq {}^{25}C_{15}$ is.
10. The largest value of x for which the fourth term in the expansion, $\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}}+\frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$ is 336 is.
11. If the constant term in the binomial expansion of $\left(x^2-\frac{1}{x}\right)^n$, $n\in N$ is 15, then the value of n is equal to.
12. Let $a=3^{\frac{1}{223}}+1$ and for all $n\geq 3$, let $f(n)={}^nC_0.a^{n-1}-{}^nC_1.a^{n-2}+{}^nC_2.a^{n-3}-\cdots+(-1)^{n-1}.{}^nC_{n-1}.a^0$. If the value of $f(2007)+f(2008)=3^k$ where $k\in N$, then the value of k is.
13. Let $1+\sum_{r=1}^{10}(3^r.\,^{10}C_r+r.\,^{10}C_r)=2^{10}(\alpha\cdot 4^5+\beta)$ where $\alpha,\beta\in N$ and $f(x)=x^2-2x-k^2+1$. If α,β lies between the roots of $f(x)=0$, then find the smallest positive integral value of k.
14. Let a and b be the coefficient of x^3 in $(1+x+2x^2+3x^3)^4$ and $(1+x+2x^2+3x^3+4x^4)^4$ respectively. Then the value of $4a/b$ is
15. If R is remainder when $6^{83}+8^{83}$ is divided by 49, then the value of $R/5$ is
16. Sum of last three digits of the number $N=7^{100}-3^{100}$ is
17. Given $(1-2x+5x^2-10x^3)(1+x)^n=1+a_1x+a_2x^2+\cdots$ and that $a_1^2=2a_2$ then the value of n is.
18. The remainder, if $1+2+2^2+2^3+\cdots+2^{1999}$ is divided by 5 is.
19. The largest real value for x such that $\sum_{k=0}^{4}\left(\frac{3^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)=\frac{32}{3}$ is.
20. The value of $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\sum_{t=0}^{r-1}\frac{1}{5^n}.{}^nC_r.{}^rC_t.3^t\right)$ is equal to.

Archives

Solutions on page 6.54

Subjective Type

1. Given that $C_1+2C_2x+3C_3x^2+\cdots+2n\,C_{2n}x^{2n-1}=2n(1+x)^{2n-1}$, where $C_r=(2n)!/[r!(2n-r)!]$; $r=0,1,2,\ldots,2n$, then prove that $C_1^2-2C_2^2+3C_3^2-\cdots-2nC_{2n}^2=(-1)^n nC_n$. **(IIT-JEE, 1979)**
2. If $(1+x)^n=C_0+C_1x+C_2x^2+\cdots+C_nx^n$, then show that the sum of the products of the coefficient taken two at a time, represented by $\sum\sum_{0\leq i<j\leq n}{}^nC_i\,{}^nC_j$ is equal to $2^{2n-1}-\frac{(2n)!}{2(n!)^2}$. **(IIT-JEE, 1983)**
3. Given,
$s_n=1+q+q^2+\cdots+q^n$, $S_n=1+\frac{q+1}{2}+\left(\frac{q+1}{2}\right)^2+\cdots+\left(\frac{q+1}{2}\right)^n$, $q\neq 1$
Prove that $^{n+1}C_1+{}^{n+1}C_2\,s_1+{}^{n+1}C_3\,s_2+\cdots+{}^{n+1}C_{n+1}\,s_n=2^nS_n$. **(IIT-JEE, 1984)**
4. Prove that
$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\cdots\text{up to } m \text{ terms}\right]$$
$$=\frac{2^{mn}-1}{2^{mn}(2^n-1)}$$ **(IIT-JEE, 1985)**
5. Let $R=(5\sqrt{5}+11)^{2n+1}$ and $f=R-[R]$ where [ ] denotes the greatest integer function, prove that $Rf=4^{2n+1}$. **(IIT-JEE, 1988)**
6. Prove that $C_0-2^2C_1+3^2C_2-4^2C_3+\cdots+(-1)^n(n+1)^2\times C_n=0$ where $C_r={}^nC_r$. **(IIT-JEE, 1989)**
7. If $\sum_{r=0}^{2n}a_r(x-2)^r=\sum_{r=0}^{2n}b_r(x-3)^r$ and $a_k=1$ for all $k\geq n$, then show that $b_n={}^{2n+1}C_{n+1}$. **(IIT-JEE, 1992)**
8. Prove that $\sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1}=0$, where $k=3n/2$ and n is an even integer. **(IIT-JEE, 1993)**
9. Let n be a positive integer and $(1+x+x^2)^n=a_0+a_1x+\cdots+a_{2n}x^{2n}$. Show that $a_0^2-a_1^2+a_2^2+\cdots+a_{2n}^2=a_n$. **(IIT-JEE, 1994)**

10. Prove that

$$\frac{3!}{2(n+3)} = \sum_{r=0}^{n} (-1)^r \left(\frac{{}^nC_r}{{}^{r+3}C_r}\right)$$ **(IIT-JEE, 1997)**

11. For any positive integer m, n (with $n \geq m$), let

$$\binom{n}{m} = {}^nC_m$$

Prove that

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \cdots + \binom{m}{m} = \binom{n+1}{m+1}.$$

Hence or otherwise, prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \cdots + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}$$

(IIT-JEE, 2000)

12. Prove that $(25)^{n+1} - 24n + 5735$ is divisible by $(24)^2$ for all $n = 1, 2, \ldots$ **(IIT-JEE, 2002)**

13. If n and k are positive integers, show that

$$2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2}$$

$$- \cdots + (-1)^k\binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$$ where $\binom{n}{k}$ stands for ${}^nC_k$. **(IIT-JEE, 2003)**

Objective Type

Fill in the blanks

1. The larger of $99^{50} + 100^{50}$ and 101^{50} is ______. **(IIT-JEE, 1982)**
2. The sum of the coefficients of the polynomial $(1 + x - 3x^2)^{2163}$ is ______. **(IIT-JEE, 1982)**
3. If $(1 + ax)^n = 1 + 8x + 24x^2 + \cdots$, then $a =$ ______ and $n =$ ______. **(IIT-JEE, 1983)**
4. The sum of the rational terms in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$ is ______. **(IIT-JEE, 1997)**

Multiple choice questions with one correct answer

1. Given positive integers $r > 1, n > 2$ and that the coefficient of $(3r)^{\text{th}}$ and $(r+2)^{\text{th}}$ terms in the binomial expansion of $(1 + x)^{2n}$ are equal. Then **(IIT-JEE, 1983)**

a. $n = 2r$ b. $n = 2r + 1$
c. $n = 3r$ d. none of these

2. The coefficient of x^4 in $(x/2 - 3/x^2)^{10}$ is **(IIT-JEE, 1983)**

a. $\frac{405}{256}$ b. $\frac{504}{259}$
c. $\frac{450}{263}$ d. none of these

3. If C_r stands for ${}^nC_r$, then the sum of the series

$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2 - 2C_1^2 + 3C_2^2 - \cdots + (-1)^n (n+1) C_n^2],$$

where n is an even positive integer is equal to

a. 0 b. $(-1)^{n/2}(n+1)$
c. $(-1)^n(n+2)$ d. $(-1)^n n$ **(IIT-JEE, 1986)**

4. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals

a. $(n-1)a_n$ b. na_n
c. $(1/2)na_n$ d. none of the above

(IIT-JEE, 1988)

5. The expression $\left(x + (x^3 - 1)^{\frac{1}{2}}\right)^5 + \left(x - (x^3 - 1)^{\frac{1}{2}}\right)^5$ is a polynomial of degree

a. 5 b. 6
c. 7 d. 8 **(IIT-JEE, 1992)**

6. For $2 \leq r \leq n$,

$$\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$$

a. $\binom{n+1}{r-1}$ b. $2\binom{n+1}{r+1}$
c. $2\binom{n+2}{r}$ d. $\binom{n+2}{r}$ **(IIT-JEE, 2000)**

7. In the binomial expansion of $(a - b)^n$ $n \geq 5$, the sum of the 5^{th} and 6^{th} terms is zero. Then a/b equals

a. $(n-5)/6$ b. $(n-4)/5$
c. $n/(n-4)$ d. $6/(n-5)$ **(IIT JEE, 2001)**

8. The sum

$$\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$$

where $\binom{p}{q} = 0$

if $p < q$ is maximum when m is

a. 5 b. 10
c. 15 d. 20 **(IIT-JEE, 2002)**

9. The coefficient of t^{24} in $(1 + t^2)^{12}(1 + t^{12})(1 + t^{24})$ is

a. ${}^{12}C_6 + 3$ b. ${}^{12}C_6 + 1$
c. ${}^{12}C_6$ d. ${}^{12}C_6 + 2$ **(IIT-JEE, 2003)**

10. If ${}^{n-1}C_r = (k^2 - 3)\,{}^nC_{r+1}$, then $k \in$

a. $(-\infty, -2]$ b. $[2, \infty)$
c. $[-\sqrt{3}, \sqrt{3}]$ d. $(\sqrt{3}, 2]$ **(IIT-JEE, 2004)**

11. The value of

$$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \cdots + \binom{30}{20}\binom{30}{30},$$

where

$\binom{n}{r} = {}^nC_r$ is

a. $\binom{30}{10}$ b. $\binom{30}{15}$
c. $\binom{60}{30}$ d. $\binom{31}{10}$

(IIT-JEE, 2005)

ANSWERS AND SOLUTIONS

Subjective Type

1. We have,

$$\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)^2$$

$$=\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}\right)$$

Now, x^n term is generated if terms of the two brackets are multiplied as shown by loops above.

Hence, the coefficient of x^n is

$$1\times\frac{1}{n!}+\frac{1}{1!}\times\frac{1}{(n-1)!}+\frac{1}{2!}\times\frac{1}{(n-2)!}+\cdots+\frac{1}{n}$$

$$=\frac{1}{n!}\left(\frac{n!}{n!}+\frac{n!}{(n-1)!1!}+\frac{n!}{(n-2)!2!}+\cdots+\frac{n!}{n!}\right)$$

$$=\frac{1}{n!}({}^nC_0+{}^nC_1+{}^nC_2+\cdots+{}^nC_n)$$

$$=\frac{2^n}{n!}$$

2. Let $a-x-1=t$, so that

$$\sum_{r=0}^{n}a_r(1-t)^r=\sum_{r=0}^{n}b_rt^r$$

$\Rightarrow$ b_n = Coefficient of t^n in $\sum_{r=0}^{n}a_r(1-t)^r$

= Coefficient of t^n in $a_n(1-t)^n$

$=(-1)^na_n$

3. $S={}^nC_0{}^nC_2+2\,{}^nC_1{}^nC_3+3\,{}^nC_2{}^nC_4+\cdots+(n-1)\,{}^nC_{n-2}{}^nC_n$

$={}^nC_0{}^nC_{n-2}+2\,{}^nC_1{}^nC_{n-3}+3\,{}^nC_2{}^nC_{n-4}+\cdots+(n-1)\,{}^nC_{n-2}{}^nC_0$

$$=\sum_{r=1}^{n}r\,{}^nC_{r-1}{}^nC_{n-r-1}$$

$$=\sum_{r=1}^{n}((r-1)+1)\,{}^nC_{r-1}{}^nC_{n-r-1}$$

$$=\sum_{r=1}^{n}\left[(r-1)\,{}^nC_{r-1}{}^nC_{n-r-1}+{}^nC_{r-1}{}^nC_{n-r-1}\right]$$

$$=\sum_{r=1}^{n}\left[n\,{}^{n-1}C_{r-2}{}^nC_{n-r-1}+{}^nC_{r-1}{}^nC_{n-r-1}\right]$$

$$=n\,{}^{2n-1}C_{n-3}+{}^{2n}C_{n-2}$$

4. $\sum_{r=0}^{n}{}^nC_r(-1)^r\left[i^r+i^{2r}+i^{3r}+i^{4r}\right]$

$$=\sum_{r=0}^{n}{}^nC_r(-1)^r\left[i^r+(-1)^r+(-i)^r+1\right]$$

$$=\sum_{r=0}^{n}\left[{}^nC_ri^r+{}^nC_r+{}^nC_r(-i)^r+{}^nC_r(-1)^r\right]$$

$$=(1+i)^n+(1+1)^n+(1-i)^n+(1-1)^n$$

$$=(\sqrt{2})^n\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^n+(\sqrt{2})^n\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^n+2^n$$

$$=(\sqrt{2})^n\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^n+(\sqrt{2})^n\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)^n+2^n$$

$$=(\sqrt{2})^n\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)+(\sqrt{2})^n\left(\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right)+2^n$$

$$=2\left(\sqrt{2}\right)^n\cos\frac{n\pi}{4}+2^n$$

5. $b_n={}^nC_1+{}^nC_2\,a+{}^nC_3\,a^2+\cdots+{}^nC_na^{n-1}$

$$=\frac{1}{a}[a\,{}^nC_1+{}^nC_2a^2+{}^nC_3a^3+\cdots+{}^nC_na^n]$$

$$=\frac{1}{a}[{}^nC_0+{}^nC_1a+{}^nC_2\,a^2+\cdots+{}^nC_na^n-1]$$

$$=\frac{1}{a}((1+a)^n-1)$$

$$b_{2006}-b_{2005}=\frac{1}{a}[(1+a)^{2006}-1)]-\frac{1}{a}[(1+a)^{2005}-1)]$$

$$=\frac{1}{a}[(1+a)^{2006}-1-(1+a)^{2005}+1]$$

$$=\frac{1}{a}(1+a)^{2005}(1+a-1)$$

$$=(1+a)^{2005}$$

$$=\left(1+4^{\frac{1}{401}}-1\right)^{2005}$$

$$=4^{\frac{2005}{401}}$$

6. Let $A=1-2n+\dfrac{2n(2n-1)}{2!}-\cdots+(-1)^{n-1}\dfrac{2n(2n-1)\cdots(n+2)}{(n-1)!}$

$$={}^{2n}C_0-{}^{2n}C_1+{}^{2n}C_2-\cdots+(-1)^{n-1}\,{}^{2n}C_{n-1}$$

$$=\frac{1}{2}\left[2\,{}^{2n}C_0-2\,{}^{2n}C_1+2\,{}^{2n}C_2-\cdots+(-1)^n2\,{}^{2n}C_{n-1}\right]$$

$$=\frac{1}{2}\left[\left({}^{2n}C_0+{}^{2n}C_{2n}\right)-\left({}^{2n}C_1+{}^{2n}C_{2n-1}\right)+\left({}^{2n}C_2+{}^{2n}C_{2n-2}\right)-\cdots+\left((-1)^{n-1}\,{}^{2n}C_{n-1}+(-1)^{n+1}\,{}^{2n}C_{n+1}\right)\right]$$

$$=\frac{1}{2}\left[{}^{2n}C_0-{}^{2n}C_1+{}^{2n}C_2-{}^{2n}C_3+\cdots(-1)^n\,{}^{2n}C_n+\cdots+{}^{2n}C_{2n}-(-1)^n\,{}^{2n}C_n\right]$$

$$=\frac{1}{2}\left[(1-1)^{2n}-(-1)^n\,{}^{2n}C_n\right]$$

$$=\frac{1}{2}\left[(-1)^{n+1}\,{}^{2n}C_n\right]$$

$$=(-1)^{n+1}\frac{(2n)!}{2n!n!}$$

7. Let $f(x)=\dfrac{n!}{x(x+1)(x+2)\cdots(x+n)}$

$$=\frac{A_0}{x}+\frac{A_1}{x+1}+\frac{A_2}{x+2}+\cdots+\frac{A_n}{x+n}\text{ (by partial fractions)}$$

Then,

$$A_0=\left[x\,f(x)\right]_{x=0}=\frac{n!}{1\times2\times3\times\cdots\times n}=1={}^nC_0$$

$$A_1=[(x+1)f(x)]_{x=-1}$$

$$=\frac{n!}{(-1)\{1\times2\times\cdots\times(n-1)\}}$$

$$=\frac{-(n!)}{(n-1)!}=-{}^{n}C_1$$

$$A_2=[(x+2)f(x)]_{x=-2}$$

$$=\frac{n!}{(-2)\times(-1)\times 1\times 2\times\cdots\times(n-2)}$$

$$=\frac{n!}{2!(n-2)!}={}^{n}C_2 \text{ and so on}$$

Thus,

$$\frac{n!}{x(x+1)(x+2)\cdots(x+n)}$$

$$=\frac{{}^{n}C_0}{x}-\frac{{}^{n}C_1}{x+1}+\frac{{}^{n}C_2}{x+2}-\cdots+(-1)^n\frac{{}^{n}C_n}{x+n} \quad (1)$$

8. To find $S={}^{100}C_0\,{}^{100}C_2+{}^{100}C_2\,{}^{100}C_4+{}^{100}C_4\,{}^{100}C_6+\cdots+{}^{100}C_{98}\,{}^{100}C_{100}$

Consider,

$${}^{100}C_0\,{}^{100}C_2+{}^{100}C_1\,{}^{100}C_3+{}^{100}C_2\,{}^{100}C_4+{}^{100}C_3\,{}^{100}C_5+\cdots+{}^{100}C_{98}\,{}^{100}C_{100}$$

$$={}^{100}C_0\,{}^{100}C_{98}+{}^{100}C_1\,{}^{100}C_{97}+{}^{100}C_2\,{}^{100}C_{96}+{}^{100}C_3\,{}^{100}C_{95}+\cdots+{}^{100}C_{98}\,{}^{100}C_0$$

= Coefficient of x^{98} in $(1+x)^{100}(1+x)^{100}$

= Coefficient of x^{98} in $(1+x)^{200}$

$$={}^{200}C_{98} \quad (1)$$

Also,

$${}^{100}C_0\,{}^{100}C_2-{}^{100}C_1\,{}^{100}C_3+{}^{100}C_2\,{}^{100}C_4-{}^{100}C_3\,{}^{100}C_5+\cdots+{}^{100}C_{98}\,{}^{100}C_{100}$$

= Coefficient of x^{98} in $(1+x)^{100}(1-x)^{100}$

= Coefficient of x^{98} in $(1-x^2)^{100}$

$$=-{}^{100}C_{49} \quad (2)$$

Adding (1) and (2), we have

$$2({}^{100}C_0\,{}^{100}C_2+{}^{100}C_2\,{}^{100}C_4+{}^{100}C_4\,{}^{100}C_6+\cdots+{}^{100}C_{98}\,{}^{100}C_{100})=\left[{}^{200}C_{98}-{}^{100}C_{49}\right]$$

$$\Rightarrow {}^{100}C_0\,{}^{100}C_2+{}^{100}C_2\,{}^{100}C_4+{}^{100}C_4\,{}^{100}C_6+\cdots+{}^{100}C_{98}\,{}^{100}C_{100}=\frac{1}{2}\left[{}^{200}C_{98}-{}^{100}C_{49}\right]$$

9. $$\sum_{r=1}^{m-1}\frac{2r^2-r(m-2)+1}{(m-r)\,{}^{m}C_r}$$

$$=\sum_{r=1}^{m-1}\frac{(r+1)^2-r(m-r)}{(m-r)\,{}^{m}C_2}$$

$$=\sum_{r=1}^{m-1}\left(\frac{(r+1)^2}{(m-r)\,{}^{m}C_r}-\frac{r}{{}^{m}C_r}\right)$$

$$=\sum_{r=1}^{m-1}\left(\frac{(r+1)}{(m-r)\left(\frac{{}^{m}C_r}{r+1}\right)}-\frac{r}{{}^{m}C_r}\right)$$

$$=\sum_{r=1}^{m-1}\left(\frac{(r+1)}{{}^{m}C_{r+1}}-\frac{r}{{}^{m}C_r}\right)$$

$$=\sum_{r=1}^{m-1}(t_{r+1}-t_r),\text{ where } t_r=\frac{r}{{}^{m}C_r}$$

$$=t_m-t_1$$

$$=\frac{m}{{}^{m}C_m}-\frac{1}{{}^{m}C_1}$$

$$=m-\frac{1}{m}$$

10. The given series is an A.G.P. Let us first find its sum.

Writing t for $1+x$, let

$$S=t^{1000}+2xt^{999}+3x^2t^{998}+\cdots+1001x^{1000}$$

$$\therefore\ (x/t)\,S=xt^{999}+2x^2t^{998}+\cdots+1001x^{1001}/t$$

Subtracting, we get

$$S(1-x/t)=t^{1000}+xt^{999}+x^2t^{998}+\cdots+x^{1000}-1001x^{1001}/t$$

$$=\frac{t^{1000}[1-(x/t)^{1001}]}{1-x/t}-1001x^{1001}/t$$

or $S=(1+x)^{1002}-x^{1001}(1+x)-1001x^{1001}$

(putting $t=1+x$ and simplifying)

Therefore, coefficient of x^{50} in the expansion is equal to the coefficient of x^{50} in the expansion of $(1+x)^{1002}$, which is equal to ${}^{1002}C_{50}$.

11. Let $$S={}^{n}C_1-\left(1+\frac{1}{2}\right){}^{n}C_2+\left(1+\frac{1}{2}+\frac{1}{3}\right){}^{n}C_3+\cdots+(-1)^{n-1}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right){}^{n}C_n.$$

The r^{th} term of the series is

$$T(r)=(-1)^{r-1}\,{}^{n}C_r\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{r}\right)$$

Let us consider a series whose general term is

$$T_1(r)=(-1)^{r-1}C_r(1+x+x^2+\cdots+x^{r-1})=(-1)^{r-1}C_r\left(\frac{1-x^r}{1-x}\right)$$

$$=\frac{(-1)^{r-1}C_r}{1-x}+\frac{(-1)^r x^r C_r}{1-x}$$

$$\Rightarrow \sum_{r=1}^{n}T_1(r)=\frac{1}{(x-1)}\sum_{r=1}^{n}(-1)^rC_r+\frac{1}{(1-x)}\sum_{r=1}^{n}(-1)^rC_rx^r$$

$$\Rightarrow \sum_{r=1}^{n}T_1(r)=\frac{1}{(x-1)}(0-1)+\frac{1}{(1-x)}((1-x)^n-1)=(1-x)^{n-1}$$

$$=L\text{ (say)}$$

Clearly, $$S=\int_0^1(1-x)^{n-1}dx=\frac{1}{n}$$

12. Consider the series $S={}^{n}C_1+2\,{}^{n}C_2+3\,{}^{n}C_3+\cdots+n\,{}^{n}C_n$

For the series $T_r=r\,{}^{n}C_r=n\,{}^{n-1}C_{r-1}$

$$\therefore\ S=\sum_{r=1}^{n}T_r=\sum_{r=1}^{n}n\,{}^{n-1}C_{r-1}=n\cdot 2^{n-1} \quad (1)$$

Now, we have

A.M. $\geq$ G.M.

$$\Rightarrow \frac{{}^{n}C_1+2\,{}^{n}C_2+3\,{}^{n}C_3+\cdots+n\,{}^{n}C_n}{\left(\frac{n(n+1)}{2}\right)}\geq\left[({}^{n}C_1)({}^{n}C_2)^2\times({}^{n}C_3)^3\cdots({}^{n}C_n)^n\right]^{\frac{1}{\frac{n(n+1)}{2}}}$$

$$\Rightarrow \quad \frac{n\times 2^{n-1}}{\left(\frac{n(n+1)}{2}\right)} \ge \left[({}^nC_1)({}^nC_2)^2({}^nC_3)^2 \cdots ({}^nC_n)^n\right]^{\frac{2}{n(n+1)}}$$

$$\Rightarrow \quad ({}^nC_1)\,({}^nC_2)^2\,({}^nC_3)^3 \cdots ({}^nC_n)^n \le \left(\frac{2^n}{n+1}\right)^{\frac{n(n+1)}{2}}$$

$$\text{or} \quad ({}^nC_1)\,({}^nC_2)^2\,({}^nC_3)^3 \cdots ({}^nC_n)^n \le \left(\frac{2^n}{n+1}\right)^{{}^{n+1}C_2}$$

13. $\frac{1}{m!}{}^nC_0+\frac{n}{(m+1)!}{}^nC_1+\frac{n(n-1)}{(m+2)!}{}^nC_2+\cdots+\frac{n(n-1)\times\cdots\times 2\times 1}{(m+n)!}{}^nC_n$

$$=\frac{n!}{(m+n)!}\left(\frac{(m+n)!}{m!n!}{}^nC_0+\frac{(m+n)!n}{(m+1)!n!}{}^nC_1+\frac{(m+n)!n(n-1)}{(m+2)!n!}{}^nC_2\right.$$

$$\left.+\cdots+\frac{(m+n)!}{n!}\frac{n(n-1)\times\cdots\times 2\times 1}{(m+n)!}{}^nC_n\right)$$

$$=\frac{n!}{(m+n)!}\left({}^{m+n}C_n\,{}^nC_0+\frac{(m+n)!}{(m+1)!(n-1)!}{}^nC_1+\frac{(m+n)!}{(m+2)!(n-2)!}\right.$$

$$\left.\times\,{}^nC_2+\cdots+\frac{(m+n)!}{1}\frac{1}{(m+n)!}{}^nC_n\right)$$

$$=\frac{n!}{(m+n)!}\left({}^{m+n}C_n{}^nC_0+{}^{m+n}C_{n-1}{}^nC_1+{}^{m+n}C_{n-2}{}^nC_2+\cdots\right.$$

$$\left.+\,{}^{m+n}C_0\,{}^nC_n\right)$$

$$=\frac{n!}{(m+n)!}[\text{coefficient of } x^n \text{ in } (1+x)^{m+n}(1+x)^n]$$

$$=\frac{n!}{(m+n)!}[\text{coefficient of } x^n \text{ in } (1+x)^{m+2n}]$$

$$=\frac{n!}{(m+n)!}\,{}^{m+2n}C_n$$

$$=\frac{n!}{(m+n)!}\,\frac{(m+2n)!}{(m+n)!n!}$$

$$=\frac{(m+2n)!}{(m+n)!(m+n)!}$$

$$=\frac{(m+n+1)(m+n+2)(m+n+3)\cdots(m+2n)}{(m+n)!}$$

14. ${}^nC_0-\frac{{}^nC_2}{(2+\sqrt{3})^2}+\frac{{}^nC_4}{(2+\sqrt{3})^4}-\frac{{}^nC_6}{(2+\sqrt{3})^6}+\cdots=(-1)^m\left(\frac{2\sqrt{2}}{1+\sqrt{3}}\right)^n$

$$=\text{Real part of }\left(1+\frac{i}{2+\sqrt{3}}\right)^n$$

$$=\text{Real part of }\left(1+i(2-\sqrt{3})\right)^n$$

$$=\text{Real part of }\left(1+i\tan\frac{\pi}{12}\right)^n$$

$$=\text{Real part of }\frac{\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)^n}{\cos^n\frac{\pi}{12}}$$

$$=\text{Real part of }\frac{\left(\cos\frac{n\pi}{12}+i\sin\frac{n\pi}{12}\right)}{\cos^n\frac{\pi}{12}}$$

$$=\frac{\cos\frac{n\pi}{12}}{\cos^n\frac{\pi}{12}}$$

$$=\frac{\cos m\pi}{\cos^n\frac{\pi}{12}}$$

$$=(-1)^m\left(\frac{2\sqrt{2}}{1+\sqrt{3}}\right)^n \quad \left[\because \cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}\right]$$

15. The given expansion can be written as

$$\{\underbrace{(1+x)(1+x)(1+x)\cdots(1+x)}_{n\text{ factors}}\}\{\underbrace{(1+y)(1+y)(1+y)\cdots(1+y)}_{n\text{ factors}}\}$$

$$\{\underbrace{(1+z)(1+z)(1+z)\cdots(1+z)}_{n\text{-factors}}\}$$

There are $3n$ factors in this product. To get a term of degree r, we choose r brackets out of these $3n$ brackets and then multiply second terms in each bracket. There are ${}^{3n}C_r$ such terms each having the coefficient 1. Hence, the sum of the coefficients is ${}^{3n}C_r$.

Objective Type

1. c. Coefficient of T_5 is ${}^nC_4$, that of T_6 is ${}^nC_5$ and that of T_7 is ${}^nC_6$. According to the condition, $2\,{}^nC_5={}^nC_4+{}^nC_6$. Hence,

$$2\left[\frac{n!}{(n-5)!5!}\right]=\left[\frac{n!}{(n-4)!4!}+\frac{n!}{(n-6)!6!}\right]$$

$$\Rightarrow \quad 2\left[\frac{1}{(n-5)5}\right]=\left[\frac{1}{(n-4)(n-5)}+\frac{1}{6\times 5}\right]$$

After solving, we get $n = 7$ or 14.

2. c. $T_{r+1}={}^{2n}C_r x^{2n-r}\left(\frac{1}{x^2}\right)^r={}^{2n}C_r x^{2n-3r}$

This contains x^m. If $2n-3r=m$, then

$$r=\frac{2n-m}{3}$$

$$\Rightarrow \quad \text{Coefficient of } x^m={}^{2n}C_r,\ r=\frac{2n-m}{3}$$

$$=\frac{2n!}{(2n-r)!r!}=\frac{2n!}{\left(2n-\frac{2n-m}{3}\right)!\left(\frac{2n-m}{3}\right)!}$$

$$=\frac{2n!}{\left(\frac{4n+m}{3}\right)!\left(\frac{2n-m}{3}\right)!}$$

3. c. $(1+x)^{21}+(1+x)^{22}+\cdots+(1+x)^{30}$

$$=(1+x)^{21}\left[\frac{(1+x)^{10}-1}{(1+x)-1}\right]=\frac{1}{x}\left[(1+x)^{31}-(1+x)^{21}\right]$$

$\Rightarrow$ Coefficient of x^5 in the given expression

$$=\text{Coefficient of } x^5 \text{ in }\left\{\frac{1}{x}\left[(1+x)^{31}-(1+x)^{21}\right]\right\}$$

$$=\text{Coefficient of } x^6 \text{ in } [(1+x)^{31}-(1+x)^{21}]$$

$= {}^{31}C_6 - {}^{21}C_6$

4. b. c.e. of x^{-1} in $(1+x)^n\left(1+\frac{1}{x}\right)^n$

$=$ c.e. of x^{-1} in $\frac{(1+x)^{2n}}{x^n}$

$=$ c.e. of x^{n-1} in $(1+x)^{2n}$

$= {}^{2n}C_{n-1}$

$= \frac{(2n)!}{(n-1)!(n+1)!}$

5. d. $(1+3x+2x^2)^6 = [1+x(3+2x)]^6$

$= 1 + {}^6C_1\, x(3+2x) + {}^6C_2 x^2(3+2x)^2 + {}^6C_3 x^3(3+2x)^3 + {}^6C_4 x^4(3+2x)^4 + {}^6C_5 x^5(3+2x)^5 + {}^6C_6 x^6(3+2x)^6$

We get x^{11} only from ${}^6C_6\, x^6(3+2x)^6$. Hence, coefficient of x^{11} is ${}^6C_5 \times 3 \times 2^5 = 576$.

6. b. We have $T_{r+1} = {}^{29}C_r 3^{29-r}(7x)^r = ({}^{29}C_r \times 3^{29-r} \times 7^r)\, x^r$

Coefficient of $(r+1)^{\text{th}}$ term is ${}^{29}C_r \times 3^{29-r} \times 7^r$

and coefficient of r^{th} term is ${}^{29}C_{r-1} \times 3^{30-r} \times 7^{r-1}$

From given condition,

${}^{29}C_r \times 3^{29-r} \times 7^r = {}^{29}C_{r-1} \times 3^{30-r} \times 7^{r-1}$

$\Rightarrow \frac{{}^{29}C_r}{{}^{29}C_{r-1}} = \frac{3}{7} \Rightarrow \frac{30-r}{r} = \frac{3}{7} \Rightarrow r = 21$

7. c. Let $(r+1)^{\text{th}}$, $(r+2)^{\text{th}}$ and $(r+3)^{\text{th}}$ be three consecutive terms.

Then,

${}^nC_r : {}^nC_{r+1} : {}^nC_{r+2} = 1:7:42$

Now,

$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{1}{7} \Rightarrow \frac{r+1}{n-r} = \frac{1}{7} \Rightarrow n - 8r = 7$ (i)

$\frac{{}^nC_{r+1}}{{}^nC_{r+2}} = \frac{7}{42} \Rightarrow \frac{r+2}{n-r-1} = \frac{1}{6} \Rightarrow n - 7r = 13$ (ii)

Solving (i) and (ii), we get $n = 55$.

8. c. $(1+2x+x^2)^n = \sum_{r=0}^{2n} a_r x^r \Rightarrow [(1+x)^2]^n = \sum_{r=0}^{2n} a_r x^r$

$\Rightarrow (1+x)^{2n} = \sum_{r=0}^{2n} a_r x^r$

$\Rightarrow \sum_{r=0}^{2n} {}^{2n}C_r x^r = \sum_{r=0}^{2n} a_r x^r$

$\Rightarrow a_r = {}^{2n}C_r$

9. b. Here $a = {}^nC_r$, $b = {}^nC_{r+1}$ and $c = {}^nC_{r+2}$.

Put $n = 2$, $r = 0$, then option (b) holds the condition, i.e.,

$n = \frac{2ac+ab+bc}{b^2-ac}$

10. c. Middle term of $(1+\alpha x)^4$ is T_3.

Its coefficient is ${}^4C_2(\alpha)^2 = 6\alpha^2$.

Middle term of $(1-\alpha x)^6$ is T_4.

Its coefficient is ${}^6C_3(-\alpha)^3 = -20\alpha^3$.

According to question,

$6\alpha^2 = -20\alpha^3$

$\Rightarrow 3\alpha^2 + 10\alpha^3 = 0$

$\Rightarrow \alpha^2(3+10\alpha) = 0$

$\Rightarrow \alpha = -\frac{3}{10}$

11. b. $(1-x)(1-x)^n$

$= (1-x)[1 + n(-x) + \cdots + {}^nC_{n-1}(-x)^{n-1} + {}^nC_n(-x)^n]$

Therefore, coefficient of x^n is

${}^nC_n(-1)^n - {}^nC_{n-1}(-1)^{n-1} = (-1)^n + (-1)^n n$

$= (-1)^n(1+n)$

12. d. Here, the coefficients of T_r, T_{r+1} and T_{r+2} in $(1+y)^m$ are in A.P.

$\Rightarrow {}^mC_{r-1}, {}^mC_r$ and ${}^mC_{r+1}$ are in A.P.

$\Rightarrow 2\,{}^mC_r = {}^mC_{r-1} + {}^mC_{r+1}$

$\Rightarrow 2\frac{m!}{r!(m-r)!} = \frac{m!}{(r-1)!(m-r+1)!} + \frac{m!}{(r+1)!(m-r-1)!}$

$\Rightarrow \frac{2}{r(m-r)} = \frac{1}{(m-r+1)(m-r)} + \frac{1}{(r+1)r}$

$\Rightarrow m^2 - m(4r+1) + 4r^2 - 2 = 0$

13. c. For $\left(ax^2 + \left(\frac{1}{bx}\right)\right)^{11}$, $T_{r+1} = {}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r$

$= {}^{11}C_r\, a^{11-r}\frac{1}{b^r}x^{22-3r}$

For x^7,

$22 - 3r = 7$

$\Rightarrow 3r = 15$

$\Rightarrow r = 5$

$\Rightarrow T_6 = {}^{11}C_5\, a^6\frac{1}{b^5}x^7$

$\Rightarrow$ Coefficient of x^7 is ${}^{11}C_5\frac{a^6}{b^5}$

Similarly, coefficient of x^{-7} in $\left(ax - \left(\frac{1}{bx^2}\right)\right)^{11}$ is ${}^{11}C_6\frac{a^5}{b^6}$.

Given that

${}^{11}C_5\frac{a^6}{b^5} = {}^{11}C_6\frac{a^5}{b_6}$

$\Rightarrow a = \frac{1}{b}$

$\Rightarrow ab = 1$

14. c. $a^{10}b^{10}c^{10}d^{10}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^{10}$

Therefore the required coefficient is equal to the coefficient of $a^{-2}b^{-6}c^{-1}d^{-1}$ in $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^{10}$, which is given by

$\frac{10!}{2!6!1!1!} = \frac{10\times9\times8\times7}{2} = 2520$

15. d. $(x-2)^5(x+1)^5$

$= [{}^5C_0 x^5 - {}^5C_1 x^4 \times 2 + \cdots][{}^5C_0 + {}^5C_1 x + \cdots]$

$\Rightarrow$ Coefficient of x^5

$= {}^5C_0{}^5C_5 - {}^5C_1 \times 2 \times {}^5C_4 + {}^5C_2 \times 2^2 \times {}^5C_3 - {}^5C_3 \times 2^3 \times {}^5C_2 + {}^5C_4 \times 2^4 \times {}^5C_1 - {}^5C_5 \times 2^5 \times {}^5C_0$

$= 1 - 5\times5\times2 + 10\times10\times4 - 10\times10\times8 + 5\times5\times16 - 32$

$= -81$

16. d. The general term in the expansion of $(1-x+y)^{20}$ is

$\frac{20!}{r!s!t!}1^r(-x)^s(y)^t$, where $r+s+t=20$

For x^2y^3, we have the term

$$\frac{20!}{15!2!3!}1^{15}(-x)^2(y)^3$$

Hence, the coefficient of x^2y^3 is

$$\frac{20!}{15!2!3!}$$

17. b. $t_{r+1} = {}^{10}C_r(\sqrt{x})^{10-r}\left(\frac{-k}{x^2}\right)^r = {}^{10}C_r x^{5-5r/2}(-k)^r$

For this to be independent of x, r must be 2, so that

${}^{10}C_2 k^2 = 405 \Rightarrow k = \pm 3$

18. a. We rewrite the given expression as $[1 + x^2(1-x)]^8$ and expand by using the binomial theorem. We have,

$$[1 + x^2(1-x)]^8$$
$$= {}^8C_0 + {}^8C_1 x^2(1-x) + {}^8C_2 x^4(1-x)^2 + {}^8C_3 x^6(1-x)^3 + {}^8C_4 x^8(1-x)^4 + {}^8C_5 x^{10}(1-x)^5 + \cdots$$

The two terms which contain x^{10} are ${}^8C_4 x^8(1-x)^8$ and ${}^8C_5 x^{10}(1-x)^5$.

Thus, the coefficient of x^{10} in the given expression is given by ${}^8C_4$ [coefficient of x^2 in the expansion of $(1-x)^4$] + ${}^8C_5$

$$= {}^8C_4(6) + {}^8C_5 = \frac{8!}{4!4!}(6) + \frac{8!}{3!5!}$$
$$= (70)(6) + 56 = 476$$

19. c. We have,

$$(1+x)^{101}(1-x+x^2)^{100} = (1+x)((1+x)(1-x+x^2))^{100}$$
$$= (1+x)(1+x^3)^{100} = (1+x)\{C_0 + C_1x^3 + C_2x^6 + \cdots + C_{100}x^{300}\}$$
$$= (1+x)\sum_{r=0}^{n} {}^nC_r x^{3r} = \sum_{r=0}^{n} {}^nC_r x^{3r} + \sum_{r=0}^{n} {}^nC_r x^{3r+1}$$

Hence, there will be no term containing $3r + 2$.

20. b. $(1 + x^3 - x^6)^{30}$

$$= \{1 + x^3(1-x^3)\}^{30}$$
$$= {}^{30}C_0 + {}^{30}C_1 x^3(1-x^3) + {}^{30}C_2 x^6(1-x^3)^2 + \cdots$$

Obviously, each term will contain x^{3m}, $m \in N$. But 28 is not divisible by 3. Therefore, there will be no term containing x^{28}.

21. b. $\left(1 + \sqrt{a} + \frac{1}{\sqrt{a}-1}\right)^{-30}$

$$= \left(\frac{a}{\sqrt{a}-1}\right)^{-30}$$
$$= \left(\frac{\sqrt{a}-1}{a}\right)^{30}$$
$$= \frac{1}{a^{30}}(1-\sqrt{a})^{30}$$
$$= \frac{1}{a^{30}}\{{}^{30}C_0 - {}^{30}C_1\sqrt{a} + \cdots + {}^{30}C_{30}(\sqrt{a})^{30}\}$$

There is no term independent of a.

22. c. The given sigma is the expansion of

$$[(x-3)+2]^{100} = (x-1)^{100} = (1-x)^{100}$$

Therefore, x^{53} will occur in T_{54}.

$$T_{54} = {}^{100}C_{53}(-x)^{53}$$

Therefore, the coefficient is $-{}^{100}C_{53}$.

23. a. We have,

$$\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}$$
$$= \frac{(x^{1/3})^3 + 1^3}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x^{1/2}(x^{1/2}-1)}$$
$$= \frac{(x^{1/3}+1)(x^{2/3} - x^{1/3} + 1)}{x^{2/3} - x^{1/3} + 1} - \frac{x^{1/2}+1}{x^{1/2}}$$
$$= x^{1/3} + 1 - 1 - x^{-1/2} = x^{1/3} - x^{-1/2}$$

$$\therefore \left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10} = (x^{1/3} - x^{-1/2})^{10}$$

Let T_{r+1} be the general term in $(x^{1/3} - x^{-1/2})^{10}$. Then,

$$T_{r+1} = {}^{10}C_r(x^{1/3})^{10-r}(-1)^r(x^{-1/2})^r$$

For this term to be independent of x, we must have

$$\frac{10-r}{3} - \frac{r}{2} = 0 \Rightarrow 20 - 2r - 3r = 0 \Rightarrow r = 4$$

So, the required coefficient is ${}^{10}C_4(-1)^4 = 210$.

24. d. $(1 + x + x^3 + x^4)^{10} = (1+x)^{10}(1+x^3)^{10}$

$$= (1 + {}^{10}C_1x + {}^{10}C_2x^2 + {}^{10}C_3x^3 + {}^{10}C_4x^4 \cdots)(1 + {}^{10}C_1x^3 + {}^{10}C_2x^6 + \cdots)$$

Therefore, coefficient of x^4 is ${}^{10}C_1{}^{10}C_1 + {}^{10}C_4 = 310$.

25. a. $(1.0002)^{3000} = (1 + 0.0002)^{3000}$

$$= 1 + (3000)(0.0002) + \frac{(3000)(2999)}{1.2}(0.0002)^2 + \cdots$$
$$= 1 + (3000)(0.0002) = 1.6$$

26. d. $3^{400} = 81100 = (1+80)^{100}$

$$= {}^{100}C_0 + {}^{100}C_1 80 + \cdots + {}^{100}C_{100} 80^{100}$$

$\Rightarrow$ Last two digits are 01

27. a. We have,

$$\frac{2}{\sqrt{2x^2+1} + \sqrt{2x^2-1}}$$
$$= \frac{2\left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)}{(2x^2+1) - (2x^2-1)} = \sqrt{2x^2+1} - \sqrt{2x^2-1}$$

Thus, the given expression can be written as

$$\left(\sqrt{2x^2+1} + \sqrt{2x^2-1}\right)^6 + \left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)^6$$

But $(a+b)^6 + (a-b)^6 = 2[a^6 + {}^6C_2a^4b^2 + {}^6C_4a^2b^4 + b^6]$

Therefore, $\left(\sqrt{2x^2+1} + \sqrt{2x^2-1}\right)^6 + \left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)^6$

$$= 2[(2x^2+1)^3 + 15(2x^2+1)^2(2x^2-1) + 15(2x^2+1) \times (2x^2-1)^2 + (2x^2-1)^3]$$

which is a polynomial of degree 6.

28. b. We have,

$$(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + \cdots + (x+2)^{n-1}$$
$$= \frac{(x+3)^n - (x+2)^n}{(x+3) - (x+2)} = (x+3)^n - (x+2)^n$$
$$\left(\because \frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2}a^1 + x^{n-3}a^2 + \cdots + a^{n-1}\right)$$

Therefore, coefficient of x^r in the given expression is equal to coefficient of x^r in $[(x+3)^n - (x+2)^n]$, which is given by ${}^nC_r3^{n-r} - {}^nC_r2^{n-r} = {}^nC_r\,(3^{n-r} - 2^{n-r})$.

29. **b.** a_1 = coefficient of x in $(1+2x+3x^2)^{10}$

= coefficient of x in $((1+2x)+3x^2)^{10}$

= coefficient of x in

$({}^{10}C_0(1+2x)^{10} + {}^{10}C_1\,(1+2x)^9(3x^2) + \cdots)$

= coefficient of x in ${}^{10}C_0(1+2x)^{10}$

$= {}^{10}C_0 2.{}^{10}C_1 = 20$

30. **b.** $T_{r+1} = {}^{1024}C_r(5^{1/2})^{1024-r}(7^{1/8})^r$

Now this term is an integer if $1024 - r$ is an even integer, for which $r = 0, 2, 4, 6, \ldots, 1022, 1024$ of which $r = 0, 8, 16, 2424, \ldots, 1024$ are divisible by 8 which makes $r/8$ an integer.

For A.P., $r = 0, 8, 16, 24, \ldots, 1024$,

$1024 = 0 + (n-1)8 \Rightarrow n = 129$

31. **c.** $\left(x^2 - 2 + \dfrac{1}{x^2}\right)^n = \dfrac{1}{x^{2n}}(x^4 - 2x^2 + 1)^n = \dfrac{(x^2-1)^{2n}}{x^{2n}}$

Total number of terms that are dependent on x is equal to number of terms in the expansion of $(x^2-1)^{2n}$ that have degree of x different from $2n$, which is given by $(2n+1) - 1 = 2n$.

32. **c.** It is given that 6th term in the expansion of $\left(\dfrac{1}{x^{8/3}} + x^2\log_{10}x\right)^8$ is 5600, therefore

$${}^8C_5\left(x^2\log_{10}x\right)^5\left(\frac{1}{x^{8/3}}\right)^3 = 5600$$

$\Rightarrow\ 56x^{10}(\log_{10}x)^5\dfrac{1}{x^8} = 5600$

$\Rightarrow\ x^2(\log_{10}x)^5 = 100$

$\Rightarrow\ x^2\,(\log_{10}x)^5 = 10^2\,(\log_{10}10)^5$

$\Rightarrow\ x = 10$

33. **b.** $n!\,(21-n)! = 21!\dfrac{n!\ \ (21-n)!}{21!} = \dfrac{21!}{{}^{21}C_n}$ which is minimum

when ${}^{21}C_n$ is maximum which occurs when $n = 10$

34. **a.** To get sum of coefficients put $x = 0$. Given that sum of coefficients is

$2^n = 64$

$\Rightarrow\ n = 6$

The greatest binomial coefficient is ${}^6C_3$.

Now given that

$T_4 - T_3 = 6 - 1 = 5$

$\Rightarrow\ {}^6C_3(3^{-x/4})^3(3^{5x/4})^3 - {}^6C_2(3^{-x/4})^2\,(3^{5x/4})^4 = 5$

which is satisfied by $x = 0$.

35. **a.** Last term of $\left(2^{1/3} - \dfrac{1}{\sqrt2}\right)^n$ is

$$T_{n+1} = {}^nC_n(2^{1/3})^{n-n}\left(-\frac{1}{\sqrt2}\right)^n = {}^nC_n\,(-1)^n\,\frac{1}{2^{n/2}} = \frac{(-1)^n}{2^{n/2}}$$

Also, we have

$$\left(\frac{1}{3^{5/3}}\right)^{\log_3 8} = \frac{1}{(3^{5/3})^{3\log_3 2}} = 3^{-(5/3)\log_3 2^3} = 2^{-5}$$

Thus,

$$\frac{(-1)^n}{2^{n/2}} = 2^{-5}$$

$\Rightarrow\ \dfrac{(-1)^n}{2^{n/2}} = \dfrac{(-1)^{10}}{2^5}$

$\Rightarrow\ n/2 = 5$

$\Rightarrow\ n = 10$

Now,

$$T_5 = T_{4+1} = {}^{10}C_4\,(2^{1/3})^{10-4}\left(-\frac{1}{\sqrt2}\right)^4$$

$$= \frac{10!}{4!6!}(2^{1/3})^6(-1)^4(2^{-1/2})^4$$

$= 210\,(2^2)\,(1)\,(2^{-2}) = 210$

36. **d.** Given,

$$\frac{{}^{n+1}C_{r+1}}{{}^nC_r} = \frac{11}{6} \Rightarrow \frac{\frac{n+1}{r+1}\times{}^nC_r}{{}^nC_r} = \frac{11}{6}$$

$\Rightarrow\ 6n + 6 = 11r + 11 \Rightarrow 6n - 11r = 5$ (1)

Also,

$$\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{6}{3} \Rightarrow \frac{\frac{n}{r}\times{}^{n-1}C_{r-1}}{{}^{n-1}C_{r-1}} = \frac{6}{3} \Rightarrow n = 2r \quad (2)$$

From (1) and (2), $r = 5$ and $n = 10$,

$\therefore\ nr = 50$

37. **b.** By the given condition,

$84 = T_6 = T_{5+1}$

$$= {}^7C_5\left(2^{\log_2\sqrt{9^{x-1}+7}}\right)^2\left(\frac{1}{2^{\frac15\log_2(3^{x-1}+1)}}\right)^5$$

$= 21\ 2^{\log_2(9^{x-1}+7)}\,2^{-\log_2(3^{x-1}+1)}$

$\Rightarrow\ 4 = 2^{\log_2\frac{9^{x-1}+7}{3^{x-1}+1}} = \dfrac{9^{x-1}+7}{3^{x-1}+1}$

$\Rightarrow\ (3^{x-1})^2 - 4\times3^{x-1} + 3 = 0$

$\Rightarrow\ (3^{x-1} - 1)\,(3^{x-1} - 3) = 0$

$\Rightarrow\ 3^{x-1} = 1$ or 3

$\Rightarrow\ 3^{x-1} = 3^0$ or 3^1

$\Rightarrow\ x - 1 = 0$ or 1

$\Rightarrow\ x = 1, 2$

38. **a.** General term,

$T_{r+1} = {}^{256}C_r\,(\sqrt3)^{256-r}\,(\sqrt[8]{5})^r$

$= {}^{256}C_r\ 3^{\frac{256-r}{2}}\ 5^{\frac{r}{8}}$

The terms are integral if $\dfrac{256-r}{2}$ and $\dfrac{r}{8}$ are both positive integers.

$\therefore\ r = 0, 8, 16, 24, \ldots, 256$

Hence, there are 33 integral terms.

39. **a.** $T_{r+1} = {}^{4n-2}C_r(ix)^r$

T_{r+1} is negative, if i^r is negative and real.

$i^r = -1 \Rightarrow r = 2, 6, 10, \ldots$ which form an A.P.

$0 \le r \le 4n - 2$

$4n - 2 = 2 + (r-1)\,4 \Rightarrow r = n$

The required number of terms is n.

40. **b.** $T_5 = {}^nC_4\,a^{n-4}\,(-2b)^4$

and $T_6 = {}^nC_5\, a^{n-5}\,(-2b)^5$

As $T_5 + T_6 = 0$, we get

${}^nC_4\, 2^4\, a^{n-4}\, b^4 = {}^nC_5\, 2^5\, a^{n-5}\, b^5$

$$\Rightarrow \frac{a^{n-4}b^4}{a^{n-5}\ b^5} = \frac{n!2^5}{5!(n-5)!}\cdot\frac{4!\,(n-4)!}{n!2^4}$$

$$\Rightarrow \frac{a}{b} = \frac{2(n-4)}{5}$$

41. b. $\left(x + \frac{1}{x} + x^2 + \frac{1}{x^2}\right)^{15}$

$$= \left(\frac{x^3 + x + x^4 + 1}{x^2}\right)^{15}$$

$$= \frac{a_0 + a_1x + a_2x^2 + \cdots + a_{60}x^{60}}{x^{30}}$$

Hence, the total number of terms is 61.

42. c. $(1 + x + x^2 + x^3)^5 = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{15}x^{15}$

Putting $x = 1$ and $x = -1$ alternatively, we have

$a_0 + a_1 + a_2 + a_3 + \cdots + a_{15} = 4^5$ (1)

$a_0 - a_1 + a_2 - a_3 + \cdots - a_{15} = 0$ (2)

Adding (1) and (2), we have

$2(a_0 + a_2 + a_4 + \cdots + a_{14}) = 4^5$

$\Rightarrow a_0 + a_2 + a_4 + \cdots + a_{14} = 2^9 = 512$

43. c. Sum of coefficients in $(1 - x\sin\theta + x^2)^n$ is $(1 - \sin\theta + 1)^n$ (putting $x = 1$)

This sum is greatest when $\sin\theta = -1$, then maximum sum is 3^n.

44. a. We know that the sum of the coefficients in a binomial expansion is obtained by replacing each variable by unit in the given expression. Therefore, sum of the coefficients in $(a + b)^n$ is given by $(1 + 1)^n$.

$\therefore$ $4096 = 2^n \Rightarrow 2^n = 2^{12} \Rightarrow n = 12$

Hence, n is even. So, the greatest coefficient is ${}^nC_{n/2}$, i.e., ${}^{12}C_6 = 924$.

45. b. We have, a = sum of the coefficients in the expansion of $(1 - 3x + 10x^2)^n = (1 - 3 + 10)^n = (8)^n = (2)^{3n}$ (putting $x = 1$)

Now, b = sum of the coefficients in the expansion of $(1 + x^2)^n$ $= (1 + 1)^n = 2^n$. Clearly, $a = b^3$.

46. b. $(1 + x - 2x^2)^6 = 1 + a_1x + a_2x^2 + \cdots$

Putting $x = 1$, we get

$0 = 1 + a_1 + a_2 + a_3 + \cdots + a_{12}$ (1)

Putting $x = -1$, we get

$64 = 1 - a_1 + a_2 - a_3 + \cdots + a_{12}$ (2)

(1) + (2) gives

$64 = 2[1 + a_2 + a_4 + \cdots a_{12}]$

$\Rightarrow 1 + a_2 + a_4 + \cdots + a_{12} = 32$

$\Rightarrow a_2 + a_4 + \cdots + a_{32} = 31$

47. a. $\frac{2^{4n}}{15} = \frac{(15+1)^n}{15}$

$$= \frac{({}^nC_0 15^n + {}^nC_1 15^{n-1} + \cdots + {}^nC_{n-1}15 + {}^nC_n)}{15}$$

$$= \text{Integer} + \frac{1}{15}$$

Hence, the fractional part of $\frac{2^{4n}}{15}$ is $\frac{1}{15}$.

48. c. As we know that ${}^nC_0 - {}^nC_1^2 + {}^nC_2^2 - {}^nC_3^2 + \cdots + (-1)^n\, {}^nC_n^2 = 0$ (if n is odd) and in the question $n = 15$ (odd). Hence, sum of given series is 0.

49. b. $(1-x)^{30} = {}^{30}C_0x^0 - {}^{30}C_1x^1 + {}^{30}C_2x^2 + \ldots + (-1)^{30}\, {}^{30}C_{30}x^{30}$ (1)

$(x + 1)^{30} = {}^{30}C_0x^{30} + {}^{30}C_1x^{29} + {}^{30}C_2x^{28} + \cdots + {}^{30}C_{10}x^{20}$
$+ \cdots + {}^{30}C_{30}x^0$ (2)

Multiplying (1) and (2) and equating the coefficient of x^{20} on both sides, we get required sum is equal to coefficient of x^{20} in $(1 - x^2)^{30}$, which is given by ${}^{30}C_{10}$.

50. b. We have, $f(x) = x^n$. So,

$f^1(x) = nx^{n-1} \Rightarrow f'(1) = n$

$f^2(x) = n(n-1)x^{n-2} \Rightarrow f^2(1) = n(n-1)$

$f^3(x) = n(n-1)(n-2)x^{n-3} \Rightarrow f^3(1) = n(n-1)(n-2)$

$\vdots$

$f^n(x) = n(n-1)(n-2)\cdots 1 \Rightarrow f^n(1) = n(n-1)(n-2)\cdots 1$

$$\Rightarrow f(1) + \frac{f^1(1)}{1} + \frac{f^2(1)}{2!} + \cdots + \frac{f^n(1)}{n!}$$

$$= 1 + \frac{n}{1} + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} \cdots + \frac{n(n-1)(n-2)\cdots 1}{n!}$$

$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$

$= 2^n$

51. b. Given series is ${}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \cdots + {}^{20}C_8$

$$= \frac{1}{2}(2\cdot{}^{20}C_0 + 2\,{}^{20}C_1 + \cdots 2\cdot{}^{20}C_8)$$

$$= \frac{1}{2}\Big[\big({}^{20}C_0 + {}^{20}C_1 + \cdots + {}^{20}C_8 + {}^{20}C_9 + {}^{20}C_{10} + {}^{20}C_{11} + \cdots + {}^{20}C_{20}\big) - ({}^{20}C_9 + {}^{20}C_{10} + {}^{20}C_{11})\Big]$$

$$= \frac{1}{2}\left[2^{20} - 2\cdot{}^{20}C_9 - {}^{20}C_{10}\right]$$

$$= 2^{19} - \frac{(2\cdot{}^{20}C_9 + {}^{10}C_{10})}{2}$$

$$= \frac{(2^{20} - {}^{20}C_{10})}{2} - {}^{20}C_9 = 2^{19} - \frac{({}^{20}C_{10} + 2\times{}^{20}C_9)}{2}$$

52. b. Let,

$$S = \frac{{}^nC_0}{n} + \frac{{}^nC_1}{n+1} + \frac{{}^nC_2}{n+2} + \cdots + \frac{{}^nC_n}{2n}$$

$$= {}^nC_0\int_0^1 x^{n-1}dx + {}^nC_1\int_0^1 x^n dx + \cdots + {}^nC_n\int_0^1 x^{2n-1}dx$$

$$= \int_0^1 [{}^nC_0x^{n-1} + {}^nC_1x^n + \cdots + {}^nC_nx^{2n-1}]dx$$

$$= \int_0^1 x^{n-1}(1 + x)^n\,dx$$

$$= \int_1^2 x^n(x-1)^{n-1}dx$$

53. b. $(1 + x)^n = C_0 + C_1x + C_2x + C_2x^2 + C_3x^3 + \cdots + C_nx^n$

$(1 - x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + \cdots + (-1)^nC_nx^n$

$\Rightarrow [(1 + x)^n - (1 - x)^n] = 2\,[C_1x + C_3x^3 + C_5x^5 + \cdots]$

$\Rightarrow \frac{1}{2}[(1 + x)^n - (1 - x)^n] = C_1x + C_3x^3 + C_5x^5 + \cdots$

Putting $x = 2$, we have

$$2\,C_1 + 2^3\,C_3 + 2^5\,C_5 + \cdots = \frac{3^n - (-1)^n}{2}$$

54. d. $\displaystyle\sum_{r=1}^{n}(-1)^{r+1}\frac{{}^nC_r}{(r+1)} = \frac{1}{n+1}\sum_{r=1}^{n}(-1)^{r+1}\,{}^{n+1}C_{r+1}$

$$=\frac{1}{n+1}(0-1+(n+1)) = \frac{n}{n+1}$$

55. c. $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots + C_{n-1}x^{n-1} + C_nx^n \quad (1)$

$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_{n-1}x + C_n \quad (2)$

Multiplying Eqs. (1) and (2) and equating the coefficient of x^{n-2}, we get

$C_0C_2 + C_1C_3 + C_2C_4 + \cdots + C_{n-2}C_n$

$=$ Coefficient of x^{n-2} in $(1+x)^{2n}$

$= {}^{2n}C_{n-2}$

$$= \frac{(2n)!}{(n-2)!(n+2)!}$$

56. a. We know that

$(1-1)^{20} = {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \cdots + {}^{20}C_{10} - {}^{20}C_{11} + {}^{20}C_{12} - \cdots + {}^{20}C_{20} = 0$

$2\,({}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \cdots - {}^{20}C_9) + {}^{20}C_{10} = 0$

$[\because\ {}^{20}C_{20} = {}^{20}C_0,\ {}^{20}C_{19} = {}^{20}C_1$, etc.]

$\Rightarrow\ {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \cdots - {}^{20}C_9 + {}^{20}C_{10}$

$$= -\frac{1}{2}\,{}^{20}C_{10} + {}^{20}C_{10} = \frac{1}{2}\,{}^{20}C_{10}$$

57. b. The given expression is the coefficient of x^4 in

${}^4C_0(1+x)^{404} - {}^4C_1(1+x)^{303} + {}^4C_2(1+x)^{202} - {}^4C_3(1+x)^{101} + {}^4C_4$

$=$ Coefficient of x^4 in $[(1+x)^{101} - 1]^4$

$=$ Coefficient of x^4 in $({}^{101}C_1\,x + {}^{101}C_2x^2 + \cdots)^4$

$= (101)^4$

58. c. Put $x = \omega, \omega^2$

$(3 + \omega + \omega^2)^{2010} = a_0 + a_1\omega + a_2\omega^2 + \cdots$

$\Rightarrow\ 2^{2010} = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega + \cdots \quad (1)$

and

$2^{2010} = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega + \cdots \quad (2)$

Adding (1) and (2), we have

$2 \times 2^{2010} = 2a_0 - a_1 - a_2 + 2a_3 - a_4 - a_5 + 2a_6 - \cdots$

$$\Rightarrow\ 2^{2010} = a_0 - \frac{1}{2}a_1 - \frac{1}{2}a_2 + a_3 - \frac{1}{2}a_4 - \frac{1}{2}a_5 + a_6 \cdots$$

59. d. $\displaystyle\sum_{r=0}^{10} r\,{}^{10}C_r 3^r(-2)^{10-r}$

$$= 10\sum_{r=0}^{10}{}^9C_{r-1}3^r(-2)^{10-r}$$

$$= 10 \times 3\sum_{r=0}^{10}{}^9C_{r-1}3^{r-1}(-2)^{10-r}$$

$= 30(3-2)^{10}$

$= 30$

60. a. $\displaystyle\sum_{r=0}^{40} r\,{}^{40}C_r\,{}^{30}C_r$

$$= 40\sum_{r=0}^{40}{}^{39}C_{r-1}\,{}^{30}C_r$$

$$= 40\sum_{r=0}^{40}{}^{39}C_{r-1}\,{}^{30}C_{30-r}$$

$= 40\,{}^{39+30}C_{r-1+30-r}$

$= 40\,{}^{69}C_{29}$

61. a. $\displaystyle\frac{r \times 2^r}{(r+2)!} = \frac{(r+2-2)2^r}{(r+2)!}$

$$= \frac{2^r}{(r+1)!} - \frac{2^{r+1}}{(r+2)!}$$

$$= -\left(\frac{2^{r+1}}{(r+2)!} - \frac{2^r}{(r+1)!}\right)$$

$= -(V(r) - V(r-1))$

$$\Rightarrow\ \sum_{r=1}^{15}\frac{r \times 2^r}{(r+2)!} = -(V(15) - V(0))$$

$$= -\left(\frac{2^{16}}{17!} - \frac{2}{2!}\right)$$

$$= 1 - \frac{2^{16}}{(17)!}$$

62. c. $t_{r+1} = (-1)^r(n-r+2)\,{}^nC_r\,2^{n-r+1}$

$$= (n+2)\,2^{n+1}(-1)^r\,{}^nC_r\left(\frac{1}{2}\right)^r - 2^{n+1}(-1)^r\,r\,{}^nC_r\left(\frac{1}{2}\right)^r$$

$$= (n+2)\,2^{n+1}\,{}^nC_r\left(-\frac{1}{2}\right)^r + 2^n\,n\,{}^{n-1}C_{r-1}\left(-\frac{1}{2}\right)^{r-1}$$

$$\therefore\ \text{Sum} = (n+2)\,2^{n+1}\left\{{}^nC_0 - {}^nC_1 \times \frac{1}{2} + {}^nC_2 \times \left(\frac{1}{2}\right)^2 - \cdots\right\}$$

$$+\,n\,2^n\left\{{}^{n-1}C_0 - {}^{n-1}C_1 \times \frac{1}{2} + {}^{n-1}C_2 \times \left(\frac{1}{2}\right)^2 + \cdots\right\}$$

$$= (n+2)\,2^{n+1}\left(1 - \frac{1}{2}\right)^n + n\,2^n\left(1 - \frac{1}{2}\right)^{n-1}$$

$= 2(n+2) + 2n$

$= 4n + 4$

63. c. Here,

$$T_r = (-1)^r\frac{{}^{50}C_r}{r+2}$$

$$= (-1)^r(r+1)\frac{{}^{50}C_r}{(r+1)(r+2)}$$

$$= (-1)^r(r+1)\frac{{}^{52}C_{r+2}}{51 \times 52}$$

$$= (-1)^r\frac{[(r+2) - 1]\,{}^{52}C_{r+2}}{51 \times 52}$$

$$= (-1)^r\frac{[52\,{}^{51}C_{r+1} - {}^{52}C_{r+2}]}{51 \times 52}$$

$$= \frac{[-52\,{}^{51}C_{r+1}(-1)^{r+1} - {}^{52}C_{r+2}(-1)^{r+2}]}{51 \times 52}$$

$$\sum_{r=0}^{50}(-1)^r\frac{{}^{50}C_r}{r+2}$$

$$= \sum_{r=0}^{50}\frac{[-52\,{}^{51}C_{r+1}(-1)^{r+1} - {}^{52}C_{r+2}(-1)^{r+2}]}{51 \times 52}$$

$$= -52\frac{(1-1)^{51}-{}^{51}C_0}{51\times 52}-\frac{(1-1)^{52}-{}^{52}C_0+{}^{52}C_1}{51\times 52}$$

$$=\frac{1}{51}-\frac{1}{52}$$

$$=\frac{1}{51\times 52}$$

Alternative solution:

$$(1-x)^n=\sum_{r=0}^{n}{}^nC_r(-1)^r x^r$$

$$\Rightarrow \quad x(1-x)^n=\sum_{r=0}^{n}(-1)^r\,{}^nC_r x^{r+1}$$

Integrating both sides within the limits 0 to 1, we get

$$\int_0^1 x(1-x)^n dx=\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{r+2}$$

$$\Rightarrow \quad \sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{r+2}=\int_0^1 x(1-x)^n dx$$

$$=\int_0^1(1-x)x^n dx \quad \text{(Replace } x \text{ by } 1-x)$$

$$=\left.\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right|_0^1$$

$$=\frac{1}{n+1}-\frac{1}{n+2}$$

$$=\frac{1}{(n+1)(n+2)}$$

Now put $n = 50$.

64. a. Given term can be written as

$(1+x)^2(1-x)^{-2}=(1+2x+x^2)[1+2x+3x^2+\cdots+(n-1)$
$\times x^{n-2}+nx^{n-1}+(n+1)x^n+\cdots]$

Coefficient of x^n is $(n+1+2n+n-1)=4n$.

65. a. $1+n\left(1-\frac{1}{x}\right)+\frac{n(n+1)}{2!}\left(1-\frac{1}{x}\right)^2+\cdots\infty$

$$=1-n\left[-\left(1-\frac{1}{x}\right)\right]+\frac{-n(-n-1)}{2!}\left[-\left(1-\frac{1}{x}\right)\right]^2+\cdots\infty$$

$$=\left[1-\left(1-\frac{1}{x}\right)\right]^{-n}$$

$$=x^n$$

66. c. $\sum_{k=1}^{n}k\left(1-\frac{1}{n}\right)^{k-1}$

$$=1+2\left(1-\frac{1}{n}\right)^1+3\left(1+\frac{1}{n}\right)^2+\cdots$$

$$=1+2t+3t^2+\cdots$$

$$=(1-t)^{-2}$$

$$\left[1-\left(1-\frac{1}{n}\right)\right]^{-2}=\left(\frac{1}{n}\right)^{-2}=n^2$$

67. d. $\left[\sqrt{1+x^2}-x\right]^{-1}=\frac{1}{\sqrt{1+x^2}-x}\times\frac{\left(\sqrt{1+x^2}+x\right)}{\left(\sqrt{1+x^2}+x\right)}$

$$=\frac{\sqrt{1+x^2}+x}{1+x^2-x^2}=x+\sqrt{1+x^2}=x+\left(1+x^2\right)^{1/2}$$

$$=x+1+\frac{1}{2}x^2+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{x^4}{2!}+\cdots$$

Obviously, the coefficient of x^4 is $-1/8$.

68. d. Let,

$$(1+y)^n=1+\frac{1}{3}x+\frac{1\times 4}{3\times 6}x^2+\frac{1\times 4\times 7}{3\times 6\times 9}x^3+\cdots$$

$$=1+ny+\frac{n(n-1)}{2!}y^2+\cdots$$

Comparing the terms, we get

$$ny=\frac{1}{3}x,\ \frac{n(n-1)}{2!}y^2=\frac{1\times 4}{3\times 6}x^2$$

Solving, $n=-1/3$, $y=-x$. Hence, the given series is $(1-x)^{-1/3}$.

69. a. Let the given series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{1\times 2}x^2+\cdots\infty$$

$$\Rightarrow \quad nx=\frac{1}{4}\Rightarrow n^2x^2=\frac{1}{16}$$

Also,

$$\frac{n(n-1)}{2}x^2=\frac{3}{32}\ \Rightarrow\ \frac{2n}{n-1}=\frac{\frac{1}{16}}{\frac{3}{32}}=\frac{2}{3}$$

$\Rightarrow \quad 3n=n-1$

$\Rightarrow \quad 2n=-1$

$\Rightarrow \quad n=-\frac{1}{2}$

$\Rightarrow \quad x=-\frac{1}{2}$

$\Rightarrow$ Required sum $=\left(1-\frac{1}{2}\right)^{-\frac{1}{2}}=\left(\frac{1}{2}\right)^{-\frac{1}{2}}$

$$=(2)^{\frac{1}{2}}=\sqrt{2}$$

70. d. Required value is

$$\left(1-\frac{2x}{1+x}\right)^{-n}=\left(\frac{1+x-2x}{1+x}\right)^{-n}=\left(\frac{1-x}{1+x}\right)^{-n}=\left(\frac{1+x}{1-x}\right)^n$$

71. d. $(1+2x+3x^2+\cdots)^{-3/2}=[(1-x)^{-2}]^{-3/2}$
$=(1-x)^3=1-3x+3x^2-x^3$

Therefore, coefficient of x^5 is 0.

72. d. $(1+x+x^2+\cdots)^2=((1-x)^{-1})^2=(1-x)^{-2}$
$=1+2x+3x^2+\cdots$

Therefore, coefficient of x^n is $n+1$.

73. b. T_{r+1} in $(1+x)^n$ is

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$$

For first negative term,

$n-r+1<0$

$\Rightarrow \frac{27}{5}-r+1<0$

$\Rightarrow r>\frac{32}{5}$

Thus, first negative term occurs when $r=7$.

74. d. $\dfrac{(1+x)^{3/2}-\left(1+\frac{1}{2}x\right)^3}{(1-x)^{1/2}}$

$=\dfrac{\left(1+\frac{3}{2}x+\frac{3}{8}x^2\right)-\left(1+\frac{3}{2}x+3\frac{x^2}{4}\right)}{(1-x)^{1/2}}$

$=\frac{-3}{8}x^2(1-x)^{-1/2}$

$=-\frac{3}{8}x^2\left(1+\frac{x}{2}\right)$

$=-\frac{3}{8}x^2$

75. d. $\dfrac{1}{(1-ax)(1-bx)}=a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$

But $(1-ax)^{-1}(1-bx)^{-1}=(1+ax+a^2x^2+\cdots)(1+bx+b^2x^2+\cdots)$

$\Rightarrow$ Coefficient of x^n is $b^n+ab^{n-1}+a^2b^{n-2}+\cdots+a^{n-1}b+a^n$

$=\dfrac{b^{n+1}-a^{n+1}}{b-a}$

$\Rightarrow a_n=\dfrac{b^{n+1}-a^{n+1}}{b-a}$

76. c. $\sum_{k=1}^{\infty}\sum_{r=0}^{k}\frac{1}{3^k}({}^kC_r)$

$=\sum_{k=1}^{\infty}\left(\frac{1}{3^k}\left(\sum_{r=0}^{k}{}^kC_r\right)\right)$

$=\sum_{k=1}^{\infty}\left(\frac{2^k}{3^k}\right)$

$=\frac{2}{3}+\left(\frac{2}{3}\right)^2+\cdots\infty$

$=\dfrac{2/3}{1-\frac{2}{3}}=2$

77. d. General term in the expansion of $(\sqrt{2}+\sqrt[3]{3}+\sqrt[6]{5})^{10}$ is

$\frac{10!}{a!b!c!}(\sqrt{2})^a(\sqrt[3]{3})^b(\sqrt[6]{5})^c$ where $a+b+c=10$.

For rational term, we have the following:

Value of a, b, c	**Value of term**
$a=4, b=0, c=6$	$\frac{10!}{4!0!6!}(\sqrt{2})^4(\sqrt[3]{3})^0(\sqrt[6]{5})^6=4200$
$a=10, b=0, c=0$	$\frac{10!}{10!0!0!}(\sqrt{2})^{10}(\sqrt[3]{3})^0(\sqrt[6]{5})^0=32$
$a=4, b=6, c=0$	$\frac{10!}{4!6!0!}(\sqrt{2})^4(\sqrt[3]{3})^6(\sqrt[6]{5})^0=7560$

78. b. $f(x)=1-x+x^2-x^3+\cdots-x^{15}+x^{16}-x^{17}=\dfrac{1-x^{18}}{1+x}$

$\Rightarrow f(x-1)=\dfrac{1-(x-1)^{18}}{x}$

Therefore, required coefficient of x^2 is equal to coefficient of x^3 in $1-(x-1)^{18}$, which is given by ${}^{18}C_3=816$.

79. a. $p=(8+3\sqrt{7})^n={}^nC_0 8^n+{}^nC_1 8^{n-1}(3\sqrt{7})+\cdots$

Let, $p_1=(8-3\sqrt{7})^n={}^nC_0 8^n-{}^7C_1 8^{n-1}(3\sqrt{7})+\cdots$

$p_1+p_2=2({}^nC_0 8^n+{}^nC_2 8^{n-2}(3\sqrt{7})^2+\cdots)$ = even integer

p_1 clearly belongs to (0, 1)

$\Rightarrow [p]+f+p_1$ = even integer

$\Rightarrow f+p_1$ = integer

$f\in(0,1), p_1\in(0,1)$

$\Rightarrow f+p\in(0,2)$

$\Rightarrow f+p_1=1$

$\Rightarrow p_1=1-f$

Now, $p(1-f)=pp_1=[(8+3\sqrt{7})^n(8-3\sqrt{7})]^n=1$

80. a. $\sum_{r=0}^{10}(r)\,{}^{20}C_r=\sum_{r=1}^{10}20\times{}^{19}C_{r-1}$

$=20({}^{19}C_0+{}^{19}C_1+\cdots+{}^{19}C_{10})$

$=20({}^{19}C_0+{}^{19}C_1+\cdots+{}^{19}C_{10})$

$=20\left(\frac{1}{2}\times2^{19}+{}^{19}C_{10}\right)$

$=20(2^{18}+{}^{19}C_{10})$

81. c. $(23)^{14}=(529)^7=(530-1)^7$

$={}^7C_0(530)^7-{}^7C_1(530)^6+\cdots-{}^7C_5(530)^2+{}^7C_6\,530-1$

$={}^7C_0(530)^7-{}^7C_1(530)^6+\cdots+3710-1=100m+3709$

Therefore, last two digits are 09.

82. b. $\dfrac{f(x)}{1-x}=b_0+b_1x+b_2x^2+\cdots+b_nx^n+\cdots$

$\Rightarrow a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$

$=(1-x)(b_0+b_1x+b_2x^2+\cdots+b_nx^n+\cdots)$

Comparing the coefficient of x^n on both the sides,

$a_n=b_n-b_{n-1}$.

83. d. $(1-x)^n(1+x)^n=\sum_{r=0}^{n}a_rx^r(1-x)^n(1-x)^{n-r}$

$\Rightarrow (1-x+2x)^n=\sum_{r=0}^{n}a_rx^r(1-x)^{n-r}$

$\Rightarrow \sum_{r=0}^{n}{}^nC_r(1-x)^{n-r}(2x)^r=\sum_{r=0}^{n}a_rx^r(1-x)^{n-r}$

Comparing general term, we get $a_r={}^nC_r2^r$.

84. d. $(1+\omega)^n = {}^nC_0 + {}^nC_1\,\omega + \cdots$

$$= ({}^nC_0 + {}^nC_3 + \cdots) + ({}^nC_1 + {}^nC_4 + \cdots)\left(\frac{-1+\sqrt{3}i}{2}\right)$$

$$+ ({}^nC_2 + {}^nC_5 + \cdots)\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$= ({}^nC_0 + {}^nC_3 + \cdots) - \frac{1}{2}({}^nC_1 + {}^nC_2 + {}^nC_4 + {}^nC_5 \ldots)$$

$$+\frac{i\sqrt{3}}{2}({}^nC_1 - {}^nC_2 + {}^nC_4 - {}^nC_5 + \cdots)$$

Equating the modulus, we get $|(-\omega^2)^n| = 1$.

85. c. $\dfrac{(x^2+x+1)(1-x)}{(1-x)^2} = (1-x^3)(1-x)^{-2}$

$$= (1-x^3)(1+2x+3x^2+\cdots)$$

Now, $a_r = (r+1) - (r-2) = 3$

But $a_1 = 2$

So,

$$\sum_{r=1}^{50} a_r = 2 + 49 \times 3 = 149$$

86. a. $N = {}^{2n}C_n = \dfrac{(2n)!}{(n!)^2} = \dfrac{(n+1)(n+2)\cdots(n+n)}{(n!)}$

$\Rightarrow$ $(n!)\,N = (n+1)(n+2)\cdots(n+n)$

Since $n < p < 2n$, so p divides $(n+1)(n+2)\cdots(n+n)$.

87. c. $\sum_{r=0}^{300} a_r \times x^r = (1+x+x^2+x^3)^{100}$

Clearly, 'a_r' is the coefficient of x^r in the expansion of $(1+x+x^2+x^3)^{100}$.

Replacing x by $1/x$ in the given equation, we get

$$\sum_{r=0}^{300} a_r\left(\frac{1}{x}\right)^r = \frac{1}{x^{300}}(x^3+x^2+x+1)^{100}$$

$$\Rightarrow \sum_{r=0}^{300} a_r x^{300-r} = (1+x+x^2+x^3)^{100}$$

Here, a_r represents the coefficient of x^{300-r} in $(1+x+x^2+x^3)^{100}$.

Thus, $a_r = a_{300-r}$

Let $I = \sum_{r=0}^{300} r \times a_r$

$$= \sum_{r=0}^{300}(300-r)a_{300-r}$$

$$= \sum_{r=0}^{300}(300-r)a_r$$

$$= 300\sum_{r=0}^{300} a_r - \sum_{r=0}^{300} ra_r$$

$\Rightarrow$ $2I = 300a$

$\Rightarrow$ $I = 150a$

88. a. $\sum_{r=1}^{n+1}\left(\sum_{k=1}^{n} {}^kC_{r-1}\right)$

$$= \sum_{r=1}^{n+1}\left(\sum_{k=1}^{n}\left({}^{k+1}C_r - {}^kC_r\right)\right)$$

$$= \sum_{r=1}^{n+1}\left({}^{n+1}C_r - {}^1C_r\right)$$

$$= 2^{n+1} - 2$$

89. b. We have,

$$(1-x)^{-n} = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \cdots$$

and

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots$$

Hence,

$a_0 + a_1 + a_2 + \cdots + a_r$

= Coefficient of x^r in the product of the two series

= Coefficient of x^r in $(1-x)^{-n}(1-x)^{-1}$

= Coefficient of x^r in $(1-x)^{-(n+1)}$

$$= \frac{(n+1)(n+2)\cdots(n+r)}{r!}$$

$$= {}^{r+n+1-1}C_{n+1-1} = {}^{n+r}C_n$$

90. d. $\sum_{r=0}^{20} r(20-r)\times({}^{20}C_r)^2 = \sum_{r=0}^{20} r\times{}^{20}C_r(20-r)\times{}^{20}C_{20-r}$

$$\Rightarrow \sum_{r=0}^{20} 20\,{}^{19}C_{r-1}\times 20\times{}^{19}C_{19-r}$$

$$= 400\sum_{r=0}^{20} {}^{19}C_{r-1}\times{}^{19}C_{19-r}$$

$= 400 \times$ coefficient of x^{18} in $(1+x)^{19}(1+x)^{19}$

$= 400 \times {}^{38}C_{18}$

$= 400 \times {}^{38}C_{20}$

Multiple Correct Answers Type

1. a, c.

Inclusion of $\log x$ implies $x > 0$.

Now, 3rd term in the expansion is

$$T_{2+1} = {}^5C_2\, x^{5-2}\,(x^{\log_{10}x})^2 = 1000000 \text{ (given)}$$

or

$$x^{3+2\log_{10}x} = 10^5$$

Taking logarithm of both sides, we get

$$(3 + 2\log_{10}x)\log_{10}x = 5$$

or

$$2y^2 + 3y - 5 = 0, \text{ where } \log_{10}x = y$$

or

$$(y-1)(2y+5) = 0 \text{ or } y = 1 \text{ or } -5/2$$

or

$$\log_{10}x = 1 \text{ or } -5/2$$

$\therefore$ $x = 10^1 = 10$ or $10^{-5/2}$

2. a, d.

Coefficients of r^{th}, $(r+1)^{th}$ and $(r+2)^{th}$ terms are ${}^{14}C_{r-1}$, ${}^{14}C_r$ and ${}^{14}C_{r+1}$.

If these coefficients are in A.P., then

$$2({}^{14}C_r) = {}^{14}C_{r-1} + {}^{14}C_{r+1}$$

$$\Rightarrow \frac{2(14)!}{r!(14-r)!} = \frac{(14)!}{(r-1)!(15-r)!} + \frac{(14)!}{(r+1)!(13-r)}$$

$$\Rightarrow \frac{2(14)!}{r!(14-r)!} = \frac{(14)![(r+1)r + (15-r)(14-r)]}{(r+1)!(15-r)!}$$

$\Rightarrow$ $2(15-r)(r+1) = 2r^2 - 28r + 210$

$\Rightarrow$ $r^2 - 14r + 45 = 0$ or $(r-5)(r-9) = 0$

$\Rightarrow$ $r = 5$ or 9

3. a, b, c.

We have,

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_n a^n$$

$$= [{}^nC_0x^n + {}^nC_2x^{n-2}a^2 + \cdots] + [{}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \cdots]$$

or

$(x + a)^n = P + Q$ (1)

Similarly,

$(x - a)^n = P - Q$ (2)

(i) (1) × (2) $\Rightarrow P^2 - Q^2 = (x^2 - a^2)^n$

(ii) Squaring (1) and (2) and subtracting (2) from (1), we get

$4PQ = (x + a)^{2n} - (x - a)^{2n}$

(iii) Squaring (1) and (2) and adding,

$2(P^2 + Q^2) = (x + a)^{2n} + (x - a)^{2n}$

4. a, c, d.

$$I + f = \left(4 + \sqrt{15}\right)^n$$

Let $f' = \left(4 - \sqrt{15}\right)^n$. Then $0 < f' < 1$

$$I + f = {}^nC_0\,4^n + {}^nC_1\,4^{n-1}\sqrt{15} + {}^nC_2\,4^{n-2}\,15 + {}^nC_3\,4^{n-3}\left(\sqrt{15}\right)^3 + \cdots$$

$$f' = {}^nC_0\,4^n - {}^nC_1\,4^{n-1}\sqrt{15} + {}^nC_2\,4^{n-2}\cdot 15 - {}^nC_3\,4^{n-3}\left(\sqrt{15}\right)^3 + \cdots$$

$\therefore\ I + f + f' = 2\,({}^nC_0\,4^n + {}^nC_2\,4^{n-2} \times 15 + \cdots) =$ even integer

$\because\ 0 < f + f' < 2 \Rightarrow f + f' = 1 \Rightarrow 1 - f = f'$

Thus, I is an odd integer. Now,

$1 - f = f' = (4 - \sqrt{15})^n$

$(I + f)(1 - f) = (I + f) f' = 1$

5. c, d.

$${}^{69}C_{3r-1} + {}^{69}C_{3r} = {}^{69}C_{r^2-1} + {}^{69}C_{r^2}$$

$\Rightarrow\ {}^{70}C_{3r} = {}^{70}C_{r^2}$

Thus, $r^2 = 3r$ or $70 - 3r = r^2$ so that $r = 0, 3$ or $7, -10$.

Hence, $r = 3$ and 7 (as the given equation is not defined for $r = 0$ and -10).

6. a, d.

It is given that the fourth term in the expansion of

$\left(ax + \frac{1}{x}\right)^n$ is $\frac{5}{2}$, therefore

$${}^nC_3\,(ax)^{n-3}\left(\frac{1}{x}\right)^3 = \frac{5}{2} \Rightarrow {}^nC_3\,a^{n-3}\,x^{n-6} = \frac{5}{2} \quad \text{(i)}$$

[$\because$ R.H.S. is independent of x]

Putting $n = 6$ in (i), we get ${}^6C_3\,a^3 = \frac{5}{2} \Rightarrow a^3 = \frac{1}{8} \Rightarrow a = \frac{1}{2}$

7. a, b, c, d.

We know that to get the sum of coefficients, we put $x = 1$.

Then, sum of coefficients is $(1 + ax - 2x^2)^n$ is $(a - 1)^n$.

Obviously, when $a > 1$, sum is positive for any n.

8. a, b, d.

$$f(m) = \sum_{i=0}^{m}\binom{30}{30-i}\binom{20}{m-i} = \sum_{i=0}^{m}\binom{30}{i}\binom{20}{m-i} = {}^{50}C_m$$

$f(m)$ is greatest when $m = 25$. Also,

$f(0) + f(1) + \cdots + f(50)$

$= {}^{50}C_0 + {}^{50}C_1 + {}^{50}C_2 + \cdots + {}^{50}C_{50} = 2^{50}$

Also, ${}^{50}C_m$ is not divisible by 50 for any m as 50 is not a prime number

$$\sum_{m=0}^{50}(f(m))^2 = ({}^{50}C_0)^2 + ({}^{50}C_1)^2 + ({}^{50}C_2)^2 + \cdots + ({}^{50}C_{50})^2 = {}^{100}C_{50}$$

9. a, b, d.

$(1 + z^2 + z^4)^8 = C_0 + C_1 z^2 + C_2 z^4 + \cdots + C_{16} z^{32}$ (1)

Putting $z = i$, where $i = \sqrt{-1}$,

$(1 - 1 + 1)^8 = C_0 - C_1 + C_2 - C_3 + \cdots + C_{16}$

$\Rightarrow\ C_0 - C_1 + C_2 - C_3 + \cdots + C_{16} = 1$

Also, putting $z = \omega$,

$(1 + \omega^2 + \omega^4)^8 = C_0 + C_1\omega^2 + C_2\omega^4 + \cdots + C_{16}\omega^{32}$

$\Rightarrow\ C_0 + C_1\omega^2 + C_2\omega + C_3 + \cdots + C_{16}\omega^2 = 0$ (2)

Putting $x = \omega^2$,

$(1 + \omega^4 + \omega^8)^8 = C_0 + C_1\omega^4 + C_2\omega^8 + \cdots + C_{16}\omega^{64}$

$\Rightarrow\ C_0 + C_1\omega + C_2\omega^2 + \cdots + C_{16}\omega = 0$ (3)

Putting $x = 1$,

$3^8 = C_0 + C_1 + C_2 + \cdots + C_{16}$ (4)

Adding (2), (3) and (4), we have

$3(C_0 + C_3 + \cdots + C_{15}) = 3^8$

$\Rightarrow\ C_0 + C_3 + \cdots + C_{15} = 3^7$

Similarly, first multiplying (1) by z and then putting $1, \omega, \omega^2$ and adding, we get

$C_1 + C_4 + C_7 + C_{10} + C_{13} + C_{16} = 3^7$

Multiplying (1) by z^2 and then putting $1, \omega, \omega^2$ and adding, we get

$C_2 + C_5 + C_8 + C_{11} + C_{14} = 3^7$

10. a, b, c.

General term is ${}^{6561}C_r\,7^{\frac{6561-r}{3}}\,11^{\frac{r}{9}}$.

To make the term free of radical sign, r should be a multiple of 9.

$\therefore\ r = 0, 9, 18, 27, \ldots, 6561$.

Hence, there are 730 terms. The greatest binomial coefficients are

${}^{6561}C_{\frac{6561-1}{2}}$ and ${}^{6561}C_{\frac{6561-3}{2}}$ or ${}^{6561}C_{3280}$ and ${}^{6561}C_{3279}$.

Now, 3280 are 3279 are not a multiple of 3; hence, both terms involving greatest binomial coefficients are irrational.

11. b, c.

For $n = 2m$, the given expression is

$$C_0 - (C_0 + C_1) + (C_0 + C_1 + C_2) - (C_0 + C_1 + C_2 + C_3) + \cdots (-1)^{n-1}(C_0 + C_1 + \cdots + C_{n-1})$$

$$= C_0 - (C_0 + C_1) + (C_0 + C_1 + C_2) - (C_0 + C_1 + C_2 + C_3) + \cdots - (C_0 + C_1 + \cdots + C_{2m-1})$$

$= -(C_1 + C_3 + C_5 + \cdots + C_{2m-1})$

$= -(C_1 + C_3 + C_5 + \cdots + C_{n-1}) = -2^{n-1}$

12. a, c, $\left(x^2 + 1 + \frac{1}{x^2}\right)$

$$= {}^nC_0 + {}^nC_1\left(x^2 + \frac{1}{x^2}\right) + {}^nC_2\left(x^2 + \frac{1}{x^2}\right)^2 + \ldots + {}^nC_n\left(x^2 + \frac{1}{x^2}\right)^n$$

This contains each of the term $x^0, x^2, x^4, \ldots x^2n, x^{-2}, x^{-4}, \ldots, x^{-2n}$

coefficient of constant term $= nC_0 + (nC_2)(2) + (nC_4)(4C_2) + (nC_6)(6C_3) + \ldots \neq 2^{n-1}$ coefficient of x^{2n-2} in $nC_{n-1} = n$

coefficient of x^2 is ${}^nC_1 + ({}^nC_3)({}^3C_1) + (nC_5)({}^5C_2) + \ldots > n$

13. a, c, d.

${}^nC_1 + {}^{n+1}C_2 + {}^{n+2}C_3 + \cdots + {}^{n+m-1}C_m$

$= {}^nC_{n-1} + {}^{n+1}C_{n-1} + {}^{n+2}C_{n-1} + \cdots + {}^{n+m-1}C_{n-1}$

$=$ Coefficient of x^{n-1} in $(1 + x)^n + (1 + x)^{n+1} + (1 + x)^{n+2} + \cdots + (1 + x)^{n+m-1}$

$= \text{Coefficient of } x^{n-1} \text{ in } (1+x)^n\left[\frac{(1+x)^m-1}{(1+x)-1}\right]$

$= \text{Coefficient of } x^{n-1} \text{ in } \frac{(1+x)^{m+n}-(1+x)^n}{x}$

$= \text{Coefficient of } x^n \text{ in } \left[(1+x)^{m+n}-(1+x)^n\right]$

$= {}^{m+n}C_n - 1$

Similarly, we can prove

$${}^mC_1 + {}^{m+1}C_2 + {}^{m+2}C_3 + \cdots + {}^{m+n-1}C_n = {}^{m+n}C_m - 1$$

14. a, b, d.

$$\frac{(n-1)(n-2)\cdots(n-m+1)}{(m-1)!}$$

$$= \frac{(n-1)(n-2)\cdots(n-m+1)(n-m)\cdots 2\cdot 1}{(n-m)!(m-1)!}$$

$= {}^{n-1}C_{m-1}$

$= \text{Coefficient of } x^{m-1} \text{ in } (1+x)^{n-1}$

$= \text{Coefficient of } x^{m-1} \text{ in } (1+x)^n (1+x)^{-1}$

Now,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_{m-1}x^{m-1} + \cdots + C_nx^n \quad (1)$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^{m-1}x^{m-1} + \cdots \quad (2)$$

Collecting the coefficients of x^{m-1} in the product of (1) and (2), we get

$(-1)^{m-1} C_0 + (-1)^{m-2} C_1 + \cdots + C_{m-1}$

$= \text{Coefficient of } x^{m-1} \text{ in } (1+x)^{n-1}$

$= {}^{n-1}C_{m-1}$

$\therefore \quad C_0 - C_1 + C_2 - \cdots + (-1)^{m-1} C_{m-1}$

$= {}^{n-1}C_{m-1}(-1)^{m-1}$

$$= \frac{(n-1)(n-2)\cdots(n-m+1)}{(m-1)!}(-1)^{m-1}$$

15. a

We have,

$$\frac{17}{4} + 3\sqrt{2} = \frac{1}{4}(9+8+12\sqrt{2})$$

$$= \frac{1}{4}(3+2\sqrt{2})^2$$

$$\therefore \quad 3 - \sqrt{\frac{17}{4}+3\sqrt{2}} = 3 - \frac{1}{2}(3+2\sqrt{2})$$

$$= \frac{3}{2} - \sqrt{2}$$

Hence, the 10th term of $\left(3-\sqrt{\frac{17}{4}+3\sqrt{2}}\right)^{20} = \left(\frac{3}{2}-\sqrt{2}\right)^{20}$ is

$${}^{20}C_9\left(\frac{3}{2}\right)^{20-9}(-\sqrt{2})^9$$

which is an irrational number.

16. a, b, c.

$$(x\sin p + x^{-1}\cos p)^{10}$$

The general term in the expansion is

$$T_{r+1} = {}^{10}C_r (x\sin p)^{10-r}(x^{-1}\cos p)^r$$

For the term independent of x, we have $10 - 2r = 0$ or $r = 5$.

Hence, the independent term is

$${}^{10}C_5 \sin^5 p \cos^5 p = {}^{10}C_5 \frac{\sin^5 2p}{32}$$

which is the greatest when $\sin 2p = 1$.

The least value of ${}^{10}C_5 \frac{\sin^5 2p}{32}$ is $-\frac{10!}{2^5(5!)^2}$ when $\sin 2p$

$= -1$ or $p = (4n-1)\frac{\pi}{4}, n \in Z$.

Sum of coefficient is $(\sin p + \cos p)^{10}$, when $x = 1$ or $(1 + \sin 2p)^5$, which is least when $\sin 2p = -1$.

Hence, least sum of coefficients is zero. Greatest sum of coefficient occurs when $\sin 2p = 1$. Hence, greatest sum is $2^5 = 32$.

17. b, c, d.

L.H.S. $= (1 + 2x^2 + x^4)(1 + C_1x + C_2x^2 + C_3x^3 + \cdots)$

R.H.S. $= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

Comparing the coefficients of $x, x^2, x^3, \ldots$

$$a_1 = C_1,\ a_2 = C_2 + 2,\ a_3 = C_3 + 2C_1 \quad (1)$$

Now, $2a_2 = a_1 + a_3$ (A.P.)

$\Rightarrow \quad 2({}^nC_2 + 2) = {}^nC_1 + ({}^nC_3 + 2\,{}^nC_1)$ [Using (1)]

$\Rightarrow \quad 2\frac{n(n-1)}{2} + 4 = 3n + \frac{n(n-1)(n-2)}{6}$

$\Rightarrow \quad n^3 - 9n^2 + 26n - 24 = 0$

$\Rightarrow \quad (n-2)(n^2 - 7n + 12) = 0$

$\Rightarrow \quad (n-2)(n-3)(n-4) = 0$

$\Rightarrow \quad n = 2, 3, 4$

18. a, d.

Middle term is $\left(\frac{n}{2}+1\right)^{\text{th}}$ or $(4+1)^{\text{th}}$ or T_5

$\Rightarrow \quad T_5 = {}^8C_4\left(\frac{x}{2}\right)^4 \times 2^4 = 1120$

$\Rightarrow \quad \frac{8\times7\times6\times5}{1\times2\times3\times4}x^4 = 1120$

$\Rightarrow \quad x^4 = \frac{1120}{70} = 16$

$\Rightarrow \quad (x^2+4)(x^2-4) = 0$

$\Rightarrow \quad x = \pm 2\ (\because\ x \in R)$

19. a, b, c, d.

Let T_5 be numerically the greatest term in the expansion of $(1 + x/3)^{10}$. Then,

$$\left|\frac{T_5}{T_4}\right| \ge 1 \text{ and } \left|\frac{T_6}{T_5}\right| \le 1$$

Now,

$$\frac{T_{r+1}}{T_r} = \frac{10-r+1}{r}\frac{x}{3}$$

$\Rightarrow \quad \left|\frac{7}{4}\times\frac{x}{3}\right| \ge 1 \text{ and } \left|\frac{6}{5}\times\frac{x}{3}\right| \le 1$

$\Rightarrow \quad |x| \ge \frac{12}{7} \text{ and } |x| \le \frac{5}{2} \quad (1)$

$\Rightarrow \quad \frac{12}{7} \le |x| \le \frac{5}{2}$

$\Rightarrow \quad x \in \left[-\frac{5}{2}, -\frac{12}{7}\right] \cup \left[\frac{12}{7}, \frac{5}{2}\right]$

20. a, c.

$(1-y)^m(1+y)^n$

$= (1 - {}^mC_1 y + {}^mC_2 y^2 - \cdots)(1 + {}^nC_1 y + {}^nC_2 y^2 + \cdots)$

$= 1 + (n-m)y + \left\{\dfrac{m(m-1)}{2} + \dfrac{n(n-1)}{2} - mn\right\}y^2 + \cdots$

Given,

$a_1 = 10$

$\Rightarrow a_1 = n - m = 10$ (1)

$a_2 = \dfrac{m^2 + n^2 - m - n - 2mn}{2} = 10$

$(m-n)^2 - (m+n) = 20$

$\Rightarrow m + n = 80$ (2)

Solving (1) and (2), we get $m = 35$, $n = 45$.

Reasoning Type

1. a. $({}^{10}C_0) + ({}^{10}C_0 + {}^{10}C_1) + ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2)$

$+ \cdots + ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + \cdots + {}^{10}C_9)$

$= 10\,{}^{10}C_0 + 9\,{}^{10}C_1 + 8\,{}^{10}C_2 + \cdots + {}^{10}C_9$

$= {}^{10}C_1 + 2\,{}^{10}C_2 + 3\,{}^{10}C_3 + \cdots + 10\,{}^{10}C_{10}$

$= \sum_{r=1}^{10} r\,{}^{10}C_r = 10\sum_{r=1}^{10} {}^9C_{r-1} = 10 \times 2^9$

2. b. $\dfrac{T_{r+1}}{T_r} = \dfrac{12 - r + 1}{r}\,\dfrac{11}{10}$

Let,

$T_{r+1} \ge T_r \Rightarrow 13 - r \ge 1.1x$

$\Rightarrow 13 \ge 2.1r$

$\Rightarrow r \le 6.19$

Hence, the greatest term occurs for $r = 6$. Hence, 7th term is the greatest term. Also, the binomial coefficient of 7th term is ${}^{12}C_6$ which is the greatest binomial coefficient.

But this is not the reason for which T_7 is the greatest. Here, it is co-incident that the greatest term has the greatest binomial coefficient

Hence, statement 1 is true, statement 2 is true; but statement 2 is not a correct explanation of statement 1.

3. a. $3456^{2222} = (7 \times 493 + 5)^{2222}$

$= (7k + 5)^{2222}$

$= 7m + 5^{2222}$

Now,

$5^{2222} = 5^2(5^3)^{740}$

$= 25(125)^{740}$

$= 25(126 - 1)^{740}$

$= 25[7n + 1]$

$= 175n + 25$

Remainder when $175n + 25$ is divided by 7 is 4.

Hence, both the statements are correct and statement 2 is a correct explanation of statement 1.

4. d. Statement 2 is true as it is the property of binomial coefficients. But statement 1 is false as three consecutive binomial coefficients may be in A.P. but not always.

5. a. $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$ (1)

We know that

$(1-x)^n = \sum_{r=0}^{n} (-1)^{n-r}\,{}^nC_r x^r = \sum_{r=0}^{n} (-1)^{n-r}\,{}^nC_r x^{n-r}$ (2)

Multiplying (1) and (2), we get

$\sum_{r=0}^{n} (-1)^{n-r}\,{}^nC_r a_r =$ coefficient of x^n in $(1-x^3)^n$

Since $n \ne 3k$, therefore

$\sum_{r=0}^{n} (-1)^{n-r} a_r\,{}^nC_r = 0$

$\Rightarrow \sum_{r=0}^{n} (-1)^r a_r\,{}^nC_r = 0$

Hence, both the statements are correct and statement 2 is a correct explanation of statement 1.

6. a. $(1+x)^n - nx - 1 = (1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n) - nx - 1$ (1)

$= {}^nC_2 x^2 + \cdots + {}^nC_n x^n$

$= x^2({}^nC_2 + {}^nC_3 x + \cdots + {}^nC_n x^{n-2})$

Hence, $(1+x)^n - nx - 1$ is divisible by x^2.

Now in (1), replace x by 8 and n by $n + 1$. Then, we have

$(1+8)^{n+1} - (n+1)8 - 1 = 8^2({}^nC_2 + {}^nC_3 8 + \cdots + {}^nC_n 8^{n-2})$

$\Rightarrow 9^{2n+2} - 8n - 9 = 8^2({}^nC_2 + {}^nC_3 8 + \cdots + {}^nC_n 8^{n-2})$

which is divisible by 64.

Hence, both the statements are correct and statement 2 is a correct explanation of statement 1.

7. b. $(1 + x + x^2 + x^3 + x^4)^{1000} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{4000} x^{4000}$

Clearly, there are 4001 terms. Also, number of term in the expansion $(a_1 + a_2 + \cdots + a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

Clearly, statement 2 has nothing to do with statement 1.

8. a. Coefficient of x^n in $\left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots \dfrac{x^n}{n!}\right)^3$

$=$ Coefficient of x^n in $\left(1 + x + \dfrac{x^2}{2!} + \cdots \dfrac{x^n}{n!} + \cdots\right)^3$

as higher powers of x are not counted while calculating the coefficient of x^n)

$=$ Coefficient of x^n in $e^{3x} = \dfrac{3^n}{n!}$

9. b. $(1+x)^{41}(1 - x + x^2)^{40}$

$= (1+x)(1+x)^{40}(1 - x + x^2)^{40}$

$= (1+x)(1+x^3)^{40}$

$= (1+x^3)^{40} + x(1+x^3)^{40}$

$= (1 + {}^{40}C_1 x^3 + {}^{40}C_2 x^6 + \cdots + {}^{40}C_{40} x^{120}) + ({}^{40}C_0 + {}^{40}C_1 x^4 + {}^{40}C_2 x^7 + \cdots + {}^{40}C_{40} x^{121})$

Hence, the coefficient of x^{85} is zero as there is no term in the above expansion which has x^{85}.

Also, statement 2 is correct but it is not a correct explanation of statement 1.

10. b. We know that the total number of terms in $(x_1 + x_2 + \cdots + x_r)^n$ is ${}^{n+r-1}C_{r-1}$. So, the total number of term in $(x_1 + x_2 + \cdots + x_n)^3$ is

${}^{3+n-1}C_{n-1} = {}^{n+2}C_{n-1} = {}^{n+2}C_3 = \dfrac{(n+2)(n+1)n}{6}$

and the total number of terms in $(x_1 + x_2 + x_3)^n$ is

${}^{n+3-1}C_{n-1} = {}^{n+2}C_3 = \dfrac{(n+2)(n+1)n}{6}$

11. **a.** We have,

$$\left(2+\sqrt{5}\right)^p+\left(2-\sqrt{5}\right)^p = 2[2^p + {}^pC_2\, 2^{p-5}\, 5$$
$$+ {}^pC_4\, 2^{p-4}\, 5^2 + \cdots + {}^pC_{p-1}\, 2\times 5^{(p-1)/2}] \qquad (1)$$

From (1), $\left(2+\sqrt{5}\right)^p+\left(2-\sqrt{5}\right)^p$ is an integer and

$$-1<\left(2-\sqrt{5}\right)^p<0 \quad (\because p \text{ is odd})$$

So,

$$\left[\left(2+\sqrt{5}\right)^p\right] = (2+\sqrt{5})^p + (2-\sqrt{5})^p$$
$$= 2^{p+1} + {}^pC_2\, 2^{p-1}\, 5 + \cdots + {}^pC_{p-1}\, 2^2\, 5^{(p-1)/2}$$

$$\therefore \quad \left[\left(2+\sqrt{5}\right)^p\right] - 2^{p+1} = 2[{}^pC_2\, 2^{p-2}\, 5 + {}^pC_4 2^{p-4}\, 5^2$$
$$+ \cdots + {}^pC_{p-1}\, 2\times 5^{(p-1)/2}]$$

Now, all the binomial coefficients

$${}^pC_2 = \frac{p(p-1)}{1\times 2},$$

$${}^pC_4 = \frac{p(p-1)(p-2)(p-3)}{1\times 2\times 3\times 4}, \ldots, {}^pC_{p-1} = p$$

are divisible by the prime p. Thus, R.H.S. is divisible by p.

12. **a.** Statement 2 is true (can be checked easily) and that is why ${}^{2n}C_0 < {}^{2n}C_1 < {}^{2n}C_2 < \cdots < {}^{2n}C_{n-1} < {}^{2n}C_n > {}^{2n}C_{n+1} \cdots > {}^{2n}C_{2n}$.

13. **b.** Obviously, statement 2 is true. But to get the sum of coefficients in the expansion of $(3^{-x/4}+3^{5x/4})^n$, we must put $x=0$.

14. **a.** We know that

${}^mC_r + {}^mC_{r-1}\, {}^nC_1 + {}^mC_{r-2}\, {}^nC_2 + \cdots + {}^nC_r$

$=$ Coefficient of x^r in $(1+x)^m(1+x)^n$

$=$ Coefficient of x^r in $(1+x)^{m+n}$

$= {}^{m+n}C_r$

$= 0$ as $m+n<r$

15. **a.** $$S = \sum\sum_{0\le i<j\le n}\left(\frac{i}{{}^nC_i}+\frac{j}{{}^nC_j}\right)$$

$$= \sum\sum_{0\le i<j\le n}\left(\frac{n-i}{{}^nC_{n-i}}+\frac{n-j}{{}^nC_{n-j}}\right)$$

$$= n\sum\sum_{0\le i<j\le n}\left(\frac{1}{{}^nC_i}+\frac{1}{{}^nC_j}\right) - S$$

$$\Rightarrow \quad S = \frac{n}{2}\sum\sum_{0\le i<j\le n}\left(\frac{1}{{}^nC_i}+\frac{1}{{}^nC_j}\right)$$

$$= \frac{n}{2}\left(\sum_{r=0}^{n-1}\frac{n-r}{{}^nC_r}+\sum_{r=1}^{n}\frac{r}{{}^nC_r}\right)$$

$$= \frac{n}{2}\left(\sum_{r=0}^{n}\frac{n}{{}^nC_r}\right)$$

$$= \frac{n^2}{2}a$$

Linked Comprehension Type

For Problems 1–3

1. b, 2. d, 3. c.

Sol. The coefficient of the 2nd, 3rd and 4th terms in the expansion are ${}^mC_1$, ${}^mC_2$ and ${}^mC_3$, which are given in A.P. Hence,

$$2\,{}^mC_2 = {}^mC_1 + {}^mC_3$$

$$\Rightarrow \quad \frac{2m(m-1)}{2!} = m + \frac{m(m-1)(m-2)}{3!}$$

$$\Rightarrow \quad m(m^2-9m+14)=0$$

$$\Rightarrow \quad m(m-2)\,(m-7)=0$$

$\Rightarrow \quad m=7$ ($\because m\ne 0$ or 2 as 6th term is given equal to 21)

Now, 6th term in the expansion, when $m=7$, is

$${}^7C_5\left[\sqrt{\left\{2^{\log(10-3^x)}\right\}}\right]^{7-5}\times\left[\sqrt[5]{\left\{2^{(x-3)\log 3}\right\}}\right]^5 = 21$$

$$\Rightarrow \quad \frac{7\times 6}{2!}\,2^{\log(10-3^x)}\times 2^{(x-2)\log 3} = 21$$

$$\Rightarrow \quad 2^{\log(10-3^x)+(x-2)\log 3} = 1 = 2^0$$

$$\Rightarrow \quad \log(10-3^x)+(x-2)\log 3 = 0$$

$$\Rightarrow \quad \log(10-3^x)\,(3)^{(x-2)} = 0$$

$$\Rightarrow \quad (10-3^x)\times 3^x\times 3^{-2} = 1$$

$$\Rightarrow \quad 10\times 3^x-(3^x)^2 = 9$$

$$\Rightarrow \quad (3^x)^2-10\times 3^x+9=0$$

$$\Rightarrow \quad (3^x-1)\,(3^x-9)=0$$

$$\Rightarrow \quad 3^x-1=0 \Rightarrow 3^x=1=3^0 \Rightarrow x=0$$

$$\Rightarrow \quad 3^x-9=0 \Rightarrow 3^x=3^2 \Rightarrow x=2$$

Hence, $x=0$ or 2. When $x=2$,

$$\left[\sqrt{\left\{2^{\log(10-3^x)}\right\}}+\sqrt[5]{\left\{2^{(x-2)\log 3}\right\}}\right]^m$$

$$=[1+1]^7=128$$

When $x=0$,

$$\left[\sqrt{\left\{2^{\log(10-3^x)}\right\}}+\sqrt[5]{\left\{2^{(x-2)\log 3}\right\}}\right]^m$$

$$=\left[\sqrt{\left\{2^{\log 9}\right\}}+\sqrt[5]{\left\{2^{-2\log 3}\right\}}\right]^7$$

$$=\left[2^{\frac{\log 9}{2}}+\frac{1}{2^{\frac{\log 9}{5}}}\right]^7 > 2^7$$

Hence, the minimum value is 128.

For Problems 4–6

4. b, 5. c, 6. c.

Sol. 2nd term is ${}^nC_1 x^{n-1} a = 240$ (1)

3rd term is ${}^nC_2\, x^{n-2}\, a^2 = 720$ (2)

4th term is ${}^nC_3\, x^{n-3}\, a^3 = 1080$ (3)

Multiplying (1) and (3) and dividing by the square of (2), we get

$$\frac{{}^nC_1\times {}^nC_3}{({}^nC_2)^2} = \frac{240\times 1080}{(720)^2}$$

$$\Rightarrow \quad \frac{n\times n(n-1)(n-2)(2!)^2}{n^2(n-1)^2\times 3!} = \frac{1}{2}$$

$\Rightarrow$ $4(n-2) = 3(n-1)$ $(\because n \neq 1)$

$\Rightarrow$ $n = 5$

Putting $n = 5$, from (1) and (2), we get

$5x^4a = 240$ and $10x^3a^2 = 720$

$$\Rightarrow \frac{(5x^4a)^2}{10x^3a^2} = \frac{(240)^2}{720}$$

or

$x^5 = 32$

$\therefore$ $x = 2$

$$\therefore \quad a = \frac{240}{5x^4} = \frac{48}{2^4} = 3$$

Hence, $x = 2$, $a = 3$ and $n = 5$.

$(x-a)^n = (2-3)^5 = -1$

Also,

$$(2+3)^5 = 2^5 + {}^5C_1 2^4 \times 3 + {}^5C_2 2^3 \times 3^2 + {}^5C_3 2^2 \times 3^3 + {}^5C_4 2 \times 3^4 + {}^5C_5 3^5$$
$$= 32 + 240 + 720 + 1080 + 810 + 243$$

Hence, least value of the term is 32.

Sum of odd-numbered terms is $32 + 720 + 810 = 1562$.

For Problems 7–9

7. b, 8. a, 9. c.

Sol. Let,

$$(1 + x + x^2)^{20} = \sum_{r=0}^{40} a_r x^r \quad (1)$$

Replacing x by $1/x$, we get

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^{20} = \sum_{r=0}^{40} a_r \left(\frac{1}{x}\right)^r$$

$$\Rightarrow \quad (1 + x + x^2)^{20} = \sum_{r=0}^{40} a_r x^{40-r} \quad (2)$$

Since (1) and (2) are same series, coefficient of x^r in (1) = coefficient of x^r in (2)

$\Rightarrow$ $a_r = a_{40-r}$

In (1), putting $x = 1$, we get

$$3^{20} = a_0 + a_1 + a_2 + \cdots + a_{40}$$
$$= (a_0 + a_1 + a_2 + \cdots + a_{19}) + a_{20} + (a_{21} + a_{n+2} + \cdots + a_{40})$$
$$= 2(a_0 + a_1 + a_2 + \cdots + a_{19}) + a_{20} \quad (\because \quad a_r = a_{40-r})$$

$$\Rightarrow \quad a_0 + a_1 + a_2 + \cdots + a_{19} = \frac{1}{2}(3^{20} - a_{20}) = \frac{1}{2}(9^{10} - a_{20})$$

Also,

$$a_0 + 3a_1 + 5a_2 + \cdots + 81a_{40}$$
$$= (a_0 + 81a_{40}) + (3a_1 + 79a_{39}) + \cdots + (39a_{19} + 43a_{21}) + 41a_{20}$$
$$= 82(a_0 + a_1 + a_2 + \cdots + a_{19}) + 41a_{20}$$
$$= 41(9^{10} - a_{20}) + 41a_{20}$$
$$= 41 \times 3^{20}$$

$a_0^2 - a_1^2 + a_2^2 - a_3^2 + \cdots$ suggests that we have to multiply the two expansions.

Replacing x by $-1/x$ in (1), we get

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^{20} = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \cdots + \frac{a_{40}}{x^{40}}$$

$$\Rightarrow \quad (1 - x + x^2)^{20} = a_0 x^{40} - a_1 x^{39} + a_2 x^{38} - \cdots + a_{40} \quad (3)$$

Clearly,

$a_0^2 - a_1^2 + a_2^2 + \cdots + a_{40}^2$ is the coefficient of x^{40} in

$(1 + x + x^2)^{20}(1 - x + x^2)^{20}$

= Coefficient of x^{40} in $(1 + x^2 + x^4)^{20}$

In $(1 + x^2 + x^4)^{20}$, replace x^2 by y, then the coefficient of y^{20} in $(1 + y + y^2)^{20}$ is a_{20}. Hence,

$$a_0^2 - a_1^2 + a_2^2 - \cdots + a_{40}^2 = a_{20}$$

$$\Rightarrow \quad (a_0^2 - a_1^2 + a_2^2 - \cdots - a_{19}^2) + a_{20}^2 + (-a_{21}^2 + \cdots + a_{40}^2) = a_{20}$$

$$\Rightarrow \quad 2(a_0^2 - a_1^2 + a_2^2 - \cdots - a_{19}^2) + a_{20}^2 = a_{20}$$

$$\Rightarrow \quad a_0^2 - a_1^2 + a_2^2 - \cdots - a_{19}^2 = \frac{a_{20}}{2}[1 - a_{20}]$$

For Problems 10–12

10. c, 11. a, 12. c.

Sol. $a_0 + a_1x + a_2x^2 + \cdots + a_{99}x^{99} + x^{100} = 0$ has roots ${}^{99}C_0$, ${}^{99}C_1$, ${}^{99}C_2$, …, ${}^{99}C_{99}$.

$$\Rightarrow \quad a_0 + a_1x + a_2x^2 + \cdots + a_{99}x^{99} + x^{100} = (x - {}^{99}C_0)(x - {}^{99}C_1)(x - {}^{99}C_2)\cdots(x - {}^{99}C_{99})$$

Now, sum of roots is

$${}^{99}C_0 + {}^{99}C_1 + {}^{99}C_2 + \cdots + {}^{99}C_{99} = -\frac{a_{99}}{\text{coefficient of } x^{100}}$$

$\Rightarrow$ $a_{99} = -2^{99}$

Also, sum of product of roots taken two at a time is

$$\frac{a_{99}}{\text{coefficient of } x^{100}}$$

$$\therefore \quad \sum\sum_{0 \le i < j \le 99} {}^{99}C_i\,{}^{99}C_j = \frac{\left(\sum_{i=0}^{99}\sum_{j=0}^{99} {}^{99}C_i\,{}^{99}C_j\right) - \sum_{i=0}^{99}({}^{99}C_i)^2}{2}$$

$$= \frac{\left(\sum_{i=0}^{99} {}^{99}C_i 2^{99}\right) - \sum_{i=0}^{99}({}^{99}C_i)^2}{2}$$

$$= \frac{2^{99}2^{99} - \sum_{i=0}^{99}({}^{99}C_i)^2}{2}$$

$$= \frac{2^{198} - {}^{198}C_{99}}{2}$$

$$({}^{99}C_0)^2 + ({}^{99}C_1)^2 + \cdots + ({}^{99}C_{99})^2$$
$$= ({}^{99}C_0 + {}^{99}C_1 + {}^{99}C_2 \cdots + {}^{99}C_{99})^2 - 2\sum\sum_{0 \le i < j \le 99} {}^{99}C_i\,{}^{99}C_j$$
$$= (-a_{99})^2 - 2a_{98}$$
$$= a_{99}^2 - 2a_{98}$$

For Problems 13–15

13. a, 14. c, 15. a.

Sol. a. $$\sum_{r=0}^{100} {}^{100}C_r \sin rx = \text{Im}\left(\sum_{r=0}^{100} {}^{100}C_r e^{irx}\right) \quad (\text{Im} = \text{imaginary part})$$

$$= \text{Im}\left(\sum_{r=0}^{100} {}^{100}C_r (e^{ix})^r\right)$$

$$= \text{Im}((1 + e^{ix})^{100})$$

$$= \text{Im}(1 + \cos x + i\sin x)^{100}$$

$$= \text{Im}\left(2\cos^2\frac{x}{2} + 2i\sin\frac{x}{2}\times\cos\frac{x}{2}\right)^{100}$$

$$= \text{Im}\left(2\cos\frac{x}{2}\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)\right)^{100}$$

$$= 2^{100}\cos^{100}\frac{x}{2}\sin(50x)$$

$$\sum_{r=0}^{50} {}^{50}C_r a^r \times b^{50-r} \times \cos(rB - (50-r)A)$$

$$= \text{Re}\left(\sum_{r=0}^{50} {}^{50}C_r a^r \times b^{50-r} \times e^{i(rB-(50-r)A)}\right)$$

$$= \text{Re}\left(\sum_{r=0}^{50} {}^{50}C_r (a\times e^{iB})^r \times (b\times e^{-iA})^{50-r}\right)$$

$$= \text{Re}(ae^{iB} + b\,e^{-iA})^{50}$$

$$= \text{Re}(a\cos B + ia\sin B + b\cos A - ib\sin A)^{50}$$

$$= \text{Re}(a\cos B + b\cos A)^{50} = c^{50}\ (\because\ a\sin B = b\sin A)$$

$$\frac{\sum_{r=0}^{50} {}^{50}C_r \sin 2rx}{\sum_{r=0}^{50} {}^{50}C_r \cos 2rx} = \frac{\sum_{r=0}^{50} {}^{50}C_{50-r} \sin 2(50-r)x}{\sum_{r=0}^{50} {}^{50}C_{50-r} \cos 2(50-r)x}$$

$$= \frac{\sum_{r=0}^{50} {}^{50}C_r[\sin 2rx + \sin 2(50-r)x]}{\sum_{r=0}^{50} {}^{50}C_r[\cos 2rx + \cos 2(50-r)x]} \left(\because \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}\right)$$

$$= \frac{\sum_{r=0}^{50} {}^{50}C_r 2\sin(50x)\cos(2r-50)x}{\sum_{r=0}^{50} {}^{50}C_r 2\cos(50x)\cos(2r-50)x}$$

$$= \tan(50x)$$

$$\Rightarrow f(\pi/8) = \tan(25\pi/4) = \tan(6\pi + \pi/4) = 1$$

For Problems 16–18

16. b, 17. b, 18. c.

Sol.

General term of the series is

$$T(r) = \sum_{r=1}^{50} \frac{{}^{50+r}C_r(2r-1)}{{}^{50}C_r(50+r)}$$

$$= \frac{{}^{50+r}C_r}{{}^{50}C_r}\left(1 - \frac{50-r+1}{50+r}\right)$$

$$= \frac{{}^{50+r}C_r}{{}^{50}C_r} - \frac{{}^{50+r}C_r}{{}^{50}C_r}\left(\frac{50-r+1}{50+r}\right)$$

Now,

$$\frac{{}^{50+r}C_r}{{}^{50}C_r}\left(\frac{50-r+1}{50+r}\right)$$

$$= \frac{(50-r+1)(50+r)!\,r!\,(50-r)!}{r!\,50!\,(50+r)\,50!}$$

$$= \frac{(50-r+1)!\,(50+r-1)!}{50!\,50!}$$

$$= \frac{{}^{50+r-1}C_{r-1}}{{}^{50}C_{r-1}}$$

$$\Rightarrow T(r) = \frac{{}^{50+r}C_r}{{}^{50}C_r} - \frac{{}^{50+r-1}C_{r-1}}{{}^{50}C_{r-1}} = V(r) - V(r-1)$$

where $V(r) = \dfrac{{}^{50+r}C_r}{{}^{50}C_r}$

Now, sum of the given series

$$P = \sum_{r=1}^{50} T(r) = V(50) - V(0)$$

$$= \frac{{}^{100}C_{50}}{{}^{50}C_{50}} - \frac{{}^{50}C_0}{{}^{50}C_0} = {}^{100}C_{50} - 1$$

Also,

$$Q = \sum_{r=0}^{50} ({}^{50}C_r)^2 = {}^{50}C_0^2 + {}^{50}C_1^2 + {}^{50}C_2^2 + \cdots + {}^{50}C_{50}^2 = {}^{100}C_{50}$$

$$\Rightarrow P - Q = -1$$

We know that

$$C_0^2 - C_1^2 + C_2^2 + \cdots + (-1)^n C_n^2$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^n\, {}^nC_{n/2}, & \text{if } n \text{ is even} \end{cases}$$

$$\Rightarrow \sum_{r=0}^{100} (-1)^r ({}^{100}C_r)^2 = (-1)^{100}\,{}^{100}C_{50} = {}^{100}C_{50}$$

$$\Rightarrow P - R = -1$$

$$Q + R = 2^{100}C_{50} = 2P + 2$$

For Problems 19–21

19. a, 20. b, 21. c.

Sol. Suppose A contains r $(0 \le r \le n)$ elements.

Then, B is constructed by selecting some elements from the remaining $n - r$ elements. Here, A can be chosen in ${}^nC_r$ ways and B in ${}^{n-r}C_0 + {}^{n-r}C_1 + \cdots + {}^{n-r}C_{n-r} = 2^{n-r}$ ways.

So, the total number of ways of choosing A and B is ${}^nC_r \times 2^{n-r}$.

But r can vary from 0 to n. So, total number of ways is

$$\sum_{r=0}^{n} {}^nC_r \times 2^{n-r} = (1+2)^n = 3^n$$

If A contains r elements, then B contains $(r+1)$ elements.

Then, the number of ways of choosing A and B is ${}^nC_r \times {}^nC_{r+1}$ $= C_r C_{r+1}$.

But r can vary from 0 to $(n-1)$.

So, the number of ways is

$$\sum_{r=0}^{n-1} C_r C_{r+1} = C_0C_1 + C_1C_2 + \cdots + C_{n-1}C_n = {}^{2n}C_{n-1}$$

Let A contains r $(0 \le r \le n)$ elements.

Then, A can be chosen in ${}^nC_r$ ways. The subset B of A can have at most r elements, and the number of ways of choosing B is 2^r.

Therefore, the number of ways of choosing A and B is ${}^nC_r \times 2^r$.

But r can vary from 0 to n.

So, the total number of ways is

$$\sum_{r=0}^{n} {}^nC_r \times 2^r = (1+2)^n = 3^n$$

Matrix-Match Type

1. a → q, r, s; b → p, q, r, s; c → p, q, r; d → p, q.

a. ${}^{(n+1)}C_4 + {}^{(n+1)}C_3 + {}^{(n+2)}C_3 = {}^{(n+3)}C_4$

$\Rightarrow {}^{(n+3)}C_4 > {}^{(n+3)}C_3 \Rightarrow \frac{{}^{n+3}C_4}{{}^{n+3}C_3} > 1$

$\Rightarrow n > 4$ or $n \ge 5$

b. $(3053)^{456} - (2417)^{333}$

$= (339 \times 9 + 2)^{456} - (269 \times 9 - 4)^{333}$

Remainder of given number is same as remainder of $2^{456} + 4^{333}$

and

$2^{456} + 4^{333} = (64)^{76} + (64)^{111}$

$= (1 + 63)^{76} + (1 + 63)^{111}$

$= (1 + 9 \times 7)^{76} + (1 + 9 \times 7)^{111}$

Hence, the remainder is 2.

c. We know that $n!$ terminates in 0 for $n \ge 5$ and 3^{4n} terminates in 1 ($\because\ 3^4 = 81$).

Therefore, $3^{180} = (3^4)^{45}$ terminates in 1.

Also, $3^3 = 27$ terminates in 7.

Hence, $183! + 3^{183}$ terminates in 7.

That is, the digit in the unit place is 7.

d. We are given

${}^mC_0 + {}^mC_1 + {}^mC_2 = 46$

$\Rightarrow 2m + m(m-1) = 90$

$\Rightarrow m^2 + m - 90 = 0$

$\Rightarrow m = 9$ as $m > 0$

Now, $(r+1)^{\text{th}}$ term of $\left(x^2 + \frac{1}{x}\right)^m$ is

$${}^mC_r (x^2)^{m-r}\left(\frac{1}{x}\right)^r = {}^mC_r\, x^{2m-3r}$$

For this to be independent of x, $2m - 3r = 0 \Rightarrow r = 6$.

2. a → q; b → s; c → p; d → r.

We know that

${}^nC_0^2 + {}^nC_1^2 + \cdots + {}^nC_n^2 = {}^{2n}C_n$

and

$${}^nC_0^2 - {}^nC_1^2 + \cdots + (-1)^n\, {}^nC_n^2 = \begin{cases} 0, & \text{if } n \text{ is odd} \\ {}^nC_{n/2}(-1)^n, & \text{if } n \text{ is even} \end{cases}$$

From this, ${}^{31}C_0^2 - {}^{31}C_1^2 + {}^{31}C_2^2 - \cdots - {}^{31}C_{31}^2 = 0$

${}^{32}C_0^2 - {}^{32}C_1^2 + {}^{32}C_2^2 - \cdots + {}^{32}C_{32}^2 = {}^{32}C_{16}$

${}^{32}C_0^2 + {}^{32}C_1^2 + {}^{32}C_2^2 - \cdots + {}^{32}C_{32}^2 = {}^{64}C_{32}$

Also, $(1/32)(1 \times {}^{32}C_1^2 + 2 \times {}^{32}C_2^2 - \cdots + 32 \times {}^{32}C_{32}^2)$

$$= \frac{1}{32}\sum_{r=1}^{32} r\left({}^{32}C_r\right)^2$$

$$= \frac{1}{32}\sum_{r=1}^{32} r\, {}^{32}C_r\, {}^{32}C_{32-r}$$

$$= \frac{1}{32}\sum_{r=1}^{32} 32\, {}^{31}C_{r-1}\, {}^{32}C_{32-r}$$

$$= {}^{63}C_{31} = {}^{63}C_{32}$$

3. a → q; b → s; c → p; d → r.

In the sum of series $\sum_{i=1}^{n}\sum_{j=1}^{n} f(i) \times f(j) = \sum_{i=1}^{n}\left(f(i)\left(\sum_{j=1}^{n} f(j)\right)\right)$ i and j are independent. In this summation, three types of terms occur, for which $i < j$, $i > j$ and $i = j$. Also, the sum of terms when $i < j$ is equal to the sum of the terms when $i > j$ if $f(i)$ and $f(j)$ are symmetrical. So, in that case

$$\sum_{i=0}^{n}\sum_{j=0}^{n} f(i) \times f(j) = \sum\sum_{0 \le i < j \le n} f(i) \times f(j) + \sum\sum_{0 \le j < i \le n} f(i) f(j) + \sum\sum_{i=j} f(i) f(j)$$

$$= 2\sum\sum_{0 \le i < j \le n} f(i) f(j) + \sum\sum_{i=j} f(i) f(j)$$

$$\Rightarrow \sum\sum_{0 \le i < j \le n} f(i) f(j) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} f(i) \times f(j) - \sum\sum_{i=j} f(i) f(j)}{2}$$

a. $$\sum\sum_{i \ne j} {}^{10}C_i\, {}^{10}C_j = \sum_{i=0}^{10}\sum_{j=0}^{10} {}^{10}C_i\, {}^{10}C_j - \sum_{i=0}^{10} {}^{10}C_i^2 = 2^{20} - {}^{20}C_{10}$$

b. $$\sum\sum_{0 \le i \le j \le 10} {}^{10}C_i\, {}^{10}C_j = \frac{\sum_{i=0}^{10}\sum_{j=0}^{10} {}^{10}C_i\, {}^{10}C_j + \sum_{i=0}^{10} {}^{10}C_i^2}{2} = \frac{2^{20} + {}^{20}C_{10}}{2}$$

c. $$\sum\sum_{0 \le i < j \le 10} {}^{10}C_i\, {}^{10}C_j = \frac{\sum_{i=0}^{10}\sum_{j=0}^{10} {}^{10}C_i\, {}^{10}C_j - \sum_{i=0}^{10} {}^{10}C_i^2}{2}$$

$$= \frac{2^{20} - {}^{20}C_{10}}{2}$$

d. $$\sum_{i=0}^{10}\sum_{j=0}^{10} {}^{10}C_i\, {}^{10}C_j = \sum_{i=0}^{10} {}^{10}C_i \sum_{j=0}^{10} {}^{10}C_j = 2^{20}$$

4. a → p, q, s; b → p, q, r, s; c → p, q, r, s; d → p, q, s.

a. In $(1+x)^{41} = {}^{41}C_0 + {}^{41}C_1 x + {}^{41}C_2 x^2 + \cdots + {}^{41}C_{20} x^{20} + {}^{41}C_{21} x^{21} + \cdots + {}^{41}C_{41} x^{41}$

$\Rightarrow {}^{41}C_{21} + {}^{41}C_{22} + \cdots + {}^{41}C_{41} = 2^{40}$

b. $(1 + \sqrt{2})^{42} = {}^{42}C_0 + {}^{42}(C_1\sqrt{2}) + {}^{42}C_2(\sqrt{2})^2 + {}^{42}C_3(\sqrt{2})^3 + \cdots + {}^{42}C_{42}(\sqrt{2})^{42}$

Sum of binomial coefficients of rational terms is

${}^{42}C_0 + {}^{42}C_2 + {}^{42}C_4 + \cdots + {}^{42}C_{42} = 2^{41}$

c. $$\left(x + \frac{1}{x} + x^2 + \frac{1}{x^2}\right)^{21} = \left(\frac{x^3 + x + x^4 + 1}{x^2}\right)^{21}$$

$$= \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_{82} x^{82}}{x^{42}} \qquad (1)$$

Now, putting $x = 1$, we get

$4^{21} = a_0 + a_1 + a_2 + \cdots + a_{82}$

Putting $x = -1$, we get

$0 = a_0 - a_1 + a_2 - a_3 + \cdots + a_{82}$

Adding, we get

$4^{21} = 2(a_0 + a_2 + \cdots + a_{82})$

$\Rightarrow a_0 + a_2 + \cdots + a_{82} = 2^{41}$

d. We know that

$$^nC_0 - {^nC_2} + {^nC_4} - {^nC_6} + \cdots = 2^{n/2}\cos\frac{n\pi}{4} \quad (1)$$

and

$$^nC_0 + {^nC_2} + {^nC_4} + {^nC_6} + \cdots = 2^{n-1} \quad (2)$$

$$\Rightarrow \quad {^nC_0} + {^nC_4} + {^nC_8} + \cdots = \frac{1}{2}\left(2^{n/2} \times \cos\frac{n\pi}{4} + 2^{n-1}\right)$$

For $n = 42$,

$$^{42}C_0 + {^{42}C_4} + {^{42}C_8} + \cdots = \frac{1}{2}\left(2^{21} \times \cos\frac{21\pi}{2} + 2^{41}\right) = 2^{40}$$

5. a → p, r; b → p, r. → p, q; d → q, r, s.

a. Let consecutive coefficients be ^nC_r and $^nC_{r+1}$. Then,

$$\frac{n!}{(n-r)!r!} = \frac{n!}{(n-r-1)!(r+1)!}$$

$$\Rightarrow \quad \frac{1}{(n-r)(n-r-1)!r!} = \frac{1}{(n-r-1)!(r+1)r!}$$

$$\Rightarrow \quad r + 1 = n - r$$

$$\Rightarrow \quad n = 2r + 1$$

Hence, n is odd.

b. $E = (19-4)^n + (19+4)^n$

$$2\left[{^nC_0}19^n + {^nC_2}19^{n-2}4^2 + \cdots + {^nC_n}4^n\right] \text{ when } n \text{ is even}$$

or

$$2\left[{^nC_0}19^n + {^nC_2}\cdot 19^{n-2}\cdot 4^2 + \cdots + {^nC_{n-1}}19\cdot 4^{n-1}\right] \text{ then } n \text{ is odd}$$

$\Rightarrow$ E is divisible by 19 when n is odd

c. $^{10}C_0\,{^{20}C_{10}} - {^{10}C_1}\,{^{18}C_{10}} + {^{10}C_2}\,{^{16}C_{10}} - \cdots$

$= $ Coefficient of x^{10} in $[{^{10}C_0}(1+x)^{20} - {^{10}C_1} \times (1+x)^{18} + {^{10}C_2}(1+x)^{16} - \cdots]$

$= $ Coefficient of x^{10} in $[{^{10}C_0}((1+x)^2)^{10} - {^{10}C_1} \times ((1+x)^2)^9 + {^{10}C_2}((1+x)^2)^8 - \cdots]$

$= $ Coefficient of x^{10} in $[(1+x)^2 - 1]^{10}$

$= $ Coefficient of x^{10} in $[2x + x^2]^{10}$

$= 2^{10}$

d. $T_r = {^{14}C_{r-1}}x^{r-1}$; $T_{r+1} = {^{14}C_r}x^r$; $T_{r+2} = {^{14}C_{r+1}}x^{r+1}$

By the given condition,

$$2\,{^{14}C_r} = {^{14}C_{r-1}} + {^{14}C_{r+1}} \quad (1)$$

$$\Rightarrow \quad 2 = \frac{^{14}C_{r-1}}{^{14}C_r} + \frac{^{14}C_{r+1}}{^{14}C_r}$$

$$\Rightarrow \quad 2 = \frac{r}{14-r+1} + \frac{14-(r+1)+1}{r+1}$$

$$\Rightarrow \quad 2 = \frac{r}{15-r} + \frac{14-r}{r+1}$$

$$\Rightarrow \quad r = 9$$

Integer Type

1.(1) Let x^7 occurs in T_{r+1} term, then

$$T_{r+1} = {^nC_r}(ax^2)^{n-r}\left(\frac{1}{bx}\right)^r$$

$$= {^{11}C_r}\frac{a^{11-r}}{b^r}.x^{22-2r-r}$$

for $x^7 \Rightarrow 22 - 3r = 7 \Rightarrow r = 5$

Hence, coefficients of x^7 is $^{11}C_5\dfrac{a^6}{b^5}$

Let x^{-7} occur in T_{r+1} term, then

$$T_{r+1} = {^{11}C_r}(ax)^{11-r}\left(-\frac{1}{bx^2}\right)^r$$

$$= {^{11}C_r}\frac{a^{11-r}}{(-b)^r}x^{11-3r}$$

For $x^{-7} \Rightarrow 11 - 3r = -7 \Rightarrow r = 6$

Hence, coefficient of x^{-7} is $^{11}C_6\dfrac{a^5}{b^6}$

Now $^{11}C_5\dfrac{a^5}{b^6} = {^{11}C_6}\dfrac{a^6}{b^5}$

$$\Rightarrow \quad {^{11}C_5}\,a = {^{11}C_6}\frac{a^5}{b^6}$$

$$\Rightarrow \quad {^{11}C_5}\,a = {^{11}C_{11-6}}\frac{1}{b}$$

$$\Rightarrow \quad {^{11}C_5}\,a = {^{11}C_5}\frac{1}{b}$$

$$\Rightarrow \quad ab = 1$$

2.(6) Coefficients of $(2r+4)^{\text{th}}$ and $(r-2)^{\text{th}}$ terms are equal.

$\Rightarrow$ $^{18}C_{2r+3} = {^{18}C_{r-3}}$ (when $^nC_x = {^nC_y}$, then $x = y$ or $x + y = n$)

$\Rightarrow$ $2r + 3 + r - 3 = 18 \Rightarrow r = 6$

3.(9) According to the question,

$^{14}C_{r-1}$, $^{14}C_r$, $^{14}C_{r+1}$ are in A.P., so $\left\{b = \dfrac{a+c}{2}\right\}$

$$\Rightarrow \quad 2.\,{^{14}C_r} = {^{14}C_{r-1}} + {^{14}C_{r+1}}$$

$$\Rightarrow \quad \frac{2.14!}{(14-r)!r!} = \frac{14!}{(14-r+1)!(r-1)!} + \frac{14!}{(14-r-1)!(r+1)!}$$

$$\Rightarrow \quad \frac{2}{(14-r)(13-r)!r(r-1)!} = \frac{1}{(15-r)(14-r)(13-r)!(r-1)!} + \frac{1}{(13-r)!(r+1)r(r-1)!}$$

$$\Rightarrow \quad \frac{2}{(14-r)r} = \frac{1}{(15-r)(14-r)} + \frac{1}{r(r+1)}$$

$$\Rightarrow \quad \frac{2}{(14-r)r} - \frac{1}{r(r+1)} = \frac{1}{(15-r)(14-r)}$$

$$\Rightarrow \quad \frac{3r-12}{r(r+1)} = \frac{1}{(15-r)}$$

$$\Rightarrow \quad r = 5 \text{ or } 9$$

4.(8) Let the three consecutive coefficients be $^nC_{r-1} = 28$, $^nC_r = 56$ and $^nC_{r+1} = 70$,

so that $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r} = \dfrac{56}{28} = 2$ and

$$\frac{^nC_{r+1}}{^nC_r} = \frac{n-r}{r+1} = \frac{70}{56} = \frac{5}{4}$$

This gives $n + 1 = 3r$ and $4n - 5 = 9r$

$$\therefore \quad \frac{4n-5}{n+1} = 3 \Rightarrow n = 8$$

5.(8) $= [\sqrt{x^2+1} + \sqrt{x^2-1}]^8 + \left[\sqrt{x^2+1} - \sqrt{x^2-1}\right]^8$

$$= 2\left[{^8C_0}\left(\sqrt{x^2+1}\right)^8 + {^8C_2}\left(\sqrt{x^2+1}\right)^6\left(\sqrt{x^2-1}\right)^2\right.$$

$$+{}^8C_4\left(\sqrt{x^2+1}\right)^4\left(\sqrt{x^2-1}\right)^4$$

$${}^8C_6\left(\sqrt{x^2+1}\right)^2\left(\sqrt{x^2-1}\right)^6+{}^8C_8\left(\sqrt{x^2-1}\right)^8\Big]$$

which has degree 8.

6.(3) $(1+0.00002)^{50000}=\left(1+\dfrac{1}{50000}\right)^{50000}$.

Now we know that $2\le\left(1+\dfrac{1}{n}\right)^n<3\ \forall n\ge 1 \Rightarrow$ Least integer is 3

7.(0) Middle term is $\left(\dfrac{n}{2}+1\right)^{\text{th}}$, i.e., $(4+1)^{\text{th}}$, i.e., T_5

$\therefore\ T_5={}^8C_4\left(\dfrac{x}{2}\right)^4.2^4=1120 \Rightarrow x^4=\dfrac{8.7.6.5}{1.2.3.4}x^4=1120$

$\Rightarrow x^4=\dfrac{1120}{70}=16$

$\Rightarrow (x^2+4)(x^2-4)=0$

$\therefore\ x=\pm 2$ only as $x\in R$

8.(4) $T_2={}^nC_1\,(a^{1/13})^{n-1}\cdot a\sqrt{a}=14a^{5/2}$

$\Rightarrow n\cdot a^{\frac{n-1}{13}}=14a$

$\Rightarrow n\cdot a^{\frac{n-14}{13}}=14$

$\Rightarrow \dfrac{n-14}{13}=0$

$\Rightarrow n=14$

$\Rightarrow \dfrac{{}^{14}C_3}{{}^{14}C_2}=\dfrac{14!}{3!\cdot 11!}\cdot\dfrac{2!\cdot 12!}{14!}=\dfrac{12}{3}=4$

9.(5) ${}^{23}C_r+2.\,{}^{23}C_{r+1}+{}^{23}C_{r+2}={}^{24}C_{r+1}+{}^{24}C_{r+2}={}^{25}C_{r+2}\ge{}^{25}C_{15}$

$\therefore$ $(r+2)$ can be 10, 11, 12, 13 and 15

so 5 elements.

10.(4)

$$\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}}+\frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$$

$$=\left((\sqrt{4^x+44})^{2/5}+\left(\frac{1}{\sqrt[3]{2^{x-1}+7}}\right)\right)^8$$

$$=\left((4^x+44)^{1/5}+\frac{1}{(2^{x-1}+7)^{1/3}}\right)^8$$

Now $T_4=T_{3+1}={}^8C_3\,((4^x+44)^{1/5})^{8-3}\dfrac{1}{((2^{x-1}+7)^{1/3})^3}$

Given $336={}^8C_3\left(\dfrac{4^x+44}{2^{x-1}+7}\right)$

Let $2^x=y$

$\Rightarrow\ 336={}^8C_3\left(\dfrac{y^2+44}{(y/2)+7}\right)$

$\Rightarrow\ 336=\dfrac{8\times7\times6}{3\times2\times1}\left(\dfrac{2(y^2+44)}{y+14}\right)$

$\Rightarrow\ y^2-3y+2=0\ \Rightarrow\ y=0,2$

11.(6) $T_{r+1}={}^nC_r\,(x^2)^{n-r}(-1)^r x^{-r}$

$={}^nC_r\,x^{2n-3r}(-1)^r$

Constant term $={}^nC_r\,(-1)^r$ if $2n=3r$

i.e., coefficient of $x=0$

hence, ${}^nC_{2n/3}\,(-1)^{2n/3}=15={}^6C_4\ n=6$

12.(9) $f(n)={}^nC_0a^{n-1}-{}^nC_1\,a^{n-2}+{}^nC_2a^{n-3}+\cdots+(-1)^{n-1}\,{}^nC_{n-1}a^0$

$=\dfrac{1}{a}\left({}^nC_0\,a^n-{}^nC_1\,a^{n-1}+{}^nC_2\,a^{n-2}+\cdots+(-1)^{n-1}\,{}^nC_{n-1}\,a'\right)$

$=\dfrac{1}{a}\left((a-1)^n-(-1)^n\,{}^nC_n\right)$

$=\dfrac{1}{a}\left(\left(\dfrac{1}{3^{223}}-(-1)^n\right)\right)$

$f(x)=\dfrac{3^{\frac{n}{223}}-(-1)^n}{\left(3^{\frac{1}{223}}+1\right)}$

$\Rightarrow f(2007)=\dfrac{3^{\frac{2007}{223}}+1}{3^{\frac{1}{223}}+1}$

$\Rightarrow f(2008)=\dfrac{3^{\frac{2008}{223}}-1}{3^{\frac{1}{223}}+1}$

$\Rightarrow f(2007)+f(2008)=\dfrac{3^{\frac{2007}{223}}+3^{\frac{2008}{223}}}{3^{\frac{1}{223}}+1}$

$=\dfrac{3^9+3^{9+\frac{1}{223}}}{3^{\frac{1}{223}}+1}$

$=3^9\dfrac{\left(1+3^{\frac{1}{223}}\right)}{\left(1+3^{\frac{1}{223}}\right)}=3^9$

$\Rightarrow\ 3^9=3^k$ then $k=9$

13.(5) We have $1+\sum_{r=1}^{10}(3^r.\,{}^{10}C_r+r.\,{}^{10}C_r)$

$=1+\sum_{r=1}^{10}3^r.{}^{10}C_r+10\sum_{r=1}^{10}{}^9C_{r-1}$

$=1+4^{10}-1+10\cdot 2^9$

$=4^{10}+5.2^{10}=2^{10}(4^5+5)$

$=2^{10}(\alpha\cdot 4^5+\beta)$, so $\alpha=1$ and $\beta=5$

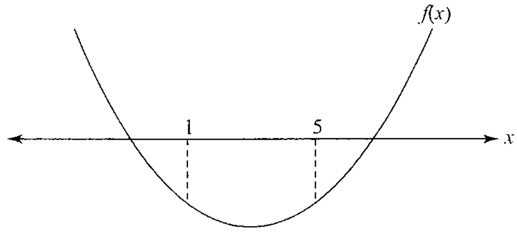


Fig. 6.1

Now $f(1)<0$ and $f(5)<0$

$f(1) < 0 \Rightarrow -k^2 < 0 \Rightarrow k \neq 0$

and $f(5) < 0$

$\Rightarrow 16 - k^2 < 0$

$\Rightarrow k^2 - 16 > 0$

$\Rightarrow k \in (-\infty, 4) \cup (4, \infty)$

Hence, the smallest positive integral value of $k = 5$.

14.(4) We have b = coefficient of x^3 in

$$\left((1 + x + 2x^2 + 3x^3) + 4x^4\right)^4$$

= coefficient of x^3 in $[{}^4C_0(1 + x + 2x^2 + 3x^3)^4 (4x^4)^0$

$+ {}^4C_1(1 + x + 2x^2 + 3x^3)^3 (4x^4)^1 + \cdots]$

= coefficient of x^3 in $(1 + x + 2x^2 + 3x^3)^4$ =

Hence, $4a/b = 4$.

15.(7) $(1 + 7)^{83} + (7 - 1)^{83} = (1 + 7)^{83} - (1 - 7)^{83}$

$= 2[{}^{83}C_1 \cdot 7 + {}^{83}C_3 \cdot 7^3 + \cdots + {}^{83}C_{83} \cdot 7^{83}] = (2 \cdot 7 \cdot 83) + 49I$

where I is an integer

Now $14 \times 83 = 1162$

$\therefore \dfrac{1162}{49} = 23\dfrac{35}{49}$

$\therefore$ Remainder is 35

16.(0) Consider $(5 + 2)^{100} - (5 - 2)^{100}$

$= 2\,[{}^{100}C_1\, 5^{99} \cdot 2 + {}^{100}C_3\, 5^{97} \cdot 2^3 + \cdots + {}^{100}C_{99} 5 \cdot 2^{99}]$

$= 2\,[1000 \cdot 5^{98} + 1000 \cdot {}^{100}C_3 \cdot 5^{94} + \cdots + 1000 \cdot 2^{98}]$

$\Rightarrow$ Minimum 000 as last three digits.

17.(6) $(1 - 2x + 5x^2 - 10x^3)\,[{}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots]$

$= 1 + a_1 x + a_2 x^2 + \cdots$

$\Rightarrow a_1 = n - 2$ and $a_2 = \dfrac{n(n-1)}{2} - 2n + 5$

Given that $a_1^2 = 2a_2$

$\Rightarrow (n - 2)^2 = n(n - 1) - 4n + 10$

$\Rightarrow n^2 - 4n + 4 = n^2 - 5n + 10$

$\Rightarrow n = 6$

18.(0) $1 + 2 + 2^2 + 2^3 + \cdots + 2^{1999}$

$= \dfrac{1(2^{2000} - 1)}{1}$

$= 2^{2000} - 1$

$= (1 - 5)^{1000} - 1$

$= 1 - {}^{1000}C_1 \cdot 5 + {}^{1000}C_2 \cdot 5^2 + \cdots + {}^{1000}C_{1000} \cdot 5^{1000} - 1$

which is divisible by 5.

19.(1) $= \displaystyle\sum_{k=0}^{4} \left(\frac{3^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)$

$= \displaystyle\sum_{k=0}^{4} \left(\frac{3^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)\frac{4!}{4!}$

$= \displaystyle\sum_{k=0}^{4} \frac{{}^4C_k \cdot 3^{4-k} \cdot x^k}{4!} = \frac{(3 + x)^4}{4!}$

According to the question,

$\dfrac{(3 + x)^4}{4!} = \dfrac{32}{3}$

$\Rightarrow (3 + x)^4 = 256$

$\Rightarrow x + 3 = 4 \Rightarrow x = 1$

20.(1) $\displaystyle\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{5^n} \cdot {}^nC_r \left(\sum_{t=0}^{r-1} {}^rC_t \cdot 3^t\right) = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{5^n} \cdot {}^nC_r\,(4^r - 3^r)$

$\displaystyle = \lim_{n\to\infty} \frac{1}{5^n}\left(\sum_{r=1}^{n} {}^nC_r\, 4^r - \sum_{r=1}^{n} {}^nC_r\, 3^r\right) = \lim_{n\to\infty} \frac{1}{5^n}(5^n - 4^n) = 1$

Archives

Subjective Type

1. Given that

$C_1 + 2C_2 x + 3C_3 x^2 + \cdots + 2n\, C_{2n} x^{2n-1} = 2n(1 + x)^{2n-1}$

where

$$C_r = \frac{2n!}{r!(2n - r)!} \quad (1)$$

Integrating both sides with respect to x, under the limits 0 to x, we get

$[C_1 x + C_2 x^2 + C_3 x^3 + \cdots + C_{2n} x^{2n}]_0^x = \left[(1 + x)^{2n}\right]_0^x$

$\Rightarrow C_1 x + C_2 x^2 + C_3 x^3 + \cdots + C_{2n} x^{2n} = (1 + x)^{2n} - 1$

$\Rightarrow C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots + C_{2n} x^{2n} = (1 + x)^{2n} \quad (2)$

Changing x by $-1/x$, we get

$$C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \frac{C_3}{x^3} + \cdots + (-1)^{2n}\frac{C_{2n}}{x^{2n}} = \left(1 - \frac{1}{x}\right)^{2n}$$

$\Rightarrow C_0 x^{2n} - C_1 x^{2n-1} + C_2 x^{2n-2} - C_3 x^{2n-3} + \cdots + C_{2n} = (x - 1)^{2n} \quad (3)$

Multiplying Eqs. (1) and (3) and equating the coefficient of x^{2n-1} on both sides, we get

$-C_1^2 + 2\,C_2^2 - 3C_3^2 + \cdots + 2n\, C_{2n}^2$

= Coefficient of x^{2n-1} in $(x^2 - 1)^{2n-1}(x - 1)$

$= 2n$[coefficient of x^{2n-2} in $(x^2 - 1)^{2n-1}$ − coefficient of x^{2n-1} in $(x^2 - 1)^{2n-1}$]

$= 2n[{}^{2n-1}C_{n-1}(-1)^{n-1} - 0]$

$= (-1)^{n-1}\, 2n\,{}^{2n-1}C_{n-1}$

$\Rightarrow C_1^2 - 2\,C_2^2 + 3C_3^2 + \cdots + 2nC_{2n}^2$

$= (-1)^n\, 2n\,{}^{2n-1}C_{n-1}$

$= (-1)^n\, n \times \left(\dfrac{2n}{n} \times {}^{2n-1}C_{n-1}\right)$

$= (-1)^n\, n\, {}^{2n}C_n$

$= (-1)^n\, n\, C_n \quad (\because {}^{2n}C_n = C_n)$

2. See solved example 6.81

3. s_n is in geometric progression, hence

$s_n = \dfrac{q^{n+1} - 1}{q - 1},\ q \neq 1$

$$S_n = \frac{\left(\frac{q+1}{2}\right)^{n+1} - 1}{\left(\frac{q+1}{2}\right) - 1} = \frac{(q + 1)^{n+1} - 2^{n+1}}{2^n(q - 1)} \quad (A)$$

Consider

${}^{(n+1)}C_1 + {}^{(n+1)}C_2 s_1 + {}^{(n+1)}C_3 s_3 + \cdots + {}^{(n+1)}C_{n+1} s_n$

$$= {}^{(n+1)}C_1\left(\frac{q-1}{q-1}\right) + {}^{(n+1)}C_2\,\frac{q^2-1}{q-1} + \cdots + {}^{(n+1)}C_{n+1}\,\frac{q^{n+1}-1}{q-1}$$

$$= \left(\frac{1}{q-1}\right)\left[\left\{{}^{(n+1)}C_1 q + {}^{(n+1)}C_2 q^2 + \cdots + {}^{(n+1)}C_{n+1} q^{n+1}\right\}\right.$$

$$\left. - \left\{{}^{(n+1)}C_1 + {}^{(n+1)}C_2 + \cdots + {}^{(n+1)}C_{n+1}\right\}\right]$$

$$= \left(\frac{1}{q-1}\right)\left[\left\{(1+q)^{n+1}-1\right\} - \left\{2^{n+1}-1\right\}\right] \qquad \text{(B)}$$

$$= \frac{(1+q)^{n+1} - 2^{n+1}}{q-1}$$

Thus,

$${}^{(n+1)}C_1 + {}^{(n+1)}C_2 s_1 + \cdots + {}^{(n+1)}C_{n+1} s_n = \frac{(1+q)^{n+1} - 2^{n+1}}{q-1}$$

But from (A), we have

$${}^{n+1}C_1 + {}^{(n+1)}C_2 s_1 + \cdots + {}^{(n+1)}C_{n+1} s_{n+1} = 2^n S_n$$

4. $$\sum_{r=0}^{n} (-1)^r \times {}^nC_r \left(\frac{1}{2^r} + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + \cdots m \text{ terms}\right)$$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \left(1-\frac{15}{16}\right)^n + \cdots m \text{ terms}$$

$$= \frac{1}{2^n} + \frac{1}{2^{2n}} + \frac{1}{2^{3n}} + \frac{1}{2^{4n}} + \cdots m \text{ terms}$$

$$= \frac{\frac{1}{2^n}\left[1-\left(\frac{1}{2^n}\right)^m\right]}{1-\frac{1}{2^n}} = \frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

5. Here $f = R - [R]$ is the fractional part of R. Thus, if I is the integral part of R, then

$$R = I + f = (5\sqrt{5} + 11)^{2n+1}, \text{ and } 0 < f < 1$$

Let $f' = (5\sqrt{5} - 11)^{2n+1}$. Then $0 < f' < 1$ (as $5\sqrt{5} - 11 < 1$)

Now, $I + f - f' = (5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1}$

$$= 2[{}^{2n+1}C_1 (5\sqrt{5})^{2n} \times 11 + {}^{2n+1}C_3 (5\sqrt{5})^{2n-2} \times 11^3 + \cdots] \qquad (1)$$

= an even integer

$\Rightarrow$ $f - f'$ must also be an integer

$\Rightarrow$ $f - f' = 0, \quad \because \quad 0 < f < 1, 0 < f' < 1$

$\Rightarrow$ $f = f'$

$\therefore$ $Rf = Rf' = (5\sqrt{5} + 11)^{2n+1}(5\sqrt{5} - 11)^{2n+1}$

$$= (125 - 121)^{2n+1} = 4^{2n+1}$$

6. $S = C_0 - 2^2C_1 + 3^2C_2 - \cdots + (-1)^n (n+1)^2 C_n$

$$T_r = (-1)^r r^2\, {}^nC_r$$
$$= (-1)^r r(r\,{}^nC_r)$$
$$= (-1)^r r(n\,{}^{n-1}C_{r-1})$$
$$= n(-1)^r ((r-1)+1)({}^{n-1}C_{r-1})$$
$$= n(-1)^r ((r-1)\,{}^{n-1}C_{r-1} + {}^{n-1}C_{r-1})$$
$$= n(-1)^r ((n-1)\,{}^{n-2}C_{r-2} + {}^{n-1}C_{r-1})$$
$$= n(n-1)\,{}^{n-2}C_{r-2}(-1)^{r-2} - n\,{}^{n-1}C_{r-1}(-1)^{r-1}$$

$$\Rightarrow \quad S = \sum_{r=0}^{n} T_r$$

$$= n(n-1)(1-1)^{n-2} - n(1-1)^{n-1}$$
$$= 0$$

7. Given that

$$\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r \qquad (1)$$

and

$$a_k = 1, \forall\, k \ge n$$

In Eq. (1), let us put $x - 3 = y$ or $x - 3 = y$ or $x - 2 = y + 1$ and we get

$$\sum_{r=0}^{2n} a_r (1+y)^r = \sum_{r=0}^{2n} b_r (y)^r$$

$$\Rightarrow \quad a_0 + a_1(1+y) + \cdots + a_{n-1}(1+y)^{n-1}$$

$$+ (1+y)^n + (1+y)^{n+1} + \cdots + (1+y)^{2n} = \sum_{r=0}^{2n} b_r y^r$$

[Using $a_k = 1, \forall\, k \ge n$]

Equating the coefficients of y^n on both the sides, we get

$${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \cdots + {}^{2n}C_n = b_n$$

$$\Rightarrow \quad ({}^{n+1}C_{n+1} + {}^{n+1}C_n) + {}^{n+2}C_n + \cdots + {}^{2n}C_n = b_n$$

[Using ${}^nC_n = {}^{n+1}C_{n+1} = 1$]

$\Rightarrow$ $b_n = {}^{n+2}C_{n+1} + {}^{n+2}C_n + \cdots + {}^{2n}C_n$ [Using ${}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r$]

Combining the terms in similar way, we get

$$b_n = {}^{2n}C_{n+1} + {}^{2n}C_n$$

$$\Rightarrow \quad b_n = {}^{2n+1}C_{n+1}$$

8. $S = \sum_{r=1}^{k} (-3)^{r-1}\, {}^{3n}C_{2r-1}$, $k = \dfrac{3n}{2}$ and n is even

$$\Rightarrow k = \frac{3(2m)}{2} = 3m$$

$$\Rightarrow S = \sum_{r=1}^{3m} (-3)^{r-1} \times {}^{6m}C_{2r-1} = {}^{6m}C_1 - 3\,{}^{6m}C_3 + 3^2\,{}^{6m}C_5 - \cdots$$

$$(-3)^{3m-1}\,{}^{6m}C_{6m-1}$$

$$= \frac{1}{\sqrt{3}}\left[\sqrt{3}\,{}^{6m}C_1 - (\sqrt{3})^3\,{}^{6m}C_3 + (\sqrt{3})^5\,{}^{6m}C_5\right.$$

$$\left. - \cdots + (-1)^{3m-1}(\sqrt{3})^{6m-1}\,{}^{6m}C_{6m-1}\right]$$

There is an alternate sign series with odd binomial coefficients. Hence, we should replace x by $\sqrt{3}i$ in $(1+x)^{6m}$. Therefore,

$$(1+\sqrt{3}i)^{6m} = {}^{6m}C_0 + {}^{6m}C_1(\sqrt{3}i) + {}^{6m}C_2(\sqrt{3}i)^2 + {}^{6m}C_3(\sqrt{3}i)^3$$

$$+ \cdots + {}^{6m}C_{6m}(\sqrt{3}i)^{6m}$$

$$\Rightarrow \quad \sqrt{3} \times {}^{6m}C_1 - (\sqrt{3})^3\,{}^{6m}C_3 + (\sqrt{3})^5\,{}^{6m}C_5 + \cdots$$

$$= \text{Imaginary part in } (1+\sqrt{3}i)^{6m}$$

$$= \text{Im}\left[2^{6m}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{6m}\right]$$

$$= \text{Im}\left[2^{6m}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{6m}\right]$$

$$= \text{Im}\left[2^{6m}(\cos 2m\pi + i\sin 2m\pi)\right] = \text{Im}\,[2^{6m}] = 0$$

$$\Rightarrow \quad S = 0$$

9. $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \cdots$ suggests that we have to multiply two expansions

$$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n} \quad (1)$$

Replacing x by $-1/x$, we get

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \cdots + \frac{a_{2n}}{x^{2n}}$$

$$\Rightarrow \quad (1 - x + x^2)^n = a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} - \cdots + a_{2n} \quad (2)$$

Clearly,

$a_0^2 - a_1^2 + a_2^2 \cdots + a_{2n}^2$ is the coefficient of x^{2n} in

$$(1 + x + x^2)^n (1 - x + x^2)^n$$

$$\Rightarrow \quad a_0^2 - a_1^2 + a_2^2 - \cdots + a_{2n}^2$$

$= $ Coefficient of x^{2n} in $(1 + x^2 + x^4)^n$

In $(1 + x^2 + x^4)^n$, replace x^2 by y, then the coefficient of y^n in $(1 + y + y^2)^n$ is a_n. Hence,

$$a_0{}^2 - a_1{}^2 + a_2{}^2 - \cdots + a_{2n}{}^2 = a_n$$

10. $\displaystyle\sum_{r=0}^{n} (-1)^r \left(\frac{{}^nC_r}{{}^{r+3}C_3}\right)$

$$= \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!r!} \frac{3!r!}{(r+3)!}$$

$$= 3!\sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r+3)!}$$

$$= \frac{3!}{(n+1)(n+2)(n+3)} \sum_{r=0}^{n} (-1)^r\, {}^{n+3}C_{r+3}$$

$$= -\frac{3!}{(n+1)(n+2)(n+3)} \sum_{r=0}^{n} (-1)^{r+3}\, {}^{n+3}C_{r+3}$$

$$= -\frac{3!}{(n+1)(n+2)(n+3)}\left[-{}^{n+3}C_3 + {}^{n+3}C_4 - \cdots (-1)^{n+3}\, {}^{n+3}C_{n+3}\right]$$

$$= -\frac{3!}{(n+1)(n+2)(n+3)}\left[({}^{n+3}C_0 - {}^{n+3}C_1 + {}^{n+3}C_2 - {}^{n+3}C_3 + \cdots + (-1)^{n+3}\, {}^{n+3}C_{n+3}) - ({}^{n+3}C_0 - {}^{n+3}C_1 + {}^{n+3}C_2)\right]$$

$$= -\frac{3!}{(n+1)(n+2)(n+3)}\left[(1-1)^{n+3} - (1 - (n+3) - \frac{(n+3)(n+2)}{2}\right]$$

$$= \frac{3!}{(n+1)(n+2)(n+3)}\left[(1 - n - 3 + \frac{(n+3)(n+2)}{2}\right]$$

$$= \frac{3!}{(n+1)(n+2)(n+3)} \frac{(n^2 + 3n + 2)}{2} = \frac{3!}{2(n+3)}$$

11. We know that the coefficient of x^r in the binomial expansion of $(1 + x)^n$ is ${}^nC_r$.

$$\therefore \quad {}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \cdots + {}^mC_m$$

$= $ Coefficient of x^m in the expansion of $[(1 + x)^n + (1 + x)^{n+1} + (1 + x)^{n-2} + \cdots + (1 + x)^m]$

$= $ Coefficient of x^m in $[(1 + x)^m + (1 + x)^{m-1} + (1 + x)^{m-2} + \cdots + (1 + x)^n]$ (writing in reserve order)

$= $ Coefficient of x^m $\left[(1+x)^m \dfrac{\{(1+x)^{n-m+1} - 1\}}{1+x-1}\right]$ [sum of G.P.]

$= $ Coefficient of x^m in $\dfrac{\left[(1+x)^{n+1} - (1+x)^m\right]}{x}$

$= $ Coefficient of x^{m+1} in $[(1 + x)^{n+1} - (1 + x)^m]$

$= {}^{n+1}C_{m+1} - 0$

$= {}^{n+1}C_{m+1}$

Now, we have to prove

$${}^nC_m + 2\,{}^{n-1}C_m + 3\,{}^{n-2}C_m + \cdots + (n - m + 1)\,{}^mC_m = {}^{n+2}C_{m+2}$$

Let us consider

$$S = (1 + x)^n + 2(1 + x)^{n-1} + 3(1 + x)^{n-2} + \cdots + (n - m + 1) \times (1 + x)^m \quad (1)$$

$$(1 + x)\, S = (1 + x)^{n+1} + 2(1 + x)^n + 3(1 + x)^{n-1} + \cdots + (n - m + 1)(1 + x)^{m+1} \quad (2)$$

Subtracting (1) from (2), we get

$$x\,S = (1 + x)^{n+1} + (1 + x)^n + (1 + x)^{n-1} + (1 + x)^{n-2} + \cdots + (1 + x)^m + (n - m + 1)(1 + x)^m$$

$$= \frac{(1+x)^{n+1}\left[1 - \left(\frac{1}{1+x}\right)^{n+1-m}\right]}{1 - \frac{1}{1+x}} + (n - m + 1)(1 + x)^m$$

$$= \frac{(1+x)^{n+1}\left[(1+x)^{n+1-m} - 1\right]}{x(1+x)^{n-m}} + (n - m + 1)(1 + x)^m$$

$$= \frac{(1+x)^{m+1}\left[(1+x)^{n+1-m} - 1\right]}{x} + (n - m + 1)(1 + x)^m$$

$$\Rightarrow \quad S = \frac{(1+x)^{n+2} - (1+x)^{m+1}}{x^2} + \frac{(n-m+1)(1+x)^m}{x}$$

Now,

$${}^nC_m + 2 \times {}^{n-1}C_m + 3 \times {}^{n-2}C_m + \cdots + (n - m + 1)\,{}^mC_m$$

$= $ Coefficient of x^m is S

$= $ Coefficient of x^m in

$$\left[\frac{(1+x)^{n+2} - (1+x)^{m+1}}{x^2} + \frac{(n-m+1)(1+x)^m}{x}\right]$$

$= $ Coefficient of x^{m+2} in $[(1 + x)^{n+2} - (1 + x)^{m+1}]$

$= {}^{n+2}C_{m+2}$

12. $(25)^{n+1} - 24n + 5735$

$= (1 + 24)^{n+1} - 24n + 5735$

$= {}^{n+1}C_0 + {}^{n+1}C_1\, 24 + {}^{n+1}C_2 24^2 + \cdots - 24n + 5735$

$= 1 + 24(n + 1) + {}^{n+1}C_2 24^2 + \cdots + {}^{n+1}C_{n+1} 24^{n+1} - 24n + 5735$

$= 5760 + 24^2({}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1} 24^{n-1})$

$= 24^2[10 + ({}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1} 24^{n-1})]$

which is divisible by 24^2.

13. $S = 2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - \cdots + (-1)^k\binom{n}{k}\binom{n-k}{0}$

$= 2^k\,{}^nC_0\,{}^nC_k - 2^{k-1}\,{}^nC_1\,{}^{n-1}C_{k-1} + \cdots + (-1)^k\,{}^nC_k\,{}^{n-k}C_0$

$= \sum_{r=0}^{k}(-1)^r 2^{k-r}\;{}^nC_r\;{}^{n-r}C_{k-r}$

$= \sum_{r=0}^{k}(-1)^r 2^{k-r}\dfrac{n!}{(n-r)!\,r!}\,\dfrac{(n-r)!}{(k-r)!(n-k)!}$

$= \dfrac{n!}{k!(n-k)!}\sum_{r=0}^{k}(-1)^r 2^{k-r}\dfrac{k!}{r!(k-r)!}$

$= {}^nC_k\sum_{r=0}^{k}(-1)^r 2^{k-r}\;{}^kC_r$

$= {}^nC_k\,[{}^kC_0 2^k - {}^kC_1 2^{k-1} + {}^kC_2 2^{k-2} - \cdots + (-1)^k\,{}^kC_k]$

$= {}^nC_k\,(2-1)^k = {}^nC_k$

Objective Type

Fill in the blanks

1. We have,

$101^{50} = (100+1)^{50} = 100^{50} + 50\times 100^{49} + \dfrac{50\times 49}{2\times 1}100^{48} + \cdots$ (1)

$99^{50} = (100-1)^{50} = 100^{50} - 50\times 100^{49} + \dfrac{50\times 49}{2\times 1}100^{48} - \cdots$ (2)

Subtracting (2) from (1), we get

$101^{50} - 99^{50} = 100^{50} + 2\,\dfrac{50\times 49\times 48}{1\times 2\times 3}100^{47} + \cdots > 100^{50}$

Hence, $101^{50} > 100^{50} + 99^{50}$.

2. If we put $x = 1$ in the expansion of $(1 + x - 3x^2)^{2163} = A_0 + A_1x + A_2x^2 + \cdots$ we will get the sum of coefficients of the given polynomial, which is equal to -1.

3. $(1+ax)^n = 1 + 8x + 24x^2 + \cdots$

$\Rightarrow\; 1 + nxa + \dfrac{n(n-1)}{2!}a^2x^2 + \cdots = 1 + 8x + 24x^2 + \cdots$

Comparing like powers of x, we get

$nax = 8x \Rightarrow na = 8$ (1)

$\dfrac{n(n-1)a^2}{2} = 24 \Rightarrow n(n-1)a^2 = 48$ (2)

Solving (1) and (2), $n = 4$, $a = 2$.

4. Let T_{r+1} be the general term in the expansion of $\left(\sqrt{2} + 3^{1/5}\right)^{10}$

$\therefore\; T_{r+1} = {}^{10}C_r\,(\sqrt{2})^{10-r}(3^{1/5})^r\;\; (0 \le r \le 10)$

$= \dfrac{10!}{r!(10-r)!}\,2^{5-r/2}\,3^{r/5}$

T_{r+1} will be rational if $2^{5-r/2}$ and $3^{r/5}$ are rational numbers. Hence, $5 - r/2$ and $r/5$ are integers. So, $r = 0$ and $r = 10$. Therefore, T_1 and T_{11} are rational terms. Now, sum of T_1 and T_{11} is

${}^{10}C_0\,2^{5-0}\times 3^0 + {}^{10}C_{10}\,2^{5-5}\times 3^2 = 32 + 9 = 41.$

Multiple choice questions with one correct answer

1. a. Given that r and n are +ve integers such that $r > 1$, $n > 2$.
Also, in the expansion of $(1+x)^{2n}$,
Coefficient of $3r^{\text{th}}$ term = coefficient of $(r+2)^{\text{th}}$ term

$\Rightarrow\; {}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$\Rightarrow\; 3r - 1 = r + 1$ or $3r - 1 + r + 1 = 2n$

[using ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$]

$\Rightarrow\; r = 1$ or $2r = n$.

But $r > 1$

$\therefore\; n = 2r$

2. a. $\left(\dfrac{x}{2} - \dfrac{3}{x^2}\right)^{10}$

General term in this expansion is

$T_{r+1} = {}^{10}C_r\left(\dfrac{x}{2}\right)^{10-r}\left(\dfrac{-3}{x^2}\right)^r = {}^{10}C_r\,x^{10-3r}\dfrac{(-1)^r 3^r}{2^{10-r}}$

For coefficient of x^4, we should have $r = 2$.

Therefore, coefficient of x^4 is ${}^{10}C_2\,\dfrac{(-1)^2 3^2}{2^8} = \dfrac{405}{256}$

3. c. Since n is even, let $n = 2m$. Then,

L.H.S. $= S = \dfrac{2m!m!}{(2m)!}\left[C_0^2 - 2C_1^2 + 3C_2^2 + \cdots + (-1)^{2m} \times (2m+1)C_{2m}^2\right]$ (1)

$\Rightarrow\; S = \dfrac{2m!m!}{(2m)!}\left[(2m+1)C_0^2 - 2mC_1^2 + (2m-1) \times C_2^2 + \cdots + C_0{}^2\right]$ (2) (Using $C_r = C_{n-r}$)

Adding (1) and (2), we get

$2S = 2\,\dfrac{m!m!}{(2m)!}(2m+2)\left[C_0^2 - C_1^2 + C_2^2 + \cdots + C_{2m}^2\right]$

Now keeping in mind that $C_0^2 - C_1^2 + C_2^2 - \cdots + C_n^2 = (-1)^{n/2}\,{}^nC_{n/2}$ if n is even, we get

$S = 2\,\dfrac{m!\,m!}{(2m)!}(m+1)\left[(-1)^m\,{}^{2m}C_m\right]$

$= 2\left(\dfrac{n}{2} + 1\right)(-1)^{n/2}$

$= (-1)^{n/2}(n+2)$

4. c. Let,

$b = \sum_{r=0}^{n}\dfrac{r}{{}^nC_r}$ (1)

$= \sum_{r=0}^{n}\dfrac{n-r}{{}^nC_{n-r}}$ (we can replace r by $n-r$)

$= \sum_{r=0}^{n}\dfrac{n-r}{{}^nC_r}$ (2)

Adding (1) and (2), we have

$2b = \sum_{r=0}^{n}\dfrac{r}{{}^nC_r} + \sum_{r=0}^{n}\dfrac{n-r}{{}^nC_r}$

$$= n\sum_{r=0}^{n}\frac{1}{^nC_r}$$

$$= na_n$$

$$\Rightarrow \quad b = \frac{n}{2}a_n$$

5. c. The given expression is $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$.

We know that

$$(x+a)^n+(x-a)^n = 2\,[^nC_0x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4+\cdots]$$

Therefore the given expression is equal to $2[^5C_0x^5 + {}^5C_2x^3(x^3-1) + {}^5C_4x(x^3-1)^2]$.

Maximum power of x involved here is 7, also only +ve integral powers of x are involved, therefore the given expression is a polynomial of degree 7.

6. d. $\binom{n}{r}+2\binom{n}{r-1}+\binom{n}{r-2}$

$$= \left[\binom{n}{r}+\binom{n}{r-1}\right]+\left[\binom{n}{r-1}+\binom{n}{r-2}\right]$$

$$= \binom{n+1}{r}+\binom{n+1}{r-1} = \binom{n+2}{r} \qquad [\because\ ^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$$

7. b. $(a-b)^n, n \ge 5$

In the binomial expansion,

$$T_5 + T_6 = 0$$

$$\Rightarrow \quad ^nC_4\,a^{n-4}\,b^4 - {}^nC_5\,a^{n-5}\,b^5 = 0$$

$$\Rightarrow \quad \frac{^nC_4}{^nC_5}\frac{a}{b} = 1 \Rightarrow \frac{4+1}{n-4}\frac{a}{b} = 1 \quad \left[\text{Using } \frac{^nC_r}{^nC_{r+1}} = \frac{r+1}{n-r}\right]$$

$$\Rightarrow \quad \frac{a}{b} = \frac{n-4}{5}$$

8. c. $\sum_{i=0}^{m} {}^{10}C_i\,{}^{20}C_{m-i} = {}^{10}C_0\,{}^{20}C_m + {}^{10}C_1\,{}^{20}C_{m-1} + {}^{10}C_2\,{}^{20}C_{m-2} + \cdots + {}^{10}C_m\,{}^{20}C_0$

= Coefficient of x^m in the expansion of product $(1+x)^{10}(x+1)^{20}$

= Coefficient of x^m in the expansion of $(1+x)^{30}$

$= {}^{30}C_m$

Hence, the maximum value ${}^{30}C_m$ is ${}^{30}C_{15}$.

9. d. $(1+t^2)^{12}(1+t^{12})(1+t^{24})$

$= (1+t^{12}+t^{24}+t^{36})(1+t^2)^{12}$

$\therefore$ Coefficient of t^{24}

$= 1 \times$ coefficient of t^{24} in $(1+t^2)^{12} + 1 \times$ coefficient of t^{12} in $(1+t^2)^{12} + 1 \times$ constant term in $(1+t^2)^{12}$

$= {}^{12}C_{12} + {}^{12}C_6 + {}^{12}C_0 = 1 + {}^{12}C_6 + 1 = {}^{12}C_6 + 2$

10. d. ${}^{n-1}C_r = {}^nC_{r+1}\,(k^2-3)$

$$\Rightarrow \quad k^2 - 3 = \frac{^{n-1}C_r}{^nC_{r+1}} = \frac{r+1}{n}$$

Now,

$$0 \le r \le n-1$$

$$\Rightarrow \quad 1 \le r+1 \le n$$

$$\Rightarrow \quad \frac{1}{n} \le \frac{r+1}{n} \le 1$$

$$\Rightarrow \quad \frac{1}{n} \le k^2 - 3 \le 1$$

$$\Rightarrow \quad 3 + \frac{1}{n} \le k^2 \le 4 \Rightarrow \sqrt{3+\frac{1}{n}} \le k \le 2$$

When $n \to \infty$, we have

$$\sqrt{3} < k \le 2$$

$$\Rightarrow \quad k \in (\sqrt{3}, 2]$$

11. a. Given series is

$${}^{30}C_0\,{}^{30}C_{10} - {}^{30}C_1\,{}^{30}C_{11} + {}^{30}C_2\,{}^{30}C_{12} - \cdots + {}^{30}C_{20}\,{}^{30}C_{30}$$

which is

$${}^{30}C_0\,{}^{30}C_{20} - {}^{30}C_1\,{}^{30}C_{19} + {}^{30}C_2\,{}^{30}C_{18} - \cdots + {}^{30}C_{20}\,{}^{30}C_0$$

= Coefficient of x^{20} in the expansion of $(x+1)^n(1-x)^n$

= Coefficient of x^{20} in the expansion of $(1-x^2)^n$

$= {}^{30}C_{10}$

CHAPTER 7

Determinants

INTRODUCTION

A system of equations can be expressed in the form of matrices. This means, a system of linear equations like

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

can be represented as

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Now whether this system of equations has a unique solution or not, is determined by the number $a_1b_2 - a_2b_1$. The number $a_1b_2 - a_2b_1$ which determines the uniqueness of solution is associated with the matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and is called determinant of A or det(A). Determinants have wide applications in engineering, science, economics, social science, etc.

Definition

Let a, b, c, d be any four numbers, real or complex. Then the symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ denotes $ad - bc$ and is called a determinant of second order; a, b, c, d are called elements of the determinant and $ad - bc$ is called its value. As shown above, the elements of a determinant are arranged in the form of a square in its designation. The diagonal on which the elements a and d are situated is called the principal diagonal and the diagonal on which the elements c and b are situated is called the secondary diagonal. The elements which lie in the same horizontal line constitute one row and the elements which lie in the same vertical line constitute one column.

Let a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 be any nine numbers. Then the symbol $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is another way of denoting

$$a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

i.e., $a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$.

Here we see that +, – and + signs occur before a_1, a_2 and a_3, respectively.

Minors and Cofactors

Let us consider a determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (1)$$

In the above determinant, if we leave the row and the column passing through the element a_{ij}, then the second order determinant thus obtained is called the minor of a_{ij}, and is denoted by M_{ij}. Thus we can get 9 minors corresponding to the 9 elements.

For example, in determinant (1) the minor of the element a_{21} is

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

The minor of the element a_{32} is $M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

In terms of the notation of minors if we expand the determinant along the first row, then

$$\Delta = (-1)^{1+1}a_{11}M_{11} + (-1)^{1+2}a_{12}M_{12} + (-1)^{1+3}a_{13}M_{13}$$
$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

Similarly expanding Δ along the second column, we have

$$\Delta = -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$$

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of the element a_{ij}.

If we denote the cofactor of the element a_{ij}, by C_{ij}, then cofactor of a_{ij} is $C_{ij} = (-1)^{i+j}M_{ij}$.

Cofactor of the element a_{21} is

$$C_{21} = (-1)^{2+1}M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

In terms of the notation of the cofactors,

$$\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$
$$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

Also, $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$, $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$, etc. Therefore, in a determinant the sum of the products of the elements of any row or column with the corresponding cofactors is equal to value of the determinant. Also the sum of the products of the elements of any row or column with the cofactors of the corresponding elements of any other row or column is zero.

Value of n-order determinant,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

(when expanded along first row)

Note: *For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeros.*

Sarrus Rule for Expansion

Sarrus gave a rule for a determinant of order 3.

Rule: The three diagonals sloping down to the right give the three positive terms and the three diagonals sloping down to the left the three negative terms.

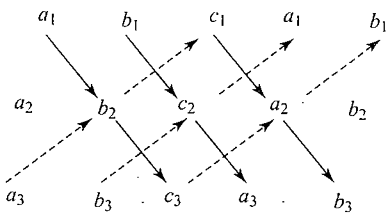


Fig. 7.1

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

Example 7.1 **A determinant of second order is made with the elements 0 and 1. Find the number of determinants with non-negative values.**

Sol. There are only three determinants of second order with negative values, viz.

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

The Number of possible determinants with elements 0 and 1 is $2^4 = 16$. Therefore number of determinants with non-negative values is 13.

Example 7.2 **Find the value of** $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$

Sol. Here in the third column, two entries are zero. So expanding along third column (C_3), we get

$$\Delta = 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$$= 4(-1 - 12) - 0 + 0$$

$$= -52$$

Example 7.3 **Find the largest value of a third-order determinant whose elements are 0 or 1.**

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant of order 3. Then,

$$\Delta = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$
$$= (a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1) - (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1)$$

Since each element of Δ is either 1 or 0, therefore the value of the determinant cannot exceed 3.

Clearly, the value of Δ is maximum when the value of each term in first bracket is 1 and the value of each term in the second bracket is zero. But $a_1b_2c_3 = a_3b_1c_1 = 1$ implies that every element of the determinant Δ is 1 and in that case $\Delta = 0$. Thus, we may have

$$\Delta = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$$

Example 7.4 **If $a, b, c \in R$, then find the number of real roots of the equation $\Delta = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix} = 0$.**

Sol. From the symmetry of the determinant, it is simple to expand by Sarrus rule.

$$\Delta = x^3 + abc - abc + (b^2x + a^2x + c^2x) = 0$$

$\Rightarrow$ $x^3 + x(a^2 + b^2 + c^2) = 0$

$\Rightarrow$ $x = 0$ or $x^2 = -(a^2 + b^2 + c^2)$

$\Rightarrow$ $x = 0$ or $x = \pm i\sqrt{a^2 + b^2 + c^2}$

Example 7.5 **If $x + y + z = 0$, prove that**

$$\begin{vmatrix} ax & by & cz \\ cy & az & bx \\ bz & cx & ay \end{vmatrix} = xyz\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Sol. Since determinant is symmetrical it is simple to expand by Sarrus rule.

$$\begin{vmatrix} ax & by & cz \\ cy & az & bx \\ bz & cx & ay \end{vmatrix} = xyz(a^3 + b^3 + c^3) - abc(x^3 + y^3 + z^3)$$

$$= xyz(a^3 + b^3 + c^3 - 3abc) - abc(x^3 + y^3 + z^3 - 3xyz)$$

$$= xyz(a^3 + b^3 + c^3 - 3abc) - abc(x + y + z) \times (x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= xyz(a^3 + b^3 + c^3 - 3abc)$$

$$= xyz\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Concept Application Exercise 7.1

1. If A, B and C are the angles of non-right angled triangle ABC, then find the value of

$$\begin{vmatrix} \tan A & 1 & 1 \\ 1 & \tan B & 1 \\ 1 & 1 & \tan C \end{vmatrix}$$

2. If $e^{i\theta} = \cos\theta + i\sin\theta$, find the value of

$$\begin{vmatrix} 1 & e^{i\pi/3} & e^{i\pi/4} \\ e^{-i\pi/3} & 1 & e^{i2\pi/3} \\ e^{-i\pi/4} & e^{-i2\pi/3} & 1 \end{vmatrix}$$

3. Find the number of real roots of the equation

$$\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0, \; a \ne b \ne c \text{ and } b(a+c) > ac$$

4. If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$ and $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = 0$, $\alpha \ne \beta \ne \gamma$, then find the equation whose roots are $\alpha + \beta - \gamma$, $\beta + \gamma - \alpha$ and $\gamma + \alpha - \beta$.

Some Operations

First, second and third rows of a determinant are denoted by R_1, R_2 and R_3, respectively and the first, second and third columns by C_1, C_2 and C_3, respectively.

(i) The interchange of its i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$.

(ii) The interchange of i^{th} column and j^{th} column is denoted by $C_i \leftrightarrow C_j$.

(iii) The addition of m-times the elements of j^{th} row of the corresponding elements of i^{th} row is denoted by $R_i \to R_i + mR_j$.

(iv) The addition of m-times the elements of j^{th} column to the corresponding elements of i^{th} column is denoted by $C_i \to C_i + mC_j$.

(v) The addition of m-times the elements of j^{th} row to n-times the elements of i^{th} row is denoted by $R_i \to nR_i + mR_j$.

PROPERTIES OF DETERMINANTS

Property I. The value of the determinant is not changed when rows are changed into corresponding columns.

Naturally when rows are changed into corresponding columns, then columns will change into corresponding rows.

Proof: Let, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding the determinant along the first row,

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \quad (1)$$

If Δ' be the value of the determinant when rows of determinant Δ are changed into corresponding columns, then

$$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1(b_2c_3 - b_3c_2) - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \quad (2)$$

From Eqs. (1) and (2), $\Delta' = \Delta$

Property II. If any two rows or columns of a determinant are interchanged, the sign of the value of the determinant is changed.

Proof: Let, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding the determinant along the first row,

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \quad (1)$$

Now,

$$\Delta' = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} \quad [R_1 \leftrightarrow R_3]$$

$$= a_3(b_2c_1 - b_1c_2) - b_3(a_2c_1 - a_1c_2) + c_3(a_2b_1 - a_1b_2)$$

$$= a_3b_2c_1 - a_3b_1c_2 - b_3a_2c_1 + a_1b_3c_2 + c_3a_2b_1 - a_1b_2c_3$$

$$= -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2) \quad (2)$$

From Eqs. (1) and (2), $\Delta' = -\Delta$

Property III. The value of a determinant is zero if any two rows of columns are identical.

Proof: Let,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

Then,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = -\Delta \quad [\text{by } R_1 \leftrightarrow R_3]$$

Thus,

$$\Delta = -\Delta$$

$$\Rightarrow 2\Delta = 0$$

$$\Rightarrow \Delta = 0$$

Property IV. A common factor of all elements of any row (or of any column) may be taken outside the sign of the determinant. In other words, if all the elements of the same row (or the same column) are multiplied by a certain number, then the determinant becomes multiplied by that number.

Proof: Let, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding the determinant along the first row, we get

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \quad (1)$$

and

$$\Delta' = \begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= ma_1(b_2c_3 - b_3c_2) - mb_1(a_2c_3 - a_3c_2) + mc_1(a_2b_3 - a_3b_2)$$

$$= m\Delta \quad [\text{from (1)}]$$

Thus,

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example: $\begin{vmatrix} 32 & 24 & 16 \\ 8 & 3 & 5 \\ 4 & 5 & 3 \end{vmatrix} = 8 \times \begin{vmatrix} 4 & 3 & 2 \\ 8 & 3 & 5 \\ 4 & 5 & 3 \end{vmatrix}$

[taking 8 common from first row]

$$= 8 \times 4 \begin{vmatrix} 1 & 3 & 2 \\ 2 & 3 & 5 \\ 1 & 5 & 3 \end{vmatrix}$$

[taking 4 common from the first column]

Property V. If every element of some column or (row) is the sum of two terms, then the determinant is equal to the sum of two determinants; one containing only the first term in place of each sum, the other only the second term. The remaining elements of both determinants are the same as in the given determinant.

Proof: We have to prove that

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Let,

$$\Delta = \begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix}$$

Then,

$$\Delta = (a_1+\alpha_1)\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - (a_2+\alpha_2)\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (a_3+\alpha_3)\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + \alpha_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$- \alpha_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + \alpha_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Note:

$$\begin{vmatrix} a_1+b_1 & c_1+d_1 & e_1 \\ a_2+b_2 & c_2+d_2 & e_2 \\ a_3+b_3 & c_3+d_3 & e_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & c_1 & e_1 \\ a_2 & c_2 & e_2 \\ a_3 & c_3 & e_3 \end{vmatrix} + \begin{vmatrix} a_1 & d_1 & e_1 \\ a_2 & d_2 & e_2 \\ a_3 & d_3 & e_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & e_1 \\ b_2 & c_2 & e_2 \\ b_3 & c_3 & e_3 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 & e_1 \\ b_2 & d_2 & e_2 \\ b_3 & d_3 & e_3 \end{vmatrix}$$

Property VI. The value of a determinant does not change when any row or column is multiplied by a number or an expression and is then added to or subtracted from any other row or column.

Here it should be noted that if the row or column which is changed is multiplied by a number, then the determinant will have to be divided by that number.

Proof: To prove $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+mb_1 & b_1 & c_1 \\ a_2+mb_2 & b_2 & c_2 \\ a_3+mb_3 & b_3 & c_3 \end{vmatrix}$

Let, $\Delta = \begin{vmatrix} a_1+mb_1 & b_1 & c_1 \\ a_2+mb_2 & b_2 & c_2 \\ a_3+mb_3 & b_3 & c_3 \end{vmatrix}$

Then,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mb_1 & b_1 & c_1 \\ mb_2 & b_2 & c_2 \\ mb_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \left[\because \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = 0\right]$$

Example: Let, $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{vmatrix} = -7$

$$\Delta' = \begin{vmatrix} 5 & 2 & 13 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{vmatrix} \; [R_1 \to R_1 + 2R_3]$$

$$= 5(15-0) - 2(10-8) + 13(0-6)$$
$$= 75 - 4 - 78 = -7$$

$$\Delta'' = \frac{1}{3}\begin{vmatrix} 7 & 6 & 19 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{vmatrix}$$

[Here Δ'' has been obtained from Δ by applying $R_1 \to 3R_1 + 2R_3$]

$$= \frac{1}{3}[7(15-0) - 6(10-8) + 19(0-6)] = \frac{1}{3}(-21) = -7$$

In obtaining Δ'' from Δ, R_1 has been changed and it has been multiplied by 3, therefore, the determinant has been divided by 3.

Note:

- *If more than one operation like $R_i \to R_i + kR_j$ is done in one step, care should be taken to see that a row that is affected in one operation should not be used in another operation. A similar remark applies to column operations.*
- *Many times we use this operation to get as many zeros as we can.*

Property VII. If $\Delta_r = \begin{vmatrix} f_1(r) & f_2(r) & f_3(r) \\ a & b & c \\ d & e & f \end{vmatrix}$ where $f_1(r), f_2(r), f_3(r)$ are functions of r and a, b, c, d, e, f are constants. Then,

$$\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} f_1(r) & \sum_{r=1}^{n} f_2(r) & \sum_{r=1}^{n} f_3(r) \\ a & b & c \\ d & e & f \end{vmatrix}$$

Also for $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ a & b & c \\ d & e & f \end{vmatrix}$ where $f_1(x), f_2(x), f_3(x)$ are functions of x and a, b, c, d, e, f are constants, we have

$$\int_p^q \Delta(x)dx = \begin{vmatrix} \int_p^q f_1(x)dx & \int_p^q f_2(x)dx & \int_p^q f_3(x)dx \\ a & b & c \\ d & e & f \end{vmatrix}$$

Note: *All the above properties are applicable for n-order determinants also.*

SOME IMPORTANT DETERMINANTS

1. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Proof:

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_2$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ x & (y-x) & (z-y) \\ x^3 & (y-x)(y+x) & (z-y)(z+y) \end{vmatrix}$$

$$= (y-x)(z-y) \times \begin{vmatrix} 1 & 1 \\ y+x & z+y \end{vmatrix}$$

(Expanding along R_1)

$$= (x-y)(y-z)(z-x)$$

2. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

3. $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

4. Circulant: Let a, b, c be positive and not all equal. Then the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.

Proof: $\Delta = a[bc - a^2] - b[b^2 - ac] + c[ab - c^2]$

$= -[a^3 + b^3 + c^3 - 3abc]$

$= -(a+b+c)[a^2 + b^2 + c^2 - ab - bc - ca]$

$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] < 0$

As $a + b + c > 0$, a, b, c are all positive and not all equal.

Example 7.6 **Without expanding at any stage, prove that the value of each of the following determinants is zero.**

a. $\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix}$ **b.** $\begin{vmatrix} 41 & 1 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix}$ **c.** $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

where w is cube root of unity

Sol.

a. Let, $\Delta = \begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & q-p & r-p \\ p-q & 0 & r-q \\ p-r & q-r & 0 \end{vmatrix}$ [Taking transpose]

$= (-1)^3 \begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix}$

[Taking (–1) common from each row]

$\therefore \Delta = -\Delta$ or $2\Delta = 0$ or $\Delta = 0$

b. $\Delta = \begin{vmatrix} 41 & 1 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix}$

Applying $C_1 \to C_1 + (-8)C_3$, we get

$\Delta = \begin{vmatrix} 1 & 1 & 5 \\ 7 & 7 & 9 \\ 5 & 5 & 3 \end{vmatrix} = 0$ [$\because$ C_1 and C_2 are identical]

c. Let, $\Delta = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 1+w+w^2 & w & w^2 \\ w+w^2+1 & w^2 & 1 \\ w^2+1+w & 1 & w \end{vmatrix}$$

$= \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix}$ [$\because 1 + w + w^2 = 0$]

$= 0$ [$\because$ C_1 consists of all zeros]

Example 7.7 **If $\Delta = \begin{vmatrix} abc & b^2c & c^2b \\ abc & c^2a & ca^2 \\ abc & a^2b & b^2a \end{vmatrix} = 0$, ($a, b, c \in R$ and are all different and non-zero) then prove that $a + b + c = 0$.**

Sol.

$$\Delta = 0 \Longrightarrow bc.ca.ab\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow \quad -a^2b^2c^2(a^3+b^3+c^3-3abc) = 0$$

(expanding by Sarrus Rule)

$$\Rightarrow \quad a^2b^2c^2(a+b+c)\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a^2)] = 0$$

$$\Rightarrow \quad a+b+c = 0$$

Example 7.8 **Prove that $a \neq 0$, $\begin{vmatrix} x+1 & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} = 0$ represents a straight line parallel to y-axis.**

Sol.

$$\Delta = 0 \Rightarrow \begin{vmatrix} x+1 & -1 & -1 \\ x & a & 0 \\ x & 0 & a^2 \end{vmatrix} = 0 \quad \begin{bmatrix} C_3 \to C_3 - C_1 \\ C_2 \to C_2 - C_1 \end{bmatrix}$$

$$\Rightarrow (a^3 + a^2 + a)x = -a^3$$

$$\Rightarrow x = \frac{-a^3}{a^3+a^2+a}$$ which is a straight line parallel to y-axis

Example 7.9 **Prove that the value of the determinant $\begin{vmatrix} -7 & 5+3i & \frac{2}{3}-4i \\ 5-3i & 8 & 4+5i \\ \frac{2}{3}+4i & 4-5i & 9 \end{vmatrix}$ is real.**

Sol. Let $$z = \begin{vmatrix} -7 & 5+3i & \frac{2}{3}-4i \\ 5-3i & 8 & 4+5i \\ \frac{2}{3}+4i & 4-5i & 9 \end{vmatrix} \quad (1)$$

To prove that this number (z) is real we have to prove that $\bar{z} = z$

Now we know that conjugate of complex number is distributive over all algebraic operations.

Hence to take conjugate of z in (1) we need not to expand determinant.

To get the conjugate of z we can take conjugate of each element of determinant.

$$\Rightarrow \quad \bar{z} = \begin{vmatrix} -7 & 5-3i & \frac{2}{3}+4i \\ 5+3i & 8 & 4-5i \\ \frac{2}{3}-4i & 4+5i & 9 \end{vmatrix} \quad (2)$$

Now interchanging rows into columns (taking transpose) in (2)

we have $$\bar{z} = \begin{vmatrix} -7 & 5+3i & \frac{2}{3}-4i \\ 5-3i & 8 & 4+5i \\ \frac{2}{3}+4i & 4-5i & 9 \end{vmatrix} \quad (3)$$

or $\bar{z} = z$ (4) (from (1) & (3))

$\Rightarrow$ z is purely real

Example 7.10 **If $a_r = (\cos 2r\pi + i\sin 2r\pi)^{\frac{1}{9}}$, then prove that $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = 0.$**

Sol. $a_r = (\cos 2r\pi + i\sin 2r\pi)^{\frac{1}{9}} = e^{i\frac{2r\pi}{9}}$

$$\Rightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$$

$$= \begin{vmatrix} e^{i\frac{2\pi}{9}} & e^{i\frac{4\pi}{9}} & e^{i\frac{6\pi}{9}} \\ e^{i\frac{8\pi}{9}} & e^{i\frac{10\pi}{9}} & e^{i\frac{12\pi}{9}} \\ e^{i\frac{14\pi}{9}} & e^{i\frac{16\pi}{9}} & e^{i\frac{18\pi}{9}} \end{vmatrix}$$

$$= e^{i\frac{6\pi}{9}} \begin{vmatrix} e^{i\frac{2\pi}{9}} & e^{i\frac{4\pi}{9}} & e^{i\frac{6\pi}{9}} \\ e^{i\frac{2\pi}{9}} & e^{i\frac{4\pi}{9}} & e^{i\frac{6\pi}{9}} \\ e^{i\frac{14\pi}{9}} & e^{i\frac{16\pi}{9}} & e^{i\frac{18\pi}{9}} \end{vmatrix}$$ [taking $e^{i\frac{6\pi}{9}}$ common from R_2]

$= 0$ [R_1 and R_2 are identical]

Example 7.11 **Without expanding the determinants, prove that**

$$\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix} + \begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix} = 0$$

Sol. $$D = \underbrace{\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix}}_{D_1} + \underbrace{\begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix}}_{D_2}$$

In D_2, interchanging C_1 and C_3,

$$D = \underbrace{\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix}}_{D_1} - \underbrace{\begin{vmatrix} 104 & 116 & 113 \\ 111 & 106 & 108 \\ 103 & 114 & 115 \end{vmatrix}}_{D_2}$$

In D_2, interchanging C_2 and C_3,

$$D = \underset{D_1}{\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix}} + \underset{D_2}{\begin{vmatrix} 104 & 113 & 116 \\ 111 & 108 & 106 \\ 103 & 115 & 114 \end{vmatrix}}$$

In D_2, interchanging R_1 and R_3,

$$D = \underset{D_1}{\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix}} - \underset{D_2}{\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix}} = 0$$

Example 7.12 **Find the value of the determinant**

$$\begin{vmatrix} \sqrt{(13)}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{(15)}+\sqrt{(26)} & 5 & \sqrt{(10)} \\ 3+\sqrt{(65)} & \sqrt{(15)} & 5 \end{vmatrix}$$

Sol. $\Delta = (\sqrt{5})^2 \begin{vmatrix} \sqrt{(13)}+\sqrt{3} & 2 & 1 \\ \sqrt{(15)}+\sqrt{(26)} & \sqrt{5} & \sqrt{2} \\ 3+\sqrt{65} & \sqrt{3} & \sqrt{5} \end{vmatrix} \; (C_2 \to \frac{1}{\sqrt{5}}C_2, C_3 \to \frac{1}{\sqrt{5}}C_3)$

Now applying $C_1 \to C_1 - \sqrt{3}C_2 - \sqrt{(13)}C_3$, we get

$$\Delta = 5\begin{vmatrix} -\sqrt{3} & 2 & 1 \\ 0 & \sqrt{5} & \sqrt{2} \\ 0 & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$= -5\sqrt{3}\,(5-\sqrt{6})$ (expanding along C_1)

Example 7.13 **Using properties of determinants, evaluate**

$$\begin{vmatrix} 18 & 40 & 89 \\ 40 & 89 & 198 \\ 89 & 198 & 440 \end{vmatrix}$$

Sol. Let $D = \begin{vmatrix} 18 & 40 & 89 \\ 40 & 89 & 198 \\ 89 & 198 & 440 \end{vmatrix}$

Let first reduce the value of elements by performing some operation.

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 5R_1$

$$D = \begin{vmatrix} 18 & 40 & 89 \\ 4 & 9 & 20 \\ -1 & -2 & -5 \end{vmatrix}$$

Applying $C_3 \to C_3 - 2C_2$

$$D = \begin{vmatrix} 18 & 40 & 9 \\ 4 & 9 & 2 \\ -1 & -2 & -1 \end{vmatrix}$$

Now applying $C_1 \to C_1 - 2C_3$ to get zeros in C_1

$$D = \begin{vmatrix} 0 & 40 & 9 \\ 0 & 9 & 2 \\ 1 & -2 & -1 \end{vmatrix} = 1\begin{vmatrix} 40 & 9 \\ 9 & 2 \end{vmatrix} = 80 - 81 = -1$$

Example 7.14 **Solve for x:** $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0.$

Sol. $\Delta = \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$ then we get

or $\Delta = \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} = 0$

or $\Delta = \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 0$

expanding along 1st row

or $(x-2).6 - (2x-3).4 + (3x-4).1 = 0$

or $x = 4$

Example 7.15 **Prove that**

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix} = 0$$

Sol. $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$

$= \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\cos\delta+\sin\delta\cos\alpha \\ \sin\beta & \cos\beta & \sin\beta\cos\delta+\sin\delta\cos\beta \\ \sin\gamma & \cos\gamma & \sin\gamma\cos\delta+\sin\delta\cos\gamma \end{vmatrix}$

$= \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix}$ $[R_3 \to R_3 - \cos\delta R_1 - \sin\delta R_2]$

$= 0$

Example 7.16 **Find the value of the determinant**

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$$

Sol. We have,

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$]

$$=\begin{vmatrix}1&2&3\\2&5&9\\3&9&19\end{vmatrix}=\begin{vmatrix}1&2&3\\0&1&3\\0&3&10\end{vmatrix}$$

[Applying $R_2\to R_2-2R_1$, $R_3\to R_3-3R_1$]

$=(10-9)=1$

Example 7.17 **Prove that** $\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}$ $=(a+b+c)^3$.

Sol. Applying $C_1\to C_1-C_2$ and $C_2\to C_2-C_3$ and taking $(a+b+c)$ common from each of C_1 and C_2, we get

$$D=(a+b+c)^2\times\begin{vmatrix}-1&0&2a\\1&-1&2a\\0&1&c-a-b\end{vmatrix}$$

Now, $R_3\to R_3+R_1+R_2$ gives

$$D=(a+b+c)^2\times\begin{vmatrix}-1&0&2a\\1&-1&2b\\0&0&(a+b+c)\end{vmatrix}$$

Expanding along R_3, we get

$$D=(a+b+c)^2(a+b+c)=(a+b+c)^3$$

Example 7.18 **Prove that**

$$\begin{vmatrix}a&b+c&a^2\\b&c+a&b^2\\c&a+b&c^2\end{vmatrix}=-(a+b+c)\times(a-b)(b-c)(c-a).$$

Sol. $\Delta=\begin{vmatrix}a&b+c&a^2\\b&c+a&b^2\\c&a+b&c^2\end{vmatrix}$

$$=(a+b+c)\times\begin{vmatrix}a&1&a^2\\b&1&b^2\\c&1&c^2\end{vmatrix}$$

[Applying $C_2\to C_2+C_1$ and taking $(a+b+c)$ common from C_2]

$$=-(a+b+c)\times\begin{vmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{vmatrix}$$

$$=-(a+b+c)(a-b)(b-c)(c-a)$$

Example 7.19 **Prove that** $\begin{vmatrix}x^2&x^2-(y-z)^2&yz\\y^2&y^2-(z-x)^2&zx\\z^2&z^2-(x-y)^2&xy\end{vmatrix}$ $=(x-y)(y-z)(z-x)(x+y+z)(x^2+y^2+z^2)$.

Sol. $D=\begin{vmatrix}x^2&x^2-(y-z)^2&yz\\y^2&y^2-(z-x)^2&zx\\z^2&z^2-(x-y)^2&xy\end{vmatrix}$

$$=\begin{vmatrix}x^2&-(x^2+y^2+z^2)&yz\\y^2&-(x^2+y^2+z^2)&zx\\z^2&-(x^2+y^2+z^2)&xy\end{vmatrix}$$

[Operating $C_2\to C_2-2C_1-2C_3$]

$$=-(x^2+y^2+z^2)\begin{vmatrix}x^2&1&yz\\y^2&1&zx\\z^2&1&xy\end{vmatrix}$$

$$=-\frac{(x^2+y^2+z^2)}{xyz}\begin{vmatrix}x^3&x&xyz\\y^3&y&xyz\\z^3&x&xyz\end{vmatrix}$$

(Multiplying R_1, R_2, R_3 by x, y, z, respectively)

$$=-(x^2+y^2+z^2)\begin{vmatrix}x^3&x&1\\y^3&y&1\\z^3&x&1\end{vmatrix}$$

$$=(x^2+y^2+z^2)\begin{vmatrix}1&x&x^3\\1&y&y^3\\1&x&z^3\end{vmatrix}$$

$$=(x-y)(y-z)(z-x)(x+y+z)(x^2+y^2+z^2)$$

Example 7.20 **If a, b, c are all different and**

$\begin{vmatrix}a&a^3&a^4-1\\b&b^3&b^4-1\\c&c^3&c^4-1\end{vmatrix}=0$, **show that** $abc(ab+bc+ca)=a+b+c$.

Sol. Expressing the given determinant as sum of two determinants, we get

$$\begin{vmatrix}a&a^3&a^4\\b&b^3&b^4\\c&c^3&c^4\end{vmatrix}-\begin{vmatrix}a&a^3&1\\b&b^3&1\\c&c^3&1\end{vmatrix}=0$$

$$\text{or } abc\begin{vmatrix}1&a^2&a^3\\0&b^2-a^2&b^3-a^3\\0&c^2-a^2&c^3-a^3\end{vmatrix}=\begin{vmatrix}a&a^3&1\\b-a&b^3-a^3&0\\c-a&c^3-a^3&0\end{vmatrix}$$

[We take a, b, c common from the first determinant and apply $R_2\to R_2-R_1$, $R_3\to R_3-R_1$ in both determinants.]

As a, b, c are all distinct, canceling out $b-a$ and $c-a$, we get

$$abc\begin{vmatrix}b+a&b^2+a^2+ab\\c+a&c^2+a^2+ac\end{vmatrix}=\begin{vmatrix}1&b^2+a^2+ab\\1&c^2+a^2+ac\end{vmatrix}$$

Applying $R_2\to R_2-R_1$ and then cancelling $c-b$ on both sides, we get

$$abc\begin{vmatrix} b+a & b^2+a^2+ab \\ 1 & a+b+c \end{vmatrix} = \begin{vmatrix} 1 & b^2+a^2+ab \\ 0 & a+b+c \end{vmatrix}$$

$\therefore$ $abc(ab + b^2 + bc + a^2 + ab + ac - b^2 - a^2 - ab) = a + b + c$

or $abc(ab + bc + ca) = a + b + c$

Hence the result.

Example 7.21 **If $x_i = a_i b_i c_i$, $i = 1, 2, 3$ are three-digit positive integers such that each x_i is a multiple of 19, then for some integer n, prove that $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is divisible by 19.**

Sol. $$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ (100a_1+10b_1+c_1) & (100a_2+10b_2+c_2) & (100a_3+10b_3+c_3) \end{vmatrix}$$

$[R_3 \to R_3 + 100R_1 + 10R_2]$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 19m_1 & 19m_2 & 19m_3 \end{vmatrix} \quad [\text{where each } m \in N]$$

$$= 19\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 19n$$

where $n = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ m_1 & m_2 & m_3 \end{vmatrix}$ is certainly an integer.

Example 7.22 **Prove that** $$\begin{vmatrix} (b+c)^2 & ba & ac \\ ba & (c+a)^2 & cb \\ ca & cb & (a+b)^2 \end{vmatrix} = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

Sol. Multiplying R_1, R_2, R_3 by a, b, c, respectively, and dividing by abc, we get

$$\Delta = \frac{1}{abc} \times \begin{vmatrix} a(b+c)^2 & ba^2 & ca^2 \\ ab^2 & b(c+a)^2 & cb^2 \\ ac^2 & bc^2 & c(a+b)^2 \end{vmatrix}$$

Taking a, b, c common from C_1, C_2 and C_3, respectively, we get

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Now, applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ gives

$$\Delta = \begin{vmatrix} (b+c)^2 & (a+b+c)(a-b-c) & (a-b-c)(a+b+c) \\ b^2 & (c+a+b)(c+a-b) & 0 \\ c^2 & 0 & (a+b+c)(a+b-c) \end{vmatrix}$$

$$= (a+b+c)^2 \times \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Applying $R_1 \to R_1 - (R_2 + R_3)$ and then taking 2 common from R_1, we get

$$\Delta = 2(a+b+c)^2 \times \begin{vmatrix} bc & -c & -b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Now, applying $C_2 \to bC_2 + C_1$, $C_3 \to C_3 + C_1$ gives

$$\Delta = \frac{2(a+b+c)^2}{bc} \times \begin{vmatrix} bc & 0 & 0 \\ b^2 & b(c+a) & b^2 \\ c^2 & c^2 & c(a+b) \end{vmatrix}$$

$$= \frac{2(a+b+c)^2}{bc} bc[(bc+ba)(ca+cb) - b^2c^2]$$

$$= 2(a+b+c)^2[bc(ac+bc+ab+a^2-bc)]$$

$$= 2abc(a+b+c)^3$$

Example 7.23 **Show that** $$\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2).$$

Sol. Let, $$\Delta = \begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{1}{a}\begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & b+a & c \end{vmatrix} \quad [\text{Multiplying first column by } a]$$

$$= \frac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & b+a & c \end{vmatrix}$$

$[\text{Applying } C_1 \to C_1 + bC_2 + cC_3]$

$$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix}$$

$[\text{Taking } a^2+b^2+c^2 \text{ common from } C_1]$

$$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix}$$

$[\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

$$= \frac{1}{a}(a^2+b^2+c^2) \times 1 \times \begin{vmatrix} c & -a-b \\ a+c & -b \end{vmatrix}$$

$[\text{Expanding along } C_1]$

$$= \frac{1}{a}(a^2+b^2+c^2)(-bc+a^2+ac+ba+bc)$$

$$= (a^2+b^2+c^2)(a+b+c)$$

Example 7.24 **Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Show that the equation**

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$$

represents a straight line.

Sol. Given,

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} (a^2+b^2+c^2)x & bx+ay & cx+a \\ (a^2+b^2+c^2)y & -ax+by-c & cy+b \\ (a^2+b^2+c^2) & cy+b & -ax-by+c \end{vmatrix} = 0$$

[Applying $C_1 \to aC_1 + bC_2 + cC_3$]

$$\Rightarrow \begin{vmatrix} x & bx+ay & cx+a \\ y & -ax+by-c & cy+b \\ 1 & cy+b & -ax-by+c \end{vmatrix} = 0 \quad [\because a^2+b^2+c^2 = 1]$$

$$\Rightarrow \begin{vmatrix} x & ay & a \\ y & -ax-c & b \\ 1 & cy & -ax-by \end{vmatrix} = 0$$

[Applying $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$]

$$\Rightarrow \begin{vmatrix} x & ay & a \\ y & -ax-c & b \\ x^2+y^2+1 & 0 & 0 \end{vmatrix} = 0$$

[Applying $R_3 \to R_3 + xR_1 + yR_2$, we get]

$\Rightarrow (x^2 + y^2 + 1)(aby + a^2x + ac) = 0 \Rightarrow ax + by + c = 0$

Example 7.25 **Prove that** $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$ **$= 1 + a^2 + b^2 + c^2$.**

Sol. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & c^2 \\ a^2 & b^2 & c(c^2+1) \end{vmatrix}$$

[Multiplying C_1, C_2, C_3 by a, b, c, respectively]

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

[Taking common a, b, c from R_1 R_2, R_3, respectively]

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= (1 + a^2 + b^2 + c^2)$

Example 7.26 **If $a^2 + b^2 + c^2 = 1$, then prove that**

$$\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & ab(1-\cos\phi) & ac(1-\cos\phi) \\ ba(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & bc(1-\cos\phi) \\ ca(1-\cos\phi) & cb(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

is independent of a, b, c.

Sol. Multiplying C_1 by a, C_2 by b and C_3 by c, we have

$$\Delta = \frac{1}{abc}\begin{vmatrix} a^3+a(b^2+c^2)\cos\phi & ab^2(1-\cos\phi) & ab^2(1-\cos\phi) \\ ba^2(1-\cos\phi) & b^3+b(c^2+a^2)\cos\phi & bc^2(1-\cos\phi) \\ ca^2(1-\cos\phi) & cb^2(1-\cos\phi) & c^3+c(a^2+b^2)\cos\phi \end{vmatrix}$$

Now take a, b and c common from R_1, R_2 and R_3, respectively, to give

$$\Delta = \frac{abc}{abc}\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ a^2(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ a^2(1-\cos\phi) & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 1 & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ 1 & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ 1 & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

$[\because a^2 + b^2 + c^2 = 1]$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$$\Delta=\begin{vmatrix}1 & b^2(1-\cos\phi) & c^2(1-\cos\phi)\\ 0 & (a^2+b^2+c^2)\cos\phi & 0\\ 0 & 0 & (a^2+b^2+c^2)\cos\phi\end{vmatrix}$$

Expanding along C_1, we get $\begin{vmatrix}\cos\phi & 0\\ 0 & \cos\phi\end{vmatrix}=\cos^2\phi$.

Example 7.27 **Let** $\Delta_r=\begin{vmatrix}r-1 & n & 6\\ (r-1)^2 & 2n^2 & 4n-2\\ (r-1)^2 & 3n^3 & 3n^2-3n\end{vmatrix}$**. Show that** $\sum\limits_{r=1}^{n}\Delta_r$ **is constant.**

Sol. Since C_1 has variable terms and C_2 and C_3 are constant, summation runs on C_1.

$$\therefore\quad \sum_{r=1}^{n}\Delta_r=\begin{vmatrix}\sum\limits_1^n(r-1) & n & 6\\ \sum\limits_1^n(r-1)^2 & 2n^2 & 4n-2\\ \sum\limits_1^n(r-1)^3 & 3n^2 & 3n^2-3n\end{vmatrix}$$

$$=\begin{vmatrix}\frac{1}{2}(n-1)n & n & 6\\ \frac{1}{6}(n-1)n(2n-1) & 2n^2 & 4n-2\\ \frac{1}{4}(n-1)^2n^2 & 3n^3 & 3n^2-3n\end{vmatrix}$$

Taking $\frac{1}{12}n(n-1)$ common from C_1 and n common from C_2, we get

$$\Sigma\Delta_r=\frac{1}{12}n^2(n-1)\times\begin{vmatrix}6 & 1 & 6\\ 2(2n-1) & 2n & 2(2n-1)\\ 3n(n-1) & 3n^2 & 3n(n-1)\end{vmatrix}$$

$= 0$, which is constant [$\because$ C_1 and C_3 are identical]

Example 7.28 **Prove that** $\begin{vmatrix}1+a & 1 & 1 & 1\\ 1 & 1+b & 1 & 1\\ 1 & 1 & 1+c & 1\\ 1 & 1 & 1 & 1+d\end{vmatrix}$ $=abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$**. Hence find the value of the determinant if a, b, c, d are the roots of the equation $px^4+qx^3+rx^2+sx+t=0$.**

Sol. Applying $R_1\to\frac{1}{a}R_1, R_2\to\frac{1}{b}R_2, R_3\to\frac{1}{c}R_3, R_4\to\frac{1}{d}R_4$, we get

$$\Delta=abcd\begin{vmatrix}1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a}\\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b}\\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c}\\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d}\end{vmatrix}$$

Applying $R_1\to R_1+R_2+R_3+R_4$ and taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$ common, we get

$$\Delta=abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\times\begin{vmatrix}1 & 1 & 1 & 1\\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b}\\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c}\\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d}\end{vmatrix}$$

Now applying $C_2\to C_2-C_1$, $C_3\to C_3-C_1$, $C_4\to C_4-C_1$,

$$\Delta=abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\times\begin{vmatrix}1 & 0 & 0 & 0\\ \frac{1}{b} & 1 & 0 & 0\\ \frac{1}{c} & 0 & 1 & 0\\ \frac{1}{d} & 0 & 0 & 1\end{vmatrix}$$

Expanding along R_1, we get

$$\Delta=abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\times\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}$$

Expanding again along R_1, we get

$$\Delta=abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

2nd part: $\Delta=abcd+(bcd+acd+abd+abc)$

$$=\frac{t}{p}-\frac{s}{p}=\frac{t-s}{p}$$

USE OF DETERMINANT IN COORDINATE GEOMETRY

Area of Triangle

The area of a triangle, the coordinates of whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\left\|\begin{matrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{matrix}\right\|$$

Condition of Concurrency of Three Lines

Three lines are said to be concurrent if they pass through a common point i.e., they meet at a point.

Let $a_1x+b_1y+c_1=0$ (1)

$a_2x+b_2y+c_2=0$ (2)

$a_3x+b_3y+c_3=0$ (3)

be three concurrent lines then $\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=0$

Condition for General 2nd Degree Equation in x and y Represent Pair of Straight Lines

The general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents pair of straight lines if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Example 7.29 **Find the area of a triangle whose vertices are $A(3, 2)$, $B(11, 8)$ and $C(8, 12)$.**

Sol. The area of triangle is $A = \frac{1}{2}\left\|\begin{matrix} 3 & 2 & 1 \\ 11 & 8 & 1 \\ 8 & 12 & 1 \end{matrix}\right\|$

Operating $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\Rightarrow A = \frac{1}{2}\left\|\begin{matrix} -8 & -6 & 0 \\ 3 & -4 & 0 \\ 8 & 12 & 1 \end{matrix}\right\|$$

$$= \frac{1}{2}\begin{vmatrix} -8 & -6 \\ 3 & -4 \end{vmatrix}$$

$= 25$ sq. units

Example 7.30 **If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in G.P. with same common ratio, then prove that the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear.**

Sol. Since points are collinear, we have to prove that area is zero. $x_2 = x_1 r$, $x_3 = x_1 r^2$ and so is $y_2 = y_1 r$, $y_3 = y_1 r^2$

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ rx_1 & ry_1 & 1 \\ r^2x_1 & r^2y_1 & 1 \end{vmatrix}$$

$$= r.r^2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \end{vmatrix}$$

$= 0$

Hence the points are collinear.

Example 7.31 **If the lines $a_1x + b_1y + 1 = 0$, $a_2x + b_2y + 1 = 0$ and $a_3x + b_3y + 1 = 0$ are concurrent, show that the points (a_1, b_1), (a_2, b_2) and (a_3, b_3) are collinear.**

Sol. The given lines are

$a_1x + b_1y + 1 = 0$ (1)

$a_2x + b_2y + 1 = 0$ (2)

and $a_3x + b_3y + 1 = 0$ (3)

If these lines are concurrent, we must have $\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$,

which is the condition of collinearity of three points (a_1, b_1), (a_2, b_2) and (a_3, b_3).

Hence, if the given lies are concurrent, the given points are collinear.

Example 7.32 **Find the values of 'a' for which the lines**

$2x + y - 1 = 0$

$ax + 3y - 3 = 0$

$3x + 2y - 2 = 0$

are concurrent.

Sol. Lines are concurrent if $\begin{vmatrix} 2 & 1 & -1 \\ a & 3 & -3 \\ 3 & 2 & -2 \end{vmatrix} = 0$

Since C_1 and C_3 are proportional, lines are concurrent for infinite values of a.

Example 7.33 **If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ (a, b, c being distinct and different from 1) are concurrent, then prove that $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$.**

Sol. If the given lines are concurrent, then $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$$

(Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$$\Rightarrow a(b-1)(c-1) - (c-1)(1-a) - (b-1)(1-a) = 0$$

$$\Rightarrow \frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

(Dividing by $(1-a)(1-b)(1-c)$)

$$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

Example 7.34 **If lines $px + qy + r = 0$, $qx + ry + p = 0$ and $rx + py + q = 0$ are concurrent then prove that $p + q + r = 0$ (where p, q, r are distinct).**

Sol. For concurrency of three lines

$px + qy + r = 0$; $qx + ry + p = 0$; $rx + py + q = 0$

We must have, $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

$\Rightarrow 3pqr - p^3 - q^3 - r^3 = 0$

$\Rightarrow (p + q + r)(p^2 + q^2 + r^2 - pq - pr - rq) = 0$

$\Rightarrow p + q + r = 0$

Example 7.35 **Find the value of λ if $2x^2 + 7xy + 3y^2 + 8x + 14y + \lambda = 0$ represent a pair of straight lines.**

Sol. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 7/2 & 4 \\ 7/2 & 3 & 7 \\ 4 & 7 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow 6\lambda + 2(7)(4)\left(\frac{7}{2}\right) - 2(7)^2 - 3(4)^2 - \lambda\left(\frac{7}{2}\right)^2 = 0$$

$$\Rightarrow 6\lambda + 196 - 98 - 48 - \frac{49\lambda}{4} = 0$$

$$\Rightarrow \frac{49\lambda}{4} - 6\lambda = 196 - 146 = 50$$

$$\Rightarrow \frac{25\lambda}{4} = 50 \therefore \lambda = \frac{200}{25} = 8$$

Concept Application Exercise 7.2

1. Prove that the value of each the following determinants is zero.

a. $\begin{vmatrix} a_1 & la_1 + mb_1 & b_1 \\ a_2 & la_2 + mb_2 & b_2 \\ a_3 & la_3 + mb_3 & b_3 \end{vmatrix}$

b. $\begin{vmatrix} a-b & b-c & c-a \\ x-y & y-z & z-x \\ p-q & q-r & r-p \end{vmatrix}$

c. $\begin{vmatrix} \log x & \log y & \log z \\ \log 2x & \log 2y & \log 2z \\ \log 3x & \log 3y & \log 3z \end{vmatrix}$

d. $\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (b^y + b^{-y})^2 & (b^y - b^{-y})^2 & 1 \\ (c^z + c^{-z})^2 & (c^z - c^{-z})^2 & 1 \end{vmatrix}$

e. $\begin{vmatrix} \sin^2\left(x + \frac{3\pi}{2}\right) & \sin^2\left(x + \frac{5\pi}{2}\right) & \sin^2\left(x + \frac{7\pi}{2}\right) \\ \sin\left(x + \frac{3\pi}{2}\right) & \sin\left(x + \frac{5\pi}{2}\right) & \sin\left(x + \frac{7\pi}{2}\right) \\ \sin\left(x - \frac{3\pi}{2}\right) & \sin\left(x - \frac{5\pi}{2}\right) & \sin\left(x - \frac{7\pi}{2}\right) \end{vmatrix}$

2. Prove that $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$.

3. If $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = k(a-b)(b-c)(c-a)$, then find the value of k.

4. Prove that $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc + a^2 & ac + b^2 & ab + c^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)$

5. Show that $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$.

6. Show that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

7. Show that $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

8. Find the value of $\begin{vmatrix} yz & zx & xy \\ p & 2q & 3r \\ 1 & 1 & 1 \end{vmatrix}$, where x, y, z are respectively p^{th}, $(2q)^{th}$ and $(3r)^{th}$ terms of an H.P.

9. Show that if $x_1, x_2, x_3 \neq 0$

$$\begin{vmatrix} x_1 + a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & x_2 + a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & x_3 + a_3b_3 \end{vmatrix} = x_1x_2x_3\left(1 + \frac{a_1b_1}{x_1} + \frac{a_2b_2}{x_2} + \frac{a_3b_3}{x_3}\right).$$

10. If $\Delta_r = \begin{vmatrix} 2^r - 1 & 2.3^r - 1 & 4.5^r - 1 \\ \alpha & \beta & \gamma \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$, then find the value of Δ.

11. Solve the equation $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$

where $a + b + c \neq 0$.

12. Solve for x, $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$.

13. If $A_1B_1C_1$, $A_2B_2C_2$, and $A_3B_3C_3$ are three three-digit numbers, each of which is divisible by k, then prove that

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \text{ is divisible by } k.$$

PRODUCT OF TWO DETERMINANTS

Let, $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

Then row by row multiplication of Δ_1 and Δ_2 is given by

$$\Delta_1 \times \Delta_2 = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Multiplication can also be performed row by column; column by row or column by column as required in the problem.

To express a determinant as product of two determinants, one requires a lots of practice and this can be done only by inspection and trial.

Property: If $A_1, B_1, C_1, \dots$ are respectively the cofactors of the elements $a_1, b_1, c_1, \dots$ of the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ \Delta \neq 0, \text{ then } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

Proof: Given,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $A_1, B_1, C_1, \dots$ are cofactors of $a_1, b_1, c_2, \dots$ Hence,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_3 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

(row by row multiplication)

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} \text{ (as } a_iA_i + b_iB_i + c_iC_i = \Delta,\ i = 1, 2, 3 \text{ and } a_iA_j + b_iB_j + c_iC_j = 0)$$

$= \Delta^3$

$$\Rightarrow \Delta \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^3 \quad \text{or} \quad \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

Note: *For n-order determinant $\Delta_c = \Delta^{n-1}$, where Δ_c is the determinant formed by the cofactors of Δ and n is order of determinant. This property is very useful in studying adjoint of matrix.*

Example 7.36 **Prove that**

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix} = 0$$

Sol. The given determinant is the product of the determinants

$$\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{vmatrix} = 0$$

Example 7.37 **If α, β, γ are real numbers, then without expanding at any stage, show that**

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} = 0$$

Sol.

$$\Delta = \begin{vmatrix} 1 & \cos\alpha\cos\beta + \sin\alpha\sin\beta & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma \\ \cos\alpha\cos\beta + \sin\alpha\sin\beta & 1 & \cos\gamma\cos\beta + \sin\gamma\sin\beta \\ \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & \cos\beta\cos\gamma + \sin\beta\sin\gamma & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$$

$= 0 \times 0 = 0$

Example 7.38 **If $\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 3$, then find the value of**

$$\begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}$$

Sol. $D_c = \begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}$ is the determinant formed by the cofactors of determinant

$$D = \begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix}.$$

Hence, $D_c = D^2 = 3^2 = 9$.

Example 7.39 **Prove that** $\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$

$$= \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix} = 2(b-c)(c-a)(a-b)$$

$\times (y-z)(z-x)(x-y)$.

Sol. $\Delta = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2-2ax+x^2 & b^2-2bx+x^2 & c^2-2cx+x^2 \\ a^2-2ay+y^2 & b^2-2by+y^2 & c^2-2cy+y^2 \\ a^2-2az+z^2 & b^2-2bz+z^2 & c^2-2cz+z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \begin{vmatrix} a^2 & 2a & 1 \\ b^2 & 2b & 1 \\ c^2 & 2c & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix} \text{ (in second determinant } C_1 \leftrightarrow C_2)$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix}$$

$$= 2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$$

Multiplying row by row, we get

$$\Delta = \begin{vmatrix} 1+2ax+a^2x^2 & 1+2bx+b^2x^2 & 1+2cx+c^2x^2 \\ 1+2ay+a^2y^2 & 1+2by+b^2y^2 & 1+2cy+c^2y^2 \\ 1+2az+a^2z^2 & 1+2bz+b^2z^2 & 1+2cz+c^2z^2 \end{vmatrix}$$

$$= \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}$$

Example 7.40 **Express** $\Delta = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$

as square of a determinant and hence evaluate it.

Sol. Keeping in mind the term $2bc - a^2$, we have

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = \Delta \quad \text{[row by row multiplication]}$$

Therefore,

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= [a(bc-a^2) + b(ac-b^2) + c(ab-c^2)]^2$$

$$= [3abc - a^3 - b^3 - c^3]^2$$

Example 7.41 **Prove without expansion that**

$$\begin{vmatrix} ah+bg & g & ab+ch \\ bf+ba & f & hb+bc \\ af+bc & c & bg+fc \end{vmatrix} = a\begin{vmatrix} ah+bg & a & h \\ bf+ba & h & b \\ af+bc & g & f \end{vmatrix}$$

Sol. Rewriting the given determinant, we have

$$\Delta = \frac{1}{c}\begin{vmatrix} ah+bg & gc & ab+ch \\ bf+ba & fc & hb+bc \\ af+bc & c^2 & bg+fc \end{vmatrix} = -\frac{a}{c}\begin{vmatrix} ah+bg & a & ch \\ bf+ba & h & bc \\ af+bc & g & fc \end{vmatrix}$$

By operating in second determinant $C_3 \to C_3 + bC_2$, we get

$$\Delta = \frac{1}{c}\Delta_1 - \frac{a}{c}\begin{vmatrix} ah+bg & a & ab+ch \\ bf+ba & h & hb+bc \\ af+bg & g & bg+fc \end{vmatrix}$$

$$= \frac{1}{c}\Delta_1 + \frac{1}{c}\begin{vmatrix} ah+bg & -a^2 & ab+ch \\ bf+ba & -ah & hb+bc \\ af+bc & -ag & bg+fc \end{vmatrix}$$

$$= \frac{1}{c}\begin{vmatrix} ah+bg & gc-a^2 & ab+ch \\ bf+ba & fc-ah & hb+bc \\ af+bc & c^2-ag & bg+fc \end{vmatrix}$$

$$= -\frac{1}{c}\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \times \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$$

Concept Application Exercise 7.3

1. Prove that

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix} = 0$$

2. Show that the determinant

$$\begin{vmatrix} a^2+b^2+c^2 & bc+ca+ab & bc+ca+ab \\ bc+ca+ab & a^2+b^2+c^2 & bc+ca+ab \\ bc+ca+ab & bc+ca+ab & a^2+b^2+c^2 \end{vmatrix}$$

is always non-negative. When is the determinant zero?

3. Prove that $\begin{vmatrix} (b+x)(c+x) & (c+x)(a+x) & (a+x)(b+x) \\ (b+y)(c+y) & (c+y)(a+y) & (a+y)(b+y) \\ (b+z)(c+z) & (c+z)(a+z) & (a+z)(b+z) \end{vmatrix}$

$= (b-c)(c-a)(a-b)(y-z)(z-x)(x-y)$.

4 Factorize the following:

$$\begin{vmatrix} 3 & a+b+c & a^3+b^3+c^3 \\ a+b+c & a^2+b^2+c^2 & a^4+b^4+c^4 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^5+b^5+c^5 \end{vmatrix}$$

DIFFERENTIATION OF A DETERMINANT

I. Let $\Delta(x)$ be a determinant of order two. If we write

$\Delta(x) = [C_1\ C_2]$, where C_1 and C_2 denote the first and second columns then

$$\Delta'(x) = [C_1'\ C_2] + [C_1\ C_2'],$$

where C_1' denotes the column which contains the derivative of all the functions in the i^{th} column C_i. In a similar fashion, if we write

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \text{ then } \Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \end{bmatrix}$$

Example: Let, $\Delta(x) = \begin{vmatrix} \sin x & \log x \\ e & 1/x \end{vmatrix}, x > 0$

Then, $\Delta'(x) = \begin{vmatrix} \cos x & \log x \\ 0 & 1/x \end{vmatrix} + \begin{vmatrix} \sin x & 1/x \\ e & -1/x^2 \end{vmatrix}$

II. Let $\Delta(x)$ be a determinant of order three. If we write $\Delta(x) = [C_1\ C_2\ C_3]$, then

$\Delta'(x) = [C_1'\ C_2\ C_3] + [C_1\ C_2'\ C_3] + [C_1\ C_2\ C_3']$ and similarly if we consider

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}, \text{ then } \Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R_3' \end{bmatrix}$$

III. If only one row (column) consists functions of x and other rows are constant, viz., let

$$\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ (say)}$$

Then

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and in general

$$\Delta^n(x) = \begin{vmatrix} f_1^n(x) & f_2^n(x) & f_3^n(x) \\ b_2 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where n is any positive integer and $f^n(x)$ denotes the n^{th} derivative of $f(x)$.

Example 7.42 **If $y = \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix}$, find $\dfrac{dy}{dx}$.**

Sol. $\dfrac{dy}{dx} = \begin{vmatrix} \cos x & -\sin x & \cos x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix} + \begin{vmatrix} \sin x & \cos x & \sin x \\ -\sin x & -\cos x & -\sin x \\ x & 1 & 1 \end{vmatrix}$

$+ \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ 1 & 0 & 0 \end{vmatrix}$

$= 0 - \begin{vmatrix} \sin x & \cos x & \sin x \\ \sin x & \cos x & \sin x \\ x & 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= 0 + (\cos^2 x + \sin^2 x)$

Example 7.43 **If $f(x) = \begin{vmatrix} x^n & n! & 2 \\ \cos x & \cos \dfrac{n\pi}{2} & 4 \\ \sin x & \sin \dfrac{n\pi}{2} & 8 \end{vmatrix}$, then find the value of $\dfrac{d^n}{dx^n}[f(x)]_{x=0} (n \in Z)$.**

Sol. $\dfrac{d^n}{dx^n}[f(x)] = \begin{vmatrix} \dfrac{d^n}{dx^n}(x^n) & n! & 2 \\ \dfrac{d^n}{dx^n}(\cos x) & \cos \dfrac{n\pi}{2} & 4 \\ \dfrac{d^n}{dx^n}(\sin x) & \sin \dfrac{n\pi}{2} & 8 \end{vmatrix}$

$= \begin{vmatrix} n! & n! & 2 \\ \cos\left(x + \dfrac{n\pi}{2}\right) & \cos \dfrac{n\pi}{2} & 4 \\ \sin\left(x + \dfrac{n\pi}{2}\right) & \sin \dfrac{n\pi}{2} & 8 \end{vmatrix} \quad (n \in Z)$

$$\Rightarrow \frac{d^n}{dx^n}[f(x)]_{x=0} = 0$$

Example 7.44 **If f, g and h are differentiable functions of x and $\Delta(x) = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$, prove that**

$$\Delta'(x) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

Sol. $(xf)' = xf' + f$ and $(x^2f)'' = [2xf + x^2f']' = 2f + 4xf' + x^2f''$

$$\Rightarrow \Delta = \begin{vmatrix} f & g & h \\ xf'+f & xg'+g & xh'+h \\ 2f+4xf'+x^2f'' & 2g+4xg'+x^2g'' & 2h+4xh'+x^2h'' \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and then $R_3 \to R_3 - 4R_2 - 2R_1$

$$\Rightarrow \Delta = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' & x^2g'' & x^2h'' \end{vmatrix}$$

Taking x common from R_2 and multiplying with R_3, we have

$$\Delta = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

$$\Rightarrow \frac{d\Delta}{dx} = \begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

$$+ \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

Example 7.45 **Let α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$, $C(x)$ be polynomials of degrees 3, 4 and 5, respectively, then show that**

$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ **is divisible by $f(x)$ where prime (′) denotes the derivatives.**

Sol. Since α is a repeated root of the quadratic equation $f(x) = 0$, $f(x)$ can be written as

$f(x) = k(x - \alpha)^2$, where k is some non-zero constant.

Let, $g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$g(x)$ is divisible by $f(x)$ if it is divisible by $(x - \alpha)^2$, i.e., $g(\alpha) = 0$ and $g'(\alpha) = 0$.

As $A(x)$, $B(x)$ and $C(x)$ are polynomials of degrees 3, 4 and 5, respectively, deg. $g(x) \geq 2$.

Now,

$$g(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$$

(R_1 and R_2 are identical)

Also

$$g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

$$\therefore \quad g'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$$

(R_1 and R_3 are identical)

This implies that $f(x)$ divides $g(x)$.

Concept Application Exercise 7.4

1. Let $f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin(2x^2) \end{vmatrix}$. Find the value of $f'(0)$.

2. If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2, then prove that

$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$ is a constant polynomial.

3. If $g(x) = \frac{f(x)}{(x-a)(x-b)(x-c)}$, where $f(x)$ is a polynomial of degree <3, then prove that

$$\frac{dg(x)}{dx} = \begin{vmatrix} 1 & a & f(a)(x-a)^{-2} \\ 1 & b & f(b)(x-b)^{-2} \\ 1 & c & f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

SYSTEM OF LINEAR EQUATIONS

System of consistent linear equations: A system of (linear) equations is said to be consistent if it has at least one solution.

Example: (i) System of equations $\left.\begin{aligned}x+y&=2\\2x+2y&=5\end{aligned}\right\}$ is inconsistent because it has no solutions, i.e., there is no value of x and y which satisfy both the equations.

Here the two straight lines are parallel.

(ii) System of equations $\left.\begin{aligned}x+y&=2\\x-y&=0\end{aligned}\right\}$ is consistent because it has a solution $x = 1$, $y = 1$.

Here the two lines intersect at one point.

Cramer's Rule

I. System of linear equations in two variables:

Let the given system of equations be

$$\left.\begin{aligned}a_1x+b_1y+c_1&=0\\a_2x+b_2y+c_2&=0\end{aligned}\right\} \quad (1)$$

where $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

Solving by cross-multiplication, we have

$$\frac{x}{b_1c_2-b_2c_1} = \frac{-y}{a_1c_2-a_2c_1} = \frac{1}{a_1b_2-a_2b_1}$$

or
$$\frac{x}{\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}} = \frac{-y}{\begin{vmatrix}a_1 & c_1\\ a_2 & c_2\end{vmatrix}} = \frac{1}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}$$

II. System of linear equations in three variables:

Let the given system of linear equations in three variables x, y and z be

$$a_1x + b_1y + c_1z = d_1 \quad (1)$$
$$a_2x + b_2y + c_2z = d_2 \quad (2)$$
$$a_3x + b_3y + c_3z = d_3 \quad (3)$$

Let, $\Delta = \begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$, $\Delta_1 = \begin{vmatrix}d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3\end{vmatrix}$, $\Delta_2 = \begin{vmatrix}a_1 & d_1 & c_1\\ a_2 & d_2 & c_2\\ a_3 & d_3 & c_3\end{vmatrix}$,

$$\Delta_3 = \begin{vmatrix}a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3\end{vmatrix}$$

Let, $\Delta \neq 0$. Now,

$$\Delta_1 = \begin{vmatrix}d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3\end{vmatrix} = \begin{vmatrix}a_1x+b_1y+c_1z & b_1 & c_1\\ a_2x+b_2y+c_2z & b_2 & c_2\\ a_3x+b_3y+c_3z & b_3 & c_3\end{vmatrix}$$

$$= \begin{vmatrix}a_1x & b_1 & c_1\\ a_2x & b_2 & c_2\\ a_3x & b_3 & c_3\end{vmatrix} = x\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} = x\Delta$$

$[C_1 \to C_1 - yC_2 - zC_3]$

$\therefore \quad x = \dfrac{\Delta_1}{\Delta}$, where $\Delta \neq 0$

Similarly, $\Delta_2 = y\Delta \qquad \therefore\ y = \dfrac{\Delta_2}{\Delta}$ and $z = \dfrac{\Delta_3}{\Delta}$

Thus $x = \dfrac{\Delta_1}{\Delta}$, $y = \dfrac{\Delta_2}{\Delta}$, $z = \dfrac{\Delta_3}{\Delta}$, where $\Delta \neq 0 \quad (4)$

The rule given in Eq. (4) to find the values of x, y, z is called the Cramer's rule.

Note:

(i) Δ_i is obtained by replacing elements of i^{th} column by d_1, d_2, d_3 where $i = 1, 2, 3$.

(ii) Cramer's rule can be used only when $\Delta \neq 0$.

Nature of Solution of System of Linear Equations

Let the given system of linear equations be

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Now there are two possibilities.

Case I: $\Delta \neq 0$

In this case, from (i), (ii) and (iii), we have

$$x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta} \text{ and } z = \frac{\Delta_3}{\Delta}$$

Hence, unique value of x, y, z will be obtained and the system of equations will have unique solution.

Case II: $\Delta = 0$

(a) When at least one of Δ_1, Δ_2, Δ_3 is non-zero

Let $\Delta_1 \neq 0$, then from (i), $\Delta_1 = x\,\Delta$ will not be satisfied for any value of x because $\Delta = 0$ and $\Delta_1 \neq 0$ and hence no value of x is possible in this case.

Similarly when $\Delta_2 \neq 0$, $\Delta_2 = y\Delta$ will not be possible for any value of y and hence no value of y will be possible when $\Delta_3 \neq 0$, $\Delta_3 = z\Delta$ will not be possible for any value of z and hence no value of z will be possible.

Thus if $\Delta = 0$ and any of Δ_1, Δ_2 and Δ_3 is non-zero, then no solution is possible and hence system of equations will be inconsistent.

(b) when $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$

In this case $\left.\begin{aligned}\Delta_1 &= x\Delta\\ \Delta_2 &= y\Delta\\ \Delta_3 &= z\Delta\end{aligned}\right\}$ will be true for all values of x, y and z.

But since $a_1x + b_2y + c_1z = d_1$, therefore, only two of x, y, z will be independent and the third will be dependent on other two.

Thus infinitely many values of x, y, z are possible and out of x, y, z only two can be given independent values.

Hence if $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, then the system of equations will be consistent and it will have infinitely many solutions.

Summary:

(i) *If $\Delta \neq 0$, then given system of equations is consistent and it has unique (one) solution.*

(ii) *If $\Delta = 0$ and any of $\Delta_1, \Delta_2, \Delta_3$ is non-zero, then given system of equations is inconsistent and it will have no solution.*

(iii) *If all of Δ_1, Δ_2 and Δ_3 are zero, then given system of equations is consistent and has infinitely many solutions.*

Conditions for Consistency of Three Linear Equations in Two Unknowns

System of three linear equations in x and y

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$
$$a_3x + b_3y + c_3 = 0$$

will be consistent if the values of x and y obtained from any two equations satisfy the third equation.

Solving first two equations by Cramer's rule, we have

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

These values of x and y will satisfy the third equation if

$$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition for consistency of three linear equations in two unknown. If such system of equations is consistent, then number of solutions is one.

System of Homogeneous Linear Equations

A system of linear equations is said to be homogeneous if the sum of powers of variable in each term is 1.

Let the three homogeneous linear equations in three unknown x, y, z be

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= 0 \quad (i) \\ a_2x + b_2y + c_2z &= 0 \quad (ii) \\ a_3x + b_3y + c_3z &= 0 \quad (iii) \end{aligned}\right\} \quad (A)$$

Clearly $x = 0$, $y = 0$, $z = 0$ is a solution of system of Eq. (A). This solution is called a trivial solution. Any other solution is called a non-trivial solution. Let, system of Eq. (A) has non-trivial solution.

Let, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

From (i) and (ii), we have

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1} = k \text{ (say)}$$

$\therefore$ $x = k(b_1c_2 - b_2c_1)$

$y = -k(a_1c_2 - a_2c_1)$

$z = k(a_1b_2 - a_2b_1)$

Putting these values of x, y, z in (iii), we get

$$k[a_3(b_1c_2 - b_2c_1) - b_3(a_1c_2 - a_2c_1) + c_3(a_1b_2 - a_2b_1)] = 0$$

or $a_3(b_1c_2 - b_2c_1) - b_3(a_1c_2 - a_2c_1) + c_3(a_1b_2 - a_2b_1) = 0$ $[\because \; k \neq 0]$

or $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

or $\Delta = 0$

This is the condition for system of Eq. (A) to have nontrivial solution.

Summary:

(i) *If $\Delta \neq 0$, then given system of equations has only trivial solution and the number of solutions in this case is one.*

(ii) *If $\Delta = 0$, then given system of equations has non-trivial solution as well as trivial solution and number of solutions in this case is infinite.*

Example 7.46 **Solve by Cramer's rule**

$$\mathbf{x + y + z = 6}$$
$$\mathbf{x - y + z = 2}$$
$$\mathbf{3x + 2y - 4z = -5}$$

Sol. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 2 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 3 & -1 & -7 \end{vmatrix} = 14$

$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ -5 & 2 & -4 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 1 \\ -4 & -2 & 0 \\ 19 & 6 & 0 \end{vmatrix} = 14$

$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 3 & -5 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 1 \\ 0 & -4 & 0 \\ 0 & -23 & -7 \end{vmatrix} = 28$

and $\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 3 & 2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 6 \\ 0 & -2 & -4 \\ 0 & -1 & -23 \end{vmatrix} = 42$

Hence by Cramer's rule,

$$x = \frac{\Delta_x}{\Delta} = 1,\; y = \frac{\Delta_y}{\Delta} = 2,\; z = \frac{\Delta_z}{\Delta} = 3$$

Example 7.47 **For what values of p and q, the system of equations $2x + py + 6z = 8, x + 2y + qz = 5, x + y + 3z = 4$ has**
(i) no solution,
(ii) a unique solution,
(iii) infinitely many solutions.

Sol. The given system of equation is

$$2x + py + 6z = 8$$
$$x + 2y + qz = 5$$
$$x + y + 3z = 4$$

$$\Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = (2-p)(3-q)$$

By Cramer's rule, if $\Delta \neq 0$, i.e., $p \neq 2$, $q \neq 3$, the system has unique solution.
If $p = 2$ or $q = 3$, $\Delta = 0$, then if $\Delta_x = \Delta_y = \Delta_z = 0$, the system has infinite solutions and if any one of $\Delta_x, \Delta_y, \Delta_z \neq 0$, system has no solution. Now,

$$\Delta_x = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix}$$

$$= 30 - 8q - 15p + 4pq = (p-2)(4q-15)$$

$$\Delta_y = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix}$$

$$= -8q + 8q = 0$$

$$\Delta_z = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix}$$

$$= p - 2$$

Thus if $p = 2$, $\Delta_x = \Delta_y = \Delta_z = 0$ for all $q \in R$, so the system has infinite solutions.

And if $p \neq 2$, $q = 3$, $\Delta_x, \Delta_z \neq 0$, the system has no solution.

Hence the system has
(i) no solution, if $p \neq 2, q = 3$,
(ii) a unique solution, if $p \neq 2, q \neq 3$,
(iii) infinitely many solutions, $p = 2, q \in R$.

Example 7.48 **Find λ for which the system of equations $x + y - 2z = 0, 2x - 3y + z = 0, x - 5y + 4z = \lambda$ is consistent and find the solutions for all such values of λ.**

Sol. The given system is

$$x - 5y + 4z = \lambda \quad (1)$$
$$x + y - 2z = 0 \quad (2)$$
$$2x - 3y + z = 0 \quad (3)$$

$$\Delta = \begin{vmatrix} 1 & -5 & 4 \\ 1 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -5 & 4 \\ 0 & 6 & -6 \\ 0 & 7 & -7 \end{vmatrix} = 0$$

Hence, system is consistent only when $\Delta_x = \Delta_y = \Delta_z = 0$. Now,

$$\Delta_x = \begin{vmatrix} \lambda & -5 & 4 \\ 0 & 1 & -2 \\ 0 & -3 & 1 \end{vmatrix} = -5\lambda = 0$$

$\Rightarrow \lambda = 0$
For $\lambda = 0$, clearly $\Delta_y = \Delta_z = 0$.

Therefore, system is consistent if $\lambda = 0$. Then on eliminating x from (1), (2) and (3), we have $y - z = 0$.

Let, $y = z = k \in R$. Then from (1), we have
$x = 5k - 4k = k$
Hence, solution is $x = y = z = k \in R$.

Example 7.49 **For what values of k, the following system of equations possesses a non-trivial solution over the set of rationals: $x + ky + 3z = 0, 3x + ky - 2z = 0, 2x + 3y - 4z = 0$. Also find the solution for this value of k.**

Sol. The system

$$x + ky + 3z = 0$$
$$3x + ky - 2z = 0$$
$$2x + 3y - 4z = 0$$

has non-trivial solution (i.e., non-zero solution) if the determinant of coefficients of x, y and z is zero. Here,

$$\Delta = \begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$$

$\Rightarrow 2k - 33 = 0$ or $k = 33/2$ (1)
Then the equations become

$$2x + 33y + 6z = 0 \quad (2)$$
$$6x + 33y - 4z = 0 \quad (3)$$
$$2x + 3y - 4z = 0 \quad (4)$$

Eliminating x, we get from (2) and (4),

$$30y + 10z = 0, \text{ i.e., } 3y + z = 0 \quad (5)$$

Let, $y = \lambda \in R$. Then $z = -3\lambda$ and so

$$2x = -33\lambda + 18\lambda = -15\lambda$$

$$\therefore \quad x = -\frac{15}{2}\lambda$$

Hence, $x = -\frac{15}{2}\lambda, y = \lambda, z = -3\lambda, \lambda \in R$.

Example 7.50 **If $2ax - 2y + 3z = 0, x + ay + 2z = 0$ and $2x + az = 0$ have a non-trivial solution, find the value of a.**

Sol. For non-trivial solution, we must have

$$\begin{vmatrix} 2a & -2 & 3 \\ 1 & a & 2 \\ 2 & 0 & a \end{vmatrix} = 0$$

$\Rightarrow 2a(a^2 - 0) + 2(a - 4) + 3(0 - 2a) = 0$
$\Rightarrow 2a^3 + 2a - 8 + 0 - 6a = 0$
$\Rightarrow 2a^3 - 4a - 8 = 0$

$\Rightarrow \quad a^3 - 2a - 4 = 0$

$\Rightarrow \quad a^3 - 2a^2 + 2a^2 - 4a + 2a - 4 = 0$

$\Rightarrow \quad a^2(a-2) + 2a(a-2) + 2(a-2) = 0$

$\Rightarrow \quad (a-2)(a^2 + 2a + 2) = 0$

$\Rightarrow \quad a = 2$

Example 7.51 **If x, y and z are not all zero and connected by the equations $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $(p_1 + \lambda q_1)x + (p_2 + \lambda q_2)y + (p_3 + \lambda q_3)z = 0$, show that**

$$\lambda = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ p_1 & p_2 & p_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

Sol. Since x, y and z are not all zero, the determinant of the coefficient of the given set of equations must satisfy

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ p_1 + \lambda q_1 & p_2 + \lambda q_2 & p_3 + \lambda q_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ p_1 & p_2 & p_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

Concept Application Exercise 7.5

1. If the equations $2x + 3y + 1 = 0$, $3x + y - 2 = 0$ and $ax + 2y - b = 0$ are consistent, then prove that $a - b = 2$.
2. If $x = cy + bz$, $y = az + cx$, $z = bx + ay$ where x, y, z are not all zeros, then find the value of $a^2 + b^2 + c^2 + 2abc$.
3. If the following system of equations is consistent,
   $(a + 1)^3x + (a + 2)^3y = (a + 3)^3$
   $(a + 1)x + (a + 2)y = a + 3$
   $x + y = 1$,
   then find the value of a.
4. Solve the system of the equations:
   $ax + by + cz = d$
   $a^2x + b^2y + c^2z = d^2$
   $a^3x + b^3y + c^3z = d^3$
   Will the solution always exist and be unique?

EXERCISES

Subjective Type

Solutions on page 7.38

1. Solve for x, $\begin{vmatrix} x^2 - a^2 & a^2 - b^2 & x^2 - c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ (x+a)^3 & (x+b)^3 & (x+c)^2 \end{vmatrix} = 0, a \neq b \neq c.$

2. Prove that $\Delta = \begin{vmatrix} a & c & c-a & a+c \\ c & b & b-c & b+c \\ a-b & b-c & 0 & a-c \\ x & y & z & 1+x+y \end{vmatrix} = 0$ implies that a, b, c are in A.P. or a, c, b are in G.P.

3. If $f(x)$ is a polynomial of degree <3, prove that
$$\begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \frac{f(x)}{(x-a)(x-b)(x-c)}$$

4. Prove that for any A.P. $a_1, a_2, a_3, \ldots$ the determinant
$$\begin{vmatrix} a_p + a_{p+m} + a_{p+2m} & 2a_p + 3a_{p+m} + 4a_{p+2m} & 4a_p + 9a_{p+m} + 16a_{p+2m} \\ a_q + a_{q+m} + a_{q+2m} & 2a_q + 3a_{q+m} + 4a_{q+2m} & 4a_q + 9a_{q+m} + 16a_{q+2m} \\ a_r + a_{r+m} + a_{r+2m} & 2a_r + 3a_{r+m} + 4a_{r+2m} & 4a_r + 9a_{r+m} + 16a_{r+2m} \end{vmatrix} = 0$$

5. Let n and r be two positive integers such that $n \geq r + 2$ and
$$\Delta(n, r) = \begin{vmatrix} {}^nC_r & {}^nC_{r+1} & {}^nC_{r+2} \\ {}^{n+1}C_r & {}^{n+1}C_{r+1} & {}^{n+1}C_{r+2} \\ {}^{n+2}C_r & {}^{n+2}C_{r+1} & {}^{n+2}C_{r+2} \end{vmatrix}.$$ Show that
$\Delta(n,r) = \dfrac{{}^{n+2}C_3}{{}^{r+2}C_3}\Delta(n-1, r-1)$. Hence or otherwise, prove that $\Delta(n, r) = \dfrac{{}^{n+2}C_3\ {}^{n+1}C_3 \ldots {}^{n-r+3}C_3}{{}^{r+2}C_3\ {}^{r+1}C_3 \ldots {}^{3}C_3}$.

6. Show that in general there are three values of t for which the following system of equations has a non-trivial solution:
   $(a - t)x + by + cz = 0$
   $bx + (c - t)y + az = 0$
   $cx + ay + (b - t)z = 0$
   Express the product of these values of t in the form of a determinant.

7. Let α_1, α_2 and β_1, β_2 be the roots of $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$, respectively. If the system of equations $\alpha_1y + \alpha_2z = 0$ and $\beta_1y + \beta_2z = 0$ has a non-trivial solution, then prove that $b^2pr = q^2ac$.

8. If A, B and C are the angles of a triangle, show that the system of equations $x \sin 2A + y \sin C + z \sin B = 0$, $x \sin C + y \sin 2B + z \sin A = 0$ and $x \sin B + y \sin A + z \sin 2C = 0$ possesses non-trivial solution. Hence, system has infinite solutions.

9. If $ax_1^2 + by_1^2 + cz_1^2 = ax_2^2 + by_2^2 + cz_2^2 = ax_3^2 + by_3^2 + cz_3^2 = d$, $ax_2x_3 + by_2y_3 + cz_2z_3 = ax_3x_1 + ax_3x_1 + by_3y_1 + cz_3z_1 = ax_1x_2 + by_1y_2 + cz_1z_2 = f$, then prove that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = (d-f)\left\{\frac{(d+2f)}{abc}\right\}^{1/2}$$

10. Let $\Delta = \begin{vmatrix} 2a_1b_1 & a_1b_2 + a_2b_1 & a_1b_3 + a_3b_1 \\ a_1b_2 + a_2b_1 & 2a_2b_2 & a_2b_3 + a_3b_2 \\ a_1b_3 + a_3b_1 & a_3b_2 + a_2b_3 & 2a_3b_3 \end{vmatrix}$. Expressing Δ as the product of two determinants, show that $\Delta = 0$.

Hence show that if $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$= (lx + my + n)(l'x + m'y + n')$, then $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

11. If in a triangle, s denotes the semi-perimeter and a, b, c denote the lengths of sides, then prove that

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3(s-a)(s-b)(s-c)$$

12. Evaluate $\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}$.

13. Prove that $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+c)(c+a)(a+b)$.

14. Prove that $\begin{vmatrix} ax-by-cz & ay+bx & cx+az \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix}$

$= (x^2 + y^2 + z^2)(a^2 + b^2 + c^2)(ax + by + cz)$.

15. If $\Delta(x) = \begin{vmatrix} a_1+x & b_1+x & c_1+x \\ a_2+x & b_2+x & c_2+x \\ a_3+x & b_3+x & c_3+x \end{vmatrix}$, show that $\Delta''(x) = 0$ and that $\Delta(x) = \Delta(0) + Sx$, where S denotes the sum of all the cofactors of all the elements in $\Delta(0)$.

Objective Type

Solutions on page 7.42

Each question has four choices a, b, c and d, out of which *only one* answer is correct. Find the correrct answer.

1. If $p + q + r = 0 = a + b + c$, then the value of the determinant

$\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix}$ is

a. 0 **b.** $pa + qb + rc$
c. 1 **d.** none of these

2. If a determinant of order 3×3 is formed by using the numbers 1 or -1, then the minimum value of the determinant is

a. -2 **b.** -4
c. 0 **d.** -8

3. If $z = \begin{vmatrix} -5 & 3+4i & 5-7i \\ 3-4i & 6 & 8+7i \\ 5+7i & 8-7i & 9 \end{vmatrix}$, then z is

a. purely real **b.** purely imaginary
c. $a + ib$, where $a \neq 0, b \neq 0$ **d.** $a + ib$, where $b = 4$

4. If α, β, γ are the roots of $px^3 + qx^2 + r = 0$, then the value of the determinant $\begin{vmatrix} \alpha\beta & \beta\gamma & \gamma\alpha \\ \beta\gamma & \gamma\alpha & \alpha\beta \\ \gamma\alpha & \alpha\beta & \beta\gamma \end{vmatrix}$ is

a. p **b.** q
c. 0 **d.** r

5. When the determinant $\begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$ is expanded in powers of $\sin x$, then the constant term in that expression is

a. 1 **b.** 0
c. -1 **d.** 2

6. If $a = \cos\theta + i\sin\theta$, $b = \cos 2\theta - i\sin 2\theta$, $c = \cos 3\theta + i\sin 3\theta$ and if $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then

a. $\theta = 2k\pi, k \in Z$ **b.** $\theta = (2k+1)\pi, k \in Z$
c. $\theta = (4k+1)\pi, k \in Z$ **d.** none of these

7. If $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = 0$, then the line $ax + by + c = 0$ passes through the fixed point which is

a. (1, 2) **b.** (1, 1)
c. (−2, 1) **d.** (1, 0)

8. If $\begin{vmatrix} x^n & x^{n+2} & x^{n+3} \\ y^n & y^{n+2} & y^{n+3} \\ z^n & z^{n+2} & z^{n+3} \end{vmatrix} = (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$, then n equals

a. 1 **b.** -1
c. 2 **d.** -2

9. If $f(x) = a + bx + cx^2$ and α, β, γ are the roots of the equation $x^3 = 1$, then $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is equal to

a. $f(\alpha)+f(\beta)+f(\gamma)$
b. $f(\alpha)f(\beta)+f(\beta)f(\gamma)+f(\gamma)f(\alpha)$
c. $f(\alpha)f(\beta)f(\gamma)$
d. $-f(\alpha)f(\beta)f(\gamma)$

10. If [ ] denotes the greatest integer less than or equal to the real number under consideration, and $-1\le x<0$, $0\le y<1$, $1\le z<2$, then the value of the determinant

$$\begin{vmatrix}[x]+1 & [y] & [z]\\ [x] & [y]+1 & [z]\\ [x] & [y] & [z]+1\end{vmatrix} \text{ is}$$

a. $[x]$ **b.** $[y]$
c. $[z]$ **d.** none of these

11. Let $a, b, c \in R$ such that no two of them are equal and satisfy $\begin{vmatrix}2a & b & c\\ b & c & 2a\\ c & 2a & b\end{vmatrix}=0$, then equation $24ax^2+4bx+c=0$ has

a. at least one root in [0, 1]
b. at least one root in $\left[-\frac{1}{2}, \frac{1}{2}\right]$
c. at least one root in $[-1, 0]$
d. at least two roots in [0, 2]

12. If p, q, r are in A.P., then the value of determinant

$$\begin{vmatrix}a^2+a^{2n+1}+2p & b^2+2^{n+2}+3q & c^2+p\\ 2^n+p & 2^{n+1}+q & 2q\\ a^2+2^n+p & b^2+2^{n+1}+2q & c^2-r\end{vmatrix} \text{ is}$$

a. 1 **b.** 0
c. $a^2b^2c^2-2^n$ **d.** $(a^2+b^2+c^2)-2^nq$

13. If $(x_1-x_2)^2+(y_1-y_2)^2=a^2$
$(x_2-x_3)^2+(y_2-y_3)^2=b^2$
$(x_3-x_1)^2+(y_3-y_1)^2=c^2$
and $k\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=(a+b+c)(b+c-a)(c+a-b)$ $\times(a+b-c)$, then the value of k is

a. 1 **b.** 2
c. 4 **d.** none of these

14. The value of the determinant $\begin{vmatrix}ka & k^2+a^2 & 1\\ kb & k^2+b^2 & 1\\ kc & k^2+c^2 & 1\end{vmatrix}$ is

a. $k(a+b)(b+c)(c+a)$
b. $k\,abc(a^2+b^2+c^2)$
c. $k(a-b)(b-c)(c-a)$
d. $k(a+b-c)(b+c-a)(c+a-b)$

15. If $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$, where a, b, c are all different, then the determinant

$$\begin{vmatrix}1 & 1 & 1\\ (x-a)^2 & (x-b)^2 & (x-c)^2\\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b)\end{vmatrix} \text{ vanishes when}$$

a. $a+b+c=0$ **b.** $x=\frac{1}{3}(a+b+c)$
c. $x=\frac{1}{2}(a+b+c)$ **d.** $x=a+b+c$

16. The determinant $\begin{vmatrix}y^2 & -xy & x^2\\ a & b & c\\ a' & b' & c'\end{vmatrix}$ is equal to

a. $\begin{vmatrix}bx+ay & cx+by\\ b'x+a'y & c'x+b'y\end{vmatrix}$ **b.** $\begin{vmatrix}ax+by & bx+cy\\ a'x+b'y & b'x+c'y\end{vmatrix}$
c. $\begin{vmatrix}bx+cy & ax+by\\ b'x+c'y & a'x+b'y\end{vmatrix}$ **d.** $\begin{vmatrix}ax+by & bx+cy\\ a'x+b'y & b'x+c'y\end{vmatrix}$

17. If $\begin{vmatrix}b^2+c^2 & ab & ac\\ ab & c^2+a^2 & bc\\ ca & cb & a^2+b^2\end{vmatrix}=ka^2b^2c^2$, then the value of k is

a. 2 **b.** 4
c. 0 **d.** none of these

18. If a, b and c are non-zero real numbers, then

$$\Delta=\begin{vmatrix}b^2c^2 & bc & b+c\\ c^2a^2 & ca & c+a\\ a^2b^2 & ab & a+b\end{vmatrix} \text{ is equal to}$$

a. abc **b.** $a^2b^2c^2$
c. $bc+ca+ab$ **d.** none of these

19. The value of $\begin{vmatrix}-1 & 2 & 1\\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1\\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1\end{vmatrix}$ is equal to

a. zero **b.** $-16\sqrt{2}$
c. $-8\sqrt{2}$ **d.** none of these

20. Let $\{D_1, D_2, D_3, ..., D_n\}$ be the set of third-order determinants that can be made with the distinct non-zero real numbers $a_1, a_2, ..., a_9$. Then

a. $\sum_{i=1}^{n} D_i=1$ **b.** $\sum_{i=1}^{n} D_i=0$
c. $D_i=D_j,\ \forall\, i, j$ **d.** None of these

21. If w is a complex cube root of unity, then value of

$$\Delta = \begin{vmatrix} a_1+b_1w & a_1w^2+b_1 & c_1+b_1\overline{w} \\ a_2+b_2w & a_2w^2+b_2 & c_2+b_2\overline{w} \\ a_3+b_3w & a_3w^2+b_3 & c_3+b_3\overline{w} \end{vmatrix}$$ is

a. 0 **b.** -1
c. 2 **d.** none of these

22. If x, y, z are in A.P., then the value of the determinant

$$\begin{vmatrix} a+2 & a+3 & a+2x \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix}$$ is

a. 1 **b.** 0
c. $2a$ **d.** a

23. If $a+b+c=0$, one root of $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ is

a. $x=1$ **b.** $x=2$
c. $x=a^2+b^2+c^2$ **d.** $x=0$

24. In triangle ABC, if

$$\begin{vmatrix} 1 & 1 & 1 \\ \cot\frac{A}{2} & \cot\frac{B}{2} & \cot\frac{C}{2} \\ \tan\frac{B}{2}+\tan\frac{C}{2} & \tan\frac{C}{2}+\frac{A}{2} & \tan\frac{A}{2}+\tan\frac{B}{2} \end{vmatrix} = 0$$, then the triangle must be

a. equilateral **b.** isosceles
c. obtuse angled **d.** none of these

25. If a, b, c, d, e and f are in G.P., then the value of

$$\begin{vmatrix} a^2 & d^2 & x \\ b^2 & e^2 & y \\ c^2 & f^2 & z \end{vmatrix}$$ depends on

a. x and y **b.** x and z
c. y and z **d.** independent of x, y and z

26. If $a^2+b^2+c^2=-2$ and $f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$,

then $f(x)$ is a polynomial of degree

a. 0 **b.** 1
c. 2 **d.** 3

27. The value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+2}C_1 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+2}C_2 \end{vmatrix}$ is equal to

a. 1 **b.** -1
c. 0 **d.** none of these

28. If $x \ne y \ne z$ and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then the value of xyz is

a. 1 **b.** 2
c. -1 **d.** -2

29. If $x \ne 0, y \ne 0, z \ne 0$ and $\begin{vmatrix} 1+x & 1 & 1 \\ 1+y & 1+2y & 1 \\ 1+z & 1+z & 1+3z \end{vmatrix} = 0$, then $x^{-1}+y^{-1}+z^{-1}$ is equal to

a. -1 **b.** -2
c. -3 **d.** none of these

30. The value of $\begin{vmatrix} yz & zx & xy \\ p & 2q & 3r \\ 1 & 1 & 1 \end{vmatrix}$, where x, y, z are, respectively, p^{th}, $(2q)^{\text{th}}$ and $(3r)^{\text{th}}$ terms of an H.P., is

a. -1 **b.** 0
c. 1 **d.** none of these

31. If $a_1b_1c_1$, $a_2b_2c_2$ and $a_3b_3c_3$ are 3-digit even natural numbers and $\Delta = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$, then Δ is

a. divisible by 2 but not necessarily by 4
b. divisible by 4 but not necessarily by 8
c. divisible by 8
d. none of these

32. The value of the determinant of n^{th} order, being given by

$$\begin{vmatrix} x & 1 & 1 & \cdots \\ 1 & x & 1 & \cdots \\ 1 & 1 & x & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$ is

a. $(x-1)^{n-1}(x+n-1)$ **b.** $(x-1)^n(x+n-1)$
c. $(1-x)^{-1}(x+n-1)$ **d.** none of these

33. If a, b, c are positive and are the p^{th}, q^{th} and r^{th} terms, respectively, of a G.P., then $\Delta = \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$ is

a. 0 **b.** $\log(abc)$
c. $-(p+q+r)$ **d.** none of these

34. If $f'(x) = \begin{vmatrix} mx & mx-p & mx+p \\ n & n+p & n-p \\ mx+2n & mx+2n+p & mx+2n-p \end{vmatrix}$, then $y=f(x)$ represents

a. a straight line parallel to x-axis
b. a straight line parallel to y-axis
c. parabola
d. a straight line with negative slope

35. If $\begin{vmatrix} x & 3 & 6 \\ 3 & 6 & x \\ 6 & x & 3 \end{vmatrix} = \begin{vmatrix} 2 & x & 7 \\ x & 7 & 2 \\ 7 & 2 & x \end{vmatrix} = \begin{vmatrix} 4 & 5 & x \\ 5 & x & 4 \\ x & 4 & 5 \end{vmatrix} = 0$, then '$x$' is equal to

a. 0 **b.** -9
c. 3 **d.** none of these

36. If $\begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^a & a \\ x^{n+5} & x^{a+6} & x^{2n+5} \end{vmatrix} = 0$, $\forall\, x \in R$, where $n \in N$, then value of 'a' is

a. n **b.** $n-1$
c. $n+1$ **d.** none of these

37. If $\Delta = \begin{vmatrix} 3 & 4 & 5 & x \\ 4 & 5 & 6 & y \\ 5 & 6 & 7 & z \\ x & y & z & 0 \end{vmatrix} = 0$, then

a. x, y, z are in A.P. **b.** x, y, z are in G.P.
c. x, y, z are in H.P. **d.** none of these

38. Let $x < 1$, then value of $\begin{vmatrix} x^2+2 & 2x+1 & 1 \\ 2x+1 & x+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$ is

a. non-negative **b.** non-positive
c. negative **d.** positive

39. Value of $\begin{vmatrix} x+y & z & z \\ x & y+z & x \\ y & y & z+x \end{vmatrix}$, where x, y, z are non-zero real numbers, is equal to

a. xyz **b.** $2xyz$
c. $3xyz$ **d.** $4xyz$

40. Which of the following is not the root of the equation

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0?$$

a. 2 **b.** 0
c. 1 **d.** -3

41. If $f(x) = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = 0$, then

a. $f'(x) = 0$ and $f''(x) = 0$ has one common root

b. $f(x) = 0$ and $f'(x) = 0$ has one common root

c. sum of roots of $f(x) = 0$ is $-3a$

d. none of these

42. Roots of the equation $\begin{vmatrix} x & m & n & 1 \\ a & x & n & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$ are

a. independent of m and n

b. independent of a, b and c

c. depend on m, n and a, b, c

d. independent of m, n and a, b, c

43. If a, b, c are different, then the value of x satisfying $\begin{vmatrix} 0 & x^2-a & x^3-b \\ x^2+a & 0 & x^2+c \\ x^4+b & x-c & 0 \end{vmatrix} = 0$ is

a. c **b.** c
c. b **d.** 0

44. If $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = k \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$, then the value of k is

a. 1 **b.** 2
c. 3 **d.** 4

45. Suppose $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

$D' = \begin{vmatrix} a_1+pb_1 & b_1+qc_1 & c_1+ra_1 \\ a_2+pb_2 & b_2+qc_2 & c_2+ra_2 \\ a_3+pb_3 & b_3+qc_3 & c_3+ra_3 \end{vmatrix}$. Then

a. $D' = D$ **b.** $D' = D(1-pqr)$
c. $D' = D(1+p+q+r)$ **d.** $D' = D(1+pqr)$

46. The value of the determinant

$\Delta = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$ is equal to

a. 1 **b.** 0
c. 2 **d.** 3

47. If $a > 0$ and discriminant of $ax^2 + 2bx + c$ is negative, then

$\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$ is

a. + ve **b.** $(ac-b)^2)(ax^2+2bx+c)$
c. $-$ve **d.** 0

(AIEEE, 2002)

48. If $a_1, a_2, \ldots, a_n, \ldots$ form a G.P. and $a_i > 0$, for all $i \geq 1$,

then $\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$ is equal to

a. 0 b. 1
c. 2 d. 3

(AIEEE, 2005)

49. Let $\vec{a}_r = x_r\hat{i} + y_r\hat{j} + z_r\hat{k}$, $r = 1, 2, 3$ be three mutually perpendicular unit vectors, then the value of $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$ is equal to

a. zero b. ± 1
c. ± 2 d. none of these

50. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\pi/4 \le x \le \pi/4$ is

a. 0 b. 2
c. 1 d. 3

51. If $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda+3 \\ \lambda^2+1 & 2-\lambda & \lambda-3 \\ \lambda^2-3 & \lambda+4 & 3\lambda \end{vmatrix}$, then p is equal to

a. -5 b. -4
c. -3 d. -2

52. If x, y, z are different from zero and $\Delta = \begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0$, then the value of the expression $\frac{a}{x} + \frac{b}{y} + \frac{c}{z}$ is

a. 0 b. -1
c. 1 d. 2

53. If A, B, C are angles of a triangle, then the value of $\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}$ is

a. 1 b. -1
c. -2 d. -4

54. For the equation $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = 0$,

a. There are exactly two distinct roots
b. There is one pair of equation real roots
c. There are three pairs of equal roots
d. Modulus of each root is 2

55. Let m be a positive integer and

$$\Delta_r = \begin{vmatrix} 2r & -1 & {}^mC_r & 1 \\ m^2 & -1 & 2^m & m+1 \\ \sin^2(m^2) & & \sin^2(m) & \sin^2(m+1) \end{vmatrix} \quad (0 \le r \le m).$$

Then the value of $\sum_{r=0}^{m} \Delta_r$ is given by

a. 0 b. $m^2 - 1$
c. 2^m d. $2^m \sin^2(2^m)$

56. If $D_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+1 & n^2+n \\ 2k-1 & n^2 & n^2+n+1 \end{vmatrix}$ and $\sum_{k=1}^{n} D_k = 56$, then n equals

a. 4 b. 6
c. 8 d. none of these

57. The value of $\sum_{r=2}^{n} (-2)^r \begin{vmatrix} {}^{n-2}C_{r-2} & {}^{n-2}C_{r-1} & {}^{n-2}C_r \\ -3 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix}$ $(n > 2)$ is

a. $2n - 1 + (-1)^n$ b. $2n + 1 + (-1)^{n-1}$
c. $2n - 3 + (-1)^n$ d. none of these

58. The value of the determinant $\begin{vmatrix} {}^nC_{r-1} & {}^nC_r & (r+1)\,{}^{n+2}C_{r+1} \\ {}^nC_r & {}^nC_{r+1} & (r+2)\,{}^{n+2}C_{r+2} \\ {}^nC_{r+1} & {}^nC_{r+2} & (r+3)\,{}^{n+2}C_{r+3} \end{vmatrix}$ is

a. $n^2 + n - 1$ b. 0
c. ${}^{n+3}C_{r+3}$ d. ${}^nC_{r-1} + {}^nC_r + {}^nC_{r+1}$

59. $\Delta_1 = \begin{vmatrix} y^5z^6(z^3-y^3) & x^4z^6(x^3-z^3) & x^4y^5(y^3-x^3) \\ y^2z^3(y^6-z^6) & xz^3(z^6-x^6) & xy^2(x^6-y^6) \\ y^2z^3(z^3-y^3) & xz^3(x^3-z^3) & xy^2(y^3-x^3) \end{vmatrix}$ and

$\Delta_2 = \begin{vmatrix} x & y^2 & z^3 \\ x^4 & y^5 & z^6 \\ x^7 & y^8 & z^9 \end{vmatrix}$. Then $\Delta_1\Delta_2$ is equal to

a. Δ_2^3 b. Δ_2^2
c. Δ_2^4 d. None of these

60. If $l_1^2 + m_1^2 + n_1^2 = 1$, etc. and $l_1l_2 + m_1m_2 + n_1n_2 = 0$, etc. and $\Delta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$, then

a. $|\Delta| = 3$ b. $|\Delta| = 2$
c. $|\Delta| = 1$ d. $\Delta = 0$

61. The value of the determinant

$$\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 & (a_1-b_4)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 & (a_2-b_4)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 & (a_3-b_4)^2 \\ (a_4-b_1)^2 & (a_4-b_2)^2 & (a_4-b_3)^2 & (a_4-b_4)^2 \end{vmatrix}$$ is

a. dependant on a_i, $i = 1, 2, 3, 4$
b. dependant on b_i, $i = 1, 2, 3, 4$
c. dependant on a_i, b_i, $i = 1, 2, 3, 4$
d. 0

62. The value of determinant $\begin{vmatrix} bc-a^2 & ac-b^2 & ab-c^2 \\ ac-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$ is

a. always positive **b.** always negative
c. always zero **d.** cannot say anything

63. Value of $\begin{vmatrix} 1+x_1 & 1+x_1x & 1+x_1x^2 \\ 1+x_2 & 1+x_2x & 1+x_2x^2 \\ 1+x_3 & 1+x_3x & 1+x_3x^2 \end{vmatrix}$ depends upon

a. x only **b.** x_1 only
c. x_2 only **d.** none of these

64. If $\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix}\begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$

$= (1 + a^2 + b^2 + c^2)^3$, then the value of λ is

a. 8 **b.** 27
c. 1 **d.** -1

65. $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$. The value of $\lim\limits_{x\to 0}\dfrac{f(x)}{x}$ is equal to

a. 1 **b.** -1
c. zero **d.** none of these

66. If the determinant $\begin{vmatrix} b-c & c-a & a-b \\ b'-c' & c'-a' & a'-b' \\ b''-c'' & c''-a'' & a''-b'' \end{vmatrix}$

$= m\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$, then the value of m is

a. 0 **b.** 2
c. -1 **d.** 1

67. Let $\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} = Ax^4 + Bx^3 + Cx^2 + Dx + E$. Then the value of

$5A + 4B + 3C + 2D + E$ is equal to

a. zero **b.** -16
c. 16 **d.** -11

68. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are the given determinants, then

a. $\Delta_1 = 3(\Delta_2)^2$ **b.** $\dfrac{d}{dx}(\Delta_1) = 3\Delta_2$
c. $\dfrac{d}{dx}(\Delta_1) = 3(\Delta_2)^2$ **d.** $\Delta_1 = 3\Delta_2^{3/2}$

69. If $y = \sin mx$, then the value of the determinant $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$,

where $y_n = \dfrac{d^n y}{dx^n}$, is

a. m^9 **b.** m^2
c. m^3 **d.** none of these

70. If the value of the determinant $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ is positive, then $(a, b, c > 0)$

a. $abc > 1$ **b.** $abc > -8$
c. $abc < -8$ **d.** $abc > -2$

71. If $A_1, B_1, C_1, \ldots$ are, respectively, the cofactors of the elements $a_1, b_1, c_1, \ldots$ of the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta \neq 0$, then

the value of $\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix}$ is equal to

a. $a_1^2\Delta$ **b.** $a_1\Delta$
c. $a_1\Delta^2$ **c.** $a_1^2\Delta^2$

72. Let $f(x) = \begin{vmatrix} 2\cos^2 x & \sin 2x & -\sin x \\ \sin 2x & 2\sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$. Then the value of

$\int_0^{\pi/2} [f(x) + f'(x)]dx$ is

a. π **b.** $\pi/2$
c. 2π **d.** $3\pi/2$

73. The number of positive integral solutions of the equation $\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 11$ is

a. 0 **b.** 3
c. 6 **d.** 12

74. a, b, c are distinct real numbers, not equal to one. If $ax + y + z = 0$, $x + by + z = 0$ and $x + y + cz = 0$ have a non-trivial solution, then the value of $\dfrac{1}{1-a} + \dfrac{1}{1-b} + \dfrac{1}{1-c}$ is equal to

a. -1 **b.** 1
c. zero **d.** none of these

75. If the system of linear equations $x + y + z = 6$, $x + 2y + 3z = 14$ and $2x + 5y + \lambda z = \mu$ ($\lambda, \mu \in R$) has a unique solution, then

a. $\lambda \neq 8$ **b.** $\lambda = 8, \mu \neq 36$
c. $\lambda = 8, \mu = 36$ **d.** none of these

76. If α, β, γ are the angles of a triangle and the system of equations

$\cos(\alpha - \beta)x + \cos(\beta - \gamma)y + \cos(\gamma - \alpha)z = 0$
$\cos(\alpha + \beta)x + \cos(\beta + \gamma)y + \cos(\gamma + \alpha)z = 0$
$\sin(\alpha + \beta)x + \sin(\beta + \gamma)y + \sin(\gamma + \alpha)z = 0$

has non-trivial solutions, then triangle is necessarily

a. equilateral **b.** isosceles
c. right angled **d.** acute angled

77. Given $a = x/(y - z)$, $b = y/(z - x)$ and $c = z/(x - y)$, where x, y and z are not all zero, then the value of $ab + bc + ca$ is

a. 0 **b.** 1
c. -1 **d.** none of these

78. If $pqr \neq 0$ and the system of equations

$(p + a)x + by + cz = 0$
$ax + (q + b)y + cz = 0$
$ax + by + (r + c)z = 0$

has a non-trivial solution, then value of $\frac{a}{p} + \frac{b}{q} + \frac{c}{r}$ is

a. -1 **b.** 0
c. 1 **d.** 2

79. The system of equations

$\alpha x - y - z = \alpha - 1$
$x - \alpha y - z = \alpha - 1$
$x - y - \alpha z = \alpha - 1$

has no solution if α is

a. either -2 or 1 **b.** -2
c. 1 **d.** not -2

80. The set of equations $\lambda x - y + (\cos\theta)z = 0$, $3x + y + 2z = 0$, $(\cos\theta)x + y + 2z = 0$, $0 \leq \theta < 2\pi$, has non-trivial solution(s)

a. for no value of λ and θ
b. for all values of λ and θ
c. for all values of λ and only two values of θ
d. for only one value of λ and all values of θ

81. If a, b, c are non-zeros, then the system of equations $(\alpha + a)x + \alpha y + \alpha z = 0$, $\alpha x + (\alpha + b)y + \alpha z = 0$, $\alpha x + \alpha y + (\alpha + c)z = 0$ has a non-trivial solution if

a. $\alpha^{-1} = -(a^{-1} + b^{-1} + c^{-1})$ **b.** $\alpha^{-1} = a + b + c$
c. $\alpha + a + b + c = 1$ **d.** none of these

82. If $c < 1$ and the system of equations $x + y - 1 = 0$, $2x - y - c = 0$ and $bx + 3by - c = 0$ is consistent, then the possible real values of b are

a. $b \in \left(-3, \frac{3}{4}\right)$ **b.** $b \in \left(-\frac{3}{2}, 4\right)$
c. $b \in \left(-\frac{3}{4}, 3\right)$ **d.** none of these

83. If a, b, c are in G.P. with common ratio r_1 and α, β, γ are in G.P. with common ratio r_2, and equations $ax + \alpha y + z = 0$, $bx + \beta y + z = 0$, $cx + \gamma y + z = 0$ have only zero solution, then which of the following is not true?

a. $r_1 \neq 1$ **b.** $r_2 \neq 1$
c. $r_1 \neq r_2$ **d.** none of these

84. If a, b, c are non-zero real numbers and if the equations $(a - 1)x = y + z$, $(b - 1)y = z + x$, $(c - 1)z = x + y$ have a non-trivial solution, then $ab + bc + ca$ equals

a. $a + b + c$ **b.** abc
c. 1 **d.** none of these

Multiple Correct Answers Type

Solutions on page 7.53

Each question has four choices a, b, c and d, out of which *one or more* answers are correct. Find the correct answer.

1. Which of the following has/have value equal to zero?

a. $\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$ **b.** $\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix}$

c. $\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$ **d.** $\begin{vmatrix} 2 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$

2. If $g(x) = \begin{vmatrix} a^{-x} & e^{x\log_e a} & x^2 \\ a^{-3x} & e^{3x\log_e a} & x^4 \\ a^{-5x} & e^{5x\log_e a} & 1 \end{vmatrix}$, then

a. graphs of $g(x)$ is symmetrical about origin
b. graphs of $g(x)$ is symmetrical about Y-axis
c. $\left.\frac{d^4 g(x)}{dx^4}\right|_{x=0} = 0$
d. $f(x) = g(x) \times \log\left(\frac{a-x}{a+x}\right)$ is an odd function

3. If $\Delta = \begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix}$, then a factor of Δ is

a. $a + b + x$
b. $x^2 - (a - b)x + a^2 + b^2 + ab$
c. $x^2 + (a + b)x + a^2 + b^2 - ab$
d. $a + b - x$

4. If $\Delta = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$ then

a. Δ is independent of θ **b.** Δ is independent of ϕ
c. Δ is a constant **d.** $\left.\frac{d\Delta}{d\theta}\right]_{\theta=\pi/2} = 0$

5. If $\phi(\alpha,\beta)=\begin{vmatrix}\cos\alpha & -\sin\alpha & 1\\ \sin\alpha & \cos\alpha & 1\\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1\end{vmatrix}$, then

a. $f(300, 200) = f(400, 200)$ **b.** $f(200, 400) = f(200, 600)$
c. $f(100, 200) = f(200, 200)$ **d.** none of these

6. The determinant $\Delta=\begin{vmatrix}a^2+x & ab & ac\\ ab & b^2+x & bc\\ ac & bc & c^2+x\end{vmatrix}$ is divisible by

a. x **b.** x^2
c. x^3 **d.** none of these

7. If $f(x)=\begin{vmatrix}a & -1 & 0\\ ax & a & -1\\ ax^2 & ax & a\end{vmatrix}$, then $f(2x)-f(x)$ is divisible by

a. x **b.** a
c. $2a+3x$ **d.** x^2

8. $\Delta=\begin{vmatrix}1 & 1+ac & 1+bc\\ 1 & 1+ad & 1+bd\\ 1 & 1+ae & 1+be\end{vmatrix}$ is independent of

a. a **b.** b
c. c, d, e **d.** none of these

9. If $g(x)=\dfrac{f(x)}{(x-a)(x-b)(x-c)}$, where $f(x)$ is a polynomial of degree < 3, then

a. $\int g(x)dx=\begin{vmatrix}1 & a & f(a)\log|x-a|\\ 1 & b & f(b)\log|x-b|\\ 1 & c & f(c)\log|x-c|\end{vmatrix}\div\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}+k$

b. $\dfrac{dg(x)}{dx}=\begin{vmatrix}1 & a & f(a)(x-a)^{-2}\\ 1 & b & f(b)(x-b)^{-2}\\ 1 & c & f(c)(x-c)^{-2}\end{vmatrix}\div\begin{vmatrix}a^2 & a & 1\\ b^2 & b & 1\\ c^2 & c & 1\end{vmatrix}$

c. $\dfrac{dg(x)}{dx}=\begin{vmatrix}1 & a & f(a)(x-a)^{-2}\\ 1 & b & f(b)(x-b)^{-2}\\ 1 & c & f(c)(x-c)^{-2}\end{vmatrix}\div\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$

d. $\int g(x)dx=\begin{vmatrix}1 & a & f(a)\log|x-a|\\ 1 & b & f(b)\log|x-b|\\ 1 & c & f(c)\log|x-c|\end{vmatrix}\div\begin{vmatrix}a^2 & a & 1\\ b^2 & b & 1\\ c^2 & c & 1\end{vmatrix}+k$

10. If $\Delta(x)=\begin{vmatrix}x^2+4x-3 & 2x+4 & 13\\ 2x^2+5x-9 & 4x+5 & 26\\ 8x^2-6x+1 & 16x-6 & 104\end{vmatrix}=ax^3+bx^2+cx+d$, then

a. $a=3$ **b.** $b=0$
c. $c=0$ **d.** None of these

11. If $f(\theta)=\begin{vmatrix}\sin\theta & \cos\theta & \sin\theta\\ \cos\theta & \sin\theta & \cos\theta\\ \cos\theta & \sin\theta & \sin\theta\end{vmatrix}$, then

a. $f(\theta)=0$ has exactly 2 real solutions in $[0, \pi]$

b. $f(\theta)=0$ has exactly 3 real solutions in $[0, \pi]$

c. range of function $\dfrac{f(\theta)}{1-\sin 2\theta}$ is $\left[-\sqrt{2},\sqrt{2}\right]$

d. range of function $\dfrac{f(\theta)}{\sin 2\theta-1}$ is $[-3, 3]$

12. If $f(\theta)=\begin{vmatrix}\sin^2 A & \cot A & 1\\ \sin^2 B & \cos B & 1\\ \sin^2 C & \cos C & 1\end{vmatrix}$, then

a. $\tan A+\tan B+C$ **b.** $\cot A\cot B\cot C$
c. $\sin^2 A+\sin^2 B+\sin^2 C$ **d.** 0

13. The roots of the equation $\begin{vmatrix}{}^xC_r & {}^{n-1}C_r & {}^{n-1}C_{r-1}\\ {}^{x+1}C_r & {}^nC_r & {}^nC_{r-1}\\ {}^{x+2}C_r & {}^{n+1}C_r & {}^{n+1}C_{r-1}\end{vmatrix}=0$ are

a. $x=n$ **b.** $x=n+1$
c. $x=n-1$ **d.** $x=n-2$

14. If $f(x)=\begin{vmatrix}3 & 3x & 3x^2+2a^2\\ 3x & 3x^2+2a^2 & 3x^3+6a^2x\\ 3x^2+2a^2 & 3x^3+6a^2x & 3x^4+12a^2x^2+2a^4\end{vmatrix}$, then

a. $f'(x)=0$
b. $y=f(x)$ is a straight line parallel to x-axis
c. $\int_0^2 f(x)dx=32a^4$
d. none of these

15. If $\begin{vmatrix}yz-x^2 & zx-y^2 & xy-z^2\\ xz-y^2 & xy-z^2 & yz-x^2\\ xy-z^2 & yz-x^2 & zx-y^2\end{vmatrix}=\begin{vmatrix}r^2 & u^2 & u^2\\ u^2 & r^2 & u^2\\ u^2 & u^2 & r^2\end{vmatrix}$, then

a. $r^2=x+y+z$ **b.** $r^2=x^2+y^2+z^2$
c. $u^2=yz+zx+xy$ **d.** $u^2=xyz$

16. Let $f(n)=\begin{vmatrix}n & n+1 & n+2\\ {}^nP_n & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2}\\ {}^nC_n & {}^{n+1}C_{n+1} & {}^{n+2}C_{n+2}\end{vmatrix}$ where the symbols have their usual meanings. Then $f(n)$ is divisible by

a. n^2+n+1 **b.** $(n+1)!$
c. $n!$ **d.** none of these

17. If a, b, c are non-zero real numbers such that

$\begin{vmatrix}bc & ca & ab\\ ca & ab & bc\\ ab & bc & ca\end{vmatrix}=0$, then

a. $\frac{1}{a}+\frac{1}{b\omega}+\frac{1}{c\omega^2}=0$
b. $\frac{1}{a}+\frac{1}{b\omega^2}+\frac{1}{c\omega}=0$
c. $\frac{1}{a\omega}+\frac{1}{b\omega^2}+\frac{1}{c}=0$
d. none of these

18. The values of $k \in R$ for which the system of equations $x + ky + 3z = 0$, $kx + 2y + 2z = 0$, $2x + 3y + 4z = 0$ admits of non-trivial solution is
a. 2
b. 5/2
c. 3
d. 5/4

19. If determinent $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ is
a. positive
b. independent of θ
c. independent of ϕ
d. none of these

Reasoning Type

Solutions on page 7.55

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.
b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.
c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** If A, B and C are the angles of a triangle and
$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}=0,$$ then triangle may not be equilateral.

Statement 2: If any two rows of a determinant are the same, then the value of that determinant is zero.

2. Consider the system of the equations $kx + y + z = 1$, $x + ky + z = k$ and $x + y + kz = k^2$.

Statement 1: System of equations has infinite solutions when $k = 1$.

Statement 2: If the determinant $\begin{vmatrix} 1 & 1 & 1 \\ k & k & 1 \\ k^2 & 1 & k \end{vmatrix}=0$, then $k = -1$.

3. Consider the determinant $f(x)=\begin{vmatrix} 0 & x^2-a & x^3-b \\ x^2+a & 0 & x^2+c \\ x^4+b & x-c & 0 \end{vmatrix}$.

Statement 1: $f(x) = 0$ has one root $x = 0$.

Statement 2: The value of skew-symmetric determinant of odd-order is always zero.

4. **Statement 1:** If the system of equations $\lambda x + (b - a)y + (c - a)z = 0$, $(a - b)x + \lambda y + (c - b)z = 0$ and $(a - c)x + (b - c)y + \lambda z = 0$ has a non-trivial solution, then the value of λ is 0.

Statement 2: The value of skew-symmetric matrix of order 3 is zero.

5. **Statement 1:** $\Delta=\begin{vmatrix} my+nz & mq+nr & mb+nc \\ kz-mx & kr-mp & kc-ma \\ -nx-ky & -np-kq & -na-kb \end{vmatrix}$ is equal to 0.

Statement 2: The value of skew-symmetric matrix of order 3 is zero.

6. **Statement 1:** If $bc + qr = ca + rp = ab + pq = -1$,
then $\begin{vmatrix} ap & a & p \\ bq & b & q \\ cr & c & r \end{vmatrix}=0 \quad (abc, pqr \neq 0)$.

Statement 2: If system of equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ has non-trivial solutions,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

7. Consider the system of equation $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$.

Statement 1: If the system has infinite number of solutions, then $\mu = 10$.

Statement 2: The determinant $\begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix}=0$ for $\mu = 10$.

8. Consider the determinant $\Delta=\begin{vmatrix} a_1+b_1x^2 & a_1x^2+b_1 & c_1 \\ a_2+b_2x^2 & a_2x^2+b_2 & c_2 \\ a_3+b_3x^2 & a_3x^2+b_3 & c_3 \end{vmatrix}=0$,
where $a_i, b_i, c_i \in R$ $(i = 1, 2, 3)$ and $x \in R$.

Statement 1: The values of x satisfying $\Delta = 0$ are $x = 1, -1$.

Statement 2: If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$, then $\Delta = 0$.

Linked Comprehension Type

Solutions on page 7.56

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

$$f(x)=\begin{vmatrix} x+c_1 & x+a & x+a \\ x+b & x+c_2 & x+a \\ x+b & x+b & x+c_3 \end{vmatrix}$$ and $g(x) = (c_1 - x)(c_2 - x)(c_3 - x)$

1. Coefficient of x in $f(x)$ is

a. $\dfrac{g(a)-f(b)}{b-a}$ b. $\dfrac{g(-a)-g(-b)}{b-a}$

c. $\dfrac{g(a)-g(b)}{b-a}$ d. none of these

2. Which of the following is not a constant term in $f(x)$?

a. $\dfrac{bg(a)-ag(b)}{(b-a)}$ b. $\dfrac{bg(a)-af(-b)}{(b-a)}$

c. $\dfrac{bf(-a)-ag(b)}{(b-a)}$ d. none of these

3. Which of the following is not true?

a. $f(-a)=g(a)$ b. $f(-a)=g(-a)$

c. $f(-b)=g(b)$ d. none of these

For Problems 4–6

Consider the function $f(x)=\begin{vmatrix} a^2+x & ab & ac \\ ab & b^2+x & bc \\ ac & bc & c^2+x \end{vmatrix}$

4. Which of the following is true?

a. $f(x)=0$ and $f'(x)=0$ have one positive common root.

b. $f(x)=0$ and $f'(x)=0$ have one negative common root.

c. $f(x)=0$ and $f'(x)=0$ have no common root.

d. None of these.

5. Which of the following is true?

a. $f(x)$ has one +ve point of maxima.

b. $f(x)$ has one –ve point of minima.

c. $f(x)=0$ has three distinct roots.

d. Local minimum value of $f(x)$ is zero.

6. In which of the following interval $f(x)$ is strictly increasing?

a. $(-\infty, \infty)$ b. $(-\infty, 0)$

c. $(0, \infty)$ d. None of these

For Problems 7–9

Given that the system of equations $x = cy + bz$, $y = az + cx$, $z = bx + ay$ has non-zero solutions and at least one of the a, b, c is a proper fraction.

7. $a^2+b^2+c^2$ is

a. >2 c. >3

d. <3 d. <2

8. abc is

a. >-1 b. >1

c. <2 d. <3

9. System has solution such that

a. $x:y:z \equiv (1-2a^2):(1-2b^2):(1-2c^2)$

b. $x:y:z \equiv \dfrac{1}{1-2a^2}:\dfrac{1}{1-2b^2}:\dfrac{1}{1-2c^2}$

c. $x:y:z \equiv \dfrac{a}{1-a^2}:\dfrac{b}{1-b^2}:\dfrac{c}{1-c^2}$

d. $x:y:z \equiv \sqrt{1-a^2}:\sqrt{1-b^2}:\sqrt{1-c^2}$

For Problems 10–12

Consider the system of equations

$x+y+z=6$
$x+2y+3z=10$
$x+2y+\lambda z=\mu$

10. The system has unique solution if

a. $\lambda \ne 3$ b. $\lambda = 3, \mu = 10$

c. $\lambda = 3, \mu \ne 10$ d. none of these

11. The system has infinite solutions if

a. $\lambda \ne 3$ b. $\lambda = 3, \mu = 10$

c. $\lambda = 3, \mu \ne 10$ d. none of these

12. The system has no solution if

a. $\lambda \ne 3$ b. $\lambda = 3, \mu = 10$

c. $\lambda = 3, \mu \ne 10$ d. none of these

For Problems 13–15

Let α, β be the roots of the equation $ax^2+bx+c=0$. Let $S_n=\alpha^n+\beta^n$ for $n \ge 1$ and $\Delta=\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$.

13. If $\Delta < 0$, then the equation $ax^2+bx+c=0$ has

a. positive real roots b. negative real roots

c. equal roots d. imaginary roots

14. If a, b, c are rational and one of the roots of the equation is $1+\sqrt{2}$, then the value of Δ is

a. 8 b. 12

c. 30 d. 32

15. If $\Delta > 0$, then

a. $f(1)>0$ b. $f(1)<0$

c. $f(1)=0$ d. cannot say anything about $f(1)$

For Problems 16–18

Let $\Delta=\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}$ and the equation $px^3+qx^2+rx+s=0$ has roots a, b, c, where $a, b, c \in R^+$.

16. The value of Δ is

a. r^2/p^2 b. r^3/p^3

c. $-s/p$ d. none of these

17. The value of Δ is

a. $\le 9r^2/p^2$ b. $\ge 27s^2/p^2$

c. $\le 27s^3/p^3$ d. none of these

18. If $\Delta = 27$ and $a^2+b^2+c^2=2$, then

a. $3p+2q=0$ b. $4p+3q=0$

c. $3p+q=0$ d. none of these

For Problems 19–21

Consider the polynomial function

$$f(x)=\begin{vmatrix}(1+x)^a & (1+2x)^b & 1\\ 1 & (1+x)^a & (1+2x)^b\\ (1+2x)^b & 1 & (1+x)^a\end{vmatrix},\ a, b \text{ being positive integers.}$$

19. The constant term in $f(x)$ is

a. 2 **b.** 1
c. -1 **d.** 0

20. The coefficient of x in $f(x)$ is

a. 2^a **b.** $2^a - 3 \times 2^b + 1$
c. 0 **d.** none of these

21. Which of the following is true?

a. All the roots of the equation $f(x) = 0$ are positive.
b. All the roots of the equation $f(x) = 0$ are negative.
c. At least one of the equation $f(x) = 0$ is repeating one.
d. None of these.

For Problems 22–24

If $x > m, y > n, z > r$ $(x, y, z > 0)$ such that $\begin{vmatrix}x & n & r\\ m & y & r\\ m & n & z\end{vmatrix} = 0.$

22. The value of $\dfrac{x}{x-m}+\dfrac{y}{y-n}+\dfrac{z}{z-r}$ is

a. 1 **b.** -1
c. 2 **d.** -2

23. The value of $\dfrac{m}{x-m}-1+\dfrac{y}{y-n}-1+\dfrac{r}{z-r}$ is

a. -2 **b.** -4
c. 0 **d.** -1

24. The greatest value of $\dfrac{xyz}{(x-m)(y-n)(z-r)}$ is

a. 27 **b.** $\dfrac{8}{27}$
c. $\dfrac{64}{27}$ **d.** none of these

For Problems 25–27

Suppose $f(x)$ is a function satisfying the following conditions:

(i) $f(0) = 2, f(1) = 1,$

(ii) f has a minimum value at $x = 5/2$

(iii) For all x, $f'(x) = \begin{vmatrix}2ax & 2ax-1 & 2ax+b+1\\ b & b+1 & -1\\ 2(ax+b) & 2ax+2b+1 & 2ax+b\end{vmatrix}$

25. The value of $f(2)$ is

a. 1/4 **b.** 1/2
c. -1 **d.** 3

26. $f(x) = 0$ has

a. both roots positive **b.** both roots negative
c. roots of opposite sign **d.** imaginary roots

27. Range of $f(x)$ is

a. $[7/16, \infty)$ **b.** $(-\infty, 15/16]$
c. $[3/4, \infty)$ **d.** none of these

Matrix-Match Type

Solutions on page 7.59

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a→p, s, b→r, c→p, q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1.

Column I	**Column II**
a. Coefficient of x in $f(x) = \begin{vmatrix}x & (1+\sin x)^3 & \cos x\\ 1 & \log(1+x) & 2\\ x^2 & 1+x^2 & 0\end{vmatrix}$	**p.** 10
b. Value of $\begin{vmatrix}1 & 3\cos\theta & 1\\ \sin\theta & 1 & 3\cos\theta\\ 1 & \sin\theta & 1\end{vmatrix}$ is	**q.** 0
c. If a, b, c are in A.P. and $f(x) = \begin{vmatrix}x+a & x^2+1 & 1\\ x+b & 2x^2-1 & 1\\ x+c & 3x^2-2 & 1\end{vmatrix}$, then $f'(0)$ is	**r.** -12
d. If $\begin{vmatrix}x & 2 & x\\ 1 & x & 6\\ x & x & x+1\end{vmatrix} = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, then a_0 is	**s.** -2

2.

Column I	**Column II**
a. The value of the determinant $\begin{vmatrix}x+2 & x+3 & x+5\\ x+4 & x+6 & x+9\\ x+8 & x+11 & x+15\end{vmatrix}$ is	**p.** 1
b. If one of the roots of the equation $\begin{vmatrix}7 & 6 & x^2-13\\ 2 & x^2-13 & 2\\ x^2-13 & 3 & 7\end{vmatrix} = 0$ is $x + 2$, then sum of all other five roots is	-6

c. The value of $\begin{vmatrix} \sqrt{6} & 2i & 3+\sqrt{6} \\ \sqrt{12} & \sqrt{3}+\sqrt{8}i & 3\sqrt{2}+\sqrt{6}i \\ \sqrt{18} & \sqrt{2}+\sqrt{12}i & \sqrt{27}+2i \end{vmatrix}$ is	**r.** 2
d. If $f(\theta) = \begin{vmatrix} \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$, then $f(\pi/3)$	**s.** -2

3.

Column I	**Column II**
a. $\begin{vmatrix} 1/c & 1/c & -(a+b)/c^2 \\ -(b+c)/a^2 & 1/a & 1/a \\ -b(b+c)/a^2c & (a+2b+c)/ac & -b(a+b)/ac^2 \end{vmatrix}$ is	**p.** independent of a
b. $\begin{vmatrix} \sin a\cos b & \sin a\sin b & \cos a \\ \cos a\cos b & \cos a\sin b & -\sin a \\ -\sin a\sin b & \sin a\cos b & 0 \end{vmatrix}$ is	**q.** independent of b
c. $\begin{vmatrix} \dfrac{1}{\sin a\cos b} & \dfrac{1}{\sin a\sin b} & \dfrac{1}{\cos a} \\ \dfrac{-\cos a}{\sin^2 a\cos b} & \dfrac{-\cos a}{\sin^2 a\sin b} & \dfrac{\sin a}{\cos^2 a} \\ \dfrac{\sin b}{\sin a\cos^2 b} & \dfrac{-\cos b}{\sin a\sin^2 b} & 0 \end{vmatrix}$ is	**r.** independent of c
d. If a, b, and c are the sides of a triangle and A, B and C are the angles opposite to a, b, and c, respectively, then $\Delta = \begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix}$	**s.** dependent on a, b

Integer Type

Solutions on page 7.60

1. Let α, β, γ are the real roots of the equation $x^3 + ax^2 + bx + c = 0$ ($a, b, c \in R$ and $a \neq 0$). If the system of equations (in u, v and w) given by
$$\alpha u + \beta v + \gamma w = 0$$
$$\beta u + \gamma v + \alpha w = 0$$
$$\gamma u + \alpha v + \beta w = 0$$
has non-trivial solutions, then the value of a^2/b is.

2. If a_1, a_2, a_3, 5, 4, a_6, a_7, a_8, a_9 are in H.P., and $D = \begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$ then the value of [D] is (where [ ] represents the greatest integer function)

3. If $\begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 & 1 \\ (\gamma+\alpha-\beta-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 & 1 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$
$= -k(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)$,
then the value of $(k)^{1/2}$ is.

4. Absolute value of sum of roots of the equation $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$ is.

5. The value of $|\alpha|$ for which the system of equation
$$\alpha x + y + z = \alpha - 1$$
$$x + \alpha y + z = \alpha - 1$$
$$x + y + \alpha z = \alpha - 1$$
has no solution, is.

6. Sum of values of p for which, the equations: $x + y + z = 1$; $x + 2y + 4z = p$ and $x + 4y + 10z = p^2$ have a solution is.

7. Three distinct points $P(3u^2, 2u^3)$; $Q(3v^2, 2v^3)$ and $R(3w^2, 2w^3)$ are collinear then $uv + vw + wu$ is equal to.

8. Let $D_1 = \begin{vmatrix} a & b & a+b \\ c & d & c+d \\ a & b & a-b \end{vmatrix}$ and $D_2 = \begin{vmatrix} a & c & a+c \\ b & d & b+d \\ a & c & a+b+c \end{vmatrix}$ then the value of $\left|\dfrac{D_1}{D_2}\right|$ is where $b \neq 0$ and $ad \neq bc$,

9. If $\Delta = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$, then the value of $(\Delta_{\max})/2$ is.

10. If $\begin{vmatrix} x & x+y & x+y+z \\ 2x & 3x+2y & 4x+3y+2z \\ 3x & 6x+3y & 10x+6y+3z \end{vmatrix} = 64$, then the real value of x is.

11. If a_1, a_2, a_3, ..., a_{12} are in A.P. and
$\Delta_1 = \begin{vmatrix} a_1a_5 & a_1 & a_2 \\ a_2a_6 & a_2 & a_3 \\ a_3a_7 & a_3 & a_4 \end{vmatrix}$ $\Delta_3 = \begin{vmatrix} a_2a_{10} & a_2 & a_3 \\ a_3a_{11} & a_3 & a_4 \\ a_3a_{12} & a_4 & a_5 \end{vmatrix}$
then $\Delta_2 : \Delta_2 =$

12. If $\begin{vmatrix} x^n & x^{n+2} & x^{n+4} \\ y^n & y^{n+2} & y^{n+4} \\ z^n & z^{n+2} & z^{n+4} \end{vmatrix} = \left(\dfrac{1}{y^2} - \dfrac{1}{x^2}\right)\left(\dfrac{1}{z^2} - \dfrac{1}{y^2}\right)\left(\dfrac{1}{x^2} - \dfrac{1}{z^2}\right)$
then $-n$ is.

13. Given $A = \begin{vmatrix} a & b & 2c \\ d & e & 2f \\ l & m & 2n \end{vmatrix}$, $B = \begin{vmatrix} f & 2d & e \\ 2n & 4l & 2m \\ c & 2a & b \end{vmatrix}$, then the value of B/A is.

14. The value of $\begin{vmatrix} 2x_1y_1 & x_1y_2 + x_2y_1 & x_1y_3 + x_3y_1 \\ x_1y_2 + x_2y_1 & 2x_2y_2 & x_2y_3 + x_3y_2 \\ x_1y_3 + x_3y_1 & x_2y_3 + x_3y_2 & 2x_3y_3 \end{vmatrix}$ is.

15. If $(1 + ax + bx^2)^4 = a_0 + a_1x + a_2x^2 + \cdots + a_8x^8$, where $a, b, a_0, a_1, \ldots, a_8 \in R$ such that $a_0 + a_1 + a_2 \neq 0$ and $\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_0 \\ a_2 & a_0 & a_1 \end{vmatrix} = 0$ then the value of $5\frac{a}{b}$ is.

Archives

Solutions on page 7.62

Subjective Type

1. For what value of k do the following system of equations posses a non-trivial (i.e., not all zero) solution over the set of rationals Q?
$x + ky + 3z = 0$
$3x + ky - 2z = 0$
$2x + 3y - 4z = 0$
For that value of k, find all the solutions for the system. **(IIT-JEE, 1979)**

2. Let a, b, c be positive and not all equal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative. **(IIT-JEE, 1981)**

3. Without expanding a determinant at any stage, show that $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = xA + B$, where A and B are determinants of order 3 not involving x. **(IIT-JEE, 1982)**

4. Show that the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ has at least one solution for any real number λ. Find the set of solutions if $\lambda = -5$. **(IIT-JEE, 1983)**

5. Show that $\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix} = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$. **(IIT-JEE, 1985)**

6. Consider the system of linear equations in x, y and z given by $(\sin 3\theta)x - y + z = 0$, $(\cos 2\theta)x + 4y + 3z = 0$, $2x + 7y + 7z = 0$. Find the values of θ for which the system has a non-trivial solution. **(IIT-JEE, 1986)**

7. Let the three-digit numbers $A28$, $3B9$ and $62C$, where A, B, C are integers between 0 and 9, be divisible by a fixed integer k. Show that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is also divisible by the same integer k. **(IIT-JEE, 1990)**

8. If $a \neq p$, $b \neq q$, $c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, then find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$. **(IIT-JEE, 1991)**

9. For a fixed positive integer n, if $\Delta = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$, then show that $\left[\frac{\Delta}{(n!)^3} - 4\right]$ is divisible by n. **(IIT-JEE, 1992)**

10. Let λ and α be real. Find the set of all values of λ for which the system of linear equations $\lambda x + (\sin \alpha)y + (\cos \alpha)z = 0$, $x + (\cos \alpha)y + (\sin \alpha)z = 0$, $-x + (\sin \alpha)y - (\cos \alpha)z = 0$. **(IIT-JEE, 1993)**

11. For all values of A, B, C and P, Q, R show that
$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$$
(IIT-JEE, 1994)

12. Let $a > 0$, $d > 0$. Find the value of the determinant
$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$
(IIT-JEE, 1996)

13. Find the value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ where a, b and c are, respectively, the p^{th}, q^{th} and r^{th} terms of a harmonic progression. **(IIT-JEE, 1997)**

14. Prove that for all values of θ, the value of the determinant
$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$$
is 0. **(IIT-JEE, 2000)**

Objective Type

Fill in the blanks

1. Let $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda+3 \\ \lambda+1 & -2\lambda & \lambda-4 \\ \lambda-3 & \lambda+4 & 3\lambda \end{vmatrix}$ be an identity in λ, where p, q, r, s and t are constants. Then, the value of t is ______________. **(IIT-JEE, 1981)**

2. The solution set of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ is ______________. **(IIT-JEE, 1981)**

3. A determinant is chosen at random from the set of all determinants of order 2 with elements 0 or 1 only. The probability that the value of determinant chosen is positive is ______________. **(IIT-JEE, 1982)**

4. Given that $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$. The other two roots are ______________ and ______________. **(IIT-JEE, 1983)**

5. The system of equation
$\lambda x + y + z = 0$
$-x + \lambda y + z = 0$
$-x - y + \lambda z = 0$
will have a non-zero solution if real values of λ are given by ______________. **(IIT-JEE, 1984)**

6. The value of the determinant $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$ is ______________. **(IIT-JEE, 1988)**

7. For positive numbers x, y and z, the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is ______________. **(IIT-JEE, 1993)**

8. If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \frac{\pi}{2}\right\}$ is ______________. **(IIT-JEE, 2011)**

True or false

1. The determinants $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ and $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ are not identically equal. **(IIT-JEE, 1983)**

2. If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$, then the two triangles with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (a_1, b_1), (a_2, b_2), (a_3, b_3) must be congruent. **(IIT-JEE, 1985)**

Multiple choice questions with one correct answer

1. Consider the set A of all determinants of order 3 with entries 0 or 1 only. Let B be the subset of A consisting of all determinants with values -1. Then
   a. C is empty
   b. B has as many elements as C
   c. $A = B \cup C$
   d. B has twice as many elements as elements as C
   (IIT-JEE, 1981)

2. If $\omega\ (\ne 1)$ is a cube root of unity, then value of the determinant $\begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -i+\omega-1 & -1 \end{vmatrix}$ is
   a. 0 **b.** 1
   c. i **d.** ω
   (IIT-JEE, 1995)

3. Let a, b, c be the real numbers. Then following system of equations in x, y and z, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, has
   a. no solution **b.** unique solution
   c. infinitely many solutions **d.** finitely many solutions
   (IIT-JEE, 1995)

4. The determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$ if
   a. x, y, z are in A.P. **b.** x, y, z are in G.P.
   c. x, y, z are in H.P. **d.** xy, yz, zx are in A.P.
   (IIT-JEE, 1995)

5. The parameter, on which the value of the determinant $\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$ does not depend, is

a. a b. p
c. d d. x

(IIT-JEE, 1997)

6. If $f(x)=\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)\ x(x-1) \end{vmatrix}$ then $f(500)$ is equal to

a. 0 b. 1
c. 500 d. -500

(IIT-JEE, 1999)

7. If the system of equations $x-ky-z=0$, $kx-y-z=0$, $x+y-z=0$ has a non-zero solution then the possible values of k are

a. $-1, 2$ b. $1, 2$
c. $0, 1$ d. $-1, 1$

(IIT-JEE, 2000)

8. Let $\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$. Then the value of the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$$ is

a. 3ω b. $3\omega(\omega-1)$
c. $3\omega^2$ d. $3\omega(1-\omega)$

(IIT-JEE, 2002)

9. The number of values of k for which the system of equations $(k+1)x+8y=4k$; $kx+(k+3)y=3k-1$ has infinitely many solutions is

a. 0 b. 1
c. 2 d. infinite

(IIT-JEE, 2002)

10. If the system of equations $x+ay=0$, $az+y=0$ and $ax+z=0$ has infinite solutions, then the value of a is

a. -1 b. 1
c. 0 d. no real values

11. Given $2x-y+2z=2$, $x-2y+z=-4$, $x+y+\lambda z=4$ then the value of λ such that the given system of equation has no solution is

a. -3 b. 1
c. 0 d. 3

(IIT-JEE, 2004)

12. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix}=x+iy$, then

a. $x=3, y=1$ b. $x=1, y=3$
c. $x=0, y=3$ d. $x=0, y=0$

(IIT-JEE, 1998)

Multiple choice questions with one or more than one correct answer

1. The determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}=0$, if

a. a, b, c are in A.P.
b. a, b, c are in G.P.
c. a, b, c are in H.P.
d. α is a root of the equation $ax^2+bx+c=0$
e. $(x-\alpha)$ is a factor of $ax^2+2bx+c$

(IIT-JEE, 1986)

Matrix-match type

This question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example. If the correct matchs are a→p, a→s, b→q, b→r, c→q, d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1. Consider the following linear equations:

$ax+by+cz=0$
$bx+cy+az=0$
$cx+ay+bz=0$

Match the expressions/statements in column I with expressions/statements in column II. **(IIT-JEE, 2007)**

Column I	**Column II**
a. $a+b+c\neq 0$ and $a^2+b^2+c^2=ab+bc+ca$	**p.** the equations represent planes meeting only at a single point
b. $a+b+c=0$ and $a^2+b^2+c^2\neq ab+bc+ca$	**q.** the equations represent the line $x=y=z$
c. $a+b+c\neq 0$ and $a^2+b^2+c^2\neq ab+bc+ca$	**r.** the equations represent identical planes
d. $a+b+c=0$ and $a^2+b^2+c^2=ab+bc+ca$	**s.** the equations represent the whole of the three-dimensional space

Integer type

1. Let ω be the complex number $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$ is equal to

(IIT-JEE, 2010)

ANSWERS AND SOLUTIONS

Subjective Type

1. Applying $R_3 \to R_3 - R_2$, we have

$$\Delta = \begin{vmatrix} x^2-a^2 & x^2-b^2 & x^2-c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ 6x^2a+2a^3 & 6x^2b+2b^3 & 6x^2c+2c^3 \end{vmatrix}$$

Applying $R_3 \to \frac{1}{2}R_3$, then $R_2 \to R_2 + R_3$, we get

$$\Delta = 2\begin{vmatrix} x^2-a^2 & x^2-b^2 & x^2-c^2 \\ x^3+3xa^2 & x^3+3xb^2 & x^3+3xc^2 \\ 3x^2a+a^3 & 3x^2b+b^3 & 3x^2c+c^3 \end{vmatrix}$$

Applying $R_2 \to (1/x)\,R_1$ and then $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we obtain

$$\Delta = 2x\begin{vmatrix} x^2-a^2 & a^2-b^2 & a^2-c^2 \\ x^2+3a^2 & 3(b^2-a^2) & 3(c^2-a^2) \\ 3x^2a+a^3 & 3x^2(b-a)+b^3-a^3 & 3x^2(c-a)+c^3-a^3 \end{vmatrix}$$

$$= 2x(b-a)(c-a)\begin{vmatrix} x^2-a^2 & -(a+b) & -(a+c) \\ x^2+3a^2 & 3(b+a) & 3(c+a) \\ 3x^2a+a^3 & 3x^2+b^2+ab+a^2 & 3x^2+c^2+a^2+ac \end{vmatrix}$$

Applying $R_2 \to R_2 + 3R_1$, and $R_3 \to R_3 + aR_1$, we get

$$\Delta = 2x\,(b-a)(c-a)\times\begin{vmatrix} x^2-a^2 & -(a+b) & -(a+c) \\ 4x^2 & 0 & 0 \\ 4x^2a & 3x^2+b^2 & 3x^2+c^2 \end{vmatrix}$$

Expanding along R_2, we have

$$\Delta = 8x^3(b-a)(c-a)\{(a+b)(3x^2+c^2)-(a+c)(3x^2+b^2)\}$$
$$= 8x^3(b-a)(c-a)\{3x^2(b-c)+ac^2+bc^2-ab^2-cb^2\}$$
$$= 8x^3(b-a)(c-a)\{3x^2(b-c)+bc(c-b)+a(c^2-b^2)\}$$
$$= 8x^3(b-a)(c-a)(b-c)\{3x^2-(bc+ac+ab)\}$$

As a, b, c are distinct, $\Delta = 0$ gives $x = 0$ or $x^2 = 1/3(bc + ca + ab)$. If $ab + bc + ca \le 0$, the only real root is $x = 0$. If $ab + bc + ca > 0$, roots are $x = 0$, $\pm\sqrt{\frac{1}{3}(bc+ca+ab)}$.

2. Applying $C_3 \to C_3 - (C_2 - C_1)$ and $C_4 \to C_4 - (C_1 + C_2)$, we get

$$\Delta = \begin{vmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ a-b & b-c & a+c-2b & 0 \\ x & y & z+x-y & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a & c & 0 \\ c & b & 0 \\ a-b & b-c & a+c-2b \end{vmatrix} \quad \text{(expanding along } C_4\text{)}$$

$= (a + c - 2b)(ab - c^2)$ (expanding along C_3)

$\therefore\ \Delta = 0$

$\Rightarrow\ a + c - 2b = 0$

or $ab - c^2 = 0$

$\Rightarrow$ a, b, c are in A.P. or a, c, b are in G.P.

3. We have $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

$$\text{L.H.S.} = \frac{1}{(a-b)(b-c)(c-a)} \times\left[\frac{f(a)(c-b)}{(x-a)}+\frac{f(b)(a-c)}{(x-b)}+\frac{f(c)(b-a)}{(x-c)}\right]$$

(expanding along C_3)

From R.H.S. by partial fractions, we get

$$\text{R.H.S.} = \frac{f(x)}{(x-a)(x-b)(x-c)}$$
$$= \frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c} \quad \text{[since degree of } f(x) \text{ is less than 3]}$$

Then,

$$A = \left[\frac{f(x)}{(x-b)(x-c)}\right]_{x=a} = \frac{f(a)}{(a-b)(a-c)}$$

Similarly,

$$B = \frac{f(b)}{(b-a)(b-c)}$$

and

$$C = \frac{f(c)}{(c-a)(c-b)}$$

$$\therefore\ \text{R.H.S.} = \frac{1}{(a-b)(b-c)(c-a)} \times\left[\frac{(c-b)f(a)}{(x-b)}+\frac{(a-c)f(b)}{(x-b)}+\frac{(b-a)f(c)}{(x-c)}\right]$$

Hence, L.H.S. = R.H.S.

4.
$$D = \begin{vmatrix} a_p+a_{p+m}+a_{p+2m} & 2a_p+3a_{p+m}+4a_{p+2m} & 4a_p+9a_{p+m}+16a_{p+2m} \\ a_q+a_{q+m}+a_{q+2m} & 2a_q+3a_{q+m}+4a_{q+2m} & 4a_q+9a_{q+m}+16a_{q+2m} \\ a_r+a_{r+m}+a_{r+2m} & 2a_r+3a_{r+m}+4a_{r+2m} & 4a_r+9a_{r+m}+16a_{r+2m} \end{vmatrix}$$

$$= \begin{vmatrix} a_p + a_{p+m} + a_{p+2m} & a_{p+m} + 2a_{p+2m} & 5a_{p+m} + 12a_{p+2m} \\ a_q + a_{q+m} + a_{q+2m} & a_{q+m} + 2a_{q+2m} & 5a_{q+m} + 12a_{q+2m} \\ a_r + a_{r+m} + a_{r+2m} & a_{r+m} + 2a_{r+2m} & 5a_{r+m} + 12a_{r+2m} \end{vmatrix}$$

[Applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 - 4C_1$]

$$= \begin{vmatrix} a_p - a_{p+2m} & a_{p+m} + 2a_{p+2m} & 2a_{p+2m} \\ a_q - a_{q+2m} & a_{q+m} + 2a_{q+2m} & 2a_{q+2m} \\ a_r - a_{r+2m} & a_{r+m} + 2a_{r+2m} & 2a_{r+2m} \end{vmatrix}$$

[Applying $C_1 \to C_1 - C_2$, $C_3 \to C_3 - 5C_2$]

Applying $C_2 \to C_2 - C_3$, then $C_1 \to C_1 + \frac{1}{2}C_3$, then taking 2 common from C_3, we get

$$D = 2\begin{vmatrix} a_p & a_{p+m} & a_{p+2m} \\ a_q & a_{q+m} & a_{q+2m} \\ a_r & a_{r+m} & a_{r+2m} \end{vmatrix}$$

$$= 2\begin{vmatrix} a_p + a_{p+2m} - 2a_{p+m} & a_{p+m} & a_{p+2m} \\ a_q + a_{q+2m} - 2a_{q+m} & a_{q+m} & a_{q+2m} \\ a_r + a_{r+2m} - 2a_{r+m} & a_{r+m} & a_{r+2m} \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_3 - 2C_2$]

$$= 2\begin{vmatrix} 0 & a_{p+m} & a_{p+2m} \\ 0 & a_{q+m} & a_{q+2m} \\ 0 & a_{r+m} & a_{r+2m} \end{vmatrix}$$

(as a_p, a_{p+m}, a_{p+2m} are in A.P. like others)

$= 0$

5. We know that

$${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$$

$$\therefore\ \Delta(n, r) = \begin{vmatrix} \frac{n}{r}\,{}^{n-1}C_{r-1} & \frac{n}{r+1}\,{}^{n-1}C_r & \frac{n}{r+2}\,{}^{n-1}C_{r+1} \\ \frac{n+1}{r}\,{}^{n}C_{r-1} & \frac{n+1}{r+1}\,{}^{n}C_r & \frac{n+1}{r+2}\,{}^{n}C_{r+1} \\ \frac{n+2}{r}\,{}^{n+1}C_{r-1} & \frac{n+2}{r+1}\,{}^{n+1}C_r & \frac{n+2}{r+2}\,{}^{n+1}C_{r+1} \end{vmatrix}$$

$$= \frac{n(n+1)(n+2)}{r(r+1)(r+2)}\Delta(n-1,\ r-1)$$

$$= \frac{{}^{n+2}C_3}{{}^{r+2}C_3}\Delta(n-1,\ r-1) \qquad (1)$$

Repeating the process, we have

$$\Delta(n, r) = \frac{{}^{n+2}C_3}{{}^{r+2}C_3}\,\frac{{}^{n+1}C_3}{{}^{r+1}C_3}\,\Delta(n-2,\ r-2)$$

$$= \frac{{}^{n+2}C_3}{{}^{r+2}C_3}\,\frac{{}^{n+1}C_3}{{}^{r+1}C_3}\,\frac{{}^{n}C_3}{{}^{r}C_3}\cdots\frac{{}^{n-r+3}C_3}{{}^{3}C_3}\,\Delta(n-r, 0) \qquad (2)$$

Now,

$$\Delta(n-r, 0) = \begin{vmatrix} {}^{n-r}C_0 & {}^{n-r}C_1 & {}^{n-r}C_2 \\ {}^{n-r+1}C_0 & {}^{n-r+1}C_1 & {}^{n-r+1}C_2 \\ {}^{n-r+2}C_0 & {}^{n-r+2}C_1 & {}^{n-r+2}C_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & n-r & \frac{1}{2}(n-r)(n-r-1) \\ 1 & n-r+1 & \frac{1}{2}(n-r+1)(n-r) \\ 1 & n-r+2 & \frac{1}{2}(n-r+2)(n-r+1) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & n-r & \frac{1}{2}(n-r)(n-r-1) \\ 0 & 1 & n-r \\ 0 & 1 & n-r+1 \end{vmatrix}$$

$(R_3 \to R_3 - R_2,\ R_2 \to R_2 - R_1)$

$$= n - r + 1 - n + r = 1 \qquad (3)$$

Hence from (2) and (3), we get

$$\Delta(n, r) = \frac{({}^{n+2}C_3)({}^{n+1}C_3)\cdots({}^{n-r+3}C_3)}{({}^{r+2}C_3)({}^{r+1}C_3)\cdots({}^{3}C_3)}$$

6. The given system of equations will have a non-trivial solution if the determinant of coefficients

$$\Delta = \begin{vmatrix} a-t & b & c \\ b & c-t & a \\ c & a & b-t \end{vmatrix} = 0 \qquad (1)$$

$\Delta = 0$ is a cubic equation (equation of degree 3) in t so it has in general 3 solutions. Let t_1, t_2 and t_3 be the solutions and

$$\Delta = p_0t^3 + p_1t^2 + p_2t + p_3 \qquad (2)$$

Clearly, coefficient of t^3 is $p_0 = -1$. So

$t_1t_2t_3 = -\frac{p_3}{(-1)} = p_3$ [constant term in the expansion of Δ, i.e. $\Delta(t = 0)$]

Hence,

$$t_1t_2t_3 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

7. Clearly $\alpha_1 + \alpha_2 = -b/a$, $\alpha_1\alpha_2 = c/a$ and $\beta_1 + \beta_2 = -q/p$, $\beta_1\beta_2 = r/p$.

System of equations $\alpha_1 y + \alpha_2 z = 0$, $\beta_1 y + \beta_2 z = 0$ has a non-trivial solution. So,

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0, \text{ i.e., } \alpha_1\beta_2 - \alpha_2\beta_1 = 0$$

or $\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2}$

$$\Rightarrow \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} = \sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}$$

$$\Rightarrow \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} = \sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}$$

$$\Rightarrow \quad \frac{-b/a}{-q/p} = \sqrt{\frac{c/a}{r/p}}$$

$$\Rightarrow \quad b^2pr = q^2ac$$

8. Determinant of coefficients is

$$\Delta = \begin{vmatrix} 2\sin A\cos A & \sin(A+B) & \sin(A+C) \\ \sin(A+B) & 2\sin B\cos B & \sin(B+C) \\ \sin(A+C) & \sin(B+C) & 2\sin C\cos C \end{vmatrix}$$

$$= \begin{vmatrix} \sin A\cos A+\sin A\cos A & \sin A\cos B+\sin B\cos A & \sin A\cos C+\sin C\cos A \\ \sin A\cos B+\sin B\cos A & 2\sin B\cos B & \sin B\cos C+\sin C\cos B \\ \sin A\cos C+\sin C\cos A & \sin B\cos C+\sin C\cos B & 2\sin C\cos C \end{vmatrix}$$

$$= \begin{vmatrix} \sin A\cos A+\sin A\cos A & \sin A\cos B+\sin B\cos A & \sin A\cos C+\sin C\cos A \\ \sin A\cos B+\sin B\cos A & \sin B\cos B+\sin B\cos B & \sin B\cos C+\sin C\cos B \\ \sin A\cos C+\sin C\cos A & \sin B\cos C+\sin C\cos B & \sin C\cos C+\sin C\cos C \end{vmatrix}$$

$$= \begin{vmatrix} \sin A & \cos A & 0 \\ \sin B & \cos B & 0 \\ \sin C & \cos C & 0 \end{vmatrix} \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix}$$

$$= 0$$

9. Let, $D = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$\therefore \quad D^2 = D \times D$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} ax_1 & by_1 & cz_1 \\ ax_2 & by_2 & cz_2 \\ ax_3 & by_3 & cz_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} ax_1^2+by_1^2+cz_1^2 & ax_1x_2+by_1y_2+cz_1z_2 & ax_3x_1+by_3y_1+cz_3z_1 \\ ax_1x_2+by_1y_2+cz_1z_2 & ax_2^2+by_2^2+cz_2^2 & ax_2x_3+by_2y_3+cz_2z_3 \\ ax_3x_1+by_3y_1+cz_3z_1 & ax_2x_3+by_2y_3+cz_2z_3 & ax_3^2+by_3^2+cz_3^2 \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} d & f & f \\ f & d & f \\ f & f & d \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} d+2f & f & f \\ d+2f & d & f \\ d+2f & f & d \end{vmatrix} \qquad [C_1 \to C_1 + C_2 + C_3]$$

$$= \frac{(d+2f)}{abc} \begin{vmatrix} 1 & f & f \\ 1 & d & f \\ 1 & f & d \end{vmatrix}$$

$$= \frac{(d+2f)}{abc} \begin{vmatrix} 1 & f & f \\ 0 & d-f & 0 \\ 1 & f & d \end{vmatrix}$$

$$= \frac{(d+2f)}{abc}(d-f) \begin{vmatrix} 1 & f \\ 1 & d \end{vmatrix} \qquad [R_2 \to R_2 - R_1]$$

$$= \frac{(d+2f)}{abc}(d-f)(d-f)$$

$$= \frac{(d+2f)(d-f)^2}{abc}$$

$$\Rightarrow D = (d-f)\left\{\frac{d+2f}{abc}\right\}^{1/2} = \text{R.H.S.}$$

10. $$\Delta = \begin{vmatrix} a_1b_1+b_1a_1 & a_1b_2+a_2b_1 & a_1b_3+b_1a_3 \\ a_2b_1+b_2a_1 & a_2b_2+a_2b_2 & a_2b_3+a_3b_2 \\ a_3b_1+b_3a_1 & a_3b_2+b_3a_2 & a_3b_3+a_3b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} b_1 & a_1 & 0 \\ b_2 & a_2 & 0 \\ b_3 & a_3 & 0 \end{vmatrix} = 0 \qquad (1)$$

(row by row multiplication)

Now,

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$= (lx + my + n)(l'x + my + n')$

$= ll'x^2 + (lm' + ml')xy + mm'y^2 + (ln' + l'n)x + (mn' + m'n)y + nn'$

Comparing the coefficients, we get

$a = ll'$, $h = \frac{1}{2}(lm' + ml')$, $b = mm'$, $g = \frac{1}{2}(ln' + l'n)$,

$f = \frac{1}{2}(mn' + m'n)$, $c = nn'$

$$\therefore \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} ll' & \frac{1}{2}(lm'+ml') & \frac{1}{2}(ln'+l'n) \\ \frac{1}{2}(lm'+ml') & mm' & \frac{1}{2}(mn'+m'n) \\ \frac{1}{2}(ln'+l'n) & \frac{1}{2}(mn'+m'n') & nn' \end{vmatrix}$$

$$= \frac{1}{8}\begin{vmatrix} 2ll' & lm'+l'm & ln'+l'n \\ lm'+l'm & 2mm' & mn'+m'n \\ ln'+l'n & mn'+m'n & 2nn' \end{vmatrix}$$

$= 0$ [From (1)]

11. Putting $s-a=\alpha$, $s-b=\beta$, $s-c=\gamma$, we get

$a = 2s-b-c = (s-b)+(s-c) = \alpha+\beta$

Similarly, $b=\gamma+\alpha$, $c=\alpha+\beta$. Also,

$\alpha+\beta+\gamma = 3s-(a+b+c) = 3s-2s = s$

$$\therefore\quad \Delta = \begin{vmatrix} (\beta+\gamma)^2 & \alpha^2 & \alpha^2 \\ \beta^2 & (\gamma+\alpha)^2 & \beta^2 \\ \gamma^2 & \gamma^2 & (\alpha+\beta)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (\beta+\gamma)^2 & \alpha^2-(\beta+\gamma)^2 & \alpha^2-(\beta+\gamma)^2 \\ \beta^2 & (\gamma+\alpha)^2-\beta^2 & 0 \\ \gamma^2 & 0 & (\alpha+\beta)^2-\gamma^2 \end{vmatrix}$$

$[C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1]$

$$= (\alpha+\beta+\gamma)^2 \times \begin{vmatrix} (\beta+\gamma)^2 & \alpha-\beta-\gamma & \alpha-\beta-\gamma \\ \beta^2 & \gamma+\alpha-\beta & 0 \\ \gamma^2 & 0 & \alpha+\beta-\gamma \end{vmatrix}$$

[Taking $(\alpha+\beta+\gamma)$ common from C_2 and C_3]

$$= 2(\alpha+\beta+\gamma)^2 \times \begin{vmatrix} \beta\gamma & -\gamma & -\beta \\ \beta^2 & \gamma+\alpha-\beta & 0 \\ \gamma^2 & 0 & \alpha+\beta-\gamma \end{vmatrix}$$

$[R_1 \to R_1 - R_2 - R_3$ and then $R_1 \to \frac{1}{2}R_1]$

Now applying $C_2 \to \beta C_2 + C_1$ and $C_3 \to \gamma C_3 + C_1$, we get

$$\Delta = \frac{2(\alpha+\beta+\gamma)^2}{\beta\gamma} \times \begin{vmatrix} \beta\gamma & 0 & 0 \\ \beta^2 & \beta(\gamma+\alpha) & \beta^2 \\ \gamma^2 & \gamma^2 & \gamma(\alpha+\beta) \end{vmatrix}$$

$$= \frac{2(\alpha+\beta+\gamma)^2}{\beta\gamma}\,\beta\gamma\,[(\beta\gamma+\beta\alpha)(\gamma\alpha+\gamma\beta)-\beta^2\gamma^2]$$

$= 2(\alpha+\beta+\gamma)^2\,[(\alpha\beta\gamma^2+\beta^2\gamma^2+\alpha^2\beta\gamma+\alpha\beta^2\gamma-\beta^2\gamma^2]$

$= 2(\alpha+\beta+\gamma)^3\,\alpha\beta\gamma$

$= 2s^3(s-a)(s-b)(s-c)$

12.
$$\Delta = \begin{vmatrix} x & \frac{1}{2}x(x-1) & \frac{1}{6}x(x-1)(x-2) \\ y & \frac{1}{2}y(y-1) & \frac{1}{6}y(y-1)(y-2) \\ z & \frac{1}{2}z(z-1) & \frac{1}{6}z(z-1)(z-2) \end{vmatrix}$$

$$= \frac{1}{12}xyz\begin{vmatrix} 1 & x-1 & x^2-3x+2 \\ 1 & y-1 & y^2-3y+2 \\ 1 & z-1 & z^2-3z+2 \end{vmatrix}$$

$$= \frac{1}{12}xyz\begin{vmatrix} 1 & x & x^2-3x+2 \\ 1 & y & y^2-3y+2 \\ 1 & z & z^2-3z+2 \end{vmatrix} \qquad [R_2 \to R_2 + R_1]$$

$$= \frac{1}{12}xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \qquad [R_3 \to R_3 + 3R_2 - 2R_1]$$

$$= \frac{1}{12}xyz(x-y)(y-z)(z-x)$$

13. Let, $\Delta = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$

Putting $a+b=0$ or $b=-a$, we get

$$\Delta = \begin{vmatrix} -2a & 0 & a+c \\ 0 & 2a & c-a \\ c+a & c-a & -2c \end{vmatrix}$$

Expanding along R_1,

$\Delta = -2a\{-4ac-(c-a)^2\} - 0 + (a+c)\{0-2a(c+a)\}$

$= 2a(c+a)^2 - 2a(c+a)^2$

$= 0$

Hence $a+b$ is a factor of Δ. Similarly $b+c$ and $c+a$ are the factors of Δ.

On expansion of determinant, we can see that each term of the determinant is a homogeneous expression in a, b, c of degree 3 and also R.H.S. is a homogeneous expression of degree 3.

$\therefore\quad \Delta = k(a+b)(b+c)(c+a)$

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = k(a+b)(b+c)(c+a)$$

Putting $a=0$, $b=1$, $c=2$, we get

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = k(0+1)(1+2)(2+0)$$

$\Rightarrow\quad 0-1(-4-6)+2(3+4)=6k$

$\Rightarrow\quad 24 = 6k$

$\therefore\quad k = 4$

Hence,

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$$

14. Applying $C_1 \to xC_1 + yC_2 + zC_3$ [to get the term $(x^2+y^2+z^2)$], we get

$$\Delta = \frac{1}{x}\begin{vmatrix} a(x^2+y^2+z^2) & ay+bx & cx+az \\ b(x^2+y^2+z^2) & by-cz-ax & bz+cy \\ c(x^2+y^2+z^2) & bz+cy & cz-ax-by \end{vmatrix}$$

Now taking $(x^2+y^2+z^2)$ common from C_1 and then applying $R_1 \to aR_1 + bR_2 + cR_2$ [to get the term $(a^2+b^2+c^2)$], we get

$$\Delta = \frac{(x^2+y^2+z^2)}{ax}$$

$$\times \begin{vmatrix} (a^2+b^2+c^2) & y(a^2+b^2+c^2) & z(a^2+b^2+c^2) \\ b & by-cz-ax & bz+cy \\ c & bz+cy & cz-ax-by \end{vmatrix}$$

Taking $(a^2+b^2+c^2)$ common from R_1 and then applying $C_2 \to C_2 - yC_1$ and $C_3 \to C_3 - zC_1$, we get

$$\Delta = \frac{(x^2+y^2+z^2)(a^2+b^2+c^2)}{ax} \times \begin{vmatrix} 1 & 0 & 0 \\ b & -cz-ax & cy \\ c & bz & -ax-by \end{vmatrix}$$

Now expanding along R_1, we get

$$\Delta = \frac{1}{ax}(x^2+y^2+z^2)(a^2+b^2+c^2)$$
$$\times [aczx + bczy + a^2x^2 + abxy - bcyz]$$
$$= (x^2+y^2+z^2)(a^2+b^2+c^2)(ax+by+cz)$$

15. $$\Delta(x) = \begin{vmatrix} a_1+x & b_1+x & c_1+x \\ a_2+x & b_2+x & c_2+x \\ a_3+x & b_3+x & c_3+x \end{vmatrix}$$

$$\therefore \quad \Delta'(x) = \begin{vmatrix} 1 & b_1+x & c_1+x \\ 1 & b_2+x & c_2+x \\ 1 & b_3+x & c_3+x \end{vmatrix} + \begin{vmatrix} a_1+x & 1 & c_1+x \\ a_2+x & 1 & c_2+x \\ a_3+x & 1 & c_3+x \end{vmatrix}$$
$$+ \begin{vmatrix} a_1+x & b_1+x & 1 \\ a_2+x & b_2+x & 1 \\ a_3+x & b_3+x & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - xC_1$ and $C_3 \to C_3 - xC_1$ in the first determinant of R.H.S., $C_1 \to C_1 - xC_2$ and $C_3 \to C_3 - xC_2$ in the second determinant and $C_1 \to C_1 - xC_3$ and $C_2 \to C_2 - xC_3$ in the third determinant, we get

$$\Delta'(x) = \begin{vmatrix} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & 1 & c_1 \\ a_2 & 1 & c_2 \\ a_3 & 1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

Now consider the cofactors of

$$\Delta(0) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which are $b_2c_3 - b_3c_2$, $c_2a_3 - c_3a_2$, $a_2b_3 - b_2a_3$, etc. Clearly,

$$\begin{vmatrix} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2) + (c_1b_3 - c_3b_1) + (b_1c_2 - b_2c_1)$$

which is the sum of cofactors of the first row elements of $\Delta(0)$. Similarly,

$$\begin{vmatrix} a_1 & 1 & c_1 \\ a_2 & 1 & c_2 \\ a_3 & 1 & c_3 \end{vmatrix} \text{ and } \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

are the sum of cofactors of secong row and third row elements, respectively, of $\Delta(0)$. Hence $\Delta'(x) = S$, where S denotes the sum of all cofactors of elements of $\Delta(0)$.

$\therefore \quad \Delta''(x) = 0$

Since $\Delta'(x) = S$, $\Delta(x) = Sx + k$. So,

$\Delta(0) = k$

Hence,

$\Delta(x) = xS + \Delta(0)$

Objective Type

1. a. $$\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix}$$

$$= pqr(a^3+b^3+c^3-3abc) - abc(p^3+q^3+r^3-3pqr)$$
$$= pqr(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$
$$- abc(p+q+r)(p^2+q^2+r^2-pq-qr-pr)$$
$$= 0$$

2. b. Let $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Applying $C_2 \to C_2 - \frac{a_{12}}{a_{11}}C_1$, $C_3 \to C_3 - \frac{a_{13}}{a_{11}}C_1$, we get

$$D = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & \left(a_{22} - \frac{a_{12}}{a_{11}} \times a_{21}\right) & \left(a_{23} - \frac{a_{13}}{a_{11}} a_{21}\right) \\ a_{31} & \left(a_{32} - \frac{a_{12}}{a_{11}} \times a_{31}\right) & \left(a_{32} - \frac{a_{13}}{a_{11}} \times a_{31}\right) \end{vmatrix}$$

which has minimum value of –4.

3. b. $$z = \begin{vmatrix} -5 & 3+4i & 5-7i \\ 3-4i & 6 & 8+7i \\ 5+7i & 8-7i & 9 \end{vmatrix}$$

$$\Rightarrow \quad \bar{z} = \begin{vmatrix} -5 & 3-4i & 5+7i \\ 3+4i & 6 & 8-7i \\ 5-7i & 8+7i & 9 \end{vmatrix} = \begin{vmatrix} -5 & 3+4i & 5-7i \\ 3-4i & 6 & 8+7i \\ 5+7i & 8-7i & 9 \end{vmatrix} = z$$

(Taking transpose)

$\Rightarrow \quad z$ is purely real

4. c. Operation $C_1 \to C_1 + C_2 + C_3$ gives $(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$\begin{vmatrix} 1 & \beta\gamma & \gamma\alpha \\ 1 & \gamma\alpha & \alpha\beta \\ 1 & \alpha\beta & \beta\gamma \end{vmatrix}$$

From the given equation, $\alpha\beta + \beta\gamma + \gamma\alpha = 0$. So, the value of determinant is 0.

5. c.

$$f(x) = \begin{vmatrix} 1-2\sin^2 x & \sin^2 x & 1-8\sin^2 x(1-\sin^2 x) \\ \sin^2 x & 1-2\sin^2 x & 1-\sin^2 x \\ 1-8\sin^2 x(1-\sin^2 x) & 1-\sin^2 x & 1-2\sin^2 x \end{vmatrix}$$

The required constant term is

$$f(0) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1(0-1) = -1$$

6. a. $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2] = 0$

$\Rightarrow \quad a+b+c=0$ or $a=b=c$

If $a+b+c=0$, we have

$\cos\theta + \cos 2\theta + \cos 3\theta = 0$ and $\sin\theta - \sin 2\theta + \sin 3\theta = 0$

$\Rightarrow \quad \cos 2\theta(2\cos\theta+1) = 0$ and $\sin 2\theta(1-2\cos\theta) = 0$ (i)

which is not possible as $\cos 2\theta = 0$ gives $\sin 2\theta \neq 0$, $\cos\theta \neq 1/2$. And $\cos\theta = -1/2$ gives $\sin 2\theta \neq 0$, $\cos\theta \neq 1/2$. Therefore, Eq. (i) does not hold simultaneously.

$\therefore \quad a+b+c \neq 0$

$\therefore \quad a=b=c$

or

$e^{i\theta} = e^{-2i\theta} = e^{3i\theta}$

which is satisfied only by $e^{i\theta} = 1$, i.e., $\cos\theta = 1$, $\sin\theta = 0$ so $\theta = 2k\pi$, $k \in Z$.

7. b. Applying $C_1 \to aC_1$ and then $C_1 \to C_1 + bC_2 + cC_3$, and taking $(a^2+b^2+c^2)$ common from C_1, we get

$$\Delta = \frac{(a^2+b^2+c^2)}{a}\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix}$$

$$= \frac{(a^2+b^2+c^2)}{a}\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix}$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$= \frac{(a^2+b^2+c^2)}{a}(-bc+a^2+ab+ac+bc)$ (expanding along C_1)

$= (a^2+b^2+c^2)(a+b+c)$

Hence, $\Delta = 0 \Rightarrow a+b+c=0$

Therefore, line $ax+by+c=0$ passes through the fixed point (1, 1).

8. b. The degree of the determinant is $n+(n+2)+(n+3) = 3n+5$ and the degree of the expression on R.H.S. is 2.

$\therefore \quad 3n+5 = 2 \Rightarrow n = -1$

9. d. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$

$= -(a+b+c)(a+b\omega^2+c\omega)(a+b\omega+c\omega^2)$

(where ω is cube roots of unity)

$= -f(\alpha)f(\beta)f(\gamma) \qquad [\because \alpha = 1, \beta = \omega, \gamma = \omega^2]$

10. c. $\because -1 \le x < 0 \quad \therefore [x] = -1$

$0 \le y < 1 \quad \therefore [y] = 0$

$1 \le z < 2 \quad \therefore [z] = 1$

Hence, the given determinant is

$$\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 1 = [z]$$

11. a. Given determinant,

$2a(bc-4a^2) + b(2ac-b^2) + c(2ab-c^2) = 0$

$\Rightarrow \quad 6abc - 8a^3 - b^3 - c^3 = 0$

$\Rightarrow \quad (2a+b+c)[(2a-b)^2+(b-c)^2+(c-2a)^2] = 0$

$\Rightarrow \quad 2a+b+c = 0 \qquad (\because b \neq c)$

Let $f(x) = 8ax^3 + 2bx^2 + cx$

$f(0) = 0$

$f\left(\frac{1}{2}\right) = a + \frac{b}{2} + \frac{c}{2} = \frac{2a+b+c}{2} = 0$

So, $f(x)$ satisfies the Rolle's theorem and hence, $f'(x) = 0$ has at least one root in $\left[0, \frac{1}{2}\right]$.

12. b. The given determinant is

$$\begin{bmatrix} 2^{n+1}-2^n+p & 2^{n+2}-2^{n+1}+q & p+r \\ 2^n+p & 2^{n+1} & p+r \\ a^2+2^n+p & b^2+2^n+2q & c^2-r \end{bmatrix}$$

(Using $R_1 \to R_1 - R_3$ and $2q = p+r$)

$$\begin{bmatrix} 2^n(2-1)+p & 2^{n+1}(2-1)+q & p+r \\ 2^n+p & 2^{n+1}+q & p+r \\ a^2+2^n+p & b^2+2^{n+1}+2q & c^2-r \end{bmatrix}$$

$$=\begin{vmatrix} 2^n+p & 2^{n+1}+q & p+r \\ 2^n+p & 2^{n+1}+q & p+r \\ a^2+2^n+p & b^2+2^{n+1}+2q & c^2-r \end{vmatrix}=0 \quad (\because R_1 \equiv R_2)$$

13. c. Consider the triangle with vertices $B(x_1, y_1)$, $C(x_2, y_2)$ and $A(x_3, y_3)$, and $AB = c$, $BC = a$ and $AC = b$. Then area of triangle is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=\sqrt{s(s-a)(s-b)(s-c)} \text{ where } 2s=a+b+c$$

Squaring and simplifying, we get

$$4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

Hence, $k=4$.

14. c. We have,

$$\begin{vmatrix} ka & k^2+a^2 & 1 \\ kb & k^2+b^2 & 1 \\ kc & k^2+c^2 & 1 \end{vmatrix}=\begin{vmatrix} ka & k^2 & 1 \\ kb & k^2 & 1 \\ kc & k^2 & 1 \end{vmatrix}+\begin{vmatrix} ka & a^2 & 1 \\ kb & b^2 & 1 \\ kc & c^2 & 1 \end{vmatrix}=0+k\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$=k(a-b)(b-c)(c-a)$$

15. b. We have,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c) \qquad (1)$$

Also,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$=abc\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ 1 & 1 & 1 \\ a^2 & b^2 & c^2 \end{vmatrix} \text{(taking } a,b,c \text{ common from } R_1, R_2, R_3)$$

$$=\begin{vmatrix} bc & ac & ab \\ 1 & 1 & 1 \\ a^2 & b^2 & c^2 \end{vmatrix} \text{(Multiplying } R_1 \text{ by } abc)$$

$$=\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ac & ab \end{vmatrix}$$

Then,

$$D=\begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(3x-a-b-c)$$

Now given that a, b, c are all different, then $D=0$.

$$\therefore \quad x=\frac{1}{3}(a+b+c)$$

16. d. Let, $\Delta=\begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix}$

Then,

$$\Delta=\frac{1}{xy}\begin{vmatrix} xy^2 & -xy & x^2y \\ ax & b & cy \\ a'x & b' & c'y \end{vmatrix} \quad \text{[Applying } C_1 \to xC_1, C_3 \to yC_3]$$

$$=\frac{1}{xy}\begin{vmatrix} 0 & -xy & 0 \\ ax+by & b & bx+cy \\ a'x+b'y & b' & b'x+c'y \end{vmatrix}$$

[Applying $C_1 \to C_1 + yC_2$, $C_3 \to C_3 + xC_2$]

$$=\frac{1}{xy}xy\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix} \quad \text{[Expanding along } R_1]$$

$$=\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$$

17. b. $\Delta=\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$

Applying $R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$, we get

$$\Delta=\frac{1}{abc}\times\begin{vmatrix} b^2+c^2 & a^2b & a^2c \\ ab^2 & b(c^2+a^2) & cb^2 \\ ac^2 & bc^2 & c(a^2+b^2) \end{vmatrix}$$

Now, applying $C_1 \to \frac{1}{a}C_1$, $C_2 \to \frac{1}{b}C_2$, $C_3 \to \frac{1}{c}C_3$, we get

$$\Delta=\frac{abc}{abc}\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$=\begin{vmatrix} 0 & -2c^2 & -2b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} \quad [R_1 \to R_1 - R_2 - R_3]$$

$$=2\begin{vmatrix} 0 & -c^2 & -b^2 \\ b^2 & a^2 & 0 \\ c^2 & 0 & a^2 \end{vmatrix}$$

(Taking 2 common from R_1 and applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$)

Evaluating along R_1, we get

$$\Delta = 2[c^2(a^2b^2) - b^2(-a^2c^2)]$$
$$= 4a^2b^2c^2$$

Hence, $k = 4$.

18. d. Applying $R_1 \to aR_1$, $R_2 \to bR_2$ and $R_3 \to cR_3$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ a^2bc^2 & abc & bc+ab \\ a^2b^2c & abc & ac+bc \end{vmatrix}$$

$$= \frac{a^2b^2c^2}{abc}\begin{vmatrix} bc & 1 & ab+ac \\ ac & 1 & bc+ab \\ ab & 1 & ac+bc \end{vmatrix}$$

Applying $C_3 \to C_3 + C_1$ and taking $(bc + ca + ab)$ common, we get

$$\Delta = abc(bc+ca+ab)\begin{vmatrix} bc & 1 & 1 \\ ac & 1 & 1 \\ ab & 1 & 1 \end{vmatrix} = 0$$

[$\because$ C_2 and C_3 are identical]

19. b. Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} -4-2\sqrt{2} & -2\sqrt{2} & 0 \\ 4\sqrt{2} & 4\sqrt{2} & 0 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$$

$$= 1(-(4+2\sqrt{2}))\,4\sqrt{2} + 2\sqrt{2} \times 4\sqrt{2}$$
$$= -16\sqrt{2}$$

20. b. The total number of third-order determinants is 9! Since the number of determinants is even and in which there are 9!/2 pairs of determinants which are obtained by changing two consecutive rows,

so $\sum_{i=1}^{n} D_i = 0$.

21. a. $$\Delta = \begin{vmatrix} a_1+b_1w & a_1w^2+b_1 & c_1+b_1\bar{w} \\ a_2+b_2w & a_2w^2+b_2 & c_2+b_2\bar{w} \\ a_3+b_3w & a_3w^2+b_3 & c_3+b_3\bar{w} \end{vmatrix}$$

Operating $C_2 \to wC_2$, we have

$$\Delta = \frac{1}{w}\begin{vmatrix} a_1+b_1w & a_1w^3+b_1w & c_1+b_1\bar{w} \\ a_2+b_2w & a_2w^3+b_2w & c_2+b_2\bar{w} \\ a_3+b_3w & a_3w^3+b_3w & c_3+b_3\bar{w} \end{vmatrix}$$

$$= \frac{1}{w}\begin{vmatrix} a_1+b_1w & a_1+b_1w & c_1+b_1\bar{w} \\ a_2+b_2w & a_2+b_2w & c_2+b_2\bar{w} \\ a_3+b_3w & a_3+b_3w & c_3+b_3\bar{w} \end{vmatrix} \quad (\because \omega^3 = 1)$$

$$= 0$$

22. b. Since x, y, z are in A.P., therefore, $x + z - 2y = 0$. Now,

$$\begin{vmatrix} a+2 & a+3 & a+2x \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2(x+z-2y) \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix}$$

[Applying $R_1 \to R_1 + R_3 - 2R_2$]

$$= \begin{vmatrix} 0 & 0 & 0 \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix} \quad [\because x+z-2y=0]$$

$$= 0$$

23. d. Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$(a+b+c-x)\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0$$

$\therefore \quad x = a + b + c = 0$

24. b. Applying $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ \cot\frac{A}{2} - \cot\frac{B}{2} & \cot\frac{B}{2} - \cot\frac{C}{2} & \cot\frac{C}{2} \\ \tan\frac{B}{2} - \tan\frac{A}{2} & \tan\frac{C}{2} - \tan\frac{B}{2} & \tan\frac{A}{2} + \tan\frac{B}{2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ \cot\frac{A}{2} - \cot\frac{B}{2} & \cot\frac{B}{2} - \cot\frac{C}{2} & \cot\frac{C}{2} \\ \dfrac{\cot\frac{A}{2} - \cot\frac{B}{2}}{\cot\frac{A}{2}\cot\frac{B}{2}} & \dfrac{\cot\frac{B}{2} - \cot\frac{C}{2}}{\cot\frac{B}{2}\cot\frac{C}{2}} & \tan\frac{A}{2} + \tan\frac{B}{2} \end{vmatrix}$$

$$= \left(\cot\frac{A}{2} - \cot\frac{B}{2}\right)\left(\cot\frac{B}{2} - \cot\frac{C}{2}\right)$$

$$\times \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & \cot\frac{C}{2} \\ \tan\frac{A}{2}\tan\frac{B}{2} & \tan\frac{B}{2}\tan\frac{C}{2} & \tan\frac{A}{2}\tan\frac{B}{2} \end{vmatrix}$$

$$= \left(\cot\frac{A}{2} - \cot\frac{B}{2}\right)\left(\cot\frac{B}{2} - \cot\frac{C}{2}\right)\left(\tan\frac{C}{2} - \tan\frac{A}{2}\right)\tan\frac{B}{2}$$

Since $\Delta = 0$, therefore

$$\cot\frac{A}{2} = \cot\frac{B}{2} \text{ or } \cot\frac{B}{2} = \cot\frac{C}{2} \text{ or } \tan\frac{A}{2} = \tan\frac{C}{2}$$

Hence, the triangle is definitely isosceles.

25. d. Since a, b, c, d, e, f are in G. P. and if r is the common ratio of the G.P., then

$b = ar$

$c = ar^2$

$d = ar^3$
$e = ar^4$
$f = ar^5$

Therefore, given determinant is

$$\begin{vmatrix} a^2 & a^2r^6 & x \\ a^2r^2 & a^2r^8 & y \\ a^2r^4 & a^2r^{10} & z \end{vmatrix}$$

$$= a^2a^2r^6 = \begin{vmatrix} 1 & 1 & x \\ r^2 & r^2 & y \\ r^4 & r^4 & z \end{vmatrix}$$

$= a^4r^6\,(0) = 0 \quad [\because C_1, C_2$ are identical]

26. c. Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$f(x) = \begin{vmatrix} 1+2x+(a^2+b^2+c^2)x & (1+b^2)x & (1+c^2)x \\ 1+2x+(a^2+b^2+c^2)x & 1+b^2x & (1+c^2)x \\ 1+2x+(a^2+b^2+c^2)x & (1+b^2)x & 1+x^2x \end{vmatrix}$$

$$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix} \quad [\because a^2+b^2+c^2 = -2]$$

$$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix}$$

[Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= (1)\,[(1-x)^2 - 0]$

$= (1-x)^2$

which is a polynomial of degree 2.

27. a. $\begin{vmatrix} 1 & 1 & 1 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+2}C_1 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+2}C_2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+1}C_0 + {}^{m+1}C_1 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+1}C_1 + {}^{m+1}C_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+1}C_0 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+1}C_1 \end{vmatrix} \quad [\text{Applying } C_3 \to C_3 - C_2]$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ {}^mC_1 & {}^mC_0 + {}^mC_1 & {}^{m+1}C_0 \\ {}^mC_2 & {}^mC_1 + {}^mC_2 & {}^{m+1}C_1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ {}^mC_1 & {}^mC_0 & {}^{m+1}C_0 \\ {}^mC_2 & {}^mC_1 & {}^{m+1}C_1 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1]$$

$= {}^mC_0\,{}^{m+1}C_1 - {}^{m+1}C_0\,{}^mC_1$
$= m + 1 - m$
$= 1$

28. c. Since each element of C_1 is the sum of two elements, putting the determinant as sum of two determinants, we get

$$\Delta = \begin{vmatrix} x^3 & x^2 & x \\ y^3 & y^2 & y \\ z^3 & z^2 & z \end{vmatrix} + \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix}$$

$$= xyz\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} + \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix}$$

$$= -(xyz+1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$= -(xyz + 1)\,(x - y)\,(y - z)\,(z - x)(x + y + z)$

Since $\Delta = 0$, x, y, z all are distinct, we have $xyz + 1 = 0$ or $xyz = -1$.

29. c. $\begin{vmatrix} 1+x & 1 & 1 \\ 1+y & 1+2y & 1 \\ 1+z & 1+z & 3+3z \end{vmatrix}$

$$= xyz\begin{vmatrix} 1+\frac{1}{x} & \frac{1}{x} & \frac{1}{x} \\ 1+\frac{1}{y} & 2+\frac{1}{y} & \frac{1}{y} \\ 1+\frac{1}{z} & 1+\frac{1}{z} & 3+\frac{1}{z} \end{vmatrix}$$

$$= xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\begin{vmatrix} 1 & 1 & 1 \\ 1+\frac{1}{y} & 2+\frac{1}{y} & \frac{1}{y} \\ 1+\frac{1}{z} & 1+\frac{1}{z} & 3+\frac{1}{z} \end{vmatrix}$$

$$= xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\begin{vmatrix} 1 & 0 & 0 \\ 1+\frac{1}{y} & 1 & -1 \\ 1+\frac{1}{z} & 0 & 2 \end{vmatrix}$$

$$= 2xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Hence, the given equation gives $x^{-1} + y^{-1} + z^{-1} = -3$.

30. b. Let a be the first term and d be the common difference of corresponding A.P. Then

$$\Delta = xyz\begin{vmatrix} 1/x & 1/y & 1/z \\ p & 2q & 3r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= xyz\begin{vmatrix} a+(p-1)d & a+(2q-1)d & a+(3r-1)d \\ p & 2q & 3r \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - aR_3$, $R_2 \to R_2 - R_3$ and then taking d common from R_1, we get

$$\Delta = xyzd\begin{vmatrix} (p-1) & (2q-1) & (3r-1) \\ (p-1) & (2q-1) & (3r-1) \\ 1 & 1 & 1 \end{vmatrix} = 0$$

31. a. As $a_1b_1c_1$, $a_2b_2c_2$ and $a_3b_3c_3$ are even natural numbers, each of c_1, c_2, c_3 is divisible by 2. Let $c_i = 2k_i$ for $i = 1, 2, 3$. Thus,

$$\Delta = 2\begin{vmatrix} k_1 & a_1 & b_1 \\ k_2 & a_2 & b_2 \\ k_3 & a_3 & b_3 \end{vmatrix} = 2m$$

where m is some natural number. Thus, Δ is divisible by 2. That Δ may not be divisible by 4 can be seen by taking the three numbers as 112, 122 and 134. Note that

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 4 & 1 & 3 \end{vmatrix} = 2$$

which is divisible by 2 but not by 4.

32. a. We have,

$$\begin{vmatrix} x & 1 & 1 & \cdots \\ 1 & x & 1 & \cdots \\ 1 & 1 & x & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} x & 1 & 1 & \cdots \\ (1-x) & (x-1) & 0 & \cdots \\ (1-x) & 0 & (x-1) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1, \cdots, R_n \to R_n - R_1$]

$= x(x-1)^{n-1} + |(x-1)^{n-1} + (x-1)^{n-1} + \cdots + (x-1)^{n-1}\ (n-1)$ times$|$

[Expanding along R_1]

$= x(x-1)^{n-1} + (n-1)(x-1)^{n-1}$

$= (x-1)^{n-1}(x+n-1)$

33. a. Let first term of G.P. is A and common ratio is R. Then,

$a = AR^{p-1} \Rightarrow \log a = \log A + (p-1)\log R$, etc.

$$\Rightarrow \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix} \quad [C_1 \to C_1 - (\log A)C_3]$$

$$= \log R\begin{vmatrix} (p-1) & p & 1 \\ (q-1) & q & 1 \\ (r-1) & r & 1 \end{vmatrix}$$

$$= \log R\begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix} (C_1 \to C_1 + C_3)$$

$= 0$

34. b. $R_3 \to R_3 - 2R_2$, hence two identical rows $\Rightarrow f(x) =$ constant.

35. b. In each determinant applying $R_1 \to R_1 + R_2 + R_3$ and then taking out $(x+9)$ common, we get

$x + 9 = 0 \Rightarrow x = -9$

36. c. Taking x^5 common from last row, we get

$$x^5\begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^a & a \\ x^n & x^{a+1} & x^{2n} \end{vmatrix} = 0, \forall x \in R$$

$\Rightarrow \quad a + 1 = n + 2 \Rightarrow a = n + 1$

(as it will make first and third row is identical)

37. a. Applying $R_1 \to R_1 + R_3 - 2R_2$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & x+z-2y \\ 4 & 5 & 6 & y \\ 5 & 6 & 7 & z \\ x & y & z & 0 \end{vmatrix}$$

$$= -(x+z-2y)\begin{vmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ x & y & z \end{vmatrix} \quad [\text{Expanding along } R_1]$$

$$= -(x+z-2y)\begin{vmatrix} 0 & -1 & 6 \\ 0 & -1 & 7 \\ x-2y+z & y-z & z \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_3 - 2C_2$ and $C_2 \to C_2 - C_3$]

$$= -(x+z-2y)^2\begin{vmatrix} -1 & 6 \\ -1 & 7 \end{vmatrix}$$

$= (x - 2y + z)^2$

Hence $\Delta = 0 \Rightarrow x, y, z$ are in A.P.

38. c. Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ reduce the determinant to

$$\begin{vmatrix} x^2-2x+1 & x-1 & 0 \\ 2x-2 & x-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$= (x-1)^3 - 2(x-1)^2 = (x-1)^2(x-1-2) = (x-1)^2(x-3)$, which is clearly negative for $x < 1$.

39. d. Applying $R_1 \to R_1 - (R_2 + R_3)$, we get

$$D = \begin{vmatrix} 0 & -2y & -2x \\ x & y+z & x \\ y & y & z+x \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & -y & -x \\ x & y+z & x \\ y & y & z+x \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & -y & -x \\ x & z & 0 \\ y & 0 & z \end{vmatrix} \quad (R_2 \to R_2 + R_1 \text{ and } R_3 \to R_3 + R_1)$$

$$= 4xyz$$

40. b. Operation $R_1 \to R_1 - R_2$, gives

$$\Delta = \begin{vmatrix} x-2 & 3(x-2) & -(x-2) \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$$

$$= (x-2)\begin{vmatrix} 1 & 3 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$$

$$= (x-2)\begin{vmatrix} 1 & 3 & -1 \\ 0 & -3(x+2) & x-1 \\ 0 & 2x+9 & x-1 \end{vmatrix}$$

$[R_2 \to R_2 - 2R_1, R_3 \to R_3 + 3R_1]$

$$= (x-2)\{-(3x+6)(x-1) - (x-1)(2x+9)\}$$

$$= -(x-2)(x-1)(5x+15)$$

Therefore, $\Delta = 0$ gives $x = 2, 1, -3$.

41. b. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x+2a & a & a \\ x+2a & x & a \\ x+2a & a & x \end{vmatrix} = (x+2a)\begin{vmatrix} 1 & a & a \\ 1 & x & a \\ 1 & a & x \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\Delta = (x+2a)\begin{vmatrix} 0 & a-x & 0 \\ 0 & x-a & a-x \\ 1 & a & x \end{vmatrix} = (x-a)^2(x+2a)$$

42. a. $\begin{vmatrix} x & m & n & 1 \\ a & x & n & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$ $[R_1 \to R_1 - R_2, R_2 \to R_2 - R_3, R_3 \to R_3 - R_4]$

$$\Rightarrow \begin{vmatrix} x-a & m-x & 0 & 0 \\ 0 & x-b & n-x & 0 \\ 0 & 0 & x-c & a \\ a & b & c & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-a & m-x & 0 \\ 0 & x-b & n-x \\ 0 & 0 & x-c \end{vmatrix} = 0$$

$$\Rightarrow (x-a)\begin{vmatrix} (x-b) & n-x \\ 0 & (x-c) \end{vmatrix} = 0$$

$\Rightarrow (x-a)(x-b)(x-c) = 0 \Rightarrow$ roots are independent of m, n

43. d. Since for $x = 0$, the determinant reduces to the determinant of a skew-symmetric matrix of odd order which is always zero, hence $x = 0$ is the solution of the given equation.

44. b. We have,

$$\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & a+b & b+c \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

[Applying $C_1 \to C_1 + (C_2 + C_3)$ on L.H.S.]

$$\Rightarrow 2\begin{vmatrix} a+b+c & -b & -c \\ a+b+c & -a & -b \\ a+b+c & -c & -a \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

[Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$ on L.H.S.]

$$\Rightarrow \begin{vmatrix} a & -b & -c \\ c & -a & -b \\ b & -c & -a \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_2 + C_3$ on L.H.S.]

$$\Rightarrow 2\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$\therefore \quad k = 2$

45. d. $D' = \begin{vmatrix} a_1+pb_1 & b_1+qc_1 & c_1+ra_1 \\ a_2+pb_2 & b_2+qc_2 & c_2+ra_2 \\ a_3+pb_3 & b_3+qc_3 & c_3+ra_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1+qc_1 & c_1+ra_1 \\ a_2 & b_2+qc_2 & c_2+ra_2 \\ a_3 & b_3+qc_3 & c_3+ra_3 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1+qc_1 & c_1+ra_1 \\ pb_2 & b_2+qc_2 & c_2+ra_2 \\ pb_3 & b_3+qc_3 & c_3+ra_3 \end{vmatrix}$$

In the first determinant, apply $C_3 \to C_3 - rC_1$ and then $C_2 \to C_2 - qC_3$.

In second determinant take p common from C_1 and then apply $C_2 \to C_2 - C_1$. Then take q common from C_2 and apply $C_3 \to C_3 - C_2$. Finally taking r common from C_3, we have ultimately $D' = (1 + pqr)D$.

46. b. $\Delta = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix}$ $(R_3 \to R_3 - R_2, R_4 \to R_4 - R_3)$

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 5 & 7 & 9 & 11 \\ 15 & 21 & 27 & 33 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 5 & 7 & 9 & 11 \\ 5 & 7 & 9 & 11 \end{vmatrix} = 0 \quad (R_4 \to R_4 - R_3)$$

47. c. Here $a > 0$ and $4b^2 - 4ac < 0$, i.e., $ac - b^2 > 0$.

$\therefore$ $ax^2 + 2bx + c > 0, \forall x \in R$

Now,

$$\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ 0 & 0 & -(ax^2+2bx+c) \end{vmatrix}$$

[Operating $R_3 \to R_3 - xR_1 - R_2$]

$= -(ax^2 + 2bx + c)(ac - b^2)$

$= -(+\text{ve})(+\text{ve}) = -\text{ve}$

48. a. We have,

$a_{n+1}^2 = a_n a_{n+2}$

$\Rightarrow$ $2 \log a_{n+1} = \log a_n + \log a_{n+2}$

Similarly,

$2 \log a_{n+4} = \log a_{n+3} + \log a_{n+5}$

$2 \log a_{n+7} = \log a_{n+6} + \log a_{n+8}$

Substituting these values in second column of determinant, we get

$$\Delta = \frac{1}{2}\begin{vmatrix} \log a_n & \log a_n + \log a_{n+2} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+3} + \log a_{n+5} & \log a_{n+5} \\ \log a_{n+6} & \log_{n+6} + \log a_{n+8} & \log a_{n+8} \end{vmatrix}$$

$= \frac{1}{2}(0) = 0$ [Using $C_2 \to C_2 - C_1 - C_3$]

49. b.
$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$\Delta^2 = \begin{vmatrix} x_1 & y_1 & z_2 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1^2+y_1^2+z_1^2 & x_1x_2+y_1y_2+z_1z_2 & x_1x_3+y_1y_3+z_1z_3 \\ x_1x_2+y_2y_1+z_2z_1 & x_2^2+y_2^2+z_2^2 & x_2x_3+y_2y_3+z_2z_3 \\ x_3x_1+y_3y_1+z_3z_1 & x_2x_3+y_2y_3+z_2z_3 & x_3^2+y_3^2+z_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \Rightarrow \Delta = \pm 1$$

50. c. Using $C_1 \to C_1 + C_2 + C_3$,

$$\Delta = \begin{vmatrix} \sin x + 2\cos x & \cos x & \cos x \\ \sin x + 2\cos x & \sin x & \cos x \\ \sin x + 2\cos x & \cos x & \sin x \end{vmatrix}$$

$$= (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix}$$

$= (\sin x + 2\cos x)(\sin x - \cos x)^2 +$

Thus, $\Delta = 0 \Rightarrow \tan x = -2$ or $\tan x = 1$

As $-\pi/4 \le x \le \pi/4$, we get $-1 \le \tan x \le 1$

$\therefore$ $\tan x = 1 \Rightarrow x = \pi/4$

51. b. We divide L.H.S. by λ^4 and C_1 by λ^2, C_2 by λ and C_3 by λ on the R.H.S. to obtain

$$p + q\left(\frac{1}{\lambda}\right) + r\left(\frac{1}{\lambda}\right)^2 + s\left(\frac{1}{\lambda}\right)^3 + t\left(\frac{1}{\lambda}\right)^4$$

$$= \begin{vmatrix} 1+3/\lambda & 1-1/\lambda & 1+3/\lambda \\ 1+1/\lambda^2 & 2/\lambda - 1 & 1-3/\lambda \\ 1-3/\lambda^2 & 1+4/\lambda & 3 \end{vmatrix}$$

Taking limit as $\lambda \to \infty$, we get

$$p = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -4$$

[Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

52. d. Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} a & b-y & c-z \\ -x & y & 0 \\ -x & 0 & z \end{vmatrix} = 0$$

Expanding along C_3, we get

$$(c-z)\begin{vmatrix} -x & y \\ -x & 0 \end{vmatrix} + z\begin{vmatrix} a & b-y \\ -x & y \end{vmatrix} = 0$$

$\Rightarrow$ $(c-z)(xy) + z(ay + bx - xy) = 0$

$\Rightarrow$ $cxy - xyz + ayz + bxz - xyz = 0$

$\Rightarrow$ $ayz + bzx + cxy = 2xyz$

$\Rightarrow$ $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

53. d. Since $A + B + C = \pi$ and $e^{i\pi} = \cos \pi + i \sin \pi = -1$,

$e^{i(B+C)} = e^{i(\pi - A)} = -e^{iA}$ and $e^{-i(B+C)} = -e^{iA}$

By taking e^{iA}, e^{iB}, e^{iC} common from R_1, R_2 and R_3, respectively, we have

$$\Delta = -\begin{vmatrix} e^{iA} & e^{-i(A+C)} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{iB} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{-i(A+C)} & e^{iC} \end{vmatrix}$$

$$= -\begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}$$

By taking e^{iA}, e^{iB}, e^{iC} common from C_1, C_2 and C_3, respectively, we have

$$\Delta = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -4$$

54. c. $\Delta = (1 + x + x^2)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1 + x + x^2)(x - 1)^2$

Therefore, $\Delta = 0$ has roots $1, 1, \omega, \omega, \omega^2, \omega^2$.

55. a. Using the sum property, we get

$$\sum_{r=0}^{m}\Delta_r=\begin{vmatrix}\sum_{r=0}^{m}(2r-1) & \sum_{r=0}^{m}{}^mC_r & \sum_{r=0}^{m}1\\ m^2-1 & 2^m & m+1\\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1)\end{vmatrix}.$$

But $\sum_{r=0}^{m}(2r-1)=\frac{1}{2}(m+1)(2m-1-1)=m^2-1$,

$\sum_{r=0}^{m}{}^mC_r=2^m$ and $\sum_{r=0}^{m}1=m+1$. Therefore,

$$\sum_{r=0}^{m}\Delta_r=\begin{vmatrix}m^2-1 & 2^m & m+1\\ m^2-1 & 2^m & m+1\\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1)\end{vmatrix}=0$$

56. d. $$\sum_{k=1}^{n}D_k=56\Rightarrow\begin{vmatrix}\sum_{k=1}^{n}1 & n & n\\ \sum_{k=1}^{n}2k & n^2+n+1 & n^2+n\\ \sum_{k=1}^{n}(2k-1) & n^2 & n^2+n+1\end{vmatrix}=56$$

$$\Rightarrow\begin{vmatrix}n & n & n\\ n(n+1) & n^2+n+1 & n^2+n\\ n^2 & n^2 & n^2+n+1\end{vmatrix}=56$$

Applying $C_3\to C_3-C_1$ and $C_2\to C_2-C_1$, we get

$$\begin{vmatrix}n & 0 & 0\\ n(n+1) & 1 & 0\\ n^2 & 0 & n+1\end{vmatrix}=56\Rightarrow n(n+1)=56\Rightarrow n=7$$

57. a. Applying $C_1\to C_1+2C_2+C_3$, we get

$$S=\sum_{r=2}^{n}(-2)^r\begin{vmatrix}{}^nC_r & {}^{n-2}C_{r-1} & {}^{n-2}C_r\\ 0 & 1 & 1\\ 0 & -1 & 0\end{vmatrix}$$

$$=\sum_{r=2}^{n}(-2)^r\,{}^nC_r$$

$$=\sum_{r=0}^{n}(-2)^r\,{}^nC_r-({}^nC_0-2\,{}^nC_1)$$

$$=(1-2)^n-(1-2n)=2n-1+(-1)^n$$

58. b. $$\Delta=\begin{vmatrix}{}^nC_{r-1} & {}^nC_r & (r+1)\,{}^{n+2}C_{r+1}\\ {}^nC_r & {}^nC_{r+1} & (r+2)\,{}^{n+2}C_{r+2}\\ {}^nC_{r+1} & {}^nC_{r+2} & (r+3)\,{}^{n+2}C_{r+3}\end{vmatrix}$$

Applying $C_1\to C_1+C_2$ and using ${}^nC_r=\frac{n}{r}\,{}^{n-1}C_{r-1}$ in C_3, we get

$$\Delta=\begin{vmatrix}{}^{n+1}C_r & {}^nC_r & (n+2)\,{}^{n+1}C_r\\ {}^{n+1}C_{r+1} & {}^nC_{r+1} & (n+2)\,{}^{n+1}C_{r+1}\\ {}^{n+1}C_{r+2} & {}^nC_{r+2} & (n+2)\,{}^{n+1}C_{r+2}\end{vmatrix}$$

$$=(n+2)\begin{vmatrix}{}^{n+1}C_r & {}^nC_r & {}^{n+1}C_r\\ {}^{n+1}C_{r+1} & {}^nC_{r+1} & {}^{n+1}C_{r+1}\\ {}^{n+1}C_{r+2} & {}^nC_{r+2} & {}^{n+1}C_{r+2}\end{vmatrix}$$

$= 0$ (as C_1 and C_3 are identical)

59. a. The given determenant Δ_1 is obtained by corresponding co-factors of determinant Δ_2; hence $\Delta_1=\Delta_2^2$. Now $\Delta_1\Delta_2=\Delta_2^2\,\Delta_2=\Delta_2^3$.

60. c. We have,

$$\Delta^2=\Delta\,\Delta=\begin{vmatrix}l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{vmatrix}\begin{vmatrix}l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{vmatrix}$$

$$=\begin{vmatrix}l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1l_3+m_1m_3+n_1n_3\\ l_1l_2+m_1m_2+n_1n_2 & l_2^2+m_2^2+n_2^2 & l_2l_3+m_2m_3+n_2n_3\\ l_1l_3+m_1m_3+n_1n_3 & l_2l_3+m_2m_3+n_2n_3 & l_3^2+m_3^2+n_3^2\end{vmatrix}$$

$$=\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}1\Rightarrow\Delta=\pm1\Rightarrow|\Delta|=1$$

61. d. The given determinant, on simplification, gives

$$\Delta_1=\begin{vmatrix}a_1^2 & -2a_1 & 1 & 0\\ a_2^2 & -2a_2 & 1 & 0\\ a_3^2 & -2a_3 & 1 & 0\\ a_4^2 & --2a_4 & 1 & 0\end{vmatrix}\times\begin{vmatrix}1 & b_1 & b_1^2 & 0\\ 1 & b_2 & b_2^2 & 0\\ 1 & b_3 & b_3^2 & 0\\ 1 & b_4 & b_4^2 & 0\end{vmatrix}$$

$=0\times0=0$

62. a. Determinant formed by the cofactors of $\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}$ is

$$\begin{vmatrix}bc-a^2 & ac-b^2 & ab-c^2\\ ac-b^2 & ab-c^2 & bc-a^2\\ ab-c^2 & bc-a^2 & ac-b^2\end{vmatrix}$$

$$\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}^2$$

63. d. $$\begin{vmatrix}1+x_1 & 1+x_1x & 1+x_1x^2\\ 1+x_2 & 1+x_2x & 1+x_2x^2\\ 1+x_3 & 1+x_3x & 1+x_3x^2\end{vmatrix}$$

$$=\begin{vmatrix}1 & x_1 & 0\\ 1 & x_2 & 0\\ 1 & x_3 & 0\end{vmatrix}\begin{vmatrix}1 & 1 & 0\\ 1 & x & 0\\ 1 & x^2 & 0\end{vmatrix}$$

$=0$

64. c. We observe that the elements in the pre-factor are the cofactors of the corresponding elements of the post-factor. Hence,

$$\begin{vmatrix}\lambda & c & -b\\ -c & \lambda & a\\ b & -a & \lambda\end{vmatrix}^3=[\lambda(\lambda^2+a^2+b^2+c^2)]^3=(1+a^2+b^2+c^2)^3$$

$\Rightarrow\ \lambda=1$

Alternative solution:

Writing $a = 0$, $b = 0$, $c = 0$ on both sides, we get

$\lambda^6\lambda^3 = 1 \Rightarrow \lambda = 1$

65. c. $f'(x) = \begin{vmatrix} -\sin x & 1 & 0 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\cos x & 2x & 2 \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \sec^2 x & 1 & 0 \end{vmatrix}$

$\Rightarrow f'(0) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$

Now, $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = \lim\limits_{x\to 0} f'(x)$ [as $f(0) = 0$]

$= f'(0) = 0$

66. a. Operating $C_1 \to C_1 + C_2 + C_3$ on the L.H.S. we get

$\Delta = \begin{vmatrix} 0 & c-a & a-b \\ 0 & c'-a' & a'-b' \\ 0 & c''-a'' & a''-b'' \end{vmatrix} = m \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$

$\Rightarrow m = 0$

67. d. Let the given determinant be equal to $\Delta(x)$. Then,

$5A + 4B + 3C + 2D + E = \Delta(1) + \Delta'(1)$

Now, $\Delta(1) = 0$ as R_2 and R_3 are identical.

$\Delta'(x) = \begin{vmatrix} 1 & 0 & 1 \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} + \begin{vmatrix} x & 2 & x \\ 2x & 1 & 0 \\ x & x & 6 \end{vmatrix} + \begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ 1 & 1 & 0 \end{vmatrix}$

$\Delta'(1) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 6 \\ 1 & 1 & 0 \end{vmatrix} = -17 + (12 + 1 - 1 - 6) = -11$

68. b. $\Delta_1 = x(x^2 - ab) - b(ax - ab) + b(a^2 - ax)$

$= x^3 - 3abx + ab^2 + a^2 b$

$\dfrac{d}{dx}(\Delta_1) = 3x^2 - 3ab = 3(x^2 - ab) = 3\Delta_2$

69. d. We have $y = \sin mx$, therefore $y_1 = m\cos mx$, $y_2 = -m^2 \sin mx$, etc.

$\therefore\ \Delta = \begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$

$= \begin{vmatrix} \sin mx & m\cos mx & -m^2 \sin mx \\ -m^3\cos mx & m^4 \sin mx & m^5 \cos mx \\ -m^6 \sin mx & -m^7 \cos mx & m^8 \sin mx \end{vmatrix}$

$= m^{12} \begin{vmatrix} \sin mx & \cos mx & -\sin mx \\ -\cos mx & \sin mx & \cos mx \\ -\sin mx & -\cos mx & \sin mx \end{vmatrix} = 0$

70. b. We have,

$\Delta = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = abc - (a+b+c) + 2$

$\therefore\ \Delta > 0 \Rightarrow abc + 2 > a + b + c$

$\Rightarrow abc + 2 > 3(abc)^{1/3}$ $\left[\because \text{A.M.} > \text{G.M.} \Rightarrow \dfrac{a+b+c}{3} > (abc)^{1/3}\right]$

$\Rightarrow x^3 + 2 > 3x$, where $x = (abc)^{1/3}$

$\Rightarrow x^3 - 3x + 2 > 0 \Rightarrow (x-1)^2(x+2) > 0$

$\Rightarrow x + 2 > 0 \Rightarrow x > -2 \Rightarrow (abc)^{1/3} > -2 \Rightarrow abc > -8$

71. b. $B_2 = a_1c_3 - a_3c_1$, $C_2 = -(a_1b_3 - a_3b_1)$

$B_3 = -(a_1c_2 - a_2c_1)$, $C_3 = a_1b_2 - a_2b_1$

$\therefore\ \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1c_3 - a_3c_1 & -a_1b_3 + a_3b_1 \\ -a_1c_2 + a_2c_1 & a_1b_2 - a_2b_1 \end{vmatrix}$

$= \begin{vmatrix} a_1c_3 & -a_1b_3 \\ -a_1c_2 & a_1b_2 \end{vmatrix} + \begin{vmatrix} a_1c_3 & a_3b_1 \\ -a_1c_2 & -a_2b_1 \end{vmatrix} + \begin{vmatrix} -a_3c_1 & -a_1b_3 \\ a_2c_1 & a_1b_2 \end{vmatrix}$

$+ \begin{vmatrix} -a_3c_1 & a_3b_1 \\ a_2c_1 & -a_2b_1 \end{vmatrix}$

$= a_1^2 \begin{vmatrix} c_3 & -b_3 \\ -c_2 & b_2 \end{vmatrix} + a_1b_1 \begin{vmatrix} c_3 & a_3 \\ -c_2 & -a_2 \end{vmatrix}$

$+ a_1c_1 \begin{vmatrix} -a_3 & -b_3 \\ a_2 & b_2 \end{vmatrix} + b_1c_1 \begin{vmatrix} -a_3 & a_3 \\ a_2 & -a_2 \end{vmatrix}$

$= a_1\{a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\}$

$= a_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\Delta$

72. a. Applying $C_1 \to C_1 - 2\sin x C_3$ and $C_2 \to C_2 + 2\cos x\, C_3$, we get

$f(x) = \begin{vmatrix} 2 & 0 & -\sin x \\ 0 & 2 & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$

$= 2\cos^2 x + 2\sin^2 x = 2$

$\therefore\ f'(x) = 0$

$\therefore\ \int_0^{\pi/2} [f(x) + f'(x)] = dx = \int_0^{\pi/2} 2dx = \pi$

73. b. $\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 11$

Multiplying R_1 by x, R_2 by y and R_3 by z, we get

$\dfrac{1}{xyz} \begin{vmatrix} x^4+x & x^3y & x^3z \\ xy^3 & y^4+y & y^3z \\ xz^3 & yz^3 & z^4+z \end{vmatrix} = 11$

Taking x, y, z common from C_1, C_2, C_3, respectively, we get

$\begin{vmatrix} x^3+1 & x^3 & x^3 \\ y^3 & y^3+1 & y^3 \\ z^3 & z^3 & z^3+1 \end{vmatrix} = 11$

Using $R_1 \to R_1 + R_2 + R_3$, we have

$(x^3 + y^3 + z^3 + 1) \begin{vmatrix} 1 & 1 & 1 \\ y^3 & y^3+1 & y^3 \\ z^3 & z^3 & z^3+1 \end{vmatrix} = 11$

Using $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$(x^3+y^3+z^3+1)\begin{vmatrix} 1 & 0 & 0 \\ y^3 & 1 & 0 \\ z^3 & 0 & 1 \end{vmatrix} = 11$$

Hence,

$$x^3+y^3+z^3 = 10$$

Therefore, the ordered triplets are (2, 1, 1), (1, 2, 1), (1, 1, 2).

74. b. Since the system has non-trivial solution,

$$\therefore \quad \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} a-1 & 1-b & 0 \\ 0 & b-1 & 1-c \\ 1 & 1 & c \end{vmatrix} = 0$$

$\Rightarrow \quad c(1-a)(1-b) + (1-b)(1-c) - (1-c)(a-1) = 0$

Dividing throughout by $(1-a)(1-b)(1-c)$, we get

$$\frac{c}{1-c} + \frac{1}{1-c} + \frac{1}{1-b} = 0$$

$$\Rightarrow \quad -1 + \frac{1}{1-c} + \frac{1}{1-b} + \frac{1}{1-a} = 0$$

$$\Rightarrow \quad \frac{1}{1-c} + \frac{1}{1-a} + \frac{1}{1-b} = 1$$

75. a. The given system of linear equations has a unique solution if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 5 & \lambda \end{vmatrix} \neq 0$$

i.e., if $\lambda - 8 \neq 0$ or $\lambda \neq 8$.

76. b. Let, $\Delta = \begin{vmatrix} \cos(\alpha-\beta) & \cos(\beta-\gamma) & \cos(\gamma-\alpha) \\ \cos(\alpha+\beta) & \cos(\beta+\gamma) & \cos(\gamma+\alpha) \\ \sin(\alpha+\beta) & \sin(\beta+\gamma) & \sin(\gamma+\alpha) \end{vmatrix}$

It is clear that either $\alpha = \beta$ or $\beta = \gamma$ or $\gamma = \alpha$ is sufficient to make $\Delta = 0$. It is not necessary that triangle is equilateral.

Also, isosceles triangle can be obtuse one.

77. c. $a = x/(y-z) \Rightarrow x - ay + az = 0$ (1)

$b = y/(z-x) \Rightarrow bx + y - bz = 0$ (2)

$c = z/(x-y) \Rightarrow -cx + cy + z = 0$ (3)

Since x, y, z are not all zero, the above system has a non-trivial solution. So,

$$\Delta = \begin{vmatrix} 1 & -a & a \\ b & 1 & -b \\ -c & c & 1 \end{vmatrix} = 0$$

$$\therefore \quad 1 + ab + bc + ca = 0$$

78. a. $\Delta = \begin{vmatrix} p+a & b & c \\ a & q+b & c \\ a & b & r+c \end{vmatrix} = 0$

Applying $R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix} p+a & b & c \\ -p & q & 0 \\ -p & 0 & r \end{vmatrix} = 0$$

$\Rightarrow \quad pqc + [q(p+a) + bp]r = 0$

Dividing by pqr, we obtain

$$\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = -1$$

79. b. For no solution or infinitely many solutions

$$\begin{vmatrix} \alpha & -1 & -1 \\ 1 & -\alpha & -1 \\ 1 & -1 & -\alpha \end{vmatrix} = 0$$

$\Rightarrow \quad \alpha(\alpha^2-1) - 1(\alpha-1) + 1(1-\alpha) = 0$

$\Rightarrow \quad \alpha(\alpha^2-1) - 2\alpha + 2 = 0$

$\Rightarrow \quad \alpha(\alpha-1)(\alpha+1) - 2(\alpha-1) = 0$

$\Rightarrow \quad (\alpha-1)(\alpha^2+\alpha-2) = 0$

$\Rightarrow \quad (\alpha-1)(\alpha+2)(\alpha-1) = 0$

$\Rightarrow \quad (\alpha-1)^2(\alpha+2) = 0$

$\Rightarrow \quad \alpha = 1, 1, -2$

But for $\alpha = 1$, there are infinite solutions. When $\alpha = -2$, we have

$-2x - y - z = -3$

$x + 2y - z = -3$

$x - y + 2z = -3$

Adding, we get $0 = -9$, which is not true. Hence there is no solution.

80. a. $D = \cos\theta - \cos^2. + 6 > 0$. Since $D > 0$ only trivial solution is possible.

81. a The given system of equations will have a non-trivial solution if

$$\begin{vmatrix} \alpha+a & \alpha & \alpha \\ \alpha & \alpha+b & \alpha \\ \alpha & \alpha & \alpha+c \end{vmatrix} = 0$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix} \alpha+a & \alpha & \alpha \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

$\Rightarrow \quad \alpha ab + c(\alpha b + ab + a\alpha) = 0 \Rightarrow \alpha(bc + ca + ab) + abc = 0$

$$\Rightarrow \quad \frac{1}{\alpha} = -\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \quad (\because a, b, c \neq 0)$$

82. c. The given system is consistent.

$$\therefore \quad \Delta = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & -c \\ -b & 3b & -c \end{vmatrix} = 0$$

$\Rightarrow \quad c + bc - 6b + b + 2c + 3bc = 0$

$\Rightarrow \quad 3c + 4bc - 5b = 0$

$$\Rightarrow \quad c = \frac{5b}{4b+3}$$

Now,

$c < 1$

$$\Rightarrow \frac{5b}{4b+3}<1$$

$$\Rightarrow \frac{5b}{4b+3}-1<0$$

$$\Rightarrow \frac{b-3}{4b+3}<0$$

$$\Rightarrow b\in\left(-\frac{3}{4},3\right)$$

83. c. As a, b, c are in G.P. with common ratio r_1 and α, β, γ are in G.P. having common ratio r_2, $a \neq 0$, $\alpha \neq 0$, $b = ar_1$, $c = ar_1^2$, $\beta = ar_2$, $\gamma = ar_2^2$.

Also the system of equations has only zero (trivial) solution.

$$\Delta = \begin{vmatrix} a & \alpha & 1 \\ b & \beta & 1 \\ c & \gamma & 1 \end{vmatrix} \neq 0$$

$$\Rightarrow a\alpha \begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & 1 \\ r_1^2 & r_2^2 & 1 \end{vmatrix} \neq 0$$

$$\Rightarrow a\alpha(r_1-1)(r_2-1)(r_1-r_2) \neq 0$$

$$\Rightarrow r_1 \neq 1, r_2 \neq 1 \text{ and } r_1 \neq r_2$$

84. b. For non-trivial solution

$$\begin{vmatrix} a-1 & -1 & -1 \\ 1 & -(b-1) & 1 \\ 1 & 1 & -(c-1) \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a-1 & -1 & 0 \\ 1 & -(b-1) & b \\ 1 & 1 & -c \end{vmatrix} = 0$$

$$\Rightarrow (a-1)(bc-c-b) + 1(-c-b) = 0$$

$$\Rightarrow abc - ac - ab - bc + b + c - c - b = 0$$

$$\Rightarrow ab + bc + ac = abc$$

Multiple Correct Answers Type

1. a, b, c.

$$\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 8 & 2 & 7 \\ 4 & 1 & -2 \\ 4 & 1 & -2 \end{vmatrix} \quad [R_3 \to R_3 - R_2 \text{ and } R_2 \to R_2 - R_1]$$

$$= 0$$

$$\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & abc \\ 1 & c^3 & abc \end{vmatrix}$$

$$[R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3]$$

$$= \frac{abc}{abc}\begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix} \quad [\text{taking } abc \text{ common from } C_3]$$

$$= 0$$

$$\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix} = \begin{vmatrix} a+b & 2a+b & 3a+b \\ a & a & a \\ 2a & 2a & 2a \end{vmatrix}$$

$$[R_3 \to R_3 - R_2, R_2 \to R_2 - R_1]$$

$$= 0$$

$$\begin{vmatrix} 2 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix} \quad [C_2 \to C_2 - 7C_3]$$

$$= \begin{vmatrix} 1 & 1 & 6 \\ 0 & 7 & 4 \\ 0 & 3 & 2 \end{vmatrix} \quad [C_1 \to C_1 - C_2]$$

$$= 2$$

2. a, c.

$$g(x) = \begin{vmatrix} a^{-x} & e^{\log_e a^x} & x^2 \\ a^{-3x} & e^{\log_e a^{3x}} & x^4 \\ a^{-5x} & e^{\log_e a^{5x}} & 1 \end{vmatrix} = \begin{vmatrix} a^{-x} & a^x & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix} \left(e^{\log a^x} = a^x\right)$$

$$\Rightarrow g(-x) = \begin{vmatrix} a^x & a^{-x} & x^2 \\ a^{3x} & a^{-3x} & x^4 \\ a^{5x} & a^{-5x} & 1 \end{vmatrix} = -\begin{vmatrix} a^{-x} & a^x & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix}$$

[interchanging 1st and 2nd columns]

$$= -g(x)$$

$$\Rightarrow g(x) + g(-x) = 0$$

$\Rightarrow$ $g(x)$ is an odd function

Hence, the graph is symmetrical about origin. Also, $g_4(x)$ is an odd function [where $g_4(x)$ is fourth derivative of $g(x)$]. Hence,

$$g_4(x) = -g_4(-x)$$

$$\Rightarrow g_4(0) = -g_4(0)$$

$$\Rightarrow g_4(0) = 0$$

3. c, d.

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b-x & a & b \\ a+b-x & -x & a \\ a+b-x & b & -x \end{vmatrix} = (a+b-x)\begin{vmatrix} 1 & a & b \\ 1 & -x & a \\ 1 & b & -x \end{vmatrix}$$

$$= (a+b-x)\begin{vmatrix} 1 & a & b \\ 0 & -x-a & a-b \\ 0 & b-a & -x-b \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= (a + b - x)\,[(x + a)(x + b) + (a - b)^2]$ [expanding along C_1]

$= (a + b - x)[x^2 + (a + b)x + a^2 + b^2 - ab]$

4. b, d.

Applying $C_1 \to C_1 - (\cot\phi)\, C_2$, we get

$$\Delta = \begin{vmatrix} 0 & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta/\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$$

$= -\dfrac{\sin\theta}{\sin\phi}[-\sin\phi\sin^2\theta - \cos^2\theta\sin\phi]$ [expanding along C_1]

$= \sin\theta$

which is independent of ϕ. Also,

$$\frac{d\Delta}{d\theta} = \cos\theta \Rightarrow \left.\frac{d\Delta}{d\theta}\right]_{\theta=\pi/2} = \cos(\pi/2) = 0$$

5. a, c.

$$\begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ 0 & 0 & 1+\sin\beta-\cos\beta \end{vmatrix}$$

[Applying $R_3 \to R_3 - R_1(\cos\beta) + R_2(\sin\beta)$]

$= (1 + \sin\beta - \cos\beta)(\cos^2\alpha + \sin^2\alpha) = 1 + \sin\beta - \cos\beta$ which is independent of α.

6. a, b,

$$\Delta = \frac{1}{a}\begin{vmatrix} a^3+ax & ab & ac \\ a^2b & b^2+x & bc \\ a^2c & bc & c^2+x \end{vmatrix}$$

Applying $C_1 \to C_1 + bC_2 + cC_3$ and taking $a^2 + b^2 + c^2 + x$ common, we get

$$\Delta = \frac{1}{a}(a^2+b^2+c^2+x)\begin{vmatrix} a & ab & ac \\ b & b^2+x & bc \\ c & bc & c^2+x \end{vmatrix}$$

Applying $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$, we get

$$\Delta = \frac{1}{a}(a^2+b^2+c^2+x)\begin{vmatrix} a & 0 & 0 \\ b & x & 0 \\ c & 0 & x \end{vmatrix}$$

$$= \frac{1}{a}(a^2+b^2+c^2+x)(ax^2) = x^2(a^2+b^2+c^2+x)$$

Thus Δ is divisible by x and x^2.

7. a, b, c.

Applying $R_3 \to R_3 - xR_2$ and $R_2 \to R_2 - xR_1$, we get

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ 0 & a+x & -1 \\ 0 & 0 & a+x \end{vmatrix} = a(a+x)^2$$

Hence,

$f(2x) - f(x) = a[(a+2x)^2 - (a+x)^2] = a(a + 2x - a - x)(a + 2x + a + x) = ax(2a + 3x)$

8. a, b, c, Operating $C_1 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & ac & bc \\ 1 & ad & bd \\ 1 & ae & be \end{vmatrix} = ab\begin{vmatrix} 1 & c & c \\ 1 & d & d \\ 1 & e & e \end{vmatrix} ab(0) = 0$$

9. a, b.

By partial fractions, we have

$$g(x) = \frac{f(a)}{(x-a)(a-b)(a-c)} + \frac{f(b)}{(b-a)(x-b)(b-c)} + \frac{f(c)}{(c-a)(c-b)(x-c)}$$

$$\Rightarrow g(x) = \frac{1}{(a-b)(b-c)(c-a)} \times \left[\frac{f(a)(c-b)}{(x-a)} + \frac{f(b)(a-c)}{(x-b)} + \frac{f(c)(b-a)}{(x-c)}\right]$$

$$\Rightarrow g(x) = \begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow \int g(x)dx = \begin{vmatrix} 1 & a & f(a)\log|x-a| \\ 1 & b & f(b)\log|x-b| \\ 1 & c & f(c)\log|x-c| \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + k$$

and

$$\frac{dg(x)}{dx} = \begin{vmatrix} 1 & a & -f(a)(x-a)^{-2} \\ 1 & b & -f(b)(x-b)^{-2} \\ 1 & c & -f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & f(a)(x-a)^{-2} \\ 1 & b & f(b)(x-b)^{-2} \\ 1 & c & f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

10. b, c.

$$\Delta'(x) = \begin{vmatrix} 2x+4 & 2x+4 & 13 \\ 4x+5 & 4x+5 & 26 \\ 16x-6 & 16x-6 & 104 \end{vmatrix} + \begin{vmatrix} x^2+4x-3 & 2 & 13 \\ 2x^2+5x-9 & 4 & 26 \\ 8x^2-6x+1 & 16 & 104 \end{vmatrix}$$

$$= 0 + 2 \times 13 \times (0) = 0$$

$\Rightarrow \Delta(x) = \text{constant} \Rightarrow a = 0, b = 0, c = 0$

11. a, c.

$f(\theta) = \sin^3\theta + \cos^3\theta - \cos\theta\sin\theta(\sin\theta + \cos\theta)$

$= (\sin\theta + \cos\theta)^3 - 4\sin\theta\cos\theta(\sin\theta + \cos\theta)$

$= (\sin\theta + \cos\theta)[1 - \sin 2\theta]$

Now,

$f(\theta) = 0$

$\Rightarrow \tan\theta = -1$ or $\sin 2\theta = 1$

$\Rightarrow f(\theta) = 0$ has 2 real solutions in $[0, \pi]$

Also, $\dfrac{f(\theta)}{1-\sin 2\theta} = \sin\theta + \cos\theta \in [-\sqrt{2}, \sqrt{2}]$

12. d. Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin(B+A)\sin(B-A) & \dfrac{\sin(A-B)}{\sin A\sin B} & 0 \\ \sin(C+A)\sin(C-A) & \dfrac{\sin(A-C)}{\sin A\sin C} & 0 \end{vmatrix}$$

$$\left[\because \cot\alpha - \cot\beta = \frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta}\right]$$

Expanding along C_3, we get

$$\Delta = \frac{\sin(A-B)\sin(A-C)}{\sin A}\left[-\frac{\sin(B+A)}{\sin C}+\frac{\sin(C+A)}{\sin B}\right]$$

$$= \frac{\sin(A-B)\sin(A-C)}{\sin A}\left[-\frac{\sin(\pi-C)}{\sin C}+\frac{\sin(\pi-B)}{\sin B}\right]$$

$$= \frac{\sin(A-B)\sin(A-C)}{\sin A}\left[-\frac{\sin C}{\sin C}+\frac{\sin B}{\sin B}\right]=0$$

13. a, c.

$$\begin{vmatrix} {}^{x}C_r & {}^{n-1}C_r & {}^{n}C_r \\ {}^{x+1}C_r & {}^{n}C_r & {}^{n+1}C_r \\ {}^{x+2}C_r & {}^{n+1}C_r & {}^{n+2}C_r \end{vmatrix} = 0 \qquad \text{(i)}$$

$$\Rightarrow \begin{vmatrix} \dfrac{x!}{r!(x-r)!} & \dfrac{(n-1)!}{r!(n-r-1)!} & \dfrac{n!}{r!(n-r)!} \\ \dfrac{(x+1)!}{r!(x+1-r)!} & \dfrac{n!}{r!(n-r)!} & \dfrac{(n+1)!}{r!(n-r+1)!} \\ \dfrac{(x+2)!}{r!(x+2-r)!} & \dfrac{(n+1)!}{r!(n+1-r)!} & \dfrac{(n+2)!}{r!(n-r+2)!} \end{vmatrix} = 0$$

Taking $\dfrac{x!}{r!(x+2-r)!}$ common from C_1, we have quadratic equation in x.

Now in (i), if we put $x = n-1$, C_1 and C_2 are the same, hence $x = n-1$ is one root of the equation.

If we put $x = n$, then C_1 and C_3 are same. Hence, $x = n$ is the other root.

14. a, b, Applying $C_3 \to C_3 - xC_2$, $C_2 \to C_2 - xC_1$, we obtain

$$\Delta(x) = \begin{vmatrix} 3 & 0 & 2a^2 \\ 3x & 2a^2 & 4a^2x \\ 3x^2+2a^2 & 4a^2x & 6a^2x^2+2a^2 \end{vmatrix}$$

Applying $C_3 \to C_3 - xC_2$, we get

$$\Delta(x) = 4a^4\begin{vmatrix} 3 & 0 & 1 \\ 3x & 1 & x \\ 3x^2+2a^2 & 2x & x^2+2a^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - 3C_3$, we get

$$\Delta(x) = 4a^4\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & x \\ -4a^2 & 2x & x^2+2a^2 \end{vmatrix} = 16a^6$$

15. b, c.

In the left-hand determinant, each element is the cofactor of the elements of the determinant

$$\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = \Delta^* \text{ (say)}$$

Hence,

$$\Delta^{*2} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$$= \begin{vmatrix} x^2+y^2+z^2 & xy+yz+zx & xz+yx+zy \\ \Sigma xy & \Sigma x^2 & \Sigma xy \\ \Sigma xy & \Sigma xy & \Sigma x^2 \end{vmatrix}$$

$$= \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix} \qquad [\text{Since } x^2+y^2+z^2 = r^2,\ xy+yz+zx = u^2]$$

16. a, c.

$$f(n) = \begin{vmatrix} n & n+1 & n+2 \\ n! & (n+1)! & (n+2)! \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} n & 1 & 1 \\ n! & nn! & (n+1)(n+1)! \\ 1 & 0 & 0 \end{vmatrix}$$

[Applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$]

$= (n+1)(n+1)! - nn! = n![(n+1)^2 - n] = n!(n^2+n+1)$

Thus, $f(n)$ is divisible by $n!$ and n^2+n+1.

17. a, b, c.

We have,

$$\begin{vmatrix} bc & ca & ab \\ ca & ab & bc \\ ab & bc & ca \end{vmatrix} = 0$$

$\Rightarrow\ (ab)^3 + (bc)^3 + (ca)^3 - 3(ab)(bc)(ca) = 0$

$\Rightarrow\ (ab + bc\omega^2 + ca\omega)(ab\omega + bc\omega^2 + ca)(ab\omega^2 + bc\omega + ca) = 0$

$\Rightarrow\ ab + bc\omega^2 + ca\omega = 0,\ ab\omega + bc\omega^2 + ca = 0,\ ab\omega^2 + bc\omega + ca = 0$

$\Rightarrow\ \dfrac{1}{c\omega^2}+\dfrac{1}{a}+\dfrac{1}{b\omega}=0,\ \dfrac{1}{c\omega}+\dfrac{1}{a}+\dfrac{1}{b\omega^2}=0,\ \dfrac{1}{c}+\dfrac{1}{a\omega}+\dfrac{1}{b\omega^2}=0$

$\Rightarrow\ \dfrac{1}{a}+\dfrac{1}{b\omega}+\dfrac{1}{c\omega^2}=0,\ \dfrac{1}{a}+\dfrac{1}{b\omega^2}+\dfrac{1}{c\omega}=0,\ \dfrac{1}{a\omega}+\dfrac{1}{b\omega^2}+\dfrac{1}{c}=0$

18. a, b. $\begin{vmatrix} 1 & k & 3 \\ k & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix} = 0$

$\Rightarrow\ 8 + 4k + 9k - 12 - 4k^2 - 6 = 0$

$\Rightarrow\ 4k^2 - 13k + 10 = 0$

$\Rightarrow\ 4k^2 - 8k - 5k + 10 = 0$

$\Rightarrow\ (2k-5)(k-2) = 0$

$\Rightarrow\ k = 5/2, 2$

19. a, b.

Applying $R_1 \to R_1 + \sin\phi\,(R_2) + \cos\phi\,(R_3)$,

$$f(x) = \Delta = \begin{vmatrix} 0 & 0 & \cos 2\phi + 1 \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\phi & \sin\theta & \cos\phi \end{vmatrix}$$

$= (\cos 2\phi + 1)(\sin^2\theta + \cos^2\theta)$

$= (1 + \cos 2\phi)$

Hence, Δ is independent of θ.

Reasoning Type

1. a. $$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0 \qquad (1)$$

Then $A = B$ or $B = C$ or $C = A$, for which any two rows are same. For (1) to hold it is not necessary that all the three rows are same or $A = B = C$.

2. b. The system of equations $kx + y + z = 1$, $x + ky + z = k$, $x + y + kz = k^2$ is inconsistent if $\Delta = \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} = 0$ and one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero where

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ k & k & 1 \\ k^2 & 1 & k \end{vmatrix}, \Delta_2 = \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & k^2 & k \end{vmatrix}, \Delta_3 = \begin{vmatrix} k & 1 & 1 \\ 1 & k & k \\ 1 & 1 & k^2 \end{vmatrix}$$

We have,

$\Delta = (k + 2)(k - 1)^2, \Delta_1 = -(k + 1)(k - 1)^2,$

$\Delta_2 = -k(k - 1)^2, \Delta_3 = (k + 1)^2(k - 1)^2$

The determinant given in statement 2 is $\Delta_1 = 0$, for which $k = 1$ or $k = -1$.

$k = 1$ makes all the determinants zero. But for $k = -1$, all the determinants are not zero.

Hence, both statements are true but statement 2 is not correct explanation of statement 1.

3. a. For $x = 0$, the determinant reduces to the determinant of a skew-symmetric matrix of odd order which is always zero. Hence, $x = 0$ is the solution of the given equation.

4. a. As the given system of equations has non-trivial solutions, hence

$$\begin{vmatrix} \lambda & b-a & c-a \\ a-b & \lambda & c-b \\ a-c & b-c & \lambda \end{vmatrix} = 0$$

When $\lambda = 0$, then the determinant becomes skew-symmetric of odd order, which is equal to zero. Thus, $\lambda = 0$.

5. a. $\Delta = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} \begin{vmatrix} 0 & m & n \\ -m & 0 & k \\ -n & -k & 0 \end{vmatrix}$ where $\begin{vmatrix} 0 & m & n \\ -m & 0 & k \\ -n & -k & 0 \end{vmatrix}$ is skew symmetric.

$\therefore \quad \Delta = 0$

6. a. We are given that

$1 + bc + qr = 0$ (i)

$1 + ca + pr = 0$ (ii)

$1 + ab + pq = 0$ (iii)

The determinant in the question involves a column consisting the elements ap, bq and cr. So multiplying (i), (ii) and (iii) by ap, bq and cr, respectively, we get

$ap + abcp + apqr = 0$ (iv)

$bq + abcq + bpqr = 0$ (v)

$cq + abcr + cpqr = 0$ (vi)

Since abc and pqr occur in all the three equations, putting $abc = x$, $pqr = y$, we get the system

$ap + px + ay = 0$

$bq + qx + by = 0$ (vii)

$cr + rx + cy = 0$

System (vii) must have a common solution (i.e., system is consistent). So,

$$\begin{vmatrix} ap & p & a \\ bq & q & b \\ cr & r & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} ap & a & p \\ bq & b & q \\ cr & c & r \end{vmatrix} = 0$$

7. b. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 3$. Now,

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix} = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & 3 \end{vmatrix} = \mu - 10$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix} = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & 3 \end{vmatrix} = 20 - 2\mu$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix} = \mu - 10$$

Clearly, for $\mu = 10$, all of $\Delta_1, \Delta_2, \Delta_3$ are zero.

8. b. $\Delta = \Delta_1 \Delta_2$ where $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} 1 & x^2 & 0 \\ x^2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Hence, both the statements are true but statement 2 is not correct explanation of statement 1.

Linked Comprehension Type

For Problems 1–3

1. c, 2. d, 3. b.

Sol. In given determinant applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$, we get

$$f(x) = \begin{vmatrix} x+c_1 & a-c_1 & 0 \\ x+b & c_2-b & a-c_2 \\ x+b & 0 & c_3-b \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & a-c_1 & 0 \\ 1 & c_2-b & a-c_2 \\ 1 & 0 & c_3-b \end{vmatrix} + \begin{vmatrix} c_1 & a-c_1 & 0 \\ b & c_2-b & a-c_2 \\ b & 0 & c_3-b \end{vmatrix}$$

So, $f(x)$ is linear. Let $f(x) = Px + Q$. Then

$f(-a) = -aP + Q, f(-b) = -bP + Q$

Then,

$$f(0) = 0 \times P + Q \Rightarrow Q = \frac{bf(-a) - af(-b)}{(b-a)} \quad (1)$$

Also,

$$f(-a)=\begin{vmatrix} c_1-a & 0 & 0 \\ b-a & c_2-a & 0 \\ b-a & b-a & c_3-a \end{vmatrix}$$

$$=(c_1-a)(c_1-a)(c_3-a)$$

Similarly,

$f(-b)=(c_1-b)(c_2-b)(c_3-b)$

$g(x)=(c_1-x)(c_2-x)(c_3-x) \Rightarrow g(a)=f(-a)$ and $g(b)=f(-b)$

Now from (1), we get

$$f(0)=\frac{bg(a)-ag(b)}{(b-a)}$$

For Problems 4–6

4. d, 5. d, 6. c.

Sol. $$\Delta=\frac{1}{a}\begin{vmatrix} a^3+ax & ab & ac \\ a^2b & b^2+x & bc \\ a^2c & bc & c^2+x \end{vmatrix}$$

Applying $C_1 \to C_1+bC_2+cC_3$ and taking $a^2+b^2+c^2+x$ common, we get

$$\Delta=\frac{1}{a}(a^2+b^2+c^2+x)\begin{vmatrix} a & ab & ac \\ b & b^2+x & bc \\ c & bc & c^2+x \end{vmatrix}$$

Applying $C_2 \to C_2-bC_1$ and $C_3 \to C_3-cC_1$, we get

$$\Delta=\frac{1}{a}(a^2+b^2+c^2+x)\begin{vmatrix} a & 0 & 0 \\ b & x & 0 \\ c & 0 & x \end{vmatrix}$$

$$=\frac{1}{a}(a^2+b^2+c^2+x)(ax^2)$$

$$=x^2(a^2+b^2+c^2+x)$$

Thus Δ is divisible by x and x^2. Also, graph of $f(x)$ is

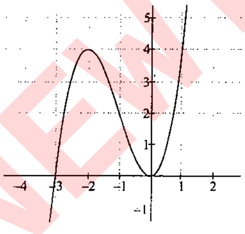


Fig. 7.2

For Problems 7–9

7. c, 8. a, 9. d.

Sol. The system of equations

$-x+cy+bz=0$ (1)

$cx-y+az=0$ (2)

$bx+ay-z=0$ (3)

has a non-zero solution if

$$\Delta=\begin{vmatrix} -1 & c & b \\ c & -1 & a \\ b & a & -1 \end{vmatrix}=0$$

$\Rightarrow \quad a^2+b^2+c^2+2abc-1=0$

$\Rightarrow \quad a^2+b^2+c^2+2abc=1$ (4)

Then clearly the system has infinitely many solutions. From (1) and (2), we have

$$\frac{x}{ac+b}=\frac{y}{bc+a}=\frac{z}{1-c^2}$$

$$\therefore \quad \frac{x^2}{(ac+b)^2}=\frac{y^2}{(bc+a)^2}=\frac{z^2}{(1-c^2)^2}$$

or $$\frac{x^2}{(1-a^2)(1-c^2)}=\frac{y^2}{(1-b^2)(1-c^2)}=\frac{z^2}{(1-c^2)^2} \quad \text{[from (4)]}$$

or $$\frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}=\frac{z^2}{1-c^2} \quad (5)$$

From (5), we see that $1-a^2$, $1-b^2$, $1-c^2$ are all positive or all negative. Given that one of a, b, c is proper fraction, so

$1-a^2>0$, $1-b^2>0$, $1-c^2>0$, which gives

$a^2+b^2+c^2<3$ (6)

Using (4) and (6), we get

$1<3+2abc$

or

$abc>-1$ (7)

For Problems 10–12

10. a, 11. b, 12. c.

Sol. $$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix}=2\lambda+3+2-2-\lambda-6=\lambda-3$$

$$\Delta_1=\begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix}=12\lambda+3\mu+20-2\mu-10\lambda-36$$

$$=2\lambda+\mu-16$$

$$\Delta_2=\begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix}=10\lambda+18+\mu-10-3\mu-6\lambda$$

$$=4\lambda-2\mu+8$$

$$\Delta_3=\begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix}=2\mu+10+12-12-\mu-20$$

$$=\mu-10$$

Thus the system has unique solutions if $\Delta \neq 0$ or $\lambda \neq 3$ and the system has infinite solutions if $\Delta=\Delta_1=\Delta_2=\Delta_3=0$ or $\lambda=3$ and $\mu=10$. System has no solution if $\Delta=0$ and at least one of Δ_1, Δ_2, Δ_3 is non-zero or $\lambda=3$ and $\mu \neq 10$.

For Problems 13–15

13. d, 14. d, 15. d.

Sol. $$\Delta=\begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}\times\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \quad \text{[multiplying row by row]}$$

$$=D^2 \text{ (say)}$$

Now,

$$D=\begin{vmatrix}1 & 1 & 1\\ 1 & \alpha & \beta\\ 1 & \alpha^2 & \beta^2\end{vmatrix}$$

$= (1-\alpha)(\alpha-\beta)(\beta-1)$

$= (\beta-\alpha)\,[\alpha\beta-\alpha-\beta+1]$

$= (\beta-\alpha)\left(\dfrac{c}{a}+\dfrac{b}{a}+1\right)=\dfrac{(\beta-\alpha)}{a}(a+b+c)$

$\therefore\quad \Delta = D^2 = \dfrac{(\beta-\alpha)^2}{a^2}(a+b+c)^2$

$= \dfrac{1}{a^2}(a+b+c)^2\left[\dfrac{b^2}{a^2}-4\dfrac{c}{a}\right]$

$= \dfrac{1}{a^4}(a+b+c)^2(b^2-4ac)$

If $\Delta < 0$, i.e., $b^2-4ac<0$, then roots are imaginary.

If one root is $1+\sqrt{2}$ and since coefficients are real, the other root is $1-\sqrt{2}$. Hence the equation is $x^2-2x-1=0$. Then the value of Δ is $(1-2-1)^2(4-4(1)(-1))=32$.

If $\Delta>0$, i.e., $b^2-4ac>0$, then roots are real and distinct but nothing can be said about $f(1)$.

For Problems 16–18

16. a, 17. b, 18. c.

Sol. Multiplying R_1, R_2, R_3 by a, b, c, respectively, and then taking a, b, c common from C_1, C_2 and C_3, we get

$$\Delta=\begin{vmatrix}-bc & ab+ac & ac+ab\\ ab+bc & -ac & bc+ab\\ ac+bc & bc+ac & -ab\end{vmatrix}$$

Now, using $C_2\to C_2-C_1$ and $C_3\to C_3-C_1$, and then taking $(ab+bc+ca)$ common from C_2 and C_3, we get

$$\Delta=\begin{vmatrix}-bc & 1 & 1\\ ab+bc & -1 & 0\\ ac+bc & 0 & -1\end{vmatrix}\times(ab+bc+ca)^2$$

Now, applying $R_2\to R_2+R_1$, we get

$$\Delta=\begin{vmatrix}-bc & 1 & 1\\ ab & 0 & 1\\ ac+bc & 0 & -1\end{vmatrix}(ab+bc+ca)^2$$

Expanding along C_2, we get

$\Delta = (ab+bc+ca)^2\,[ac+bc+ab]$

$= (ab+bc+ca)^3$

$= (r/p)^3 = r^3/p^3$

Now given a, b, c are all positive, then

A.M. $\geq$ G.M.

$\Rightarrow\quad \dfrac{ab+bc+ac}{3}\geq(ab\times bc\times ac)^{1/3}$

$\Rightarrow\quad (ab+bc+ac)^3\geq 27a^2b^2c^2$

$\Rightarrow\quad (ab+bc+ac)^3\geq 27(s^2/p^2)$

If $\Delta=27$, then $ab+bc+ca=3$, and given that $a^2+b^2+c^2=3$, from $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$, we have $a+b+c=\pm 3$

$\Rightarrow\quad a+b+c=3$ (since all the roots are positive)

$\Rightarrow\quad 3p+q=0$

For Problems 19–21

19. d, 20. c, 21. c.

Let,

$$\begin{vmatrix}(1+x)^a & (1+2x)^b & 1\\ 1 & (1+x)^a & (1+2x)^b\\ (1+2x)^b & 1 & (1+x)^a\end{vmatrix}=A+Bx+Cx^2+\cdots$$

Putting $x=0$, we get

$$A=\begin{vmatrix}1&1&1\\1&1&1\\1&1&1\end{vmatrix}=0$$

Now differentiating both sides with respect to x and putting $x=0$, we get

$$B=\begin{vmatrix}a&2b&0\\1&1&1\\1&1&1\end{vmatrix}+\begin{vmatrix}1&1&1\\0&a&2b\\1&1&1\end{vmatrix}+\begin{vmatrix}1&1&1\\1&1&1\\2b&0&a\end{vmatrix}=0$$

Hence coefficient of x is 0. Since $f(x)=0$ and $f'(0)=0$, $x=0$ is a repeating root of the equation $f(x)=0$.

For Problems 22–24

22. c, 23. d, 24. b.

$$\begin{vmatrix}x&n&r\\m&y&r\\m&n&z\end{vmatrix}=0$$

Applying $R_1\to R_1-R_2$ and $R_2\to R_2-R_3$, we get

$$\begin{vmatrix}x-m&n-y&0\\0&y-n&r-z\\m&n&z\end{vmatrix}=0$$

$\Rightarrow\quad (x-m)(y-n)z+(n-y)(r-z)m-n(r-z)(x-m)=0$

Dividing by $(x-m)(y-n)(z-r)$, we have

$\dfrac{z}{z-r}+\dfrac{m}{x-m}+\dfrac{n}{y-n}=0$

$\Rightarrow\quad \dfrac{z}{z-r}+\dfrac{m}{x-m}+\dfrac{n}{y-n}=0$

$\Rightarrow\quad \dfrac{z}{z-r}+\dfrac{m}{x-m}+1+\dfrac{n}{y-n}+1=2$

$\Rightarrow\quad \dfrac{z}{z-r}+\dfrac{x}{x-m}+\dfrac{y}{y-n}=2=2$

$\Rightarrow\quad \dfrac{z}{z-r}-1+\dfrac{x}{x-m}-1+\dfrac{y}{y-n}-1=-1$

$\Rightarrow\quad \dfrac{m}{x-m}+\dfrac{n}{y-n}+\dfrac{r}{z-r}=-1$

Now,

A.M. $\geq$ G.M.

$\Rightarrow\quad \dfrac{\dfrac{z}{z-r}+\dfrac{x}{x-m}+\dfrac{y}{y-n}}{3}\geq\left(\dfrac{z}{(z-r)}\,\dfrac{x}{(x-m)}\,\dfrac{y}{(y-n)}\right)^{1/3}$

$\Rightarrow\quad \dfrac{z}{z-r}\,\dfrac{x}{x-m}\,\dfrac{y}{y-n}\leq\dfrac{8}{27}$

For Problems 25–27

25. b, **26.** d, **27.** a.

Sol. $$f'(x)=\begin{vmatrix}2ax & 2ax-1 & 2ax+b+1\\ b & b+1 & -1\\ 2(ax+b) & 2ax+2b+1 & 2ax+b\end{vmatrix}$$

Applying $C_1\to C_1-C_3$, $C_2\to C_2-C_3$

$$f'(x)=\begin{vmatrix}-(b+1) & -(b+2) & 2ax+b+1\\ (b+1) & (b+2) & -1\\ b & b+1 & 2ax+b\end{vmatrix}$$

Applying $R_1\to R_1+R_2$ and $R_3\to R_3-R_2$, we get

$$f'(x)=\begin{vmatrix}0 & 0 & 2ax+b\\ b+1 & b+2 & -1\\ -1 & -1 & 2ax+b+1\end{vmatrix}$$

$$=(2ax+b)[-b-1+b+2]$$

$\therefore\quad f'(x)=2ax+b$

$\therefore\quad f(x)=ax^2+bx+c$

$f(0)=2\Rightarrow c=2$

$f(1)=1\Rightarrow a+b+2=1\Rightarrow a+b=-1$

$f'(5/2)=0\Rightarrow 5a+b=0$

$\Rightarrow\quad a=1/4,\ b=-5/4$

Hence, $f(x)=\frac{1}{4}x^2-\frac{5}{4}x+2$

Clearly, discriminant (D) of the equation $f(x)=0$ is less than 0.

Hence, $f(x)=0$ has imaginary roots. Also, $f(2)=1/2$. And minimum value of $f(x)$ is

$$\frac{\frac{25}{16}-4\frac{1}{4}(2)}{4\frac{1}{4}}=\frac{7}{16}$$

Hence, range of the $f(x)$ is $\left[\frac{7}{16},\infty\right)$.

Matrix-Match Type

1. a→s; b→p; c→s; d→s.

a. Coefficient of x in $f(x)$ is coefficient of x in $\begin{vmatrix}x & 1 & 1\\ 1 & x & 2\\ x^2 & 1 & 0\end{vmatrix}$

Therefore, coefficient of x is -2.

b. Let $D=\begin{vmatrix}1 & 3\cos\theta & 1\\ \sin\theta & 1 & 3\cos\theta\\ 1 & \sin\theta & 1\end{vmatrix}$

$=(3\cos\theta-\sin\theta)^2$

$\Delta_{max}=10$

c. $f'(x)=0$

$\Rightarrow\quad f'(0)=0$

d. $a_0=\begin{vmatrix}0 & 2 & 0\\ 1 & 0 & 6\\ 0 & 0 & 1\end{vmatrix}=-2(1)=-2$

2. a→s; b→r; c→q, r; d→p.

a. The given determinant is $\Delta=\begin{vmatrix}x+2 & x+3 & x+5\\ x+4 & x+6 & x+9\\ x+8 & x+11 & x+15\end{vmatrix}$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_2$, we have

$$\Delta=\begin{vmatrix}x+2 & x+3 & x+5\\ 2 & 3 & 4\\ 4 & 5 & 6\end{vmatrix}$$

$$=2\begin{vmatrix}x & x & x+1\\ 2 & 3 & 4\\ 1 & 1 & 1\end{vmatrix}\quad[\text{Applying } R_1\to R_1-R_2 \text{ and } R_3\to R_3-R_2]$$

$$=2\begin{vmatrix}x & 0 & 1\\ 2 & 1 & 1\\ 1 & 0 & 0\end{vmatrix}\quad[\text{Applying } C_2\to C_2-C_1 \text{ and } C_3\to C_3-C_2]$$

$=-2$ [Expanding along R_3]

b. $$\begin{vmatrix}7 & 6 & x^2-13\\ 2 & x^2-13 & 2\\ x^2-13 & 3 & 7\end{vmatrix}$$

Let $x^2-13=t$. Then

$t^3-67t+126=0$

$\Rightarrow\quad t=-9,2,7\Rightarrow x=\pm2,\pm\sqrt{20},\pm\sqrt{15}$

Hence sum of other five roots is 2.

c. $$\Delta=\begin{vmatrix}\sqrt{6} & 2i & 3+\sqrt{6}\\ \sqrt{12} & \sqrt{3}+\sqrt{8}i & 3\sqrt{2}+\sqrt{6}i\\ \sqrt{18} & \sqrt{2}+\sqrt{12}i & \sqrt{27}+2i\end{vmatrix}$$

Taking $\sqrt{6}$ common from C_1, we get

$$\Delta=\sqrt{6}\begin{vmatrix}1 & 2i & 3+\sqrt{6}\\ \sqrt{2} & \sqrt{3}+2\sqrt{2}i & 3\sqrt{2}+\sqrt{6}i\\ \sqrt{3} & \sqrt{2}+2\sqrt{3}i & 3\sqrt{3}+2i\end{vmatrix}$$

Applying $R_2\to R_2-\sqrt{2}R_1$ and $R_3\to R_3-\sqrt{3}R_1$, we get

$$\Delta=\sqrt{6}\begin{vmatrix}1 & 2i & 3+\sqrt{6}\\ 0 & \sqrt{3} & \sqrt{6}i-2\sqrt{3}\\ 0 & \sqrt{2} & 2i-3\sqrt{2}\end{vmatrix}$$

$$=\sqrt{6}\begin{vmatrix}\sqrt{3} & \sqrt{6}i-2\sqrt{3}\\ \sqrt{2} & 2i-3\sqrt{2}\end{vmatrix}$$

$$=\sqrt{6}\begin{vmatrix}\sqrt{3} & -2\sqrt{3}\\ \sqrt{2} & -3\sqrt{2}\end{vmatrix}\quad[\text{Applying } C_2\to C_2-\sqrt{2}i\,C_1]$$

$=\sqrt{6}(-3\sqrt{6}+2\sqrt{6})$

$=-6$, which is an integer

d. $$f(\theta)=\begin{vmatrix}\cos^2\theta & \cos\theta\sin\theta & -\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta\\ \sin\theta & -\cos\theta & 0\end{vmatrix}$$

Applying $R_1\to R_1+(\sin\theta)R_3$ and $R_2\to R_2-(\cos\theta)R_3$, we get

$$f(\theta)=\begin{vmatrix}1 & 0 & -\sin\theta\\ 0 & 1 & \cos\theta\\ \sin\theta & -\cos\theta & 0\end{vmatrix}$$

$=\sin^2\theta+\cos^2\theta=1$

$\Rightarrow\quad f(\pi/3)=1$

3. a→p, q, r; b→q; c→s; d→p, q, r.

a. Multiplying C_1 by a, C_2 by b and C_3 by c, we obtain

$$\Delta=\frac{1}{abc}\begin{vmatrix}\frac{a}{c} & \frac{b}{c} & -\frac{a+b}{c}\\ -\frac{b+c}{a} & \frac{b}{a} & \frac{c}{a}\\ -\frac{b(b+c)}{ac} & \frac{b(a+2b+c)}{ac} & -\frac{b(a+b)}{ac}\end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta=\frac{1}{abc}\begin{vmatrix}0 & \frac{b}{c} & -\frac{a+b}{c}\\ 0 & \frac{b}{a} & \frac{c}{a}\\ 0 & \frac{b(a+2b+c)}{ac} & -\frac{b(a+b)}{ac}\end{vmatrix}$$

This shows that Δ is independent of a, b and c.

b. Applying $C_1 \to C_1 - (\cot b)C_2$, we get

$$\Delta=\begin{vmatrix}0 & \sin a\sin b & \cos a\\ 0 & \cos a\sin b & -\sin a\\ -\sin a/\sin b & \sin a\cos b & 0\end{vmatrix}$$

$$=-\frac{\sin a}{\sin b}[-\sin b\sin^2 a-\cos^2 a\sin b]\ \text{[expanding along } C_1]$$

$$=\sin a$$

c. Taking $1/\sin a\cos b$, $1/\sin a\sin b$, $1/\cos a$ common from C_1, C_2, C_3, respectively, we get

$$\Delta=\frac{1}{\sin^2 a\cos a\sin b\cos b}\Delta_1$$

where $\Delta_1=\begin{vmatrix}1 & 1 & 1\\ -\cot a & -\cot a & \tan a\\ \tan b & -\cot b & 0\end{vmatrix}$

$$=\begin{vmatrix}0 & 1 & 1\\ 0 & -\cot a & \tan a\\ 1/\sin b\cos b & -\cot b & 0\end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$, we get

$$\Delta=\frac{1}{\sin b\cos b}[\tan a+\cot a]$$

$$=\frac{1}{\sin a\cos a\sin b\cos b}$$

d. $\begin{vmatrix}a^2 & b\sin A & c\sin A\\ b\sin A & 1 & \cos A\\ c\sin A & \cos A & 1\end{vmatrix}$

$$=\begin{vmatrix}a^2 & a\sin B & a\sin C\\ a\sin B & 1 & \cos A\\ a\sin C & \cos A & 1\end{vmatrix}$$

$$=a^2\begin{vmatrix}1 & \sin B & \sin C\\ \sin B & 1 & \cos A\\ \sin C & \cos A & 1\end{vmatrix}$$

$$=a^2\begin{vmatrix}1 & 0 & 0\\ \sin B & 1-\sin^2 B & \cos A-\sin B\sin C\\ \sin C & \cos A-\sin B\sin C & 1-\sin^2 C\end{vmatrix}$$

[Applying $C_2 \to C_2 - (\sin B)C_1$ and $C_3 \to C_3 - (\sin C)C_1$]

$= a^2[\cos^2 B\cos^2 C - (\cos A - \sin B\sin C)^2]$

$= a^2[\cos^2 B\cos^2 C - (\cos(B + C) + \sin B\sin C)^2]$

$= a^2[\cos^2 B\cos^2 C - \cos^2 B\cos^2 C]$

$= 0$

Integer Type

1.(3) Equation $x^3 + ax^2 + bx + c = 0$ has roots α, β, γ.

$\therefore\ \alpha+\beta+\gamma=-a$

$\alpha\beta+\beta\gamma+\gamma\alpha=b$

Since the given system of equations has non-trivial solutions, so

$$\begin{vmatrix}\alpha & \beta & \gamma\\ \beta & \gamma & \alpha\\ \gamma & \alpha & \beta\end{vmatrix}=0$$

$\Rightarrow\ \alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=0$

$\Rightarrow\ (\alpha+\beta+\gamma)[\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha]=0$

$\Rightarrow\ (\alpha+\beta+\gamma)[(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha)]=0$

$\Rightarrow\ -a[a^2-3b]=0\ \Rightarrow\ a^2/b=3$

2.(2) We have $D=\begin{vmatrix}a_1 & a_2 & a_3\\ 5 & 4 & a_6\\ a_7 & a_8 & a_9\end{vmatrix}$

Since $a_n=\frac{20}{n}$; $d=\frac{1}{20}$

Hence, $D=\begin{vmatrix}20 & \frac{20}{2} & \frac{20}{3}\\ \frac{20}{4} & \frac{20}{5} & \frac{20}{6}\\ \frac{20}{7} & \frac{20}{8} & \frac{20}{9}\end{vmatrix}=\frac{(20)^3}{4\times7}\begin{vmatrix}1 & \frac{1}{2} & \frac{1}{3}\\ 1 & \frac{4}{5} & \frac{2}{3}\\ 1 & \frac{7}{8} & \frac{7}{9}\end{vmatrix}$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$=\frac{(20)^3}{4\times7}\begin{vmatrix}0 & \frac{-3}{10} & \frac{-1}{3}\\ 0 & \frac{-3}{40} & \frac{-1}{9}\\ 1 & \frac{7}{8} & \frac{7}{9}\end{vmatrix}=\frac{50}{21}$$

$\Rightarrow\ [D]=2$

3.(8) Let $D=\begin{vmatrix}(\beta+\gamma-\alpha-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 & 1\\ (\gamma+\alpha-\beta-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 & 1\\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1\end{vmatrix}$

Applying $R_1 \to R_1 - R_3$

$R_2 \to R_2 - R_3$

$$=\begin{vmatrix}(\beta+\gamma-\alpha-\delta)^4-(\alpha+\beta-\gamma-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2-(\alpha+\beta-\gamma-\delta)^2 & 0\\ (\gamma+\delta-\beta-\delta)^4-(\alpha+\beta-\gamma-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2-(\alpha+\beta-\gamma-\delta)^2 & 0\\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1\end{vmatrix}$$

$$= 4(\beta-\delta)(\gamma-\alpha).\,4(\alpha-\delta)(\gamma-\beta)$$

$$\times\begin{vmatrix}(\beta+\gamma-\alpha-\delta)^2+(\alpha+\beta-\gamma-\delta)^2 & 1 & 0\\ (\gamma+\alpha-\beta-\delta)^2+(\alpha+\beta-\gamma-\delta)^2 & 1 & 0\\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1\end{vmatrix}$$

Apply $R_1 \to R_1 - R_2$

$$= 16(\beta-\delta)(\gamma-\alpha)(\alpha-\delta).4(\gamma-\delta)(\beta-\alpha)$$

$$\begin{vmatrix}1 & 0 & 0\\ (\gamma+\alpha-\beta-\delta)^2+(\alpha+\beta-\gamma-\delta)^2 & 1 & 0\\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1\end{vmatrix}$$

$$= -64(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)$$

4.(4) $\Delta = \begin{vmatrix}x+2 & 2x+3 & 3x+4\\ 2x+3 & 3x+4 & 4x+5\\ 3x+5 & 5x+8 & 10x+17\end{vmatrix} = 0.$

Applying $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$

$$\Delta = \begin{vmatrix}x+2 & 2x+3 & 3x+4\\ x+1 & x+1 & x+1\\ x+2 & 2(x+2) & 6(x+2)\end{vmatrix} = 0$$

$\therefore\quad \Delta = (x+1)(x+2)\begin{vmatrix}x+2 & 2x+3 & 3x+4\\ 1 & 1 & 1\\ 1 & 2 & 6\end{vmatrix} = 0$

$\therefore\quad \Delta = (x+1)(x+2)\,[(x+2).4-(2x+3).5+(3x+4).1] = 0$

$\Delta = (x+1)(x+2)(-3x-3) = 0$

or $(x+1)^2(x+2) = 0$

$\therefore\quad x = -1, -1, -2$

5.(2) System of equations

$\Rightarrow\quad \alpha x + y + z = \alpha - 1$ (1)

$x + \alpha y + z = \alpha - 1$ (2)

$x + y + \alpha z = \alpha - 1$ (3)

Since system has no solution.

Therefore, (1) $\Delta = 0$ and (2) $\alpha - 1 \neq 0$

$$\begin{vmatrix}\alpha & 1 & 1\\ 1 & \alpha & 1\\ 1 & 1 & \alpha\end{vmatrix} = 0, \alpha \neq 1$$

$R_1 \to R_1 - R_3,\ R_2 \to R_2 \to R_3$

$$\begin{vmatrix}\alpha-1 & 0 & 1-\alpha\\ 0 & \alpha-1 & 1-\alpha\\ 1 & 1 & \alpha\end{vmatrix} = 0$$

$\Rightarrow\quad (\alpha-1)[\alpha(\alpha-1)-(1-\alpha)] + (1-\alpha)[-(\alpha-1)] = 0$

$\Rightarrow\quad (\alpha-1)[\alpha(\alpha-1)+(\alpha-1)] + (\alpha-1)^2 = 0$

$\Rightarrow\quad (\alpha-1)^2[(\alpha+1)+1] = 0$

$\Rightarrow\quad \alpha = 1, 1, -2 \Rightarrow \alpha = 1, -2$

Since system has no solution, $\alpha \neq 1$.

$\therefore\quad \alpha = -2$

6.(3) $x + y + z = 1$ (1)

$x + 2y + 4z = p$ (2)

$x + 4y + 10z = p^2$ (3)

$$\Delta = \begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 4 & 10\end{vmatrix}$$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$= \begin{vmatrix}0 & -1 & -3\\ 0 & -2 & -6\\ 1 & 4 & 10\end{vmatrix} = 0$$

Since $\Delta = 0$, solution is not unique solution.

The system will have infinite solutions if $\Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0$

$$\Delta_1 = \begin{vmatrix}1 & 1 & 1\\ p & 2 & 4\\ p^2 & 4 & 10\end{vmatrix} = 0$$

$C_3 \to C_3 - C_2$

$$\Delta_1 = \begin{vmatrix}1 & 1 & 0\\ p & 2 & 2\\ p^2 & 4 & 6\end{vmatrix} = 0$$

$\Rightarrow\quad 1(12-8) - 1(6p - 2p^2) = 0$

$\Rightarrow\quad 4 - 6p + 2p^2 = 0$

$\Rightarrow\quad 2(p^2 - 3p + 2) = 0$

$\Rightarrow\quad p^2 - 3p + 2 = 0$

$\Rightarrow\quad p = 1$ or 2

Also for these values of p, $\Delta_2, \Delta_3 = 0$

7.(0) $\begin{vmatrix}3u^2 & 2u^3 & 1\\ 3v^2 & 2v^3 & 1\\ 3w^2 & 2w^3 & 1\end{vmatrix} = 0$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\Rightarrow\quad \begin{vmatrix}u^2-v^2 & u^3-v^3 & 0\\ v^2-w^2 & v^3-w^3 & 0\\ w^2 & w^3 & 1\end{vmatrix} = 0$$

$$\Rightarrow\quad \begin{vmatrix}u+v & u^2+v^2+vu & 0\\ v+w & v^2+w^2+vw & 0\\ w^2 & w^3 & 1\end{vmatrix} = 0$$

$R_1 \to R_1 - R_2$

$$\Rightarrow\quad \begin{vmatrix}u-w & (u^2-w^2)+v(u-w) & 0\\ v+w & v^2+w^2+vw & 0\\ w^2 & w^3 & 1\end{vmatrix} = 0$$

$$\Rightarrow\quad \begin{vmatrix}1 & u+w+v & 0\\ v+w & v^2+w^2+vw & 0\\ w^2 & w^3 & 1\end{vmatrix} = 0$$

$\Rightarrow\quad (v^2 + w^2 + vw) - (v+w)[(v+w)+u] = 0$

$\Rightarrow\quad v^2 + w^2 + vw - (v+w)^2 - u(v+w) = 0$

$\Rightarrow\quad uv + vw + wu = 0$

8.(2) Using $C_3 \to C_3 - (C_1 + C_2)$ in D_1 and D_2, we have

$$\therefore\quad \frac{D_1}{D_2} = \frac{-2b(ad-bc)}{b(ad-bc)} = -2$$

9.(5) $\Delta = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$

Applying $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 0 & \sin\theta - 3\cos\theta & 0 \end{vmatrix}$$

$= -(\sin\theta - 3\cos\theta)(3\cos\theta - \sin\theta)$

$= (3\cos\theta - \sin\theta)^2$

Now, $-\sqrt{9+1} \le 3\cos\theta - \sin\theta \le \sqrt{9+1}$

$\Rightarrow$ $(3\cos\theta - \sin\theta)^2 \le 10.1$

$\Rightarrow$ $\Delta_{max.} = 10$

10.(4)

$$\Delta = x\begin{vmatrix} 1 & x+y & x+y+z \\ 2 & 3x+2y & 4x+3y+2z \\ 3 & 6x+3y & 10x+6y+3z \end{vmatrix}$$

$$= x^2\begin{vmatrix} 1 & 1 & x+y \\ 2 & 3 & 4x+3y \\ 3 & 6 & 10x+6y \end{vmatrix} \begin{vmatrix} C_3 \to C_3 - zC_1 \\ C_2 \to C_2 - yC_1 \end{vmatrix}$$

$$= x^3\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} [C_3 \to C_3 - yC_2]$$

$= x^3(6 - 8 + 3) = 64$

$\Rightarrow x^3 = 64 \Rightarrow x = 4$

11.(1)

$$\Delta_1 = \begin{vmatrix} a_1^2 + 4a_1d & a_1 & d \\ a_2^2 + 4a_2d & a_2 & d \\ a_3^2 + 4a_3d & a_3 & d \end{vmatrix}, [C_3 \to C_3 - C_2]$$

where d is the common difference of A.P.

$$= d\begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} + 4d\begin{vmatrix} a_1 & a_1 & d \\ a_2 & a_2 & d \\ a_3 & a_3 & d \end{vmatrix}$$

$= d(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) = -2d^4$

Similarly, $\Delta_2 = -2d^4$.

12.(4) $\Delta = (xyz)^n \begin{vmatrix} 1 & x^2 & x^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$

$= (xyz)^n (x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$

Clearly when

$$n = -4, \Delta = \left(\frac{1}{y^2} - \frac{1}{x^2}\right)\left(\frac{1}{z^2} - \frac{1}{y^2}\right)\left(\frac{1}{x^2} - \frac{1}{z^2}\right)$$

13.(2)

$$B = 2.2\begin{vmatrix} f & d & e \\ n & l & m \\ c & a & b \end{vmatrix}$$

[Taking 2 common from R_2 and C_2]

$$= 2\begin{vmatrix} 2f & d & e \\ 2n & l & m \\ 2c & a & b \end{vmatrix}$$

$$= 2\begin{vmatrix} 2c & a & b \\ 2f & d & e \\ 2n & l & m \end{vmatrix}$$

$[R_3 \leftrightarrow R_2$, then $R_2 \leftrightarrow R_1]$

$$= 2\begin{vmatrix} a & b & 2c \\ d & e & 2f \\ l & m & 2n \end{vmatrix} = 2A$$

$[C_1 \leftrightarrow C_2$ and then $C_2 \leftrightarrow C_3]$

14.(0) $\Delta = \begin{vmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \end{vmatrix}\begin{vmatrix} y_1 & x_1 & 0 \\ y_2 & x_2 & 0 \\ y_3 & x_3 & 0 \end{vmatrix} = 0.0 = 0$

15.(8) Putting $x = 0$, $a_0 = 1$

$(1 + ax + bx^2)^4 = (1 + ax + bx^2)(1 + ax + bx^2)$
$(1 + ax + bx^2)(1 + ax + bx^2)$

Clearly $a_0 = 1$, a_1 = coefficient of $x = a + a + a + a = 4a$

a_2 = coefficient of $x^2 = 4b + 6a^2$

Now $\Delta = -(a_0^3 + a_1^3 + a_2^3 - 3a_0a_1a_2)$

$\because$ $a_0 + a_1 + a_2 \ne 0$

$\therefore$ $a_0 = a_1 = a_2$

$1 = 4a = 6a^2 + 4b \Rightarrow a = \frac{1}{4}, b = \frac{5}{32}$

Archives

Subjective Type

1. We should have

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$$

$\Rightarrow$ $1(-4k + 6) - k(-12 + 4) + 3(9 - 2k) = 0$

$\Rightarrow$ $-2k + 33 = 0 \Rightarrow k = \frac{33}{2}$

Substituting $k = \frac{33}{2}$ and putting $x = m$ where $m \in Q$, we get the system as

$33y + 6z = -2m$ (1)

$33y - 4z = -6m$ (2)

$3y - 4z = -2m$ (3)

$(1) - (2) \Rightarrow 10z = 4m \Rightarrow z = \frac{2}{5}m$

$(1) \Rightarrow 33y = -2m - \frac{12m}{5} = -\frac{22m}{5}$

$\Rightarrow$ $y = -\frac{2m}{15}$

Therefore, the solution is $x = m$, $y = \frac{-2m}{15}$, $z = \frac{2m}{5}$.

2. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

$= -(a+b+c)[a^2 + b^2 + c^2 - ab - bc - ca]$

$= -\frac{1}{2}(a+b+c)[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca]$

$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$

As $a, b, c > 0$, therefore $a + b + c > 0$. Also $a \neq b \neq c$.

$\therefore \quad (a-b)^2 + (b-c)^2 + (c-a)^2 > 0$

Hence, the given determinant is –ve.

3. $\begin{vmatrix} x^2 + x & x+1 & x-2 \\ 2x^2 + 3x - 1 & 3x & 3x-3 \\ x^2 + 2x + 3 & 2x-1 & 2x-1 \end{vmatrix} = Ax + B$

On L.H.S. operating $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$, we get

$\begin{vmatrix} x^2 + x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$

$= \begin{vmatrix} x^2 & x+1 & x-2 \\ 0 & x-2 & x+1 \\ 0 & x-2 & x+1 \end{vmatrix} + \begin{vmatrix} x & x & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$

$= 0 + \begin{vmatrix} x & x & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$

$= \begin{vmatrix} x & x+1 & x-2 \\ -1 & -2 & 3 \\ 4 & 0 & 0 \end{vmatrix}$ (Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$)

$= \begin{vmatrix} x & x & x \\ -1 & -2 & 3 \\ 4 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 4 & 0 & 0 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \\ 4 & 0 & 0 \end{vmatrix} x + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 4 & 0 & 0 \end{vmatrix}$

$= xA + B =$ R.H.S.

Hence proved.

4. $\Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} = 7\lambda + 35$

If $7\lambda + 35 \neq 0$, i.e. $\lambda \neq -5$, system has a unique solution.

[$\because \Delta \neq 0 \Rightarrow$ unique solution]

But if $\lambda = -5$, we have $\Delta = 0$. Solution exists in this case if $\Delta_x = \Delta_y = \Delta_z = 0$. Now for $\lambda = -5$,

$\Delta_x = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ -3 & 5 & -5 \end{vmatrix} = 0$

$\Delta_y = \begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & -5 \end{vmatrix} = 0$

$\Delta_z = \begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & 5 & -3 \end{vmatrix} = 0$

Thus $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$, so there exists infinite number of solutions. Now eliminating x from the equations, we have

$7y - 13z = -9$

$7y - 13z = -9$

which are same, so putting

$z = k \in R$, $y = (13k - 9)/7$ and so $x = (4 - 5k)/7$, where k is any real number.

5. Applying $C_2 \to C_2 + C_1$, $C_3 \to C_2 + C_3$ and using ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$, we have

$\Delta = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$

$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$ [Applying $C_3 \to C_3 + C_2$]

6. The system has a non-trivial solution if

$\Delta = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$

$\Rightarrow \quad 7 \sin 3\theta + 7 \cos 2\theta - 6 + 7 \cos 2\theta - 8 = 0$

$\Rightarrow \quad \sin 3\theta + 2 \cos 2\theta = 2$

$\Rightarrow \quad 3 \sin \theta - 4 \sin^3 \theta + 2 - 4 \sin^2 \theta = 2$

$\Rightarrow \quad \sin \theta (4 \sin^2 \theta + 4 \sin \theta - 3) = 0$

$\Rightarrow \quad \sin \theta (2 \sin \theta + 3)(2 \sin \theta - 1) = 0$

$\Rightarrow \quad \sin \theta = 0;\ 1/2$, since $\sin \theta \neq -3/2$

Hence, $\theta = n\pi$ or $n\pi + (-1)^n \pi/6$; $n \in Z$.

7. As $A28$, $3B9$ and $62C$ are divisible by k, there exists $m_1, m_2, m_3 \in Z$ such that

$100A + 20 + 8 = m_1 k$, $300 + 10B + 9 = m_2 k$ and $600 + 20 + C = m_3 k$.

Now,

$\Delta = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$

Applying $R_2 \to 100R_1 + R_2 + 10R_3$, we get

$\Delta = \begin{vmatrix} A & 3 & 6 \\ 100A + 20 + 8 & 300 + 10B + 9 & 600 + 20 + C \\ 2 & B & 2 \end{vmatrix}$

$$= \begin{vmatrix} A & 3 & 6 \\ m_1k & m_2k & m_3k \\ 2 & B & 2 \end{vmatrix}$$

$$= k\begin{vmatrix} A & 3 & 6 \\ m_1 & m_2 & m_3 \\ 2 & B & 2 \end{vmatrix} = k\Delta_1$$

As all elements of Δ_1 are integers, Δ_1 must be integer.

$\therefore \quad \Delta = k \times$ some integer

$\Rightarrow \quad \Delta$ is divisible by k.

8. Given,

$$\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ reduce the determinant into

$$\begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

$\Rightarrow \quad (p-a)(q-b)r + a(b-q)(c-r) - b(p-a)(c-r) = 0$

Dividing throughout by $(p-a)\,(q-b)\,(r-c)$, we get

$$\frac{r}{r-c} + \frac{a}{p-a} + \frac{b}{q-b} = 0$$

$$\Rightarrow \quad \frac{r}{r-c} + 1 + \frac{a}{p-a} + 1 + \frac{b}{q-b} = 2$$

9. Taking $n!$, $(n+1)!$, $(n+2)!$ common from R_1, R_2 and R_3, respectively, we get

$$\Delta = (n)!\,(n+1)!\,(n+2)! \times \begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 1 & (n+2) & (n+2)(n+3) \\ 1 & (n+3) & (n+3)(n+4) \end{vmatrix}$$

$$\therefore \quad \frac{\Delta}{(n!)^3} = (n+1)^2(n+2) \times \begin{vmatrix} 1 & n+1 & (n+1)(n+2) \\ 0 & 1 & 2(n+2) \\ 0 & 1 & 2(n+3) \end{vmatrix}$$

$[R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_2]$

$= (n+1)^2(n+2)2 = 2[n^3 + 4n^2 + 5n + 2]$

$$\therefore \quad \frac{\Delta}{(n!)^3} - 4 = 2n(n^2 + 4n + 5)$$

Hence, $\dfrac{\Delta}{(n!)^3} - 4$ is divisible by n.

10. The given system has a non-trivial solution if

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

By expanding the determinant along first column, we get

$\lambda = \sin 2\alpha + \cos 2\alpha$

Now,

$-\sqrt{2} \le \sin 2\alpha + \cos 2\alpha \le \sqrt{2}$

$\Rightarrow \quad -\sqrt{2} \le \lambda \le \sqrt{2}$

For $\lambda = 1$,

$\sin 2\alpha + \cos 2\alpha = 1$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin 2\alpha + \frac{1}{\sqrt{2}}\cos 2\alpha = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(2\alpha - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \cos\left(2n\pi \pm \frac{\pi}{4}\right)$$

$$\Rightarrow \quad 2\alpha = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{4},\ n \text{ being an integer}$$

11. $$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix}$$

$= 0 \times 0 = 0$

12. Taking $\dfrac{1}{a(a+d)(a+2d)}$ common from R_1, $\dfrac{1}{(a+d)(a+2d)(a+3d)}$ from R_2 an $\dfrac{1}{(a+2d)(a+3d)(a+4d)}$ from R_3, we have

$$\Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\Delta'$$

where

$$\Delta' = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix}$$

$$= \begin{vmatrix} (a+d)(a+2d) & 2d & a \\ (a+2d)(a+3d) & 2d & a+d \\ (a+3d)(a+4d) & 2d & a+2d \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_3]$$

$$= \begin{vmatrix} (a+d)(a+2d) & 2d & a \\ (a+2d)2d & 0 & d \\ (a+3d)2d & 0 & d \end{vmatrix} \quad [\text{applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_2]$$

$= -2d[(a+2d)2d^2 - (a+3d)2d^2] = 4d^4$

Hence, $\Delta = 4d^4/[a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)]$.

13. Given that a, b, c are p^{th}, q^{th} and r^{th} terms of a H.P. Hence, $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are p^{th}, q^{th} and r^{th} terms of an A.P. So,

$$\therefore \quad \left.\begin{aligned} \frac{1}{a} &= A + (p-1)D \\ \frac{1}{b} &= A + (q-1)D \\ \frac{1}{c} &= A + (r-1)D \end{aligned}\right\} \qquad (1)$$

Now given determinant is

$$\Delta = \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Substituting the values of $1/a$, $1/b$, $1/c$ from (i), we get

$$\Delta = abc\begin{vmatrix} A+(p-1)D & A+(q-1)D & A+(r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Operating $R_1 \to R_1 - (A-D)R_3 - DR_2$, we get

$$\Delta = abc\begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

14. Operating $R_1 \to R_1 + R_2 + R_3$ and using trigonometric identities, the given determinant becomes

$$\begin{vmatrix} \sin\theta+2\sin\theta\left(-\frac{1}{2}\right) & \cos\theta+2\cos\theta\left(-\frac{1}{2}\right) & \sin 2\theta+2\sin 2\theta\left(-\frac{1}{2}\right) \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0$$

Objective Type

Fill in the blanks

1. Putting $\lambda = 0$, we have $t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = 0$ (skew-symmetric determinant)

2. Clearly for $x = -1$, $R_2 \equiv R_3$ and for $x = 2$, $R_1 \equiv R_3$. Hence roots are $x = -1, 2$.

3. With 0 and 1 as elements there are $2 \times 2 \times 2 \times 2 = 16$ determinants of order 2×2 out of which only $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$ are the three determinants whose values are +ve. Therefore, the required probability is 3/16.

4. $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

Operating $R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

$$\therefore \quad (x+9)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

Operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$(x+9)\begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (x+9)(x-2)(x-7) = 0 \quad$ (Expanding along R_1)

$\Rightarrow \quad x = -9, 2, 7$

5. The given homogeneous system of equations will have non-zero solutions if

$D = 0$

$\Rightarrow \begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$

$\Rightarrow \quad \lambda(\lambda^2+1) - 1(-\lambda+1) + 1(1+\lambda) = 0$

$\Rightarrow \quad \lambda^3 + 3\lambda = 0$

$\Rightarrow \quad \lambda(\lambda^2+3) = 0$, but $\lambda^2 + 3 \neq 0$ for real λ

$\Rightarrow \quad \lambda = 0$

6. $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

[Multiplying R_1 by a, R_2 by b and R_3 by c]

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \text{ [Applying } C_1 \leftrightarrow C_2 \text{ and then } C_2 \leftrightarrow C_3\text{]}$$

$= 0$

7. $D = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix}$$

(Taking $\frac{1}{\log x}, \frac{1}{\log y}, \frac{1}{\log z}$ common from R_1, R_2 and R_3, respectively)

$$= \frac{1}{\log x \ \log y \ \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix}$$

$= 0$

8. $R_1 \to R_1 + R_3$

$$\Rightarrow \quad f(\theta) = \begin{vmatrix} 0 & 0 & 2 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$$

$= 2(\tan^2\theta + 1) = 2\sec^2\theta$

True or false

1. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} \begin{pmatrix} R_1 \to aR_1 \\ R_2 \to bR_2 \\ R_3 \to cR_3 \end{pmatrix}$$

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \begin{pmatrix} C_1 \leftrightarrow C_3 \\ C_2 \leftrightarrow C_3 \end{pmatrix}$$

Hence, the given statement is false.

2. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

$\Rightarrow$ Area of Δ_1 = Area of Δ_2

where Δ_1 is the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) and Δ_2 is the triangle with vertices (a_1, b_1), (a_2, b_2) and (a_3, b_3).

But two triangles of same area may not be congruent. Hence, the given statement is false.

Multiple choice questions with one correct answer

1. b. For every 'det. with 1' ($\in B$) we can find a det. with value -1 by changing the sign of one entry of '1'. Hence there are equal number of elements in B and C.

Therefore, (b) is the correct option.

2. b $\begin{vmatrix} 0 & 1+\omega+\omega^2 & 0 \\ 1-i & -1 & \omega^2-1 \\ -1 & -1+\omega-1 & -1 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 0 & 0 \\ 1-i & -1 & \omega^2-1 \\ -1 & -i+\omega-1 & -1 \end{vmatrix} \quad [\because 1+\omega+\omega^2 = 0]$$

(Operating $R_1 \to R_1 - R_2 + R_3$)

3. b. Let $\frac{x^2}{a^2} = X, \frac{y^2}{b^2} = Y, \frac{z^2}{c^2} = Z$

Then the given system of equations is

$X + Y - Z = 1$

$X - Y + Z = 1$

$-X + Y + Z = 1$

Coefficient determinant is

$$A = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$

$= 1(-1-1) - 1(1+1) - 1(1-1)$

$= -4 \neq 0$

Hence, the given system of equations has unique solutions.

4. b. Given,

$$\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$$

Operating $C_1 \to C_1 - pC_2 - C_3$, we get

$$\begin{vmatrix} 0 & x & y \\ 0 & y & z \\ -(xp^2+2py+z) & xp+y & yp+z \end{vmatrix} = 0$$

$\Rightarrow \quad (xz - y^2)(xp^2 + 2py + z) = 0$

$\Rightarrow \quad xz - y^2 = 0$

$\Rightarrow \quad y^2 = xz$

Hence, x, y, z are in G.P.

5. b. Let, $\Delta = \begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$

Expanding along first row, we have

$1[\cos px \sin(p+d)x - \cos(p+d)x \sin px]$

$-a[\cos(p-d)x \sin(p+d)x - \cos(p+d)x \sin(p-d)x]$

$+ a^2[\cos(p-d)x \sin px - \cos px \sin(p-d)x]$

$= \sin dx - a \sin 2dx + a^2 \sin dx$

which is independent of p.

6. a. Taking x common from R_2 and $x(x-1)$ common from R_3, we get

$$f(x) = x^2(x-1)\begin{vmatrix} 1 & x & x+1 \\ 2 & x-1 & x+1 \\ 3 & x-2 & x+1 \end{vmatrix}$$

Applying $C_3 \to C_3 - C_2$, we get

$$f(x) = x^2(x-1)\begin{vmatrix} 1 & x & 1 \\ 2 & x-1 & 2 \\ 3 & x-2 & 3 \end{vmatrix} = 0$$

Thus, $f(500) = 0$.

7. d. For the given homogeneous system of equations to have non-zero solution, determinant of coefficient matrix should be zero, i.e.,

$$\begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 1(1+1) + k(-k+1) - (k+1) = 0$$

$\Rightarrow$ $2 - k^2 + k - k - 1 = 0$

$\Rightarrow$ $k^2 = 1$

$\Rightarrow$ $k = \pm 1$

8. b. Given that $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. Also, $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$. Now given determinant is

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

[Using $\omega = -1 - \omega^2$ and $\omega^3 = 1$]

Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix} \quad [\text{as } 1 + \omega + \omega^2 = 0]$$

Expanding along C_1, we get

$3(\omega^2 - \omega^4) = 3(\omega^2 - \omega)$

$= 3\omega(\omega - 1)$

9. b. For infinitely many solutions the two equations become identical. Hence,

$$\frac{k+1}{k} = \frac{8}{k+3} = \frac{4k}{3k-1} \Rightarrow k = 1$$

10. a. The given system is

$x + ay = 0$

$az + y = 0$

$ax + z = 0$

It is a system of homogeneous equations, therefore, it will have infinitely many solutions if determinant of coefficient matrix is zero. Therefore,

$$\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $1(1 - 0) - a(0 - a^2) = 0$

$\Rightarrow$ $1 + a^3 = 0$

$\Rightarrow$ $a^3 = -1$

$\Rightarrow$ $a = -1$

11. d. Since the system has no solution

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow$ $2(-2\lambda + 1) + 1(\lambda + 1) + 2(3) = 0$

$\Rightarrow$ $-4\lambda + 2 + \lambda + 1 + 6 = 0$

$\Rightarrow$ $3\lambda = 9$

$\Rightarrow$ $\lambda = 3$

12. d. $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$ (given)

$$\Rightarrow -3i \begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} = x + iy$$

$\Rightarrow$ $x + iy = 0 + i0$

$\Rightarrow$ $x = y = 0$

Multiple choice questions with one or more than one correct answer

1. b. e. Given that

$$\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0$$

Operating $C_3 \to C_3 - C_1\alpha - C_2$, we get

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & b\alpha + c & -(a\alpha^2 + b\alpha + b\alpha + c) \end{vmatrix} = 0$$

$$\Rightarrow (a\alpha^2 + 2b\alpha + c) \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & b\alpha + c & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $(ac - b^2)(a\alpha^2 + 2b\alpha + c) = 0$

$\Rightarrow$ either $ac - b^2 = 0$ or $a\alpha^2 + 2ba + c = 0$

$\Rightarrow$ either a, b, c are in G.P. or $(x - \alpha)$ is a factor of $ax^2 + 2bx + c$

Hence, (b) and (e) are the correct answers.

Matrix-match type

1. a→r; b→q; c→p; d→s.

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= -\frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

a. If $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$

$\Rightarrow$ $\Delta = 0$ and $a = b = c \neq 0$

Therefore, the equations represent identical planes.

b. $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$

$\Rightarrow$ $\Delta = 0$

Therefore, the equations have infinitely many solutions.

$ax + by = (a + b)z$

$bx + cy = (b + c)z$

$\Rightarrow$ $(b^2 - ac)y = (b^2 - ac)z \Rightarrow y = z$

$\Rightarrow$ $ax + by + cy = 0 \Rightarrow ax = ay \Rightarrow x = y = z$

c. $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$

$\Rightarrow$ $\Delta \neq 0$

Therefore, the equations represent planes meeting at only one point.

d. $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$

$\Rightarrow$ $a = b = c = 0$

Therefore, the equations represent whole of the three-dimensional space.

Integer type

1. (1)

$\omega = e^{i2\pi/3}$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$\Rightarrow$ $z\left[(z+\omega)^2(z+\omega) - 1 - \omega(z+\omega-1) + \omega^2(1-z-\omega^2)\right] = 0$

$\Rightarrow$ $z^3 = 0$

$\Rightarrow$ $z = 0$ is only solution.

CHAPTER
8

Matrices

- Definition
- Algebra of Matrices
- Special Matrices
- Adjoint of Square Matrix
- Equivalent Matrices
- System of Simultaneous Linear Equation
- Matrices of Reflection and Rotation
- Characteristic Roots and Characteristic Vector of a Square Matrix

DEFINITION

A rectangular array of symbols (which could be real or complex numbers) along rows and columns is called a matrix.

Thus a system of $m \times n$ symbols arranged in a rectangular formation along m rows and n columns and bonded by the brackets $[\cdot]$ is called an m by n matrix (which is written as $m \times n$ matrix). Thus,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is a matrix of order $m \times n$.

In a compact form, the above matrix is represented by $A = [a_{ij}]$, $1 \le i \le m$, $1 \le j \le n$ or simply $[a_{ij}]_{m \times n}$. The numbers a_{11}, a_{12}, ..., etc., of this rectangular array are called the elements of the matrix. The element a_{ij} belongs to the i^{th} row and j^{th} column and is called the $(i, j)^{th}$ element of the matrix.

Equal Matrices

Two matrices are said to be equal if they have the same order and each element of one is equal to the corresponding element of the other.

Example 8.1 **If a matrix has 8 elements, what are the possible orders it can have?**

Sol. We know that if matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 8 elements, we will find all ordered pairs of natural numbers, whose product is 8.

Thus, all possible ordered pairs are (1, 8), (8, 1), (4, 2), (2, 4)

Hence, possible orders are 1×8, 8×1, 4×2, 2×4

Example 8.2 **Construct the matrix of order 3×2 whose elements are given by $a_{ij} = 2i - j$.**

Sol. In general 3×2 matrix is given by $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

Now $a_{ij} = 2i - j$

$\Rightarrow$ $a_{11} = 2(1) - 1 = 1$

$a_{12} = 2(1) - 2 = 0$

$a_{21} = 2(2) - 1 = 3$

$a_{22} = 2(2) - 2 = 2$

$a_{31} = 2(3) - 1 = 5$

$a_{32} = 2(3) - 2 = 4$

Hence the required matrix is $\begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$

Example 8.3 **If $\begin{bmatrix} x+y & 2x+z \\ x-y & 2z+w \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 0 & 10 \end{bmatrix}$, then find the values of x, y, z, w.**

Sol. We have $\begin{bmatrix} x+y & 2x+z \\ x-y & 2z+w \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 0 & 10 \end{bmatrix}$

Comparing the elements we have

$x + y = 4$, $2x + z = 7$, $x - y = 0$ and $2z + w = 10$

Solving these equations we get $x = 2$ and $y = 2$, $z = 3$, $w = 4$

Example 8.4 **For what values of x and y are the following matrices equal?**

$$A = \begin{bmatrix} 2x+1 & 3y \\ 0 & y^2-5y \end{bmatrix}, \quad B = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$$

Sol. We have,

$$\begin{bmatrix} 2x+1 & 3y \\ 0 & y^2-5y \end{bmatrix} = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$$

$\Rightarrow$ $2x + 1 = x + 3$ (1)

$3y = y^2 + 2$ (2)

$y^2 - 5y = -6$ (3)

$\Rightarrow$ $x = 2$ [From Eq. (1)]

$y = 1$ or 2 [From Eq. (2)]

$y = 2$ or 3 [From Eq. (3)]

$\Rightarrow$ $x = 2$ and $y = 2$

Classification of Matrices

Row Matrix

A matrix having a single row is called a row matrix, e.g., [1 3 5 7].

Column Matrix

A matrix having a single column is called a column matrix, e.g.,

$$\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Note: *Matrices consisting of only one column or row are called vectors.*

Square Matrix

An $m \times n$ matrix A is said to be a square matrix if $m = n$, i.e., number of rows = number of columns. For example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

is a square matrix of order 3×3.

Note: *The diagonal from left-hand side upper corner to right-hand side lower corner is known as leading diagonal or principal diagonal. In the above example, diagonal containing the elements 1, 3, 5 is called the leading or principal diagonal.*

Diagonal Matrix

A square matrix all of whose elements, except those in the leading diagonal, are zero is called a diagonal matrix. For a square matrix, $A = [a_{ij}]_{n \times n}$ to be a diagonal matrix, $a_{ij} = 0$, whenever $i \neq j$.

A diagonal matrix of order $n \times n$ having $d_1, d_2, \ldots, d_n$ as diagonal elements is denoted by diag $[d_1, d_2, \ldots, d_n]$. For example,

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is a diagonal matrix of order 3 × 3 to be denoted by A = diag [3 5 –1].

Scalar Matrix

A diagonal matrix whose all the leading diagonal elements are equal is called a scalar matrix.

For a square matrix $A = [a_{ij}]_{n \times n}$ to be a scalar matrix,

$$a_{ij} = \begin{cases} 0, & i \neq j \\ m, & i = j \end{cases}$$

where $m \neq 0$. For example,

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

is a scalar matrix.

Unit Matrix or Identity Matrix

A diagonal matrix of order n which has unity for all its diagonal elements, is called a unit matrix of order n and is denoted by I_n.

Thus, a square matrix $A = [a_{ij}]_{n \times n}$ is a unit matrix if

$$a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

For example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Triangular Matrix

A square matrix in which all the elements below the diagonal are zero is called upper triangular matrix and a square matrix in which all the elements above diagonal are zero is called lower triangular matrix.

Given a square matrix $A = [a_{ij}]_{n \times n}$, For upper triangular matrix, $a_{ij} = 0$, $i > j$ and for lower triangular matrix, $a_{ij} = 0$, $i < j$. For example,

$$\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -5 & 4 \end{bmatrix}$$

are, respectively, upper and lower triangular matrices.

Note:

- *Diagonal matrix is both upper and lower triangular.*
- *A triangular matrix $A = [a_{ij}]_{n\times n}$ is called strictly triangular if $a_{ii} = 0$ for $1 \leq i \leq n$.*

Null Matrix

If all the elements of a matrix (square or rectangular) are zero, it is called a null or zero matrix. For $A = [a_{ij}]$ to be null matrix, $a_{ij} = 0$, $\forall\, i, j$. For example,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ are null matrices.}$$

Trace of Matrix

The sum of the elements of a square matrix A lying along the principal diagonal is called the trace of A, i.e., tr(A). Thus, if $A = [a_{ij}]_{n \times n}$, then

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

Properties of Trace of a Matrix

Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and λ be a scalar. Then,

(i) $\text{tr}(\lambda A) = \lambda \text{tr}(A)$

(ii) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$

(iii) $\text{tr}(AB) = \text{tr}(BA)$

Determinant of Square Matrix

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, where $a_{ij} = (i, j)^{\text{th}}$ element of A.

This may be thought of as a function which associates each square matrix with a unique number (real or complex). If M is the set of square matrices, K is the set of numbers (real or complex) and $f : M \to K$ is defined by $f(A) = k$, where $A \in M$ and $k \in K$, then $f(A)$ is called the determinant of A. It is also denoted by $|A|$ or det(A) or Δ.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = det(A).

Note:

- *If $A_1, A_2, \ldots, A_n$ are square matrices of the same order then $|A_1A_2 \ldots A_n| = |A_1||A_2| \ldots |A_n|$.*
- *If k is scalar, then $|kA| = k^n|A|$, where n is order of the matrix A.*
- *If A and B are square matrices of same order then $|AB| = |BA|$ even though $AB \neq BA$*

Singular and Non-Singular Matrix

A square matrix A is said to be non-singular if $|A| \neq 0$, and a square matrix A is said to be singular if $|A| = 0$.

Example 8.5 **Find the values of x for which matrix $\begin{bmatrix} 3 & -1+x & 2 \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{bmatrix}$ singular.**

Sol. Given matrix is singular

$$\Rightarrow \begin{vmatrix} 3 & x-1 & 2 \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & x & -x \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{vmatrix} = 0 \quad [R_1 \to R_1 - R_2]$$

$$\Rightarrow \begin{vmatrix} 0 & x & -x \\ -x & 0 & x \\ x+3 & -1 & 2 \end{vmatrix} = 0 \quad [R_2 \to R_2 - R_3]$$

$$\Rightarrow \begin{vmatrix} 0 & x & 0 \\ -x & 0 & x \\ x+3 & -1 & 1 \end{vmatrix} = 0 \quad [C_3 \to C_3 + C_2]$$

$$\Rightarrow \quad -x[(-x) - x(x+3)] = 0 \Rightarrow x(x^2 + 4x) = 0 \Rightarrow x = 0, -4$$

Hence only one value of x in closed interval $[-4, -1]$ i.e. $x = -4$

ALGEBRA OF MATRICES

Addition and Subtraction of Matrices

Before we give the formal definition of addition and subtraction of matrices, we will discuss an example from a real life situation. Let the marks of the three students S_1, S_2, S_3 in maths, physics and chemistry in two tests are as follows:

$$\text{Test 1} \quad \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} P & C & M \end{array} \\ \begin{bmatrix} 40 & 40 & 60 \\ 30 & 70 & 40 \\ 25 & 50 & 55 \end{bmatrix} \end{array} \qquad \text{Test 2} \quad \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} P & C & M \end{array} \\ \begin{bmatrix} 55 & 65 & 78 \\ 40 & 65 & 35 \\ 42 & 65 & 70 \end{bmatrix} \end{array}$$

Now if we want to find the aggregate marks in both the tests, then we must have

$$\text{Aggregate marks} = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array}\!\!\begin{array}{c} \text{Test 1} \\ \begin{array}{ccc} P & C & M \end{array} \\ \begin{bmatrix} 40 & 40 & 60 \\ 30 & 70 & 40 \\ 25 & 50 & 55 \end{bmatrix} \end{array} + \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array}\!\!\begin{array}{c} \text{Test 2} \\ \begin{array}{ccc} P & C & M \end{array} \\ \begin{bmatrix} 55 & 65 & 78 \\ 40 & 65 & 35 \\ 42 & 65 & 70 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} P & C & M \end{array} \\ \begin{bmatrix} 40+55 & 40+65 & 60+78 \\ 30+40 & 70+65 & 40+35 \\ 25+42 & 50+65 & 55+70 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array}\!\!\begin{array}{c} \text{Test 1} \\ \begin{array}{ccc} P & C & M \end{array} \\ \begin{bmatrix} 95 & 105 & 138 \\ 70 & 135 & 75 \\ 67 & 115 & 125 \end{bmatrix} \end{array}$$

Thus, any two matrices can be added if they are of the same order and the resulting matrix is of the same order. If two matrices A and B are of the same order, they are said to be conformable for addition.

Let A, B be two matrices, each of order $m \times n$. Then, their sum $A + B$ is a matrix of order $m \times n$ and is obtained by adding the corresponding elements of A and B.

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices of the same order, their sum $A + B$ is defined to be the matrix of order $m \times n$ such that $(A + B)_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. For example,

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \pm \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 \pm c_1 & b_1 \pm d_1 \\ a_2 \pm c_2 & b_2 \pm d_2 \\ a_3 \pm c_3 & b_3 \pm d_3 \end{bmatrix}$$

Note:

- *Only matrices of the same order can be added or subtracted.*
- *Addition of matrices is commutative* $[A + B = B + A]$ *as well as associative* $[(A + B) + C = A + (B + C)]$.
- *Cancellation laws hold well in case of addition.*
- *The equation* $A + X = O$ *has a unique solution in the set of all* $m \times n$ *matrices (where* O *is null matrix).*

Scalar Multiplication

Before we give the formal definition of scalar multiplication, we will discuss an example from a real life situation. Let the marks of the three students S_1, S_2, S_3 in maths, physics and chemistry are as follows:

$$\begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} P & C & M \end{array} \\ \begin{bmatrix} 40 & 40 & 60 \\ 30 & 70 & 40 \\ 25 & 50 & 55 \end{bmatrix} \end{array}$$

After the result, the examination body realizes that the test papers were too difficult for the students to perform well. So they decided to give 10% grace marks to each students in each subject. Then the revised result is

$$(1.1) \times \left(\begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} P & C & M \end{array} \\ \begin{bmatrix} 40 & 40 & 60 \\ 30 & 70 & 40 \\ 25 & 50 & 55 \end{bmatrix} \end{array} \right) = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} P & C & M \end{array} \\ \begin{bmatrix} 44 & 44 & 66 \\ 33 & 77 & 44 \\ 27.5 & 55 & 60.5 \end{bmatrix} \end{array}$$

Thus, the matrix obtained by multiplying every element of a matrix A by a scalar λ is called the scalar multiple of A by λ and is denoted by λA, i.e., if $A = [a_{ij}]$, then $\lambda A = [\lambda a_{ij}]$. For example, if

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 7 & 8 \end{bmatrix}_{2\times 3}, \text{ then } 2A = \begin{bmatrix} 4 & 6 & 10 \\ 12 & 14 & 16 \end{bmatrix}_{2\times 3}$$

Note: *All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication with scalars.*

Example 8.6 **If** $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 4 \\ 7 & 2 \end{bmatrix}$, **find** $3A - 2B$.

Sol. $3A - 2B = 3\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 4 \\ 7 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 6 & -3 \\ 9 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 8 \\ 14 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6-2 & -3-8 \\ 9-14 & 3-4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -11 \\ -5 & -1 \end{bmatrix}$$

Example 8.7 **If** $A = \text{diag}\ (1\ -1\ 2)$ **and** $B = \text{diag}\ (2\ 3\ -1)$, **then find** $3A + 4B$.

Sol. $3A + 4B = 3\ \text{diag}\ (1\ -1\ 2) + 4\ \text{diag}\ (2\ 3\ -1)$

$= \text{diag}(3\ -3\ 6) + \text{diag}(8\ 12\ -4)$
$= \text{diag}(11\ 9\ 2)$

Example 8.8 **If**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{bmatrix}, \text{ then find } D = \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix} \text{ such}$$

that $A + B - D = O$.

Sol. Here $A + B - D = O$.

$\therefore \quad D = A + B$

$$\Rightarrow \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & 2-2 \\ 3+1 & 4-5 \\ 5+4 & 6+3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 9 & 9 \end{bmatrix}$$

$$\Rightarrow \quad D = \begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 9 & 9 \end{bmatrix}$$

Example 8.9 **Let $A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$ and $2A - B$ $= \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ then find $Tr(A) - Tr(B)$.**

Sol. Here to find the value of $Tr(A) - Tr(B)$, we need not to find the matrices A and B

We can find $Tr(A) - Tr(B)$ using the properties of Trace of matrix

That is

$$A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$$

$\Rightarrow \quad Tr(A + 2B) = -1$

or $\quad T_r(A) + 2T_r(B) = -1 \qquad$ (i)

$$2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$\Rightarrow \quad Tr(2A - B) = 3$

or $\quad 2T_r(A) - T_r(B) = 3 \qquad$ (ii)

Solving (i) and (ii), we get $Tr(A) = 1$ and $Tr(B) = -1$

$\Rightarrow \quad Tr(A) - Tr(B) = 2$

Multiplication of Matrices

Before we give the formal definition of how to multiply two matrices, we will discuss an example from a real life situation.

Consider a city with two kinds of population: the inner city population and the suburb population. We assume that every year 40% of the inner city population moves to the suburbs, while 30% of the suburb population moves to the inner part of the city. Let I (resp. S) be the initial population of the inner city (resp. the suburban area). So after one year, the population of the inner part is $0.6I + 0.3S$, while the population of the suburbs is $0.4I + 0.7S$.

After two years, the population of the inner city is $0.6(0.6I + 0.3S) + 0.3(0.4I + 0.7S)$ and the suburban population is given by $0.4(0.6I + 0.3S) + 0.7(0.4I + 0.7S)$.

Is there a nice way of representing the two populations after a certain number of years? Let us show how matrices may be helpful to answer this question. Let us represent the two populations in one table (meaning a column object with two entries):

$$\begin{bmatrix} I \\ S \end{bmatrix}$$

So after one year the table which gives the two populations is

$$\begin{bmatrix} 0.6I + 0.3S \\ 0.4I + 0.7S \end{bmatrix}$$

If we consider the following rule (the product of two matrices)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} I \\ S \end{bmatrix} = \begin{bmatrix} aI + bS \\ cI + dS \end{bmatrix}$$

then the populations after one year are given by the formula

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} I \\ S \end{bmatrix}$$

After two years, the populations are

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \left(\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} I \\ S \end{bmatrix} \right)$$

Combining this formula with the above result, we get

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6\times0.6+0.3\times0.4 & 0.6\times0.3+0.3\times0.7 \\ 0.4\times0.6+0.7\times0.4 & 0.4\times0.3+0.7\times0.7 \end{bmatrix}$$

In other words, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

In fact, we do not need to have two matrices of the same size to multiply them. Above, we did multiply a (2×2) matrix with a (2×1) matrix [which gave a (2×1) matrix]. In fact, the general rule says that in order to perform the multiplication AB, where A is a $m \times n$ matrix and B is a $k \times l$ matrix, then we must have $n = k$. The result will be a $m \times l$ matrix. For example, we have

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ dx+ey+fz \end{bmatrix}$$

Two matrices A and B are conformable for the product AB if the number of columns in A (pre-multiplier) is same as the number of rows in B (post-multiplier). Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices of order $m \times n$ and $n \times p$, respectively, then their product AB is of order $m \times p$ and is defined as

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$= [a_{i1}\ a_{i2} \cdots a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

$$= (i^{\text{th}} \text{ row of } A)\ (j^{\text{th}} \text{ column of } B) \quad (1)$$

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$.

Now, we define the product of a row matrix and a column matrix. Let $A = [a_1\ a_2 \ldots a_n]$ be a row matrix and

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

be a column matrix. Then,

$$AB = a_1b_1 + a_2b_2 + \cdots + a_nb_n \quad (2)$$

Thus, from (1), we have $(AB)_{ij}$ is the sum of the product of elements of i^{th} row of A with the corresponding elements of j^{th} column of B. For example,

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 4 & -3 \end{bmatrix}$$

Here A is a 3×3 matrix and B is a 3×2 matrix. Therefore, A and B are conformable for the product AB and it is of order 3×2 such that

$(AB)_{11}$ = (First row of A) (First column of B)

$$= [2\ 1\ 3] \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 2 \times 1 + 1 \times 2 + 3 \times 4 = 16$$

$(AB)_{12}$ = (First row of A) (Second column of B)

$$= [2\ 1\ 3] \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = 2 \times (-2) + 1 \times 1 + 3 \times (-3) = -12$$

$(AB)_{21}$ = (Second row of A) (First column of B)

$$= [3\ -2\ 1] \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 3 \times 1 + (-2) \times 2 + 1 \times 4 = 3$$

Similarly, we have $(AB)_{22} = -11$, $(AB)_{31} = 3$ and $(AB)_{32} = -1$.

$$\therefore \quad AB = \begin{bmatrix} 16 & -12 \\ 3 & -11 \\ 3 & -1 \end{bmatrix}$$

Properties of Matrix Multiplication

- *Commutative law does not necessarily hold for matrices.*
- *If $AB = BA$, then matrices A and B are called commutative matrices.*
- *If $AB = -BA$, then matrices A and B are called anti-commutative matrices.*
- *Matrix multiplication is associative: $A(BC) = (AB)C$.*

Proof:

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$ and $C = [c_{ij}]_{p\times q}$. Then, AB is an $m \times p$ matrix and so $(AB)C$ is a $m \times q$ matrix. Also, BC is of order $n \times q$ and so $A(BC)$ is of order $m \times q$. Thus $(AB)C$ and $A(BC)$ are of the same order. Now,

$$((AB)C)_{ij} = \sum_{r=1}^{p} (AB)_{ir}(C)_{rj}$$

$$= \sum_{r=1}^{p}\left(\sum_{s=1}^{n} a_{is}b_{sr}\right)c_{rj} = \sum_{r=1}^{p}\sum_{s=1}^{n}(a_{is}b_{sr})c_{rj}$$

$$= \sum_{r=1}^{p}\sum_{s=1}^{n} a_{is}(b_{sr}c_{rj})$$

[By association law of multiplication of numbers]

$$= \sum_{s=1}^{n} a_{is}\left(\sum_{r=1}^{p}(b_{sr}c_{rj})\right) = \sum_{s=1}^{n} a_{is}(BC)_{sj} = (A(BC))_{ij}$$

for all i, j

Thus, $(AB)C$ and $A(BC)$ are two matrices of the same order such that their corresponding elements are equal. Hence, $(AB)C = A(BC)$.

- *Matrix multiplication is distributive with respect to addition $A(B \pm C) = AB \pm AC$.*
- *If the product $AB = O$, it is not necessary that atleast one of the matrix should be zero matrix. For example, if $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ while neither A nor B is the null matrix.*
- *Cancellation law does not necessarily hold, i.e., if $AB = AC$, then in general $B \neq C$, even if $A \neq O$.*
- *Matrix multiplication $A \times A$ is represented as A^2. Thus $A^n = A\,A \cdots n$ times.*
- *If $A = diag\ (a_1, a_2, a_3, \ldots, a_n)$ and $B = diag\ (b_1, b_2, b_3, \ldots, b_n)$, then $A \times B = diag\ (a_1b_1, a_2b_2, \ldots, a_nb_n)$. Thus $A^n = dign\ (a_1^n, a_2^n, a_3^n, \ldots, a_n^n)$.*
- *If A and B are diagonal matrices of the same order, then $AB = BA$ or diagonal matrices are commutative.*
- *If A and B are commutative, then*

$$(A + B)^2 = (A + B)(A + B)$$
$$= A^2 + AB + BA + B^2$$
$$= A^2 + 2AB + B^2$$

Similarly,

$$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

In general,

$$(A + B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + \cdots + {}^nC_nB^n$$

Matrices A and I are always commutative. Hence,

$$(I + A)^n = {}^nC_0 + {}^nC_1A + {}^nC_1A^2 + \cdots + {}^nC_nA^n$$

Transpose of Matrix

The matrix obtained from any given matrix A, by interchanging rows and columns, is called the transpose of A and is denoted by A^T. If $A = [a_{ij}]_{m\times n}$ and $A^T = [b_{ij}]_{n\times m}$, then $b_{ij} = a_{ji}$, $\forall\ i, j$. For example, if

$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}_{3\times 2}$, then $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}_{2\times 3}$

Properties of Transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$, A and B being conformable matrices
- $(aA)^T = aA^T$, a being scalar
- $(AB)^T = B^T A^T$ (reversal law), A and B being conformable for multiplication

Proof:
Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ be two matrices. Then, AB is an $m \times p$ matrix and therefore $(AB)^T$ is a $p \times m$ matrix. Since A^T and B^T are $n \times m$ and $p \times n$ matrices, therefore $B^T A^T$ is a $p \times m$ matrix. Thus, the two matrices $(AB)^T$ and $B^T A^T$ are of the same order such that

$$((AB)^T)_{ij} = (AB)_{ji}$$
$$= \sum_{r=1}^{n} a_{jr} b_{ri}$$
$$= \sum_{r=1}^{n} b_{ri} a_{jr}$$
$$= \sum_{r=1}^{n} (B^T)_{ir} (A^T)_{rj}$$
$$= (B^T A^T)_{ij}$$

Hence, by the definition of equality of two matrices, we have $(AB)^T = B^T A^T$. Transpose of matrix A is also denoted by A'.

- $|A^T| = |A|$

Matrix Polynomial

If matrix A satisfies the polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, then $f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$.

Example 8.10 If $\begin{bmatrix} 1 & -2 & -3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ then find the product AB and BA.

Sol. $AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Example 8.11 **Find the value of x and y that satisfy the equations $\begin{bmatrix} 3 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} y & y \\ x & x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$.**

Sol. Given $\begin{bmatrix} 3 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} y & y \\ x & x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 3y-2x & 3y-2x \\ 3y & 3y \\ 2y+4x & 2y+4x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$$

Comparing elements we have

$3y - 2x = 3$ (1)

$2y + 4x + 10$ (2)

Solving (1) and (2) we get we get $x = 3/2$ $y = 2$

Example 8.12 **Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p \\ q \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Such that $AB = B$ and $a + d = 2$, then find the value of $(ad - bc)$.**

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p \\ q \end{bmatrix}$

given $AB = B$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} ap+bq \\ cp+dq \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

$\Rightarrow$ $ap + bq = p$ (1)

and $cp + dq = q$ (2)

Eliminating p, q from (1) and (2) we have

$$\begin{vmatrix} a-1 & b \\ c & d-1 \end{vmatrix} = 0$$

$\Rightarrow$ $(a - 1)(d - 1) - bc = 0$

$\Rightarrow$ $ad - (a + d) + 1 - bc = 0$

$\Rightarrow$ $ad - bc = (a + d) - 1 = 2 - 1 = 1$

Example 8.13 **If $A = \begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix}$, then show that**

$$A^8 = \begin{pmatrix} p^8 & q\left(\dfrac{p^8-1}{p-1}\right) \\ 0 & 1 \end{pmatrix}$$

Sol. $A^2 = \begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^2 & pq+q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^2 & q(p+1) \\ 0 & 1 \end{pmatrix}$

$$A^3 = \begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^2 & pq+q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^3 & p^2q+pq+q \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} p^3 & q(p^2+p+1) \\ 0 & 1 \end{pmatrix}$$

Similarly,

$$A^4=\begin{pmatrix} p^4 & q(p^3+p^2+p+1) \\ 0 & 1 \end{pmatrix} \text{ and so on.}$$

$$\therefore \quad A^8=\begin{pmatrix} p^8 & q(p^7+p^8+\cdots+1) \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} p^8 & q\left(\dfrac{p^8-1}{p-1}\right) \\ 0 & 1 \end{pmatrix}$$

Example 8.14 **The matrix $R(t)$ is defined by $R(t)=\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. Show that $R(s)R(t)=R(s+t)$.**

Sol. $R(s)R(t)=\begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

$$=\begin{bmatrix} \cos s\cos t-\sin s\sin t & \cos s\sin t+\sin s\cos t \\ -\sin s\cos t-\cos s\sin t & \cos s\cos t-\sin s\sin t \end{bmatrix}$$

$$=\begin{bmatrix} \cos(s+t) & \sin(s+t) \\ -\sin(s+t) & \cos(s+t) \end{bmatrix}$$

$$=R(s+t)$$

Example 8.15 **If $A=\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, show that $A^2=3A-2I$. Using this result, show that $A^8=255A-254I$.**

Sol. $A=\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

$$\Rightarrow \quad A^2=\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

Now,

$$3A-2I=3\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}-2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 3-2 & 0 \\ 3-0 & 6-2 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

Hence, $A^2=3A-2I$

Now,

$$A^4=(A^2)^2=(3A-2I)^2$$
$$=9A^2-12AI+4I^2$$
$$=9A^2-12A+4I$$
$$=9(3A-2I)-12A+4I$$
$$=15A-14I$$
$$A^8=(A^4)^2=(15A-14I)^2$$
$$=225A^2-420AI+196I^2$$
$$=225(3A-2I)-420A+196I$$
$$=255A-254I$$

Example 8.16 **Matrix A has m rows and $n+5$ columns, matrix B has m rows and $11-n$ columns. If both AB and BA exist, prove that AB and BA are square matrices.**

Sol. If both AB and BA exist, then the number of columns of A is equal to the number of rows of B.

$$\therefore \quad n+5=m \qquad (1)$$

And the number of columns of B is equal to the number of rows of A.

$$\therefore \quad 11-n=m \qquad (2)$$

Solving Eqs. (1) and (2), we get $n=3$ and $m=8$. Hence, A has order 8×8 and B has order 8×8. Hence, both AB and BA are square matrices.

Example 8.17 **If $A=\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, $n\in N$, then prove that $A^{4n}=I$.**

Sol. $A^2=\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}=\begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$A^4=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

$$\Rightarrow \quad A^{4n}=I^n=I$$

Example 8.18 **Find the value of x for which the matrix product $\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$ equals an identity matrix.**

Sol. $\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$

$$=\begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x-2 & 5x \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (given)}$$

$$\Rightarrow \quad 5x=1,\ 10x-2=0,\ \therefore\ x=\frac{1}{5}$$

Example 8.19 **Let $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are two matrices such that they are commutative and $c\neq 3b$. Then find the value of $(a-d)/(3b-c)$.**

Sol. $AB=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$=\begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix}$$

$$BA=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}=\begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix}$$

If $AB=BA$, then

$$a+2c=a+3b$$
$$\Rightarrow \quad 2c=3b \Rightarrow b\neq 0$$
$$b+2d=2a+4b$$
$$\Rightarrow \quad 2a-2d=-3b$$
$$\Rightarrow \quad \frac{a-d}{3b-c}=\frac{-\frac{3}{2}b}{3b-\frac{3}{2}b}=-1$$

Example 8.20 If $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and $n \in N$, then prove that $A^n = 2^{n-1}A$.

Sol. $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

$$\Rightarrow \quad A^2=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}$$

$$\Rightarrow \quad A^3=A^2A=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix}=4A$$

$$\Rightarrow \quad A^n=2^{n-1}A$$

Example 8.21 If $A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$, show that $A^k=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}$, where k is any positive integer.

Sol. We have,

$$A^2=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=\begin{bmatrix}5 & -8\\ 2 & -3\end{bmatrix}=\begin{bmatrix}1+2\times 2 & -4\times 2\\ 2 & 1-2\times 2\end{bmatrix}$$

and

$$A^3=\begin{bmatrix}5 & -8\\ 2 & -3\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=\begin{bmatrix}7 & -12\\ 3 & -5\end{bmatrix}=\begin{bmatrix}1+2\times 3 & -4\times 3\\ 3 & 1-2\times 3\end{bmatrix}$$

Thus, it is true for indices 2 and 3. Now, assume

$$A^k=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}$$

Then,

$$A^{k+1}=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}3+2k & -4(k+1)\\ k+1 & -1-2k\end{bmatrix}$$

$$=\begin{bmatrix}1+2(k+1) & -4(k+1)\\ k+1 & 1-2(k+1)\end{bmatrix}$$

Thus, if the law is true for A^k, it is also true for A^{k+1}. But it is true for $k = 2, 3$, etc. Hence, by induction, the required result follows.

Example 8.22 Let $A=\begin{bmatrix}1 & 2\\ 3 & -5\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ and X be a matrix such that $A = BX$. Then find X.

Sol. Let,

$$\begin{bmatrix}1 & 2\\ 3 & -5\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}a & b\\ 2c & 2d\end{bmatrix}$$

$\therefore \quad a = 1, b = 2, 2c = 3, 2d = -5$

$$\therefore \quad X=\begin{bmatrix}1 & 2\\ 3/2 & -5/2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2 & 4\\ 3 & -5\end{bmatrix}$$

Concept Application Exercise 8.1

1. Solve the following equations for X and Y:
$$2X-Y=\begin{bmatrix}3 & -3 & 0\\ 3 & 3 & 2\end{bmatrix},\ 2Y+X=\begin{bmatrix}4 & 1 & 5\\ -1 & 4 & -4\end{bmatrix}$$

2. If ω is complex cube root of unity and $A=\begin{bmatrix}\omega & 0\\ 0 & \omega\end{bmatrix}$, then calculate A^{100}.

3. If $A=\begin{bmatrix}1 & 2 & x\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & -2 & y\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $AB = I_3$, then find the value of $x + y$.

4. If $A=\begin{bmatrix}1 & 0\\ 1/2 & 1\end{bmatrix}$, then find A^{100}.

5. If $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, then find the values of θ satisfying the equation $A^T + A = I_2$.

6. If $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, then find out the values of α, β such that $(\alpha I+\beta A)^2 = A^2$.

7. Consider the matrices
$$A=\begin{bmatrix}4 & 6 & -1\\ 3 & 0 & 2\\ 1 & -2 & 5\end{bmatrix}, B=\begin{bmatrix}2 & 4\\ 0 & 1\\ -1 & 2\end{bmatrix}, C=\begin{bmatrix}3\\ 1\\ 2\end{bmatrix}$$
Out of the given matrix products, which one is not defined.
a. $(AB)^TC$ **b.** $C^TC(AB)^T$ **c.** C^TAB **d.** A^TABB^TC

8. If $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ p & q & r\end{bmatrix}$, then show that $A^3 = pI + qA + rA^2$.

9. If $A=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}, B=\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$, then find a and b.

Conjugate of Matrix

The matrix obtained from any given matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate numbers is called conjugate of A and is denoted by $\bar{A}$. For example, if

$$A=\begin{bmatrix}1+2i & 2-3i & 3+4i\\ 4-5i & 5+6i & 6-7i\\ 8 & 7+8i & 7\end{bmatrix}$$

then,

$$\bar{A}=\begin{bmatrix}1-2i & 2+3i & 3-4i\\ 4+5i & 5-6i & 6+7i\\ 8 & 7-8i & 7\end{bmatrix}$$

Properties of Conjugate

- $\overline{(\bar{A})}=A$
- $\overline{(A+B)}=\bar{A}+\bar{B}$
- $\overline{(\alpha A)}=\bar{\alpha}\bar{A}$, α being any number
- $\overline{(AB)}=\bar{A}\bar{B}$, A and B being conformable for multiplication

Transpose Conjugate of a Matrix

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^θ. The conjugate of the transpose of A is the same as the transpose of the conjugate of A, i.e., $\overline{(A')} = (\overline{A})' = A^\theta$.

If $A = [a_{ij}]_{m\times n}$, then $A^\theta = [b_{ji}]_{n\times m}$ where $b_{ji} = \overline{a}_{ij}$, i.e., the $(j, i)^{\text{th}}$ element of A^θ is equal to the conjugate of $(i, j)^{\text{th}}$ element of A. For example,

$$\text{if } A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}, \text{ then } A^\theta = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$$

Properties of Transpose Conjugate

- $(A^\theta)^\theta = A$
- $(A + B)^\theta = A^\theta + B^\theta$
- $(kA)^\theta = \bar{k}A^\theta$, k being any number
- $(AB)^\theta = B^\theta A^\theta$

SPECIAL MATRICES

Symmetric Matrix

A square matrix $A = [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$ for all i, j. For example, the matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & 5 \\ 1 & 5 & -2 \end{bmatrix}$$

is symmetric, because $a_{12} = -1 = a_{21}$, $a_{13} = 1 = a_{31}$, $a_{23} = 5 = a_{32}$.

It follows from the definition of a symmetric matrix that A is symmetric $\Leftrightarrow a_{ij} = a_{ji}$ for all i, j $\Leftrightarrow (A)_{ij} = (A^T)_{ij}$ for all $i, j \Leftrightarrow A = A^T$.

Thus, a square matrix A is a symmetric matrix if $A^T = A$. Matrices

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, B = \begin{bmatrix} 2+i & 1 & 3 \\ 1 & 2 & 3+2i \\ 3 & 3+2i & 4 \end{bmatrix}$$

are symmetric matrices, because $A^T = A$ and $B^T = B$.

Skew-Symmetric Matrix

A square matrix $A = [a_{ij}]$ is a skew-symmetric matrix if $a_{ij} = -a_{ji}$ for all i, j. For example, matrix

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix} \text{ is skew-symmetric, because}$$

$$a_{12} = 2, a_{21} = -2 \Rightarrow a_{12} = -a_{21}$$
$$a_{13} = -3, a_{31} = 3 \Rightarrow a_{13} = -a_{31}$$
$$a_{23} = 5, a_{32} = -5 \Rightarrow a_{23} = -a_{32}$$

It follows from the definition of a skew-symmetric matrix that A is skew-symmetric. This implies and is implied by

$$a_{ij} = -a_{ji} \text{ for all } i, j$$
$$\Leftrightarrow (A)_{ij} = -(A^T)_{ij} \text{ for all } i, j$$
$$\Leftrightarrow A = -A^T$$
$$\Leftrightarrow A^T = -A$$

Thus, a square matrix A is a skew-symmetric matrix if $A^T = -A$. Therefore, matrices

$$A = \begin{bmatrix} 0 & 2i & 3 \\ -2i & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ -5 & -2 & 0 \end{bmatrix}$$

are skew-symmetric matrices, because $A^T = -A$ and $B^T = -B$.

Properties of Symmetric and Skew-Symmetric Matrices

(i) If A is a symmetric matrix, then $-A$, kA, A^T, A^n, A^{-1}, B^TAB are also symmetric matrices, where $n \in N$, $k \in R$ and B is a square matrix of order that of A

(ii) If A is a skew-symmetric matrix, then
 (a) A^{2n} is a symmetric matrix for $n \in N$,
 (b) A^{2n+1} is a skew-symmetric matrix for $n \in N$,
 (c) kA is also skew-symmetric matrix, where $k \in R$,
 (d) B^TAB is also skew-symmetric matrix where B is a square matrix of order that of A.

(iii) If A, B are two symmetric matrices, then
 (a) $A \pm B$, $AB + BA$ are also symmetric matrices,
 (b) $AB - BA$ is a skew-symmetric matrix,
 (c) AB is a symmetric matrix, when $AB = BA$.

(iv) If A, B are two skew-symmetric matrices, then
 (a) $A \pm B$, $AB - BA$ are skew-symmetric matrices,
 (b) $AB + BA$ is a symmetric matrix.

(v) If A is a skew-symmetric matrix and C is a column matrix, then C^TAC is a zero matrix.

Example 8.23 **Show that the elements on the main diagonal of a skew-symmetric matrix are all zero.**

Sol. Let $A = [a_{ij}]$ be a skew-symmetric matrix. Then, $a_{ij} = -a_{ji}$ for all i, j (by definition). Hence,

$$a_{ii} = -a_{ii} \text{ for all values of } i$$
$$\Rightarrow 2a_{ii} = 0$$
$$\Rightarrow a_{ii} = 0 \text{ for all values of } i$$
$$\Rightarrow a_{11} = a_{22} = a_{33} = \cdots = a_{nn} = 0$$

Example 8.24 **Let A be a square matrix. Then prove that**

a. $A + A^T$ is a symmetric matrix,

b. $A - A^T$ is a skew-symmetric matrix and

c. AA^T and A^TA are symmetric matrices.

Sol.

a. Let $P = A + A^T$. Then,

$$P^T = (A + A^T)^T = A^T + (A^T)^T \quad [\because (A + B)^T = A^T + B^T]$$
$$\Rightarrow P^T = A^T + A \quad [\because (A^T)^T = A]$$
$$\Rightarrow P^T = A + A^T = P$$

[By commutative law of matrix addition]

Therefore, P is a symmetric matrix.

b. Let $Q = A - A^T$. Then,

$$Q^T = (A - A^T)^T = A^T - (A^T)^T \quad [\because (A + B)^T = A^T + B^T]$$
$$\Rightarrow Q^T = A^T - A \quad [\because (A^T)^T = A]$$
$$\Rightarrow Q^T = -(A - A^T) = -Q$$

Therefore, Q is skew-symmetric.

c. We have,

$(AA^T)^T = ((A^T)^T A^T)$ [By reversal law]

$= AA^T$ [$\because (A^T)^T = A$]

Therefore, AA^T is symmetric. Similarly, it can be proved that A^TA is symmetric.

Example 8.25 **Prove that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.**

Sol. Let A be a square matrix. Then,

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = P + Q \text{ (say)}$$

where $P = (1/2)(A + A^T)$ and $Q = (1/2)(A - A^T)$.

Now $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric. Therefore, P is symmetric and Q is skew-symmetric. Hence proved.

Example 8.26 **If matrix** $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} = B + C$**, where B is symmetric matrix and C is skew-symmetric matrix. Then find matrix B and C.**

Sol. Here matrix A is expressed as sum of symmetric and skew-symmetric matrix.

Then $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$

$$\text{Now } A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$$

$$\Rightarrow B = \frac{1}{2}\left(\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 5 & 2 \\ 5 & -6 & 1 \\ 2 & 1 & 8 \end{bmatrix}$$

$$\text{and } C = \frac{1}{2}\left(\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & 7 \\ 4 & -7 & 0 \end{bmatrix}$$

Example 8.27 $A = \begin{bmatrix} 3 & a & -1 \\ 2 & 5 & c \\ b & 8 & 2 \end{bmatrix}$ **is symmetric and** $B = \begin{bmatrix} d & 3 & a \\ b-a & e & -2b-c \\ -2 & 6 & -f \end{bmatrix}$ **is skew-symmetric, then find AB.**

Sol. A is symmetric

$\Rightarrow A^T = A$

$$\Rightarrow \begin{bmatrix} 3 & 2 & b \\ a & 5 & 8 \\ -1 & c & 2 \end{bmatrix} = \begin{bmatrix} 3 & a & -1 \\ 2 & 5 & c \\ b & 8 & 2 \end{bmatrix}$$

$\Rightarrow a = 2, b = -1, c = 8$

B is skew-symmetric

$\Rightarrow B^T = -B$

$$\Rightarrow \begin{bmatrix} d & b-a & -2 \\ 3 & e & 6 \\ a & -2b-c & -f \end{bmatrix} = \begin{bmatrix} -d & -3 & -a \\ a-b & -e & 2b+c \\ 2 & -6 & f \end{bmatrix}$$

$\Rightarrow d = -d, f = -f$ and $e = -e$

$\Rightarrow d = f = 0$

$$\text{So } A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 8 \\ -1 & 8 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 8 \\ -1 & 8 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 3 & -6 \\ -31 & 54 & -26 \\ -28 & 9 & -50 \end{bmatrix}$$

Example 8.28 **Let A, B, C, D be (not necessarily square) real matrices such that**

$A^T = BCD$; $B^T = CDA$; $C^T = DAB$ and $D^T = ABC$

for the matrix $S = ABCD$, prove that $S^3 = S$.

Sol. $S = ABCD = A(BCD) = AA^T$ (1)

$S^3 = (ABCD)(ABCD)(ABCD)$

$= (ABC)(DAB)(CDA)(BCD)$

$= D^TC^TB^TA^T$

$= (BCD)^TA^T = AA^T$ (2)

from (1) and (2), $S = S^3$

Example 8.29 **If** $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ **is a matrix satisfying $AA^T = 9I_3$, then find the values of a and b.**

Sol. We have,

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$$

$\therefore AA^T = 9I_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+2b+4 \\ 0 & 9 & 2a+2-2b \\ a+2b+4 & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\Rightarrow a + 2b + 4 = 0$, $2a + 2 - 2b = 0$ and $a^2 + 4 + b^2 = 9$

$\Rightarrow a + 2b + 4 = 0$, $a - b + 1 = 0$ and $a^2 + b^2 = 5$

Solving $a + 2b + 4 = 0$ and $a - b + 1 = 0$, we get

$a = -2, b = -1$. Clearly, these values satisfy $a^2 + b^2 = 5$.

Hence, $a = -2$ and $b = -1$.

Unitary Matrix

A square matrix is said to be unitary if $\bar{A}'A = I$ since $|\bar{A}'| = |A|$ and $|\bar{A}'A| = |\bar{A}'||A|$, therefore if $\bar{A}'A = I$, we have $|\bar{A}'||A| = 1$.

Thus, the determinant of unitary matrix is of unit modulus. For a matrix to be unitary it must be non-singular.

Hermitian and Skew-Hermitian Matrix

A square matrix $A = [a_{ij}]$ is said to be Hermitian matrix if $a_{ij} = \bar{a}_{ji}$, $\forall\, i, j$, i.e., $A = A^{\theta}$. For example,

$$\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}, \begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$$

are Hermitian matrices.

A square matrix $A = [a_{ij}]$ is said to be skew-Hermitian if $a_{ij} = -\bar{a}_{ji}$, $\forall\, i, j$, i.e., $A^{\theta} = -A$. For example,

$$\begin{bmatrix} 0 & -2+i \\ 2+i & 0 \end{bmatrix}, \begin{bmatrix} 3i & -3+2i & -1-i \\ 3+2i & -2i & -2-4i \\ 1-i & 2-4i & 0 \end{bmatrix}$$

are skew-Hermitian matrices.

Note:

- *If A is a Hermitian matrix, then $a_{ii} = \bar{a}_{ii} \Rightarrow a_{ii}$ is real, $\forall\, i$. Thus every diagonal element of a Hermitian matrix must be real.*
- *A Hermitian matrix over the set of real numbers is actually a real symmetric matrix.*
- *If A is a skew-Hermitian matrix, then $a_{ii} = -\bar{a}_{ii} \Rightarrow a_{ii} + \bar{a}_{ii} = 0$, i.e., a_{ii} must be purely imaginary or zero.*
- *A skew-Hermitian matrix over the set of real numbers is actually a real skew-symmetric matrix.*

Orthogonal Matrix

Any square matrix A of order n is said to be orthogonal, if $AA' = A'A = I_n$.

Idempotent Matrix

A square matrix A is called idempotent provided it satisfies the relation $A^2 = A$.

Involuntary Matrix

A square matrix such that $A^2 = I$ is called involuntary matrix.

Nilpotent Matrix

A square matrix A is called a nilpotent matrix if there exists a positive integer m such that $A^m = O$. If m is the least positive integer such that $A^m = O$, then m is called the index of the nilpotent matrix A.

Example 8.30 **Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.**

Sol. $A^2 = A\times A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$= A$

Hence, the matrix A is idempotent.

Example 8.31 **Show that $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of order 3.**

Sol. Let, $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = A\times A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$\therefore \quad A^3 = A^2\times A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore \quad A^3 = O$

Hence A is nilpotent of order 3.

Example 8.32 **Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory.**

Sol. $A^2 = A\times A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the given matrix A is involutory.

Example 8.33 **If A and B are n-rowed unitary matrices, then AB and BA are also unitary matrices.**

Sol. Let A and B be two unitary matrices. Then,

$$A^{\Theta} A = AA^{\Theta} = I \text{ and } B^{\Theta}B = BB^{\Theta} = I$$

Now,

$$(AB)^{\Theta}(AB) = (B^{\Theta}A^{\Theta})(AB)$$

$$= B^{\Theta}(A^{\Theta}A)B = B^{\Theta}IB = B^{\Theta}B = I$$

Hence, matrix AB is unitary. Similarly, we can show that

$(BA)^{\Theta}(BA) = I \Rightarrow BA$ is also unitary

Example 8.34 **If $abc = p$ and $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$, prove that A is orthogonal if and only if a, b, c are the roots of the equation $x^3 \pm x^2 - p = 0$.**

Sol. Here $AA^T = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$

$$= \begin{bmatrix} a^2+b^2+c^2 & ac+ab+bc & ab+bc+ca \\ ca+ab+bc & a^2+b^2+c^2 & cb+ba+ac \\ ab+cb+ac & bc+ca+ab & a^2+b^2+c^2 \end{bmatrix}$$

Now, $AA^T = I$ if

$$a^2 + b^2 + c^2 = 1 \text{ and } ab + bc + ca = 0$$

$$\Rightarrow \quad (a+b+c)^2 - 2(ab+bc+ca) = 1 \text{ and } ab+bc+ca = 0$$

$$\Rightarrow \quad a+b+c = \pm 1 \text{ and } ab+bc+ca = 0$$

Also, $abc = p$, so that a, b, c are the roots of the equation $x^3 \pm x^2 - p = 0$. Conversely, since a, b, c are the roots of $x^3 \pm x^2 - p = 0$, $a + b + c = \pm 1$ and $ab + bc + ca = 0$, hence $a^2 + b^2 + c^2 = 1$.

Concept Application Exercise 8.2

1. Express the matrix $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.
2. Show that all positive integral powers of a symmetric matrix are symmetric.
3. If A and B be n-rowed orthogonal matrices, then show that AB and BA are also orthogonal matrices.
4. Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $A - \lambda I$ is a singular matrix, then find the values of λ.

ADJOINT OF SQUARE MATRIX

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then, the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj A. Thus,

$$\text{adj } A = [C_{ij}]^T \Rightarrow (\text{adj } A)_{ij} = C_{ji}$$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

where C_{ij} denotes the cofactor of a_{ij} in A. For example,

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Here, $C_{11} = s$, $C_{12} = -r$, $C_{21} = -q$, $C_{22} = p$. Therefore,

$$\text{adj } A = \begin{bmatrix} s & -r \\ -q & p \end{bmatrix}^T = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$$

Example 8.35 **Find the adjoint of the matrix**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 3 \end{bmatrix}$$

Sol. Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by

$$C_{11} = \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 9$$

$$C_{12} = -\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = -3$$

$$C_{13} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1$$

$$C_{22} = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4$$

$$C_{23} = -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -3$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4$$

$$C_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = 5$$

$$C_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 9 & -3 & 5 \\ -1 & 4 & -3 \\ -4 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 9 & -1 & -4 \\ -3 & 4 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

Inverse of Matrix

A non-singular square matrix of order n is invertible if there exists a square matrix B of the same order such that

$$AB = I_n = BA$$

In such a case, we say that the inverse of A is B and we write

$$A^{-1} = B$$

Also from $A(\text{adj } A) = |A|I_n = (\text{adj } A)A$, we can conclude that

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

Properties of Adjoint and Inverse of a Matrix

1. Let A be square matrix of order n. Then $A(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$.

Proof: Let $A = [a_{ij}]$, and let C_{ij} be cofactor of a_{ij} in A. Then, $(\text{adj } A)_{ij} = C_{ji}$, $\forall\ 1 \le i, j \le n$. Now,

$$(A(\text{adj } A))_{ij} = \sum_{r=1}^{n} (A)_{ir}\, (\text{adj } A)_{rj}$$

$$= \sum_{r=1}^{n} a_{ir} C_{jr} = \begin{cases} |A|, & if \quad i = j \\ 0, & if \quad i \ne j \end{cases}$$

$$\Rightarrow \quad A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ 0 & 0 & |A| & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & |A| \end{bmatrix}$$

$$= |A|\, I_n$$

Similarly,

$$((\text{adj } A)A)_{ij} = \sum_{r=1}^{n} (\text{adj} A)_{ir} (A)_{rj}$$

$$= \sum_{r=1}^{n} C_{ri} a_{rj} = \begin{cases} |A|, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \ne j \end{cases}$$

Hence, $A(\text{adj } A) = |A|I_n = (\text{adj } A)A$

2. Every invertible matrix possesses a unique inverse.

Proof: Let A be an invertible matrix of order $n \times n$. Let B and C be two inverses of A. Then,

$$AB = BA = I_n \qquad (1)$$

$$AC = CA = I_n \qquad (2)$$

Now,

$AB = I_n$

$\Rightarrow$ $C(AB) = CI_n$ [pre-multiplying by C]

$\Rightarrow$ $(CA)B = CI_n$ [by associativity]

$\Rightarrow$ $I_n B = CI_n$ [$\because CA = I_n$ from (2)]

$\Rightarrow$ $B = C$ [$\because I_n B = B$, $CI_n = C$]

Hence, an invertible matrix possesses a unique inverse.

3. (*Reversal law*) If A and B are invertible matrices of the same order, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. In general, if A, B, C, ... are invertible matrices, then $(ABC\cdots)^{-1} = \cdots C^{-1}B^{-1}A^{-1}$.

Proof: It is given that A and B are invertible matrices.

$\therefore$ $|A| \ne 0$ and $|B| \ne 0$

$\Rightarrow$ $|A|\,|B| \ne 0$

$\Rightarrow$ $|AB| \ne 0$ [$\because |AB| = |A|\,|B|$]

Hence, AB is an invertible matrix. Now,

$(AB)\,(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ [by associativity]

$= (AI_n)A^{-1}$ [$\because BB^{-1} = I_n$]

$= AA^{-1}$ [$\because AI_n = A$]

$= I_n$ [$\because AA^{-1} = I_n$]

Also,

$(B^{-1}A^{-1})\,(AB) = B^{-1}(A^{-1}A)B$ [by associativity]

$= B^{-1}(I_nB)$ [$\because A^{-1}A = I_n$]

$= B^{-1}B$ [$\because I_nB = B$]

$= I_n$ [$\because B^{-1}B = I_n$]

Thus,

$$(AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB)$$

Hence,

$$(AB)^{-1} = B^{-1}A^{-1}$$

4. If A is an invertible square matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

Proof: A is an invertible matrix.

$\therefore$ $|A| \ne 0$

$\Rightarrow$ $|A^T| \ne 0$ [$\because |A^T| = |A|$]

Hence, A^T is also invertible. Now,

$AA^{-1} = I_n = A^{-1}A$

$\Rightarrow$ $(AA^{-1})^T = (I_n)^T = (A^{-1}A)^T$

$\Rightarrow$ $(A^{-1})^T(A^T) = I_n = A^T(A^{-1})^T$

[by reversal law for transpose]

$\Rightarrow$ $(A^T)^{-1} = (A^{-1})^T$ [by definition of inverse]

5. If A is a non-singular square matrix of order n, then $|\text{adj } A| = |A|^{n-1}$.

Proof: We have,

$$A(\text{adj } A) = |A|\, I_n$$

$$\Rightarrow \quad A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}_{n \times n}$$

$$\Rightarrow \quad |A(\text{adj } A)| = \begin{vmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & |A| \end{vmatrix} = |A|^n$$

$\Rightarrow$ $|A|\,|\text{adj } A| = |A|^n$ [$\because |AB| = |A|\,|B|$]

$\Rightarrow$ $|\text{adj } A| = |A|^{n-1}$

6. *Reversal law for adjoint*: If A and B are non-singular square matrices of the same order, then adj $(AB) = (\text{adj } B)\,(\text{adj } A)$ (using $(AB)^{-1} = B^{-1} . A^{-1}$)

7. If A is an invertible square matrix, then adj $(A^T) = (\text{adj } A)^T$ (using $(A^T)^{-2} = (A^{-1})^T$)

8. If A is a non-singular square matrix, then adj (adj A) $= |A|^{n-2} A$

Proof: We know that $B(\text{adj } B) = |B|I_n$ for every square matrix of order n. Replacing B by adj A, we get

$$(\text{adj } A)\,[\text{adj } (\text{adj } A)] = |\text{adj } A|I_n$$

$\Rightarrow$ $(\text{adj } A)[\text{adj}(\text{adj } A)] = |A|^{n-1}I_n$ $\quad[\because |\text{adj } A| = |A|^{n-1}]$

$\Rightarrow$ $A\{(\text{adj } A)(\text{adj adj } A)\} = A\{|A|^{n-1}I_n\}$
[pre-multiplying both sides by A]

$\Rightarrow$ $(A \text{ adj } A)(\text{adj adj } A) = |A|^{n-1}(AI_n)$ [by associativity]

$\Rightarrow$ $|A|I_n(\text{adj adj } A) = |A|^{n-1}A$
$[\because AI_n = A \text{ and } A \text{ adj } A = |A|I_n]$

$\Rightarrow$ $|A|(I_n(\text{adj adj } A)) = |A|^{n-1}A$

$\Rightarrow$ $|A|(\text{adj adj } A) = |A|^{n-1}A$

$\Rightarrow$ $\text{adj adj } A = |A|^{n-2}A$ $\quad\left[\text{multiplying both sides by } \dfrac{1}{|A|}\right]$

9. If A is a non-singular matrix, then $|A^{-1}| = |A|^{-1}$, i.e., $|A^{-1}| = 1/|A|$.

Proof: Since $|A| \neq 0$, therefore A^{-1} exists such that

$$AA^{-1} = I = A^{-1}A$$

$\Rightarrow$ $|AA^{-1}| = |I|$

$\Rightarrow$ $|A|\,|A^{-1}| = 1$ $\quad[\because |AB| = |A|\,|B| \text{ and } |I| = 1]$

$\Rightarrow$ $|A^{-1}| = \dfrac{1}{|A|}$ $\quad[\because |A| \neq 0]$

10. Inverse of the k^{th} power of A is the k^{th} power of the inverse of A.

Proof: We have to prove that $(A^{-1})^k = (A^k)^{-1}$.

$$(A^k)^{-1} = (A \times A \times A \cdots \times A)^{-1}$$
$$= (A^{-1}A^{-1}A^{-1}\cdots A^{-1})$$
$$= (A^{-1})^k$$

Example 8.36 **Find the values of K for which matrix $A = \begin{bmatrix} 1 & 0 & -K \\ 2 & 1 & 3 \\ K & 0 & 1 \end{bmatrix}$ is invertible.**

Sol. Matrix A is invertible if $|A| \neq 0$

i.e., $\begin{vmatrix} 1 & 0 & -K \\ 2 & 1 & 3 \\ K & 0 & 1 \end{vmatrix} \neq 0$

$\Rightarrow$ $1(1) - K(-K) \neq 0$

$\Rightarrow |A| = K^2 + 1 \neq 0$ which is true for all real K.

Hence A is invertible for all real values of K

Example 8.37 **Let p be a non-singular matrix, and $I + p + p^2 + \cdots + p^n = O$ then find p^{-1}.**

Sol. We have $I + p + p^2 + \cdots + p^n = O$ (1)

Since p is non-singular matrix, p is invertible

Multiplying (1) both sides by p^{-1},

$\Rightarrow$ $p^{-1} + I + Ip + \cdots + p^{n-1}I = O.p^{-1}$

$\Rightarrow$ $p^{-1} + I(1 + p + \cdots + p^{n-1}) = O$

$\Rightarrow$ $p^{-1} = -I(I + p + p^2 + \cdots + p^{n-1}) = -(-p^n) = p^n$.

Example 8.38 **Given the matrices A and B as $A = \begin{bmatrix} 1 & -1 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$. The two matrices X and Y are such that $XA = B$ and $AY = B$ then find the matrix $3(X + Y) = \begin{bmatrix} 4 & -1 \\ 4 & 2 \end{bmatrix}$.**

Sol. Here A is non singular but B is singular hence only A^{-1} exists

Now $XA = B$

$\Rightarrow$ $X = BA^{-1}$ (1)

And $AY = B$

$\Rightarrow$ $Y = A^{-1}B$ (2)

Also $A^{-1} = \dfrac{1}{3}\begin{bmatrix} -1 & 1 \\ -4 & 1 \end{bmatrix}$

$\Rightarrow X = BA^{-1} = \dfrac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

$\Rightarrow Y = A^{-1}B = \dfrac{1}{3}\begin{bmatrix} -1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$

$\Rightarrow 3(X + Y) = \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 4 & 2 \end{bmatrix}$

Example 8.39 **Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$. If B is the inverse of A, then find the value α.**

Sol. Here,

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

$\therefore$ $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$

$= 1(1 + 3) + 1(2 + 3) + 1(2 - 1)$

$= 4 + 5 + 1 = 10$

$\Rightarrow$ $\text{adj } A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

$\Rightarrow$ $B = A^{-1} = \dfrac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

$\Rightarrow$ $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

Hence, $\alpha = 5$.

Example 8.40 If $A=\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then find the value of $|A|\,|\text{adj } A|$.

Sol. $|A|\,|\text{adj } A| = |A \text{ adj } A| = ||A|\, I|$

$$= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

$$= |A|^3 = (a^3)^3 = a^9$$

Example 8.41 For the matrix $A=\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find x and y so that $A^2 + xI = yA$. Hence, find A^{-1}.

Sol. We have,

$$A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 9+7 & 3+5 \\ 21+35 & 7+25 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$$

Now,

$$A^2 + xI = yA$$

$$\Rightarrow \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16+x & 8+0 \\ 56+0 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

$$\Rightarrow 16 + x = 3y,\ y = 8,\ 7y = 56,\ 5y = 32 + x$$

Putting $y = 8$ in $16 + x = 3y$, we get $x = 24 - 16 = 8$. Clearly, $x = 8$ and $y = 8$ also satisfy $7y = 56$ and $5y = 32 + x$. Hence, $x = 8$ and $y = 8$. We have,

$$|A| = \begin{vmatrix} 3 & 1 \\ 7 & 5 \end{vmatrix} = 8 \neq 0$$

So, A is invertible.

Putting $x = 8$, $y = 8$ in $A^2 + xI = yA$, we get

$$A^2 + 8I = 8A$$

$$\Rightarrow A^{-1}(A^2 + 8I) = 8A^{-1}A \quad \text{[re-multiplying throughout by } A^{-1}]$$

$$\Rightarrow A^{-1}A^2 + 8A^{-1}I = 8A^{-1}A$$

$$\Rightarrow A + 8A^{-1} = 8I$$

$$[\because A^{-1}A^2 = (A^{-1}A)A = IA = A,\ A^{-1}I = A^{-1} \text{ and } A^{-1}A = I]$$

$$\Rightarrow 8A^{-1} = 8I - A$$

$$\Rightarrow A^{-1} = \frac{1}{8}(8I - A) = \frac{1}{8}\left\{\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right\}$$

$$\Rightarrow A^{-1} = \frac{1}{8}\begin{bmatrix} 8-3 & 0-1 \\ 0-7 & 8-5 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5/8 & -1/8 \\ -7/8 & 3/8 \end{bmatrix}$$

Example 8.42 By the method of matrix inversion, solve the system.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

Sol. We have,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & -5 & -1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow AX = B \qquad (1)$$

Clearly $|A| = -4 \neq 0$. Therefore,

$$\text{adj } A = \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj}.A}{|A|} = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

Now,

$$A^{-1}B = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{-1}{4}\begin{bmatrix} -4 & 4 \\ -12 & -8 \\ -20 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

From Eq. (1),

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

$$\Rightarrow x_1 = 1,\ x_2 = 3,\ x_3 = 5 \text{ or } y_1 = -1,\ y_2 = 2,\ y_3 = 1$$

Example 8.43 If A, B and C are $n \times n$ matrix and det (A) = 2, det (B) = 3 and det (C) = 5, then find the value of the det (A^2BC^{-1}).

Sol. Given that $|A| = 2$, $|B| = 3$, $|C| = 5$. Now,

$$\det(A^2BC^{-1}) = |A^2BC^{-1}| = \frac{|A|^2|B|}{|C|} = \frac{4\times 3}{5} = \frac{12}{5}$$

Example 8.44 Matrices A and B satisfy $AB = B^{-1}$, where $B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$. Find

a. without finding B^{-1}, the value of K for which $KA - 2B^{-1} + I = O$.

b. without finding A^{-1}, the matrix X satisfying $A^{-1}XA = B$.

Sol. a. $AB = B^{-1} \Rightarrow AB^2 = I$

Now,

$$KA - 2B^{-1} + I = O$$

$$\Rightarrow KAB - 2B^{-1}B + IB = O$$

$$\Rightarrow KAB - 2I + B = O$$

$$\Rightarrow KAB^2 - 2B + B^2 = O$$

$\Rightarrow KI - 2B + B^2 = O$

$\Rightarrow K\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - 2\begin{bmatrix}2 & -1\\2 & 0\end{bmatrix} + \begin{bmatrix}2 & -1\\2 & 0\end{bmatrix}\begin{bmatrix}2 & -1\\2 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}K & 0\\0 & K\end{bmatrix} - \begin{bmatrix}4 & -2\\4 & 0\end{bmatrix} + \begin{bmatrix}2 & -2\\4 & -2\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}K-2 & 0\\0 & K-2\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$\Rightarrow K = 2$

b. $A^{-1}XA = B$

$\Rightarrow AA^{-1}XA = AB$

$\Rightarrow IXA = AB$

$\Rightarrow XAB = AB^2$

$\Rightarrow XAB = I$

$\Rightarrow XAB^2 = B$

$\Rightarrow XI = B$

$\Rightarrow X = B$

Example 8.45 **If there are three square matrix A, B, C of same order satisfying the equation $A^2 = A^{-1}$ and let $B = A^{2^n}$ & $C = A^{2^{(n-2)}}$ then prove that det.$(B - C) = 0$, $n \in N$.**

Sol. $B = A^{2^n} = A^{2\cdot 2^{n-1}}$

$= (A^2)^{2^{n-1}}$

$= (A^{-1})^{2^{n-1}}$

$= (A^{2^{n-1}})^{-1}$

$= \left(A^{2\cdot 2^{n-2}}\right)^{-1}$

$= \left((A^2)^{2^{n-2}}\right)^{-1}$

$= \left((A^{-1})^{-1}\right)^{2^{n-2}} = A^{2^{(n-2)}}$

so $B = C \Rightarrow (B - C) = O \Rightarrow$ det. $(B - C) = 0$

Example 8.46 **If A is a non singular matrix satisfying $AB - BA = A$, then prove that det. $(B + I)$ = det. $(B - I)$.**

Sol. A is non-singular det $A \neq 0$

Given $AB - BA = A$

hence $AB = A + BA = A(I + B)$

$\Rightarrow$ det. A · det. B = det. A · det. $(I + B)$

$\Rightarrow$ det. B = det. $(I + B)$ (1) (as A is non singular)

again $AB - A = BA$

$\Rightarrow A(B - I) = BA$

$\Rightarrow$ (det. A) · det.$(B - I)$ = det. B · det. A

$\Rightarrow$ det. $(B - I)$ = det. (B) (2)

from (1) and (2), det. $(B - I)$ = det. $(B + I)$

Example 8.47 **If A is a symmetric and B skew symmetric matrix and $A + B$ is non singular and $C = (A + B)^{-1}(A - B)$ then prove that**

(i) $C^T(A + B)C = A + B$
(ii) $C^T(A - B)C = A - B$
(iii) $C^TAC = A$

Sol.

(i) $(A + B)C = (A + B)(A + B)^{-1}(A - B)$

$\Rightarrow (A + B)C = A - B$ (1)

$C^T = (A - B)^T((A + B)^{-1})^T$

$= (A + B)((A + B)^T)^{-1}$

{as $|A + B| \neq 0 \Rightarrow |(A + B)^T| \neq 0 \Rightarrow |A - B| \neq 0$}

$= (A + B)(A - B)^{-1}$ (2)

From (1) and (2),

$C^T(A + B)C = (A + B)(A - B)^{-1}(A - B)$

$= (A + B)$ (3)

(ii) taking transpose in (3)

$C^T(A + B)^T(C^T)^T = (A + B)^T$

$C^T(A - B)C = A - B$ (4)

(iii) adding (3) and (4)

$C^T[A + B + A - B]C = 2A$

$C^TAC = A$

EQUIVALENT MATRICES

If a matrix B is obtained from a matrix A by one or more elementary transformations, then A and B are equivalent matrices and we write $A \sim B$. Let,

$$A = \begin{bmatrix}1 & 2 & 3 & 4\\2 & 1 & 4 & 3\\3 & 1 & 2 & 4\end{bmatrix}$$

Then

$$A \sim \begin{bmatrix}1 & 2 & 3 & 4\\1 & -1 & 1 & -1\\3 & 1 & 2 & 4\end{bmatrix} \quad [\text{Applying } R_2 \to R_2 + (-1)R_1]$$

$$\sim \begin{bmatrix}1 & 2 & 3 & 1\\1 & -1 & 1 & -2\\3 & 1 & 2 & 2\end{bmatrix} \quad [\text{Applying } C_4 \to C_4 + (-1)C_3]$$

An elementary transformation is called a row transformation or a column transformation accordingly as it is applied to rows or columns.

Theorem 1

Every elementary row (column) transformation of an $m\times n$ matrix (not identity matrix) can be obtained by pre-multiplication (post-multiplication) with the corresponding elementary matrix obtained from the identity matrix $I_m(I_n)$ by subjecting it to the same elementary row (column) transformation.

Theorem 2

Let $C = AB$ be a product of two matrices. Any elementary row (column) transformation of AB can be obtained by subjecting the pre-factor A (post factor B) to the same elementary row (column) transformation.

Method of Finding the Inverse of a Matrix by Elementary Transformations:

Let A be a non singular matrix of order n. Then A can be reduced to the identity matrix I_n by a finite sequence of elementary transformation only. As we have discussed every elementary row transformation of a matrix is equivalent to pre-multiplication by the corresponding elementary matrix. Therefore there exist elementary matrices $E_1, E_2, \ldots E_k$ such that

$$(E_k E_{k-1} \ldots E_2 E_1)A = I_n$$

$$\Rightarrow (E_k E_{k-1} \ldots E_2 E_1)AA^{-1} = I_n A^{-1} \quad \text{(post multiplying by } A^{-1})$$

$$\Rightarrow (E_k E_{k-1} \ldots E_2 E_1)I_n = A^{-1} \ (\because I_n A^{-1} = A^{-1} \text{and } AA^{-1} = I_n)$$

$$\Rightarrow A^{-1} = (E_k E_{k-1} \ldots E_2 E_1)I_n$$

Algorithm for Finding the Inverse of a Non Singular Matrix by Elementary Row Transformations

Let A be non-singular matrix of order n

Step I : Write $A = I_n A$

Step II : Perform a sequence of elementary row operations successively on A on the LHS and the pre factor I_n on the RHS till we obtain the result $I_n = BA$

Step III : Write $A^{-1} = B$

The following steps will be helpful to find the inverse of a square matrix of order 3 by using elementary row transformations.

Step I : Introduce unity at the intersection of first row and first column either by interchanging two rows or by adding a constant multiple of elements of some other row to first row.

Step II : After introducing unity at (1,1) place introduce zeros at all other places in first column.

Step III : Introduce unity at the intersection of 2nd row and 2nd column with the help of 2nd and 3rd row.

Step IV : Introduce zeros at all other places in the second column except at the intersection of 2nd row and 2nd column.

Step V : Introduce unity at the intersection of 3rd row and third column.

Step VI : Finally introduce zeros at all other places in the third column except at the intersection of third row and third column.

Example 8.48 **Using elementary transformation, find the inverse of the matrix** $A = \begin{bmatrix} a & b \\ c & \left(\frac{1+bc}{-a}\right) \end{bmatrix}$.

Sol. $A = \begin{bmatrix} a & b \\ c & \left(\frac{1+bc}{a}\right) \end{bmatrix}$

We write,

$$\begin{bmatrix} a & b \\ c & \left(\frac{1+bc}{a}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{b}{a} \\ c & \left(\frac{1+bc}{a}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix} A \qquad \left(R_1 \to \frac{R_1}{a}\right)$$

$$\text{or} \quad \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{1}{a} \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ \frac{-c}{a} & 1 \end{bmatrix} A \qquad (R_2 \to R_2 - cR_1)$$

$$\text{or} \quad \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ -c & a \end{bmatrix} A \qquad (R_2 \to aR_2)$$

$$\text{or} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix} A \qquad \left(R_1 \to R_1 - \frac{b}{a}R_2\right)$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$$

Example 8.49 **Obtain the inverse of the following matrix using elementary operations** $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Sol. We have, $A = IA$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \text{ (Applying } R_1 \leftrightarrow R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \text{ (Applying } R_3 \to R_3 - 3R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \text{ (Applying } R_1 \to R_1 - 2R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \text{ (Applying } R_3 \to R_3 + 5R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \text{ (Applying } R_3 \to \frac{1}{2}R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \,(R_1 \to R_1 + R_3 \text{ and } R_2 \to R_2 - 2R_3)$$

Hence,

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

Concept Application Exercise 8.3

1. For any 2×2 matrix A, if A (adj A) $= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then find $|A|$, i.e., det A.
2. Find the multiplicative inverse of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.
3. Find the value of x for which the matrix $A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ is inverse of $B = \begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$.
4. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, $B = \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix}$ where $0 < \beta < \pi/2$, then prove that $BAB = A^{-1}$. Also find the least positive value of α for which $BA^4B = A^{-1}$.
5. By using elementary operations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.
6. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix}$ and $CB = D$. Solve the equation $AX = B$.

SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS

Consider the following system of n linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad \vdots \qquad\quad \vdots \quad &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

This system of equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1}$$

or $AX = B$

Then $n \times n$ matrix A is called the coefficient matrix of the system of linear equations.

Homogeneous and Non-Homogeneous System of Linear Equations

A system of equations $AX = B$ is called a homogeneous system if $B = O$. Otherwise, it is called a non-homogeneous system of equations.

Solution of a System of Equations

Consider the system of equations $AX = B$. A set of values of the variables $x_1, x_2, \ldots, x_n$ which simultaneously satisfy all the equations is called a solution of the system of equations.

Consistent System

If the system of the equations has one or more solutions, then it is said to be a consistent system of equations, otherwise it is an inconsistent system of equations.

Solution of a Non-Homogeneous System of Linear Equations

There are two methods of solving a non-homogeneous system of simultaneous linear equations:

a. Cramer's rule: Discussed in the chapter on determinants.

b. Matrix method:

Consider the equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad (1)$$

If

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

then, Eq. (1) is equivalent to the matrix equation

$$AX = D \qquad (2)$$

Multiplying both sides of Eq. (2) by the inverse matrix A^{-1}, we get

$$A^{-1}(AX) = A^{-1}D \Rightarrow IX = A^{-1}D \qquad [\because A^{-1}A = I]$$

$$\Rightarrow X = A^{-1}D \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad (3)$$

where A_1, B_1, etc., are the cofactors of a_1, b_1, etc., in the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad (\Delta \neq 0)$$

(i) If A is a non-singular matrix, then the system of equations given by $AX = B$ has a unique solution given by $X = A^{-1}B$.

(ii) If A is singular matrix, and (adj A) $D = O$, then the system of the equations given by $AX = D$ is consistent with infinitely many solutions.

(iii) If A is a singular matrix and (adj A) $D \neq O$, then the system of equations given by $AX = D$ is inconsistent and has no solution.

Solution of Homogeneous System of Linear Equations

Let $AX = O$ be a homogeneous system of n linear equations with n unknowns. Now if A is non-singular, then the system of equations will have a unique solution, i.e., trivial solution and if A is a singular, then the system of equations will have infinitely many solutions.

Example 8.50 **Solve the following system of equations, using matrix method: $x + 2y + z = 7, x + 3z = 11, 2x - 3y = 1$.**

Sol. The given system of equations is

$$x + 2y + z = 7$$
$$x + 0y + 3z = 11$$
$$2x - 3y + 0z = 1$$

or $$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

or $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ 17 \\ 1 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix} = 18$$

So, the given system of equations has a unique solution given by $X = A^{-1}B$.

$$\therefore \text{ adj } A = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & -2 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

Now,

$$X = A^{-1} B$$

$$\Rightarrow X = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 63 - 33 + 6 \\ 42 - 22 - 2 \\ -21 + 77 - 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \Rightarrow x = 2, y = 1, z = 3$$

Example 8.51 **Show that the following system of equations is inconsistent.**

$$2x + 4y + 6z = 8$$
$$x + 2y + 3z = 5$$
$$x + y + 3z = 4$$

Sol. The given system is

$$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix}$$

or

$$AX = B$$

where

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix}$$

Now,

$$\text{adj } A = \begin{bmatrix} 3 & 0 & -1 \\ -6 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{adj } (A)\, B = \begin{bmatrix} 3 & 0 & -1 \\ -6 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ -40 \\ 0 \end{bmatrix}$$

Thus, the system of equations is inconsistent.

MATRICES OF REFLECTION AND ROTATION

Reflection Matrix

Reflection in the x-axis

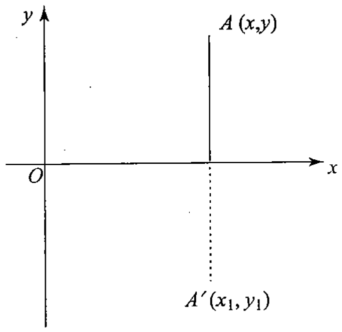


Fig. 8.1

Let A be any point and A' be its image after reflection in the x-axis.

If the coordinates of A and A' be (x, y) and (x_1, y_1), respectively, then $x_1 = x$ and $y_1 = -y$. These may be written as

$$\left.\begin{aligned} x_1 &= 1 \times x + 0 \times y \\ y_1 &= 0 \times x + (-1)y \end{aligned}\right\}$$

Thus, system of equation in the matrix form will be

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflection of a point A (x, y) in the x-axis Similarly, the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ will describe the reflection of a point (x, y) in the y-axis.

Reflection through the origin

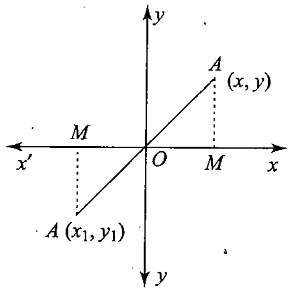


Fig. 8.2

If $A'(x_1, y_1)$ be the image of $A(x, y)$ after reflection through the origin, then

$$\left.\begin{aligned} x_1 &= -x \\ y_1 &= -y \end{aligned}\right\}$$

$\Rightarrow$ $x_1 = (-1)x + 0 \times y$ and $y_1 = 0 \times x + (-1)y$

Thus, the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflection of a point $A(x, y)$ through the origin.

Reflection in the line y = x

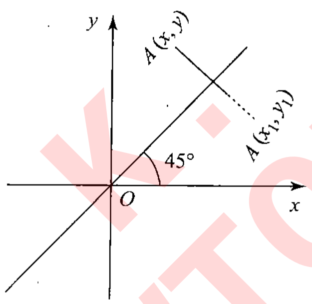


Fig. 8.3

In this case $x_1 = 0 \times x + 1 \times y$

$y_1 = 1 \times x + 0 \times y$

And the reflection matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Reflection in line y = x tan θ

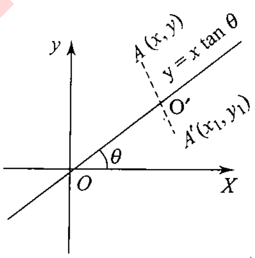


Fig. 8.4

Considering the line $y = x \tan\theta$ shown in Fig. 8.4, we have

$x_1 = x\cos 2\theta + y \sin 2\theta$ ($\because$ O' is the mid-point of AA')

$y_1 = x \sin 2\theta - y\cos 2\theta$

In matrix form, we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

describes the reflection of a point (x, y) in the line $y = \tan\theta$.

Note: *By putting $\theta = 0$, $\pi/2$, $\pi/4$ we can get the reflection matrices in x-axis, y-axis and the line $y = x$, respectively.*

Rotation Through an Angle θ

Let A (x, y) be any point such that $OA = r$ and $\angle AOX = \phi$. Let OA rotate through an angle θ in the anti-clockwise direction such that $A'(x_1, y_1)$ is the new position. Then

$OA' = r$

$x_1 = x\cos\theta - y\sin\theta$

$y_1 = x\sin\theta + y\cos\theta$

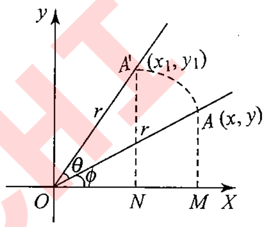


Fig. 8.5

In matrix form, we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Example 8.52 **Find the image of the point (2, –4) under the transformations $(x, y) \longrightarrow (x + 3y, y - x)$.**

Sol. Let (x_1, y_1) be the image of the point (x, y). Then by the given transformation

$x_1 = 1 \times x + 3 \times y$

$y_1 = (-1)\,x + 1 \times y$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} -10 \\ -6 \end{bmatrix}$$

Therefore, the image is (–10, –6).

Example 8.53

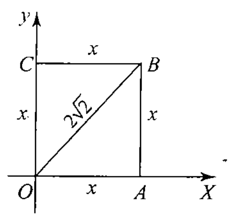


Fig. 8.6

Write down the 2 × 2 matrix A which corresponds to a counterclockwise rotation of 60° about the origin. In Fig. 8.6,

the square $OABC$ has it diagonal OB of $2\sqrt{2}$ units in length. The square is rotated counterclockwise about O through $60°$. Find the coordinates of the vertices of the square after rotating.

Sol. The matrix of rotation will be

$$\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

Since each side of the square is x,

$\therefore\ x^2 + x^2 = (2\sqrt{2})^2$

$\Rightarrow x = 2$ units

Therefore the coordinates of the vertices O, A, B, C are (0, 0), (2, 0), (2, 2), (0, 2), respectively.

Let after rotation A, B, C map into A', B', C', respectively, while O maps into itself. Then,

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

$\therefore\quad A\,(2, 0) \to A'\,(1, \sqrt{3})$

Similarly,

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{3} \\ \sqrt{3}+1 \end{bmatrix}$$

$\therefore\quad B\,(2, 2) \to B'\,(1-\sqrt{3}, \sqrt{3}+1)$

and

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$$

$\therefore\quad C\,(0, 2) \to C'\,(-\sqrt{3}, 1)$

CHARACTERISTIC ROOTS AND CHARACTERISTIC VECTOR OF A SQUARE MATRIX

Definition

Any non-zero vector, X, is said to be a characteristic vector of a matrix A, if there exists a number λ such that $AX = \lambda X$. And then λ is said to be a characteristic root of the matrix A corresponding to the characteristic vector X and vice versa. Characteristic roots (vectors) are also often called proper values, latent values or eigenvalues (vectors).

Note:
It will be useful to remember that

(i) *a characteristic vector of a matrix cannot correspond two different characteristic roots, but*

(ii) *a characteristic root of a matrix can correspond to different characteristic vectors. Thus, if*

$$AX = \lambda_1 X,\ AX = \lambda_2 X,\ \lambda_2 \neq \lambda_1$$
$$\lambda_1 X = \lambda_2 X \Rightarrow (\lambda_1 - \lambda_2)\,X = O$$

But $X \neq O$ and $(\lambda_1 - \lambda_2) \neq 0$. And therefore $(\lambda_1 - \lambda_2)$; $X \neq O$. Thus we have a contradiction and as such we see the truth of statement (i).

But if $AX = \lambda X$, then also $A(kX) = \lambda(kX)$, so that kX is also a characteristic vector of A corresponding to the same characteristic root λ. Thus we have the truth of statement (ii).

Determinant of Characteristic Roots and Vectors

If λ be a characteristic root and X, a corresponding characteristic vector of a matrix A, then we have

$$AX = \lambda X = \lambda IX \Rightarrow (A - \lambda I)X = O$$

Since $X \neq O$, we deduce that the matrix $(A - \lambda I)$ is singular so that its determinant

$$|A - \lambda I| = 0$$

Thus, every characteristic root λ of a matrix A is a root of its characteristic equation

$$|A - \lambda I| = 0 \qquad (1)$$

Conversely, if λ be any root of the characteristic equation [Eq. (1)], then the matrix equation $(A - \lambda I)X = O$ necessarily possesses a non-zero solution X so that there exists a vector $X \neq O$ such that $AX = \lambda IX = \lambda X$.

Thus, every root of the characteristic equation of a matrix is a characteristic root of the matrix.

If A be n-rowed, then the characteristic equation $|A - \lambda I| = 0$ is of n^{th} degree so that every n-rowed square matrix possesses n characteristic roots, which, of course, may not all be distinct.

Example 8.54 **Show that the two matrices A, $P^{-1}AP$ have the same characteristic roots.**

Sol. Let,

$$\begin{aligned} P^{-1}AP &= B \\ \therefore\quad B - \lambda I &= P^{-1}AP - \lambda I \\ &= P^{-1}AP - P^{-1}\lambda IP \\ &= P^{-1}(A - \lambda I)P \\ \Rightarrow\ |B - \lambda I| &= |P^{-1}|\,|A - \lambda I|\,|P| \\ &= |A - \lambda I|\,|P^{-1}|\,|P| \\ &= |A - \lambda I|\,|P^{-1}P| \\ &= |A - \lambda I|\,|I| = |A - \lambda I| \end{aligned}$$

Thus, the two matrices A and B have the same characteristic determinants and hence the same characteristic equations and the same characteristic roots. The same thing may also be seen in another way. Now,

$$AX = \lambda X$$

$\Rightarrow\quad P^{-1}AX = \lambda P^{-1}X$

$\Rightarrow\quad (P^{-1}AP)(P^{-1}X) = \lambda(P^{-1}X)$

Example 8.55 **Show that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n eigen values of a square matrix A of order n, then the eigen values of the matrix A^2 be $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$.**

Sol. $AX = \lambda X$

$\Rightarrow\quad A(AX) = A(\lambda X)$

$\Rightarrow\quad A^2X = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$

i.e.,

$$A^2X = \lambda^2 X$$

Hence, eigenvalue of A^2 is λ^2. Thus if $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of A, then $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$ are eigenvalues of A^2.

Example 8.56 **Show that the characteristic roots of an idempotent matrix are either zero or unity.**

Sol. Since A is an idempotent matrix, hence,

$$A^2 = A$$

Let X be a latent vector of the matrix A corresponding to the latent root λ so that

$$AX = \lambda X \quad (1)$$

or

$$(A - \lambda I)X = O$$

such that

$$X \neq O$$

On pre-multiplying Eq. (1) by A, we get

$$A(AX) = A(\lambda X) = \lambda(AX)$$

i.e.,

$$(AA)X = \lambda(AX)$$

$\Rightarrow$ $AX = \lambda^2 X$

$\Rightarrow$ $\lambda X = \lambda^2 X \quad (\because A^2 = A)$

$\Rightarrow$ $(\lambda^2 - \lambda)X = O \quad (\because AX = \lambda X)$

$\Rightarrow$ $\lambda^2 - \lambda = 0$

$\Rightarrow$ $\lambda(\lambda - 1) = 0 \quad (\because X \neq 0)$

$\Rightarrow$ $\lambda = 0, \lambda = 1$

Example 8.57 **Find the characteristic roots of the two-rowed orthogonal matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and verify that they are of unit modulus.**

Sol. We have,

$$|A - \lambda I| = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix}$$

$$= (\cos\theta - \lambda)^2 + \sin^2\theta$$

Therefore characteristic equation of A is

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$\Rightarrow$ $\cos\theta - \lambda = \pm i\sin\theta$

$\Rightarrow$ $\lambda = \cos\theta \pm i\sin\theta$

which are of unit modulus.

Example 8.58 **Prove that the product of the characteristic roots of a square matrix of order n is equal to the determinant of the matrix.**

Sol: Let $A = [a_{ij}]$ be a given square matrix. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the characteristic roots of A. If $\phi(\lambda)$ is the characteristic function, then

$$\phi(\lambda) = |A - \lambda I|$$

$$= \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

$$= (-1)^n [\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \cdots + p_n] \quad (1)$$

$$= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \quad (2)$$

On putting $\lambda = 0$, we have

$$\phi(0) = |A| = \lambda_1 \times \lambda_2 \times \lambda_3 \times \cdots \times \lambda_n = (-\lambda)^n p_n \quad (3)$$

Hence, the product of characteristic roots of a square matrix is equal to the determinant of the matrix.

Example 8.59 **If A is non-singular, prove that the eigen values of A^{-1} are the reciprocals of the eigen value of A.**

Sol. Let A be an eigen value of A and X be a corresponding eigenvector. Then,

$$AX = \lambda X$$

$\Rightarrow$ $X = A^{-1}(\lambda X) = \lambda(A^{-1}X)$

$\Rightarrow$ $\frac{1}{\lambda} X = A^{-1}X \quad [\because A \text{ is non-singular} \Rightarrow \lambda \neq 0]$

$\Rightarrow$ $A^{-1}X = \frac{1}{\lambda} X$

Therefore, $1/\lambda$ is an eigenvalue of A^{-1} and X is a corresponding eigenvector.

Example 8.60 **If α is a characteristic root of a non-singular matrix, then prove that $|A|/\alpha$ is a characteristic root of adj A.**

Sol. Since α is a characteristic root of a non-singular matrix, therefore $\alpha \neq 0$. Also α is a characteristic root of A implies that there exists a non-zero vector X such that

$$AX = \alpha X$$

$\Rightarrow$ $(\text{adj } A)(AX) = (\text{adj } A)(\alpha X)$

$\Rightarrow$ $[(\text{adj } A)A]X = \alpha(\text{adj } A)X$

$\Rightarrow$ $|A|IX = \alpha(\text{adj } A)X \quad [\because (\text{adj } A)A = |A|I]$

$\Rightarrow$ $|A|X = \alpha(\text{adj } A)X$

$\Rightarrow$ $\frac{|A|}{\alpha} X = (\text{adj } A)X$

$\Rightarrow$ $(\text{adj } A)X = \frac{|A|}{\alpha} X$

Since X is a non-zero vector, therefore $|A/\alpha|$ is a characteristic root of the matrix adj A.

EXERCISES

Subjective Type

Solutions on page 8.36

1. If $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then prove that $(pI + qX)^m = p^m I + mp^{m-1} qX$, $\forall$ $p, q \in R$, where I is a two-rowed unit matrix and $m \in N$.

2. If A is an upper triangular matrix of order $n \times n$ and B is a lower triangular matrix of order $n \times n$, then prove that $(A' + B) \times (A + B')$ will be a diagonal matrix of order $n \times n$ [assume all elements of A and B to be non-negative and an elements of $(A' + B) \times (A + B')$ as C_{ij}].

3. If B, C are square matrices of order n and if $A = B + C$, $BC = CB$, $C^2 = O$, then without using mathematical induction, show that for any positive integer p, $A^{p+1} = B^p[B + (p+1)C]$.

4. If $D = \text{diag}\ [d_1, d_2, \ldots, d_n]$, then prove that $f(D) = \text{diag}\ [f(d_1), f(d_2), \ldots, f(d_n)]$, where $f(x)$ is a polynomial with scalar coefficient.

5. Show that the solutions of the equation

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix}^2 = O \text{ are } \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} \pm\sqrt{\alpha\beta} & -\beta \\ \alpha & \mp\sqrt{\alpha\beta} \end{bmatrix}$$

where α, β are arbitrary.

6. If $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$, then prove that $A^2 + 3A + 2I = O$, hence find B and C matrices of order 2 with integer elements, if $A = B^3 + C^3$.

7. Find the possible square roots of the two-rowed unit matrix I.

8. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then show that $A^2 - 4A - 5I = O$, where I and O are the unit matrix and the null matrix of order 3, respectively. Use this result to find A^{-1}.

9. If S is a real skew-symmetric matrix, then prove that $I - S$ is non-singular and the matrix $A = (I + S)(I - S)^{-1}$ is orthogonal.

10. If B and C are non-singular matrices and O is null matrix, then show that $\begin{bmatrix} A & B \\ C & O \end{bmatrix}^{-1} = \begin{bmatrix} O & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}$.

11. Show that every square matrix A can be uniquely expressed as $P + iQ$, where P and Q are Hermitian matrices.

12. Express A as the sum of a Hermitian and a skew-Hermitian matrix, where $A = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix}$.

Objective Type

Solutions on page 8.38

Each question has four choices a, b, c and d, out of which *only one* is correct. Find the correct answer.

1. The inverse of a skew-symmetric matrix of odd order is
   a. a symmetric matrix **b.** a skew symmetric
   c. diagonal matrix **d.** does not exist

2. Let A and B be two 2×2 matrices. Consider the statements
   (i) $AB = O \Rightarrow A = O$ or $B = O$
   (ii) $AB = I_2 \Rightarrow A = B^{-1}$
   (iii) $(A + B)^2 = A^2 + 2AB + B^2$
   Then
   a. (i) and (ii) are false, (iii) is true
   b. (ii) and (iii) are false, (i) is true
   c. (i) is false, (ii) and (iii) are true
   d. (i) and (iii) are false, (ii) is true

3. The equation $\begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$ has

(i) for $y = 0$	**(p)** rational roots
(ii) for $y = -1$	**(q)** irrational roots
	(r) integral roots

Then

	(i)	(ii)
a.	(p)	(r)
b.	(q)	(p)
c.	(p)	(q)
d.	(r)	(p)

4. The number of diagonal matrix A of order n for which $A^3 = A$ is
   a. 1 **b.** 0
   c. 2^n **d.** 3^n

5. If $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ is n^{th} root of I_2, then choose the correct statements:
   (i) if n is odd, $a = 1$, $b = 0$
   (ii) if n is odd, $a = -1$, $b = 0$
   (iii) if n is even, $a = 1$, $b = 0$
   (iv) if n is even, $a = -1$, $b = 0$
   a. i, ii, iii **b.** ii, iii, iv
   c. i, ii, iii, iv **d.** i, iii, iv

6. A is a 2×2 matrix such that $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $A^2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The sum of the elements of A is
   a. -1 **b.** 0
   c. 2 **d.** 5

7. The product of matrices $A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ is a null matrix if $\theta - \phi =$

a. $2n\pi, n \in Z$
b. $n\frac{\pi}{2}, n \in Z$
c. $(2n+1)\frac{\pi}{2}, n \in Z$
d. $n\pi, n \in Z$

8. Let A, B be two matrices such that they commute, then for any positive integer n,
(i) $AB^n = B^n A$
(ii) $(AB)^n = A^n B^n$
a. only (i) is correct
b. Both (i) and (ii) are correct
c. only (ii) is correct
d. none of (i) and (ii) is correct

9. If $A = [a_{ij}]_{4\times4}$, such that $a_{ij} = \begin{cases} 2, & \text{when } i = j \\ 0, & \text{when } i \neq j \end{cases}$, then $\left\{\frac{\det(\operatorname{adj}(\operatorname{adj} A))}{7}\right\}$ is (where $\{\cdot\}$ represents fractional part function)
a. 1/7
b. 2/7
c. 3/7
d. none of these

10. If $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is to be the square root of two-rowed unit matrix, then α, β and γ should satisfy the relation
a. $1 - \alpha^2 + \beta\gamma = 0$
b. $\alpha^2 + \beta\gamma - 1 = 0$
c. $1 + \alpha^2 + \beta\gamma = 0$
d. $1 - \alpha^2 - \beta\gamma = 0$

11. If A is a square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is
a. $3I$
b. O
c. I
d. $2I$

12. If A and B are square matrices of order n, then $A - \lambda I$ and $B - \lambda I$ commute for every scalar λ, only if
a. $AB = BA$
b. $AB + BA = O$
c. $A = -B$
d. none of these

13. Matrix A such that $A^2 = 2A - I$, where I is the identity matrix, then for $n \geq 2$, A^n is equal to
a. $2^{n-1}A - (n-1)I$
b. $2^{n-1}A - I$
c. $nA - (n-1)I$
d. $nA - I$

14. Let $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ and $(A+1)^{50} - 50A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the value of $a + b + c + d$ is
a. 2
b. 1
c. 4
d. none of these

15. If $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then A^8 equals
a. $4B$
b. $128B$
c. $-128B$
d. $-64B$

16. If $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$, then sum of all the elements of matrix A is
a. 0
b. 1
c. 2
d. -3

17. If $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$, then $A(\bar{A}^T)$ equals
a. O
b. I
c. $-I$
d. $2I$

18. Identify the incorrect statement in respect of two square matrices A and B conformable for sum and product:
a. $t_r(A + B) = t_r(A) + t_r(B)$
b. $t_r(\alpha A) = \alpha t_r(A), \alpha \in R$
c. $t_r(A^T) = t_r(A)$
d. none of these.

19. A is an involuntary matrix given by $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$, then the inverse of $A/2$ will be
a. $2A$
b. $\frac{A^{-1}}{2}$
c. $\frac{A}{2}$
d. A^2

20. If A is a non-singular matrix such that $AA^T = A^TA$ and $B = A^{-1}A^T$, then matrix B is
a. involuntary
b. orthogonal
c. idempotent
d. none of these

21. If P is an orthogonal matrix and $Q = PAP^T$ and $x = P^TQ^{1000}P$, then x^{-1} is, where A is involutary matrix
a. A
b. I
c. A^{1000}
d. none of these

22. If n^{th}-order square matrix A is a orthogonal, then, $|\operatorname{adj}(\operatorname{adj} A)|$ is
a. always -1 if n is even
b. always 1 if n is odd
c. always 1
d. none of these

23. If both $A - \frac{1}{2}I$ and $A + \frac{1}{2}$ are orthogonal matrices, then
a. A is orthogonal
b. A is skew-symmetric matrix of even order
c. $A^2 = \frac{3}{4}I$
d. none of these

24. In which of the following type of matrix inverse does not exist always
a. idempotent
b. orthogonal
c. involuntary
d. none of these

25. If A is an orthogonal matrix, then A^{-1} equals
a. A^T
b. A
c. A^2
d. none of these

26. If Z is an idempotent matrix, then $(I + Z)^n$
a. $I + 2^nZ$
b. $I + (2^n - 1)Z$
c. $I - (2^n - 1)Z$
d. none of these

27. If A and B are two matrices such that $AB = B$ and $BA = A$, then

a. $(A^5 - B^5)^3 = A - B$ **b.** $(A^5 - B^5)^3 = A^3 - B^3$

c. $A - B$ is idempotent **d.** $A - B$ is nilpotent

28. If A is a nilpotent matrix of index 2, then for any positive integer n, $A(I + A)^n$ is equal to

a. A^{-1} **b.** A

c. A^n **d.** I_n

29. Let A be an n^{th}-order square matrix and B be its adjoint, then $|AB + KI_n|$ is (where K is a scalar quantity)

a. $(|A| + K)^{n-2}$ **b.** $(|A| + K)^n$

c. $(|A| + K)^{n-1}$ **d.** none of these

30. If $A^2 = I$, then the value of $\det(A - I)$ is (where A has order 3)

a. 1 **b.** -1

c. 0 **d.** cannot say anything

31. If $A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$, $B = \begin{bmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{bmatrix}$ and if A is invertible, then which of the following is not true?

a. $|A| = |B|$

b. $|A| = -|B|$

c. $|\text{adj } A| = |\text{adj } B|$

d. A is invertible if and only if B is invertible

32. If A and B are two non-singular matrices such that $AB = C$, then $|B|$ is equal to

a. $\dfrac{|C|}{|A|}$ **b.** $\dfrac{|A|}{|C|}$

c. $|C|$ **d.** none of these

33. If A and B are squares matrices such that $A^{2006} = O$ and $AB = A + B$, then $\det(B)$ equals

a. 0 **b.** 1

c. -1 **d.** none of these

34. If matrix A is given by $A = \begin{bmatrix} 6 & 11 \\ 2 & 4 \end{bmatrix}$, then the determinant of $A^{2005} - 6A^{2004}$ is

a. 2^{2006} **b.** $(-11)\,2^{2005}$

c. $-2^{2005} \cdot 7$ **d.** $(-9)\,2^{2004}$

35. If A is a non-diagonal involutory matrix, then

a. $A - I = O$ **b.** $A + I = O$

c. $A - I$ is non-zero singular **d.** none of these

36. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ -4 & 3 & c \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$, then the values of a and c are equal to

a. 1, 1 **b.** 1, -1

c. 1, 2 **d.** -1, 1

37. If A and B are two non-singular matrices of the same order such that $B^r = I$, for some positive integer $r > 1$. Then $A^{-1} B^{r-1} A - A^{-1} B^{-1} A =$

a. I **b.** $2I$

c. O **d.** $-I$

38. For two unimodular complex numbers z_1 and z_2, $\begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}^{-1} \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1}$ is equal to

a. $\begin{bmatrix} z_1 & z_2 \\ \bar{z}_1 & \bar{z}_2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ **d.** none of these

39. If $A(\alpha, \beta) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & e^{\beta} \end{bmatrix}$, then $A(\alpha, \beta)^{-1}$ is equal to

a. $A(-\alpha, -\beta)$ **b.** $A(-\alpha, \beta)$

c. $A(\alpha, -\beta)$ **d.** $A(\alpha, \beta)$

40. If $A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ and $a^2 + b^2 + c^2 + d^2 = 1$, then A^{-1} is equal to

a. $\begin{bmatrix} a+ib & -c-id \\ -c+id & a-ib \end{bmatrix}$ **b.** $\begin{bmatrix} a+ib & -c+id \\ -c+id & a-ib \end{bmatrix}$

c. $\begin{bmatrix} a-ib & -c-id \\ -c-id & a+ib \end{bmatrix}$ **d.** none of these

41. If $A^3 = O$, then $I + A + A^2$ equals

a. $I - A$ **b.** $(I + A^1)^{-1}$

c. $(I - A)^{-1}$ **d.** none of these

42. If A is order 3 square matrix such that $|A| = 2$, then $|\text{adj(adj(adj A))}|$ is

a. 512 **b.** 256

c. 64 **d.** none of these

43. $(-A)^{-1}$ is always equal to (where A is n^{th}-order square matrix)

a. $(-1)^n A^{-1}$ **b.** $-A^{-1}$

c. $(-1)^{n-1} A^{-1}$ **d.** none of these

44. For each real x, $-1 < x < 1$. Let $A(x)$ be the matrix $(1-x)^{-1}\begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}$ and $z = \dfrac{x+y}{1+xy}$. Then

a. $A(z) = A(x)\,A(y)$ **b.** $A(z) = A(x) - A(y)$

c. $A(z) = A(x) + A(y)$ **d.** $A(z) = A(x)\,[A(y)]^{-1}$

45. If $\begin{bmatrix} 1/25 & 0 \\ x & 1/25 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}^{-2}$, then the value of x is

a. $a/125$ **b.** $2a/125$

c. $2a/25$ **d.** none of these

46. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $f(x) = \dfrac{1+x}{1-x}$, then $f(A)$ is

a. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

c. $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ **d.** none of these

47. If A is a square matrix of order n such that $|\text{adj}(\text{adj } A)| = |A|^9$, then the value of n can be

a. 4 **b.** 2

c. either 4 or 2 **d.** none of these

48. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, then $A^T A^{-1}$ is

a. $\begin{bmatrix} -\cos 2x & \sin 2x \\ -\sin 2x & \cos 2x \end{bmatrix}$ **b.** $\begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

c. $\begin{bmatrix} \cos 2x & \cos 2x \\ \cos 2x & \sin 2x \end{bmatrix}$ **d.** none of these

49. If $A = \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$ and I is a 2×2 unit matrix, then

$(I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \sin\alpha \end{bmatrix}$ is

a. $-I + A$ **b.** $I - A$

c. $-I - A$ **d.** none of these

50. The matrix X for which $\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix} X = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$ is

a. $\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{10} & \frac{1}{5} \end{bmatrix}$

c. $\begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$ **d.** $\begin{bmatrix} 6 & 2 \\ \frac{11}{2} & 2 \end{bmatrix}$

51. If A and B are square matrices of the same order and A is non-singular, then for a positive integer n, $(A^{-1} BA)^n$ is equal to

a. $A^{-n} B^n A^n$ **b.** $A^n B^n A^{-n}$

c. $A^{-1} B^n A$ **d.** $n(A^{-1} BA)$

52. If A is singular matrix, then adj A is

a. singular **b.** non-singular

c. symmetric **d.** not defined

53. The inverse of a diagonal matrix is

a. a diagonal matrix **b.** a skew-symmetric matrix

c. a symmetric matrix **d.** none of these

54. If P is non-singular matrix, then value of adj(P^{-1}) in terms of P is

a. $P/|P|$ **b.** $P|P|$

c. P **d.** none of these

55. If adj $B = A$, $|P| = |Q| = 1$, then adj$(Q^{-1} BP^{-1})$ is

a. PQ **b.** QAP

c. PAQ **d.** $PA^{-1}Q$

56. If A is non-singular and $(A - 2I)(A - 4I) = O$, then $\frac{1}{6}A + \frac{4}{3}A^{-1}$ is equal to

a. O **b.** I

c. $2I$ **d.** $6I$

57. If $A(\alpha, \beta) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & e^\beta \end{bmatrix}$, then $A(\alpha, \beta)^{-1}$ in terms of function of A is

a. $A(\alpha, -\beta)$ **b.** $A(-\alpha, -\beta)$

c. $A(-\alpha, \beta)$ **d.** none of these

58. If A and B are two square matrices such that $B = -A^{-1} BA$, then $(A + B)^2$ is equal to

a. $A^2 + B^2$ **b.** O

c. $A^2 + 2AB + B^2$ **d.** $A + B$

59. Let a and b be two real numbers such that $a > 1$, $b > 1$. If $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, then $\lim_{n\to\infty} A^{-n}$ is

a. unit matrix **b.** null matrix

c. $2I$ **d.** none of these

60. Let $f(x) = \frac{1+x}{1-x}$. If A is matrix for which $A^3 = O$, then $f(A)$ is

a. $I + A + A^2$ **b.** $I + 2A + 2A^2$

c. $I - A - A^2$ **d.** none of these

61. If A and B are two non-zero square matrices of the same order such that the product $AB = O$, then

a. both A and B must be singular

b. exactly one of them must be singular

c. both of them are non-singular

d. none of these

62. If $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $A =$

a. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ **d.** $-\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

63. If $A^2 - A + I = 0$, then the inverse of A is

a. A^{-2} **b.** $A + I$

c. $I - A$ **d.** $A - I$

64. The number of solutions of the matrix equation $X^2 = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ is

a. more than 2 **b.** 2

c. 0 **d.** 1

65. If A and B are symmetric matrices of the same order and $X = AB + BA$ and $Y = AB - BA$, then $(XY)^T$ is equal to

a. XY **b.** YX

c. $-YX$ **d.** none of these

66. If A is a 3×3 skew-symmetric matrix, then trace of A is equal to

a. -1 **b.** 1
c. $|A|$ **d.** none of these

67. Elements of a matrix A of order 10×10 are defined as $a_{ij} = w^{i+j}$ (where w is cube root of unity), then trace (A) of the matrix is

a. 0 **b.** 1
c. 3 **d.** none of these

68. Let $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where $\alpha \in R$. Then $(F(\alpha))^{-1}$ is equal to

a. $F(\alpha^{-1})$ **b.** $F(-\alpha)$
c. $F(2\alpha)$ **d.** none of these

69. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $G(y) = \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{bmatrix}$, then $[F(x)\,G(y)]^{-1}$ is equal to

a. $F(-x)\,G(-y)$ **b.** $G(-y)\,F(-x)$
c. $F(x^{-1})\,G(y^{-1})$ **d.** $G(y^{-1})\,F(x^{-1})$

70. If A is a skew-symmetric matrix and n is odd positive integer, then A^n is

a. a skew-symmetric matrix **b.** a symmetric matrix
c. a diagonal matrix **d.** none of these

71. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (where $bc \neq 0$) satisfies the equations $x^2 + k = 0$, then

a. $a + d = 0$ **b.** $k = -|A|$
c. $k = |A|$ **d.** none of these

72. If $A, B, A + I, A + B$ are idempotent matrices, then AB is equal to

a. BA **b.** $-BA$
c. I **d.** O

73 The matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is

a. idempotent matrix **b.** involutory matrix
c. nilpotent matrix **d.** none of these

74. If A is symmetric as well as skew-symmetric matrix, then A is

a. diagonal matrix **b.** null matrix
c. triangular matrix **d.** none of these

75. If A and B are square matrices of the same order and A is non-singular, then for a positive integer n, $(A^{-1}BA)^n$ is equal to

a. $A^{-n}B^nA^n$ **b.** $A^nB^nA^{-n}$
c. $A^{-1}B^nA$ **d.** $n(A^{-1}BA)$

76. Which of the following is an orthogonal matrix?

a. $\begin{bmatrix} 6/7 & 2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ 3/7 & -6/7 & 2/7 \end{bmatrix}$ **b.** $\begin{bmatrix} 6/7 & 2/7 & 3/7 \\ 2/7 & -3/7 & 6/7 \\ 3/7 & 6/7 & -2/7 \end{bmatrix}$

c. $\begin{bmatrix} -6/7 & -2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ -3/7 & 6/7 & 2/7 \end{bmatrix}$ **d.** $\begin{bmatrix} 6/7 & -2/7 & 3/7 \\ 2/7 & 2/7 & -3/7 \\ -6/7 & 2/7 & 3/7 \end{bmatrix}$

77. If $k \in R_0$, then $\det\{\text{adj}(kI_n)\}$ is equal to

a. k^{n-1} **b.** $k^{n(n-1)}$
c. k^n **d.** k

78. Given that matrix $A = \begin{bmatrix} x & 3 & 2 \\ 1 & y & 4 \\ 2 & 2 & z \end{bmatrix}$. If $xyz = 60$ and $8x + 4y + 3z = 20$, then $A(\text{adj } A)$ is equal to

a. $\begin{bmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix}$ **b.** $\begin{bmatrix} 88 & 0 & 0 \\ 0 & 88 & 0 \\ 0 & 0 & 88 \end{bmatrix}$

c. $\begin{bmatrix} 68 & 0 & 0 \\ 0 & 68 & 0 \\ 0 & 0 & 68 \end{bmatrix}$ **d.** $\begin{bmatrix} 34 & 0 & 0 \\ 0 & 34 & 0 \\ 0 & 0 & 34 \end{bmatrix}$

79. Let $A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$ and $2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$. Then $\text{tr}(A) - \text{tr}(B)$ has the value equal to

a. 0 **b.** 1
c. 2 **d.** none

80. If $A_1, A_3, \ldots, A_{2n-1}$ are n skew symmetric matrices of same order, then $B = \sum_{r=1}^{n} (2r-1)\,(A_{2r-1})^{2r-1}$ will be

a. symmetric
b. skew-symmetric
c. neither symmetric nor skew-symmetric
d. data not adequate

81. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 0 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$. Which of the following is true?

a. $AX = B$ has a unique solution
b. $AX = B$ has exactly three solutions
c. $AX = B$ has infinitely many solutions
d. $AX = B$ is inconsistent

82. Consider three matrices $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$. Then the value of the sum $\text{tr}(A) + \text{tr}\left(\frac{ABC}{2}\right) + \text{tr}\left(\frac{A(BC)^2}{4}\right) + \text{tr}\left(\frac{A(BC)^3}{8}\right) + \cdots + \infty$ is

a. 6 **b.** 9
c. 12 **d.** none

Multiple Correct Answers Type

Solutions on page 8.46

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. If A is unimodular, then which of the following is unimodular?

a. $-A$
b. A^{-1}
c. adj A
d. ωA, where ω is cube root of unity

2. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2 + 2AB$, then

a. $a = -1$ **b.** $a = 1$
c. $b = 2$ **d.** $b = -2$

3. If $AB = A$ and $BA = B$, then which of the following is/are true?

a. A is idempotent **b.** B is idempotent
c. A^T is idempotent **d.** none of these

4. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is an orthogonal matrix, then

a. $a = -2$ **b.** $a = 2, b = 1$
c. $b = -1$ **d.** $b = 1$

5. Let A and B are two non-singular square matrices, A^T and B^T are the transpose matrices of A and B, respectively, then which of the following are correct?

a. B^TAB is symmetric matrix if A is symmetric
b. B^TAB is symmetric matrix if B is symmetric
c. B^TAB is skew-symmetric matrix for every matrix A
d. B^TAB is skew-symmetric matrix if A is skew-symmetric

6. If $A(\theta) = \begin{bmatrix} \sin\theta & i\cos\theta \\ i\cos\theta & \sin\theta \end{bmatrix}$, then which of the following is not true?

a. $A(\theta)^{-1} = A(\pi - \theta)$
b. $A(\theta) + A(\pi + \theta)$ is a null matrix
c. $A(\theta)$ is invertible for all $\theta \in R$
d. $A(\theta)^{-1} = A(-\theta)$

7. If A is a matrix such that $A^2 + A + 2I = O$, then which of the following is/are true?

a. A is non-singular **b.** A is symmetric
c. A cannot be skew-symmetric **d.** $A^{-1} = -\frac{1}{2}(A + I)$

8. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then

a. adj(adj A) = A **b.** |adj(adj A)| = 1
c. |adj A| = 1 **d.** none of these

9. If $\begin{pmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{pmatrix}\begin{pmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{pmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then

a. $a = \cos 2\theta$ **b.** $a = 1$
c. $b = \sin 2\theta$ **d.** $b = -1$

10. If $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -1/3 \end{bmatrix}$, then

a. $|A| = -1$ **b.** adj $A = \begin{bmatrix} -1 & 1 & -2 \\ 0 & -3 & -1 \\ 0 & 0 & 1/3 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 1/3 & 7 \\ 0 & 1/3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ **d.** $A = \begin{bmatrix} 1 & -1/3 & -7 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

11. If B is an idempotent matrix, and $A = I - B$, then

a. $A^2 = A$ **b.** $A^2 = I$
c. $AB = O$ **d.** $BA = O$

12. Which of the following statements is/are true about square matrix A of order n?

a. $(-A)^{-1}$ is equal to $-A^{-1}$ when n is odd only.
b. If $A^n = O$, then $I + A + A^2 + \cdots + A^{n-1} = (I - A)^{-1}$.
c. If A is skew-symmetric matrix of odd order, then its inverse does not exist.
d. $(A^T)^{-1} = (A^{-1})^T$ holds always.

13. If $A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$,

then $A_iA_k + A_kA_i$ is equal to

a. $2I$ if $i = k$ **b.** O if $i \neq k$
c. $2I$ if $i \neq k$ **d.** O always

14. If $A = (a_{ij})_{n \times n}$ and f is a function, we define $f(A) = (f(a_{ij}))_{n \times n}$.

Let $A = \begin{pmatrix} \pi/2 - \theta & \theta \\ -\theta & \pi/2 - \theta \end{pmatrix}$. Then

a. $\sin A$ is invertible **b.** $\sin A = \cos A$
c. $\sin A$ is orthogonal **d.** $\sin(2A) = 2\sin A\cos A$

15. If $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal, then

a. $a = \pm \frac{1}{\sqrt{2}}$ **b.** $b = \pm \frac{1}{\sqrt{12}}$

c. $c = \pm \frac{1}{\sqrt{3}}$ **d.** none of these

16. If A and B are two invertible matrices of the same order, then adj(AB) is equal to

a. adj(B) adj(A) **b.** $|B|\,|A|\,B^{-1}A^{-1}$

c. $|B|\,|A|\,A^{-1}B^{-1}$ **d.** $|A|\,|B|\,(AB)^{-1}$

17. If A, B and C are three square matrices of the same order, then $AB = AC \Rightarrow B = C$. Then

a. $|A| \neq 0$ **b.** A is invertible

c. A may be orthogonal **d.** A is symmetric

18. Suppose $a_1, a_2, \ldots$ are real numbers, with $a_1 \neq 0$. If $a_1, a_2, a_3, \ldots$ are in A.P., then

a. $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_5 & a_6 & a_7 \end{bmatrix}$ is singular (where $i = \sqrt{-1}$)

b. the system of equations $a_1x + a_2y + a_3z = 0$, $a_4x + a_5y + a_6z = 0$, $a_7x + a_8y + a_9z = 0$ has infinite number of solutions

c. $B = \begin{bmatrix} a_1 & ia_2 \\ ia_2 & a_1 \end{bmatrix}$ is non-singular

d. none of these

19. If α, β, γ are three real numbers and

$$A = \begin{bmatrix} 1 & \cos(\alpha-\beta) & \cos(\alpha-\gamma) \\ \cos(\beta-\alpha) & 1 & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\gamma-\beta) & 1 \end{bmatrix}$$, then which of following is/are true?

a. A is singular **b.** A is symmetric

c. A is orthogonal **d.** A is not invertible

20. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then which of following is not true?

a. $\lim_{n\to\infty} \frac{1}{n^2} A^{-n} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$ **b.** $\lim_{n\to\infty} \frac{1}{n} A^{-n} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

c. $A^{-n} = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix} \forall\, n \neq N$ **d.** none of these

21. If C is skew-symmetric matrix of order n and X is $n \times 1$ column matrix, then X^TCX is

a. singular **b.** non-singular

c. invertible **d.** non-invertible

22. If $S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} b+c & c+a & b-c \\ c-b & c+b & a-b \\ b-c & a-c & a+b \end{bmatrix}$ $(a, b, c \neq 0)$, then SAS^{-1} is

a. symmetric matrix **b.** diagonal matrix

c. invertible matrix **d.** singular matrix

23. If D_1 and D_2 are two 3×3 diagonal matrices, then which of the following is/are true?

a. D_1D_2 is diagonal matrix **b.** $D_1D_2 = D_2D_1$

c. $D_1^2 + D_2^2$ is a diagonal matrix **d.** none of these

24. If A and B are symmetric and commute, then which of the following is/are symmetric?

a. $A^{-1}B$ **b.** AB^{-1}

c. $A^{-1}B^{-1}$ **d.** None of these

25. A skew-symmetric matrix A satisfies the relation $A^2 + I = O$, where I is a unit matrix then A is

a. idempotent **b.** orthogonal

c. of even order **d.** odd order

26. If $AB = A$ and $BA = B$, then

a. $A^2B = A^2$ **b.** $B^2A = B^2$

c. $ABA = A$ **d.** $BAB = B$

27. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. Then

a. $A^2 - 4A - 5I_3 = O$ **b.** $A^{-1} = \frac{1}{5}(A - 4I_3)$

c. A^3 is not invertible **d.** A^2 is invertible

Reasoning Type

Solutions on page 8.49

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.

b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.

c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. Statement 1: $|\text{adj}(\text{adj}(\text{adj } A))| = |A|^{(n-1)^3}$, where n is order of matrix A.

Statement 2: $|\text{adj } A| = |A|^n$.

2. Statement 1: If $D = \text{diag}\,[d_1, d_2, \ldots, d_n]$, then $D^{-1} = \text{diag}\,[d_1^{-1}, d_2^{-1} \ldots, d_n^{-1}]$.

Statement 2: If $D = \text{diag}\,[d_1, d_2, \ldots, d_n]$, then $D^n = \text{diag}\,[d_1^n, d_2^n \ldots, d_n^n]$.

3. Statement 1: Matrix 3×3, $a_{ij} = \frac{i-j}{i+2j}$ cannot be expressed as a sum of symmetric and skew-symmetric matrix.

Statement 2: Matrix 3×3, $a_{ij} = \frac{i-j}{i+2j}$ is neither symmetric nor skew-symmetric.

4. **Statement 1:** If a, b, c, d are real numbers and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^3 = O$, then $A^2 = O$.
Statement 2: For matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $A^2 - (a + d)A + (ad - bc)I = O$.

5. **Statement 1:** If $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $[F(\alpha)]^{-1} = F(-\alpha)$.
Statement 2: For matrix $G(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$, we have $[G(\beta)]^{-1} = G(-\beta)$.

6. **Statement 1:** $A = \begin{bmatrix} 4 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. Then $(AB)^{-1}$ does not exist.
Statement 2: Since $|A| = 0$, $(AB)^{-1} = B^{-1}A^{-1}$ is meaningless.

7. **Statement 1:** The determinant of a matrix $A = [a_{ij}]_{5\times5}$ where $a_{ij} + a_{ji} = 0$ for all i and j is zero.
Statement 2: The determinant of a skew-symmetric matrix of odd order is zero.

8. **Statement 1:** The inverse of the matrix $A = [a_{ij}]_{n\times n}$ where $a_{ij} = 0, i \geq j$ is $B = [a_{ij}^{-1}]_{n\times n}$.
Statement 2: The inverse of singular matrix does not exist.

9. **Statement 1:** For a singular square matrix A, $AB = AC \Rightarrow B = C$.
Statement 2: If $|A| = 0$, then A^{-1} does not exist.

10. **Statement 1:** If A, B, C are matrices such that $|A_{3\times3}| = 3$, $|B_{3\times3}| = -1$ and $|C_{2\times2}| = +2$, then $|2ABC| = -12$.
Statement 2: For matrices A, B, C of the same order, $|ABC| = |A|\,|B|\,|C|$.

11. **Statement 1:** If the matrices $A, B, (A + B)$ are non-singular, then $[A(A + B)^{-1} B]^{-1} = B^{-1} + A^{-1}$.
Statement 2:
$$\begin{aligned}[A(A + B)^{-1} B]^{-1} &= [A(A^{-1} + B^{-1})B]^{-1} \\ &= [(I + AB^{-1})B]^{-1} \\ &= [(B + AB^{-1}B)]^{-1} \\ &= [(B + AI)]^{-1} \\ &= [(B + A)]^{-1} \\ &= B^{-1} + A^{-1}.\end{aligned}$$

12. **Statement 1:** Let A, B be two square matrices of the same order such that $AB = BA$, $A^m = O$ and $B^n = O$ for some positive integers m, n, then there exists a positive integer r such that $(A + B)^r = O$.
Statement 2: If $AB = BA$ then $(A + B)^r$ can be expanded as binomial expansion.

13. **Statement 1:** If $A = [a_{ij}]_{n\times n}$ is such that $a_{ij} = \bar{a}_{ji}, \forall\, i, j$ and $A^2 = O$, then matrix A null matrix.
Statement 2: $|A| = 0$.

14. **Statement 1:** If A is orthogonal matrix of order 2, then $|A| = \pm 1$.
Statement 2: Every two-rowed real orthogonal matrix is of any one of the forms $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ or $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

Linked Comprehension Type

Solutions on page 8.50

Based upon each paragraph, some multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

Let A is matrix of order 2×2 such that $A^2 = O$.

1. $A^2 - (a + d)A + (ad - bc)I$ is equal to
a. I **b.** O
c. $-I$ **d.** none of these

2. tr(A) is equal to
a. 1 **b.** 0
c. -1 **d.** none of these

3. $(I + A)^{100} =$
a. $100A$ **b.** $100(I + A)$
c. $100I + A$ **d.** $I + 100A$

For Problems 4–6

If A and B are two square matrices of order 3×3 which satisfy $AB = A$ and $BA = B$, then

4. Which of the following is true?
a. If matrix A is singular then matrix B is non-singular.
b. If matrix A is non-singular then matrix B is singular.
c. If matrix A is singular then matrix B is also singular.
d. Cannot say anything.

5. $(A + B)^7$ is equal to
a. $7(A + B)$ **b.** $7 \cdot I_{3\times3}$
c. $64\,(A + B)$ **d.** $128\,I$

6. $(A + I)^5$ is equal to (where I is identity matrix)
a. $I + 60I$ **b.** $I + 16A$
c. $I + 31A$ **d.** none of these

For Problems 7–8

Consider an arbitrary 3×3 matrix $A = [a_{ij}]$, a matrix $B = [b_{ij}]$ is formed such that b_{ij} is the sum of all the elements except a_{ij} in the i^{th} row of A. Answer the following questions.

7. If there exists a matrix X with constant elements such that $AX = B$, then X is
a. skew-symmetric **b.** null matrix
c. diagonal matrix **d.** none of these

8. The value of $|B|$ is equal to
a. $|A|$ **b.** $|A|/2$
c. $2|A|$ **d.** none of these

For Problems 9–11

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ satisfies $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$. And trace of a square matrix X is equal to the Sum of elements in its principal diagonal.

Further consider a matrix $\underset{3\times 3}{\cup}$ with its column as $\cup_1, \cup_2, \cup_3$ such that

$$A^{50}\cup_1 = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix}, A^{50}\cup_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, A^{50}\cup_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then answer the following questions.

9. The values of $|A^{50}|$ equals

a. 0 **b.** 1
c. –1 **d.** 25

10. Trace of A^{50} equals

a. 0 **b.** 1
c. 2 **d.** 3

11. The value of $|\cup|$equals

a. 0 **b.** 1
c. 2 **d.** –1

For Problems 12–14

Let A be a square matrix of order 2 or 3 and I be the identity matrix of the same order. Then the matrix $A - \lambda I$ is called characteristic matrix of the matrix A, where λ is some complex number. The determinant of the characteristic matrix is called characteristic determinant of the matrix A which will of course be a polynomial of degree 3 in λ. The equation $\det(A - \lambda I) = 0$ is called characteristic equation of the matrix A and its roots (the values of λ) are called characteristic roots or eigenvalues. It is also known that every square matrix has its characteristic equation.

12. The eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ are

a. 2, 1, 1 **b.** 2, 3, –2
c. –1, 1, 3 **d.** none of these

13. Which of the following matrices do not have eigenvalues as 1 and –1?

a. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ (where $i = \sqrt{-1}$)

c. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

14. If one of the eigenvalues of a square matrix A of order 3×3 is zero, then

a. det A must be non-zero **b.** det A must be zero
c. adj A must be a zero matrix **d.** none of these

For Problems 15–17

Let A be a $m \times n$ matrix. If there exists a matrix L of type $n \times m$ such that $LA = I_n$, then L is called left inverse of A. Similarly, if there exists a matrix R of type $n \times m$ such that $AR = I_m$, then R is called right inverse of A. For example, to find right inverse of matrix

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$, we take $R = \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}$

and solve $AR = I_3$, i.e.,

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow x - u = 1$ $\quad y - v = 0$ $\quad z - w = 0$
$x + u = 0$ $\quad y + v = 1$ $\quad z + w = 0$
$2x + 3u = 0$ $\quad 2y + 3v = 0$ $\quad 2z + 3w = 1$

As this system of equations is inconsistent, we say there is no right inverse for matrix A.

15. Which of the following matrices is NOT left inverse of matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$?

a. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & -7 & 3 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

c. $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 0 & 3 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

16. The number of right inverses for the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ is

a. 0 **b.** 1
c. 2 **d.** infinite

17. For which of the following matrices, the number of left inverses is greater than the number of right inverses?

a. $\begin{bmatrix} 1 & 2 & 4 \\ -3 & 2 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix}$ **d.** $\begin{bmatrix} 3 & 3 \\ 1 & 1 \\ 4 & 4 \end{bmatrix}$

For Problems 18–20

If e^A is defined as $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \frac{1}{2}\begin{bmatrix} f(x) & g(x) \\ g(x) & f(x) \end{bmatrix}$

where $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$ and $0 < x < 1$, then I is an identity matrix.

18. $\int \frac{g(x)}{f(x)} dx$ is equal to

a. $\log(e^x + e^{-x}) + c$ **b.** $\log(e^x - e^{-x}) + c$
c. $\log(e^{2x} - 1) + c$ **d.** none of these

19. $\int (g(x)+1)\sin x dx$ is equal to

a. $\frac{e^x}{2}(\sin x - \cos x) + c$

b. $\frac{e^{2x}}{5}(2\sin x - \cos x) + c$

c. $\frac{e^x}{5}(\sin 2x - \cos 2x) + c$

d. none of these

20. $\int \frac{f(x)}{\sqrt{g(x)}} dx$ is equal to

a. $\frac{1}{2\sqrt{e^x - 1}} - \operatorname{cosec}^{-1}(e^x) + c$

b. $\frac{2}{\sqrt{e^x - e^{-x}}} - \sec^{-1}(e^x) + c$

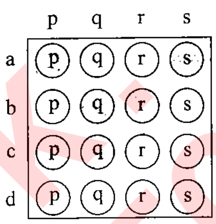
c. $\frac{1}{2\sqrt{e^{2x} - 1}} + \sec^{-1}(e^x) + c$

d. none of these

Matrix-Match Type

Solutions on page 8.52

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a→p, a→s, b→q, b→r, c→p, c→q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I	Column II
a. $(I-A)^n$ is if A is idempotent	**p.** $2^{n-1}(I-A)$
b. $(I-A)^n$ is if A is involuntary	**q.** $I - nA$
c. $(I-A)^n$ is if A is nilpotent of index 2	**r.** A
d. If A is orthogonal, then $(A^T)^{-1}$	**s.** $I - A$

2.

Column I	Column II
a. If A is an idempotent matrix and I is an identity matrix of the same order, then the value of n, such that $(A+I)^n = I + 127$ is	**p.** 9
b. If $(I-A)^{-1} = I + A + A^2 + \cdots + A^7$, then $A^n = O$ where n is	**q.** 10
c. If A is matrix such that $a_{ij} = (i+j)(i-j)$, then A is singular if order of matrix is	**r.** 7
d. If a non-singular matrix A is symmetric, show that A^{-1} is also symmetric, then order of A can be	**s.** 8

3.

Column I (A, B, C are matrices)	Column II
a. If $\|A\| = 2$, then $\|2A^{-1}\| =$ (where A is of order 3)	**p.** 1
b. If $\|A\| = 1/8$, then $\|\text{adj}(\text{adj}(2A))\| =$ (where A is of order 3)	**q.** 4
c. If $(A+B)^2 = A^2 + B^2$, and $\|A\| = 2$, then $\|B\| =$ (where A and B are of odd order)	**r.** 24
d. $\|A_{2\times 2}\| = 2$, $\|B_{3\times 3}\| = 3$ and $\|C_{4\times 4}\| = 4$, then $\|ABC\|$ is equal to	**s.** 0
	t. does not exist

Integer Type

Solutions on page 8.53

1. $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$ and $(A^8 + A^6 + A^4 + A^2 + I)V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$ (where I is the 2 × 2 identity matrix), then the product of all elements of matrix V is.

2. If $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix and $f(x) = x - x^2$ and $bc = 1/4$ then the value of $1/f(a)$ is.

3. Let X be the solution set of the equation $A^x = I$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and I is the corresponding unit matrix and $x \subseteq N$ then the minimum value of $\sum(\cos^x\theta + \sin^x\theta)$, $\theta \in R$.

4. $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ and $f(x)$ is defined as $f(x) = \det.(A^TA^{-1})$ then the value of $\underbrace{f(f(f(f\ldots\ldots\ldots f(x))))}_{n\text{-times}}$ is $(n \geq 2)$.

5. The equation $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 3 & 4 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a solution for (x, y, z) besides (0, 0, 0). Then the value of k is.

6. If A is an idempotent matrix satisfying, $(I - 0.4A)^{-1} = I - \alpha A$ where I is the unit matrix of the same order as that of A then the value of $|9\alpha|$ is equal to.

7. Let $A = \begin{bmatrix} 3x^2 \\ 1 \\ 6x \end{bmatrix}$, $B = [a\ b\ c]$ and $C = \begin{bmatrix} (x+2)^2 & 5x^2 & 2x \\ 5x^2 & 2x & (x+2)^2 \\ 2x & (x+2)^2 & 5x^2 \end{bmatrix}$ be three given matrices, where a, b, c and $x \in R$, Given that $\text{tr}\cdot(AB) = \text{tr}\cdot(C)$ $x \in R$, where $\text{tr}\cdot(A)$ denotes trace of A. If $f(x) = ax^2 + bx + c$ then the value of $f(1)$ is.

8. Let A be the set of all 3 × 3 skew symmetric matrices whose entries are either −1, 0 or 1. If there are exactly three 0's, three 1's and three (−1)'s, then the number of such matrices, is.

9. Let $A = [a_{ij}]_{3\times3}$ be a matrix such that $AA^T = 4I$ and $a_{ij} + 2c_{ij} = 0$ where c_{ij} is the cofactor of a_{ij} and I is the unit matrix of order 3.
$$\begin{vmatrix} a_{11}+4 & a_{12} & a_{13} \\ a_{21} & a_{22}+4 & a_{23} \\ a_{31} & a_{32} & a_{33}+4 \end{vmatrix} + 5\lambda \begin{vmatrix} a_{11}+1 & a_{12} & a_{13} \\ a_{21} & a_{22}+1 & a_{23} \\ a_{31} & a_{32} & a_{33}+1 \end{vmatrix} = 0$$
then the value of 10λ is.

10. Let S be the set which contains all possible values of l, m, n, p, q, r for which
$$A = \begin{bmatrix} l^2-3 & p & 0 \\ 0 & m^2-8 & q \\ r & 0 & n^2-15 \end{bmatrix}$$ be a non-singular idempotent matrix. Then the sum of all the elements of the set S is.

11. If A is a diagonal matrix of order 3×3 is commutative with every square matrix of order 3×3 under multiplication and tr(A) = 12, then the value of $|A|^{1/2}$ is.

12. If A is a square matrix of order3 such that $|A| = 2$ then $|(\text{adj } A^{-1})^{-1}|$ is.

Archives

Solutions on page 8.54

Subjective Type

1. Given a matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ where a, b, c are real positive numbers, $abc = 1$ and $A^TA = I$, then find the value of $a^3 + b^3 + c^3$. **(IIT-JEE, 2003)**

2. If M is a 3×3 matrix, where det $M = 1$ and $MM^T = I$, where 'I' is an identity matrix, prove that det $(M - I) = 0$. **(IIT-JEE, 2004)**

3. If $A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}, B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}, U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}, V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $AX = U$ has infinitely many solutions, prove that $BX = V$ has no unique solution. **(IIT-JEE, 2004)**

4. Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew-symmetric, and $(A + B)(A - B) = (A - B)(A + B)$. If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then find the possible values of k. **(IIT-JEE, 2008)**

Objective Type

Multiple choice questions with one correct answer

1. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, then value of α for which $A^2 = B$ is
a. 1 **b.** −1
c. 4 **d.** no real values
(IIT-JEE, 2003)

2. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then the value of α is
a. ± 1 **b.** ± 2 **c.** ± 3 **d.** ± 5
(IIT-JEE, 2004)

3. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^{-1} = \left[\frac{1}{6}(A^2 + cA + dI)\right]$. The values of c and d are
a. (−6, −11) **b.** (6, 11)
c. (−6, 11) **d.** (6, −11)
(IIT-JEE, 2005)

4. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$ and $x = P^TQ^{2005}P$, then X is equal to
a. $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$
b. $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$
c. $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$
d. $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$
(IIT-JEE, 2005)

5. The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is
a. 0 **b.** $2^9 - 1$ **c.** 168 **d.** 2
(IIT-JEE, 2010)

6. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \theta & 1 \end{bmatrix}$, where each of $a, b,$ and c is either ω or ω^2. Then the number of distinct matrices in the set S is
a. 2 **b.** 6 **c.** 4 **d.** 8
(IIT-JEE, 2011)

Multiple choice questions with one or more correct answer

1. Let M and N be two 3×3 non-singular skew symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to
a. M^2 **b.** $-N^2$ **c.** $-M^2$ **d.** MN
(IIT-JEE, 2011)

Comprehension

Read the passages given below and answer the questions that follow.

For Problems 1–3

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, and U_1, U_2 and U_3 are columns of a 3 × 3 matrix U. If column matrices U_1, U_2 and U_3 satisfy

$$AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

then answer the following questions. **(IIT-JEE, 2006)**

1. The value $|U|$ is

   a. 3 **b.** −3 **c.** 3/2 **d.** 2

2. The sum of the elements of the matrix U^{-1} is

   a. −1 **b.** 0 **c.** 1 **d.** 3

3. The value of $[3\ 2\ 0]\ U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is

   a. 5 **b.** 5/2 **c.** 4 **d.** 3/2

For Problems 4–6

Let A be the set of all 3 × 3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0. **(IIT-JEE, 2009)**

4. The number of matrices in A is

   a. 12 **b.** 6 **c.** 9 **d.** 3

5. The number of matrices A in A for which the system of linear equations

   $$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

   has a unique solution is

   a. less than 4 **b.** at least 4 but less than 7
   c. at least 7 but less than 10 **d.** at least 10

6. The number of matrices A in A for which the system of linear equations

   $$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

   is inconsistent is

   a. 0 **b.** more than 2
   c. 2 **d.** 1

For Problems 7–9

Let P be an odd prime number and T_p be the following set of 2 × 2 matrices:

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, \ldots, p-1\} \right\}$$ **(IIT-JEE, 2010)**

7. The number of A in T_p such that A is either symmetric or skew – symmetric or both, and det(A) divisible by p is

   a. $(p-1)^2$ **b.** $2(p-1)$
   c. $(p-1)^2 + 1$ **d.** $2p - 1$

8. The number of A in T_p such that the trace of A is not divisible by p but det(A) divisible by p is [Note : The trace of matrix is the sum of its diagonal entries].

   a. $(p-1)(p^2-p+1)$ **b.** $p^3 - (p-1)^2$
   c. $(p-1)^2$ **d.** $(p-1)(p^2-2)$

9. The number of A in T_p such that det(A) is not divisible by p is

   a. $2p^2$ **b.** $p^3 - 5p$
   c. $p^3 - 3p$ **d.** $p^3 - p^2$

For Problems 10–12

Let a, b and c be three real numbers satisfying $[a\ b\ c] \begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ 0\ 0]$ **(IIT-JEE, 2011)**

10. If the point $P(a, b, c)$ with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is

    a. 0 **b.** 12 **c.** 7 **d.** 6

11. Let ω be a solution of $x^3 - 1 = 0$ with Im(ω) > 0. If $a = 2$ with b and c satisfying (E), then the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to

    a. −2 **b.** 2 **c.** 3 **d.** −3

12. Let $b = 6$, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty} \left(\frac{1}{\alpha} + \frac{1}{\beta} \right)^n$ is

    a. 6 **b.** 7 **c.** $\frac{6}{7}$ **d.** ∞

Integer type

1. Let K be a positive real number and

   $$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2 \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

   If det (adj A) + det (adj B) = 10^6, then $[k]$ is equal to.

   [Note: adj M denotes the adjoint of a square matrix M and $[k]$ denotes the largest integer less than or equal to k]. **(IIT-JEE, 2010)**

2. Let M be a 3 × 3 matrix satisfying

   $$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \text{ and } M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

   Then the sum of the diagonal entries of M is. **(IIT-JEE, 2011)**

ANSWERS AND SOLUTIONS

Subjective Type

1. We have

$$X^2 = X \times X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\Rightarrow \quad X^3 = X^2 \times X = O \times X = O$

Similarly, $X^4 = X^5 = X^6 = \cdots = O$

Now, by binomial theorem, we have

$$(pI + qX)^m = (pI)^m + {}^mC_1 (pI)^{m-1}(qX) + {}^mC_2 (pI)^{m-2}(qX)^2 + \cdots + {}^mC_m (qX)^m$$

$$= p^m I + mp^{m-1} qX + \frac{m(m-1)}{2!} p^{m-2}(qX)^2 + \cdots + q^m X^m$$

$$= p^m I + mp^{m-1} qX + O + O + \cdots + O \; (\because X^2 = X^3 = \cdots = X^m = O)$$

$\Rightarrow \quad (pI + qX)^m = p^m + mp^{m-1}qX, \quad \forall\, p, q \in R$

2. A is an upper triangular matrix. So,

$$A = \begin{cases} a_{ij} = 0, & i > j \\ a_{ij} \neq 0, & i \le j \end{cases} \quad \text{and} \quad A' = \begin{cases} a_{ij} = 0, & i < j \\ a_{ij} \neq 0, & i \ge j \end{cases}$$

B is lower triangular matrix. So,

$$B = \begin{cases} b_{ij} = 0, & i < j \\ b_{ij} \neq 0, & i \ge j \end{cases}$$

$$\text{and } B' \begin{cases} b_{ij} = 0, & i > j \\ b_{ij} \neq 0, & i \le j \end{cases}$$

$$\text{Let, } C' = A' + B = \begin{cases} c'_{ij} = 0, & i < j \\ c'_{ij} \neq 0, & i \ge j \end{cases}$$

$$C'' = (A + B') = \begin{cases} c''_{ij} = 0, & i > j \\ c''_{ij} \neq 0, & i \le j \end{cases}$$

$\therefore \quad (A' + B) \times (A + B') = C$ (let) for C, $c_{ij} \neq 0$ (for all i and j)

Therefore, $(A' + B) \times (A + B')$ is a matrix of order $n \times n$, $\forall\, c_{ij} \neq 0$ in any case.

3. $A^{p+1} = (B + C)^{p+1}$

We can expand $(B + C)^{p+1}$ like binomial expansion as $BC = CB$.

$$\therefore \quad (B + C)^{p+1} = {}^{p+1}C_0 B^{p+1} + {}^{p+1}C_1 B^p C + {}^{p+1}C_2 B^{p-1} C^2 + \cdots + {}^{p+1}C_{p+1} C^{p+1}$$

$$= {}^{p+1}C_0 B^{p+1} + {}^{p+1}C_1 B^p C + O + O + \cdots + O \; (\because C^2 = O \Rightarrow C^3 = C^4 = \cdots = O)$$

$$= B^{p+1} + (p+1)B^p C$$

$$= B^p[B + (p+1)C]$$

4. Let, $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$\Rightarrow \quad f(D) = a_0 I + a_1 D + a_2 D^2 + \cdots + a_n D^n$

$$= a_0 \times \text{diag}(1, 1, \ldots, 1)$$
$$+ a_1 \times \text{diag}(d_1, d_2, \ldots, d_n)$$
$$+ a_2 \times \text{diag}(d_1^2, d_2^2, \ldots, d_n^2)$$
$$+ \cdots$$
$$\vdots$$
$$+ a_n \times \text{diag}(d_1^n, d_2^n, \ldots, d_n^n)$$

$$= \text{diag}(a_0 + a_1 d_1 + a_2 d_1^2 + \cdots + a_n d_1^n,$$
$$a_0 + a_1 d_2 + a_2 d_2^2 + \cdots + a_n d_2^n,$$
$$a_0 + a_1 d_3 + a_2 d_3^2 + \cdots + a_n d_3^n$$
$$\vdots$$
$$a_0 + a_1 d_n + a_2 d_n^2 + \cdots + a_n d_n^n)$$
$$= \text{diag}(f(d_1), f(d_2), \ldots, f(d_n))$$

5. Given equation is

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x^2 + yz & xy + yt \\ zx + tz & zy + t^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad x^2 + yz = 0 \qquad (1)$

$y(x + t) = 0 \qquad (2)$

$z(x + t) = 0 \qquad (3)$

$yz + t^2 = 0 \qquad (4)$

From Eqs. (1) and (4), we have $x^2 = t^2$ or $x = \pm t$.

Case I:

If $x = t$, from Eqs. (2) and (3), $y = 0$, $z = 0$. Then from Eq. (1), $x = 0 = t$.

Case II:

If $x = -t$, then Eqs. (2) and (3) are satisfied for all values of y and z. If we take $y = -\beta$, $z = \alpha$, then from Eq. (1),

$$x = \pm\sqrt{\alpha\beta} = -t$$

Obviously, Case I is included in Case II ($\alpha = 0 = \beta$). Hence, the general solution of the given equation is

$$x = -t = \pm\sqrt{\alpha\beta},\; y = -\beta,\; z = \alpha$$

$$\Rightarrow \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} \pm\sqrt{\alpha\beta} & -\beta \\ \alpha & \mp\sqrt{\alpha\beta} \end{bmatrix}$$

where α, β are arbitrary.

6. $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$

$$\Rightarrow \quad A^2 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix}$$

$\Rightarrow \quad A^2 + 3A + 2I$

$$= \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix} + 3\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad A^2 + 3A + 2I = O \qquad (1)$

From Eq. (1),

$A^3 + 3A^2 + 2A = O$

$\Rightarrow \quad (A + I)^3 - A = I^3$

$\Rightarrow \quad A = (A + I)^3 - I^3 = (A + I)^3 + (-I)^3$

$\Rightarrow \quad B = A + I$ and $C = -I$

$$\therefore \quad B = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\text{and } C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

7. Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square root of the matrix.

$$I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $A^2 = I$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow a^2+bc = 1 \quad (1)$$
$$ab+bd = 0 \quad (2)$$
$$ac+cd = 0 \quad (3)$$
$$cb+d^2 = 1 \quad (4)$$

If $a+d=0$, the above four equations hold simultaneously if $d = -a$ and $a^2+bc=1$. Hence, one possible square root of I is

$$A=\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$$

where α, β, γ are the three numbers related by the condition $\alpha^2+\beta\gamma=1$.

If $a+d \neq 0$, then above four equations hold simultaneously if $b=0, c=0, a=1, d=1$ or if $b=0, c=0, a=-1, d=-1$.

Hence, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

i.e., $\pm I$ are other possible square roots of I.

8. Given $A=\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

$$\therefore \quad A^2 = A\times A=\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\times\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{pmatrix}$$

$$=\begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$$

$$\therefore \quad A^2-4A-5I=\begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}-4\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}-5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}-\begin{pmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{pmatrix}-\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$=\begin{pmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \quad A^2-4A-5I=O$$

or

$$5I = A^2-4A$$

Multiplying by A^{-1}, we get

$$5A^{-1}=A-4I$$

$$=\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}-4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}-\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} 1-4 & 2-0 & 2-0 \\ 2-0 & 1-4 & 2-0 \\ 2-0 & 2-0 & 1-4 \end{pmatrix}$$

$$=\begin{pmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{pmatrix}$$

$$\therefore \quad A^{-1}=\frac{1}{5}\begin{pmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{pmatrix}=\begin{pmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{pmatrix}$$

9. First, we will show that $I-S$ is non-singular. The equality $|I-S|=0$ implies that I is a characteristic root of the matrix S, but this is not possible, for a real skew-symmetric matrix can have zero or purely imaginary numbers as its characteristic roots. Thus $|I-S| \neq 0$, i.e., $I-S$ is non-singular. We have,

$$A^T = [(I-S)^{-1}]^T (I+S)^T$$
$$= [(I-S)^T]^{-1} (I+S)^T$$

But

$$(I-S)^T = I^T - S^T = I+S$$

and

$$(I+S)^T = I^T + S^T = I-S$$

$$\therefore \quad A^T = (I+S)^{-1}(I-S)$$

$$\therefore \quad A^TA = (I+S)^{-1}(I-S)(I+S)(I-S)^{-1}$$
$$= (I+S)^{-1}(I+S)(I-S)(I-S)^{-1}$$
$$= I$$

Thus, A is orthogonal.

10. Let $P=\begin{bmatrix} A & B \\ C & O \end{bmatrix}$

Consider the matrix equation $PX = Q$

or

$$\begin{bmatrix} A & B \\ C & O \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

which is equivalent to the following system of equations

$$Ax_1 + Bx_2 = q_1 \quad (1)$$
$$Cx_1 + O = q_2 \quad (2)$$

From Eq. (2), we have

$$Cx_1 = q_2$$

or

$$C^{-1}Cx_1 = C^{-1}q_2$$
$$\Rightarrow x_1 = q_2C^{-1} \quad (3)$$

Putting the value of x_1 in Eq. (1), we get

$$q_2AC^{-1} + Bx_2 = q_1$$

or

$$q_2B^{-1}AC^{-1} + B^{-1}Bx_2 = B^{-1}q_1$$
$$\Rightarrow x_2 = B^{-1}q_1 - q_2B^{-1}AC^{-1} \quad (4)$$

Both Eqs. (3) and (4) are equivalent to the matrix equation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\Rightarrow \quad X = \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix} Q$$

$$\Rightarrow \quad P^{-1} = \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}$$

11. Let $P = 1/2(A + A^\theta)$

and $Q = 1/2i(A - A^\theta)$. Then,

$$A = P + iQ \qquad (1)$$

Now,

$$P^\theta = \left\{\frac{1}{2}(A + A^\theta)\right\}^\theta$$

$$= \frac{1}{2}(A^\theta + (A^\theta)^\theta)$$

$$= \frac{1}{2}(A^\theta + A) = \frac{1}{2}(A + A^\theta) = P$$

Therefore p is a Hermitian matrix. Also,

$$Q^\theta = \left\{\frac{1}{2}(A - A^\theta)\right\}^\theta$$

$$= \left(\frac{\bar{1}}{2i}\right)(A - A^\theta)^\theta$$

$$= -\frac{1}{2i}\{A^\theta - (A^\theta)^\theta\}$$

$$= -\frac{1}{2i}(A^\theta - A)$$

$$= \frac{1}{2i}(A - A^\theta)$$

$$= Q$$

Therefore, Q is also Hermitian matrix.

Thus A can be expressed in the form (1). Since A is unique, let $A = R + iS$ where R and S are both Hermitian matrices. We have,

$$A^\theta = (R + iS)^\theta$$

$$= R^\theta + (IS)^\theta$$

$$= R^\theta - iS^\theta$$

$$= R - iS \qquad \text{(since } R \text{ and } S \text{ are both Hermitian)}$$

$$\therefore \quad A + A^\theta = (R + iS) + (R - iS) = 2R$$

$$\Rightarrow \quad R = \frac{1}{2}(A + A^\theta) = P$$

Also,

$$A - A^\theta = (R + iS) - (R - iS) = 2iS$$

$$\Rightarrow \quad S = \frac{1}{2i}(A - A^\theta) = Q$$

Hence expression (1) for A is unique.

12. We have,

$$A = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix}$$

$$\Rightarrow \quad A^T = \begin{bmatrix} 2+3i & -3-i & 3-2i \\ 2 & 7 & i \\ 5 & 3-i & 2+i \end{bmatrix}$$

$$\Rightarrow \quad \bar{A}^T = \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix}$$

or

$$A^\ominus = \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix}$$

$$\therefore \quad A + A^\ominus = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix} + \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1+i & 8+2i \\ -1-i & 14 & 3-2i \\ 8-2i & 3+2i & 4 \end{bmatrix} \qquad (1)$$

and

$$A - A^\ominus = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix} - \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix}$$

$$= \begin{bmatrix} 6i & 5-i & 2-2i \\ -5-i & 0 & 3 \\ -2-2i & -3 & 2i \end{bmatrix} \qquad (2)$$

Adding Eqs. (1) and (2), we get

$$2A = \begin{bmatrix} 4 & -1+i & 8+2i \\ -1-i & 14 & 3-2i \\ 8-2i & 3+2i & 4 \end{bmatrix} + \begin{bmatrix} 6i & 5-i & 2-2i \\ -5-i & 0 & 3 \\ -2-2i & -3 & 2i \end{bmatrix}$$

$$\text{Hence, } A = \begin{bmatrix} 2 & -\frac{1}{2}+\frac{i}{2} & 4+i \\ -\frac{1}{2}-\frac{i}{2} & 7 & \frac{3}{2}-i \\ 4-i & \frac{3}{2}+i & 2 \end{bmatrix} + \begin{bmatrix} 3i & \frac{5}{2}-\frac{i}{2} & 1-i \\ -\frac{5}{2}-\frac{i}{2} & 0 & \frac{3}{2} \\ -1-i & -\frac{3}{2} & i \end{bmatrix}$$

Objective Type

1. d. Let A be a skew-symmetric matrix of order n. By definition,

$$A' = -A$$

$$\Rightarrow \quad |A'| = |-A|$$

$$\Rightarrow \quad |A| = (-1)^n \, |A|$$

$$\Rightarrow \quad |A| = -|A| \qquad [\because n \text{ is odd}]$$

$$\Rightarrow \quad 2|A| = 0$$

$$\Rightarrow \quad |A| = 0$$

Hence, A^{-1} does not exist.

2. d. (i) is false.

If $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

(ii) is true as the product AB is an identity matrix, if and only if B is inverse of the matrix A.

(iii) is false since matrix multiplication in not commutative.

3. c. $[1 \quad x \quad y]\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = 10$

$\Rightarrow \quad [1 \quad 3+2x \quad 1-x+y]\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = [0]$

$\Rightarrow \quad [1 + 3x + 2x^2 + y - xy + y^2] = [0]$

$\Rightarrow \quad 2x^2 + y^2 + y + 3x - xy + 1 = 0$

If $y = 0$, $2x^2 + 3x + 1 = 0$

$\Rightarrow \quad (2x - 1)(x + 1) = 0$

$\Rightarrow \quad x = -1/2, -1$ (rational roots)

If $\quad y = -1, 2x^2 + 4x + 1 = 0$

$\Rightarrow \quad x = \dfrac{-4 \pm \sqrt{12}}{4} = \dfrac{-2 \pm \sqrt{3}}{2}$ (irrational roots)

4. d. $A = \text{diag}(d_1, d_2, ..., d_n)$

Given, $A^3 = A$

$\Rightarrow \quad \text{diag}(d_1^3, d_2^3, ..., d_n^3) = \text{diag}(d_1, d_2, ..., d_n)$

$\Rightarrow \quad d_1^3 = d_1, d_2^3 = d_2, ..., d_n^3 = d_n$

Hence, all $d_1, d_2, d_3, ..., d_n$ have three possible values $\pm 1, 0$. Each diagonal element can be selected in three ways. Hence, the number of different matrices is 3^n.

5. d. If A is n^{th} root of I_2, then $A^n = I_2$. Now,

$$A^2 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$$

Thus,

$$A^n = \begin{bmatrix} a^n & na^{n-1}b \\ 0 & a^n \end{bmatrix}$$

Now,

$$A^n = I \Rightarrow \begin{bmatrix} a^n & na^{n-1}b \\ 0 & a^n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad a^n = 1, b = 0$

6. d. $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (1)

$A^2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (2)

Let A be given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The first equation gives

$a - b = -1$ (3)

$c - d = 2$ (4)

For second equation gives

$$A^2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = A\left(A\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = A\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This gives

$-a + 2b = 1$ (5)

$-c + 2d = 0$ (6)

Eqs. (3) + (5) $\Rightarrow b = 0$ and $a = -1$

Eqs. (4) + (6) $\Rightarrow d = 2$ and $c = 4$

So the sum $a + b + c + d = 5$.

7. c. $AB = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\cos\phi\sin\theta\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos\theta\cos\phi\sin\theta\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi(\cos(\theta-\phi)) & \cos\theta\sin\phi(\cos(\theta-\phi)) \\ \sin\theta\cos\phi(\cos(\theta-\phi)) & \sin\theta\sin\phi(\cos(\theta-\phi)) \end{bmatrix}$$

$$= (\cos(\theta-\phi))\begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{bmatrix}$$

Now, $AB = O \Rightarrow \cos(\theta - \phi) = 0 \Rightarrow \theta - \phi = (2n + 1)\pi/2, n \in Z$.

8. b. $AB^n = ABBBBB\cdots B$

$= (AB)BBB\cdots B$

$= B(AB)BBB\cdots B$

$= BB(AB)BB\cdots B$

$\vdots$

$= B^nA$

$(AB)^n = (AB)(AB)(AB)\cdots(AB)$

$= A(BA)(BA)(BA)\cdots(BA)B$

$= A(AB)(AB)(AB)\cdots(AB)B$

$= A^2(BA)(BA)(BA)\cdots(BA)B^2$

$= A^2(AB)(AB)(AB)\cdots(AB)B^2$

$= A^3(BA)(BA)(BA)\cdots(BA)B^3$

$\vdots$

$= A^nB^n$

9. a. From given data $|A| = 2^4$

$\Rightarrow \quad |\text{adj}(\text{adj } A)| = (2^4)^9 = 2^{36}$

$\Rightarrow \quad \left\{\dfrac{\det(\text{adj}(\text{adj } A))}{7}\right\} = \left\{\dfrac{2^{36}}{7}\right\} = \left\{\dfrac{(7+1)^{12}}{7}\right\} = \dfrac{1}{7}$

10. b. We have,

$$\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \alpha^2 + \beta\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad \alpha^2 + \beta\gamma - 1 = 0$

11. c. Given $A^2 = A$. Now,

$(I + A)^3 - 7A = I^3 + 3I^2A + 3IA^2 + A^3 - 7A$

$= I + 3A + 3A + A - 7A$

$= I + O$

$= I$

12. a. $(A - \lambda I)(B - \lambda I) = (B - \lambda I)(A - \lambda I)$

$\Rightarrow \quad AB - \lambda(A + B)I + \lambda^2I^2 = BA - \lambda(B + A)I + \lambda^2I^2$

$\Rightarrow \quad AB = BA$

13. c. Given, $A^2 = 2A - I$

Now, $A^3 = A(A^2)$

$= A(2A - I)$

$= 2A^2 - A$

$= 2(2A - I) - A$

$= 3A - 2I$

$A^4 = A(A^3)$

$= A(3A - 2I)$

$= 3A^2 - 2A$

$= 3(2A - I) - 2A$

$= 4A - 3I$

Following this, we can say $A^n = nA - (n-1)I$.

14. a. We have, $A^2 = O, A^k = O, \forall\, k \geq 2$

Thus,

$(A + I)^{50} = I + 50A$

$\Rightarrow \quad (A + I)^{50} = I + 50A$

$\Rightarrow \quad a = 1, b = 0, c = 0, d = 1$

15. b. We have,

$A = iB$

$$\Rightarrow \quad A^2 = (iB)^2 = i^2B^2 = -B^2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = -2B$$

$\Rightarrow \quad A^4 = (-2B)^2 = 4B^2 = 4(2B) = 8B$

$\Rightarrow \quad (A^4)^2 = (8B)^2$

$\Rightarrow \quad A^8 = 64B^2 = 128B$

16. b. Since the product matrix is 3×3 matrix and the pre-multiplier of A is a 3×2 matrix, therefore A is 2×3 matrix. Let,

$A = \begin{bmatrix} l & m & n \\ x & y & z \end{bmatrix}$. Then the given equation becomes

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} l & m & n \\ x & y & z \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2l - x & 2m - y & 2n - z \\ l & m & x \\ -3l + 4x & -3m + 4y & -3n + 4z \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$\Rightarrow \quad 2l - x = -1, 2m - y = -8, 2n - z = -10, l = 1, m = -2, n = -5$

$\Rightarrow \quad x = 3, y = 4, z = 0, l = 1, m = -2, n = -5$

$$\Rightarrow \quad A = \begin{bmatrix} l & m & n \\ x & y & z \end{bmatrix} = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$$

17. b. Let, $A = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$\therefore \quad A^T = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1-i & -1 \end{bmatrix}$$

$$\Rightarrow \quad (\bar{A}^T) = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\therefore \quad A(\bar{A}^T) = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

18. d. See theory.

19. a. A is involuntary. Hence,

$A^2 = I \Rightarrow A = A^{-1}$

Also,

$$(kA)^{-1} = \frac{1}{k}(A)^{-1}$$

$$\Rightarrow \quad \left(\frac{1}{2}A\right)^{-1} = 2(A)^{-1} \Rightarrow 2A$$

20. b. Given,

$B = A^{-1}A^T$

$\Rightarrow \quad B^T = (A^{-1}A^T)^T = A \times (A^{-1})^T$

$\Rightarrow \quad B \times B^T = A^{-1}A^T \times A \times (A^{-1})^T = A^{-1} \times (A^T \times A)(A^{-1})^T$

$= A^{-1}(A \times A^T)(A^{-1})^T$

$= (A^{-1}A) \times (A^{-1}A)^T = I \times I^T = I$

21. b. $P^TP = I$

$Q = PAP^T$

$\therefore \quad x = P^TQ^{1000}P = P^T(PAP^T)^{1000}P$

$= P^TPAP^T(PAP^T)^{999}P$

$= IAP^T \cdot PAP^T(PAP^T)^{998}P$

$= AIAP^T(PAP^T)^{998}P$

$= A^2P^TPAP^T(PAP^T)^{997}P$

$= A^3P^T((PAP^T)^{997}P$

$\vdots$

$= A^{1000} = I \quad (\because A$ is involuntary)

Hence, $x^{-1} = I$.

22. b. Since A is orthogonal, hence

$AA^T = I$

$\Rightarrow \quad |AA^T| = 1$

$\Rightarrow \quad |A^2| = 1$

$\Rightarrow \quad |A| = \pm 1$

Now, $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

23. b. $\left(A' - \dfrac{1}{2}I\right)\left(A - \dfrac{1}{2}I\right) = I$ and $\left(A' + \dfrac{1}{2}I\right)\left(A + \dfrac{1}{2}I\right) = I$

$\Rightarrow \quad A + A' = 0$ (subtracting the two results)

$\Rightarrow \quad A' = -A$

$$\Rightarrow \quad A^2 = -\frac{3}{4}I$$

$$\Rightarrow \quad \left(\frac{-3}{4}\right)^n = (\det(A))^2$$

$\Rightarrow \quad n$ is even

24. a. For involuntary matrix,

$A^2 = I$

$\Rightarrow \quad |A^2| = |I| \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

For idempotent matrix,

$A^2 = A$

$\Rightarrow \quad |A^2| = |A| \Rightarrow |A|^2 = |A| \Rightarrow |A| = 0$ or 1

For orthogonal matrix,

$AA^T = I$

$\Rightarrow \quad |AA^T| = |I| \Rightarrow |A|\,|A^T| = 1 \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

Thus if matrix A is idempotent it may not be invertible.

25. a. $A \times A^T = I$

$\Rightarrow \quad |A \times A^T| = |I|$

$\Rightarrow \quad |A|^2 = 1$

$\Rightarrow \quad |A| = \pm 1$

$\Rightarrow \quad A^{-1}$ exists

$\Rightarrow \quad A^{-1} \times A \times A^T = A^{-1} \times I$

$\Rightarrow \quad A^{-1} = A^T$

26. b. Z is idempotent, then

$Z^2 = Z \Rightarrow Z^3, Z^4, \ldots, Z^n = Z$

$$\therefore \quad (I+Z)^n = {}^nC_0 I^n + {}^nC_1 I^{n-1}Z + {}^nC_2 I^{n-2}Z^2 + \cdots + {}^nC_n Z^n$$
$$= {}^nC_0 I + {}^nC_1 Z + {}^nC_2 Z + {}^nC_3 Z + \cdots + {}^nC_n Z$$
$$= I + ({}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n)Z$$
$$= I + (2^n - 1)Z$$

27. d. Since $AB = B$ and $BA = A$, so

$BAB = B^2$

$\Rightarrow \quad (BA)B = B^2$

$\Rightarrow \quad AB = B^2$

$\Rightarrow \quad B = B^2$

Hence, B is idempotent and similarly A.

$(A - B)^2 = A^2 - AB - BA + B^2 = A - B - A + B = O$

Therefore, $A - B$ is nilpotent.

28. b. $A^2 = O, A^3 = A^4 = \cdots = A^n = O$

Then, $A(I + A)^n = A(I + nA) = A + nA^2 = A$

29. b. We have, $AB = A(\text{adj } A) = |A|\, I_n$

$\therefore \quad AB + KI_n = |A|I_n + KI_n$

$\Rightarrow \quad AB + KI_n = (|A| + k)I_n$

$\Rightarrow \quad |AB + KI_n| = |(|A| + k)I_n| \quad (\because |aI_n| = a^n)$

$= (|A| + K)^n$

30. d. $\det(A - I) = \det(A - A^2)$

$= \det A(I - A)$

$= \det A \times \det(I - A)$

$= -\det A \times \det(A - I)$

Now,

$A^2 = I$

$\Rightarrow \quad \det(A^2) = \det(I)$

$\Rightarrow \quad (\det A)^2 = 1$

$\Rightarrow \quad \det(A) = \pm 1$

Thus, $\det(A)$ can be 1 or -1, from which we cannot say anything about $\det(A - I)$.

31. a. $|B| = \begin{vmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{vmatrix}$ (Multiplying R_2 by -1)

$= -\begin{vmatrix} q & -b & y \\ p & -a & x \\ r & -c & z \end{vmatrix}$ (Multiplying C_2 by -1)

$= \begin{vmatrix} q & b & y \\ p & a & x \\ r & c & z \end{vmatrix}$ (Changing R_1 with R_2)

$= -\begin{vmatrix} p & a & x \\ q & b & y \\ r & c & z \end{vmatrix}$

$= -\begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$

Hence $|A| = -|B|$, obviously when $|A| \neq 0$, $|B| \neq 0$. Also, $|\text{adj } B| = |B|^2 = (-|A|)^2 = |A|^2$.

32 a. $AB = C$

$\Rightarrow \quad |AB| = |C|$

$\Rightarrow \quad |A||B| = |C|$

$\Rightarrow \quad |B| = \dfrac{|C|}{|A|}$

33 a. $AB = A + B$

$\Rightarrow \quad B = AB - A = A(B - I)$

$\Rightarrow \quad \det(B) = \det(A)\det(B - I) = 0$

$\Rightarrow \quad \det(B) = 0$

34. b. $|A^{2005} - 6A^{2004}| = |A|^{2004}|A - 6I|$

$$= 2^{2004}\begin{vmatrix} 0 & 11 \\ 2 & -2 \end{vmatrix} = (-22)\, 2^{2004} = (-11)\,(2)^{2005}$$

35. c. $A^2 = I$

$\Rightarrow \quad A^2 - I = O$

$\Rightarrow \quad (A + I)(A - I) = 0$

Therefore, either $|A + I| = 0$ or $|A - I| = 0$. If $|A - I| \neq 0$, then $(A + I)(A - I) = O \Rightarrow A - I = O$ which is not so.

$\therefore \quad |A - I| = 0$ and $A - I \neq O$

36. b. We have,

$I = AA^{-1}$

$$= \frac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -08 & 6 & 2c \\ 5 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & c+1 \\ 0 & 1 & 2\ (c+1) \\ 4(1-a) & 3\ (a-1) & 2+ac \end{bmatrix}$$

Comparing the elements of AA^{-1} with those of I, we have

$c + 1 = 0 \Rightarrow c = -1$

$\Rightarrow \quad c = -1$ and $a - 1 = 0 \Rightarrow a = 1$

37. c. Given $B^r = I \Rightarrow B^r B^{-1} = IB^{-1} \Rightarrow B^{r-1} = B^{-1}$

$\Rightarrow \quad A^{-1}B^{r-1}A - A^{-1}B^{-1}A = A^{-1}B^{-1}A - A^{-1}B^{-1}A = O$

38. c. $\begin{vmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_1 & z_1 \end{vmatrix}^{-1}\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1}$

$$= \left(\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}\begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} z_1\bar{z}_1 + z_2\bar{z}_2 & 0 \\ 0 & z_2\bar{z}_2 + z_1\bar{z}_1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} |z_1|^2 + |z_2|^2 & 0 \\ 0 & |z_1|^2 + |z_2|^2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

39. a. We have,

$$A(\alpha, \beta)^{-1} = \frac{1}{e^\beta}\begin{bmatrix} e^\beta \cos\alpha & -e^\beta \sin\alpha & 0 \\ e^\beta \sin\ \alpha & e^\beta \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= A(-\alpha, -\beta)$

40. a. We have,

$$|A| = (a+ib)(a-ib) - (-c+id)(c+id)$$
$$= a^2+b^2+c^2+d^2 = 1$$

and $\text{adj}(A) = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$

Then $A^{-1} = \begin{bmatrix} a-ib & -c-id \\ -c+id & a-ib \end{bmatrix}$

41. c. Given

$A^3 = O$

Now,

$$(I-A)(I+A+A^2)$$
$$= I^2 + IA + IA^2 - AI - A^2 - A^3$$
$$= I - A^3$$
$$= I$$

$\Rightarrow (I-A)^{-1} = I + A + A^2$

42. b. We know that $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

$\Rightarrow |\text{adj}(\text{adj}(\text{adj } A))| = |\text{adj } A|^{(n-1)^2}$

$$= |A|^{(n-1)^3}$$
$$= 2^8 = 256$$

43. b. $(-A)^{-1} = \dfrac{\text{adj}(-A)}{|-A|} = \dfrac{(-1)^{n-1}\text{adj}(A)}{(-1)^n |A|} = \dfrac{\text{adj}(A)}{-|A|} = -A^{-1}$

44. a. $A(x)A(y) = (1-x)^{-1}(1-y)^{-1}\begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}\begin{bmatrix} 1 & -y \\ -y & 1 \end{bmatrix}$

$$= (1+xy-(x+y))^{-1}\begin{bmatrix} 1+xy & -(x+y) \\ -(x+y) & 1+xy \end{bmatrix}$$

$$= \left(1 - \frac{x+y}{1+xy}\right)^{-1}\begin{bmatrix} 1 & -\dfrac{x+y}{1+xy} \\ -\dfrac{x+y}{1+xy} & 1 \end{bmatrix} = A(z)$$

45. b. Let, $A = \begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}$

$\Rightarrow \text{adj}(A) = \begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}$

$\Rightarrow A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix} = \dfrac{1}{25}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}$

$\Rightarrow A^{-2} = (A^{-1})^2 = \dfrac{1}{25}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}\dfrac{1}{25}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}$

$$= \frac{1}{625}\begin{bmatrix} 25 & 0 \\ 10a & 25 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{25} & 0 \\ \dfrac{2a}{125} & \dfrac{1}{25} \end{bmatrix}$$

Now,

$$\begin{bmatrix} 1/25 & 0 \\ x & 1/25 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{25} & 0 \\ \dfrac{2a}{125} & \dfrac{1}{25} \end{bmatrix}$$

$\Rightarrow x = 2a/125$

46. c. $f(x) = \dfrac{1+x}{1-x}$

$\Rightarrow (1-x)f(x) = 1+x$

$\Rightarrow (I-A)f(A) = (I+A)$

$\Rightarrow f(A) = (I-A)^{-1}(I+A)$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right)$$

$\Rightarrow f(A) = \left(\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right)$

$$= \frac{\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}}{-4}$$

$$= \frac{\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}}{-4}$$

$$= \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

47. a. We know that in a square matrix of order n,

$|\text{adj } A| = |A|^{n-1}$

$\Rightarrow |\text{adj }(\text{adj } A)| = |\text{adj } A|^{n-1} = |A|^{(n-1)^2}$

$\Rightarrow n^2 - 2n - 8 = 0$

$\Rightarrow n = 4$ as $n = -2$ is not possible

48. b. $|A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix} = 1 + \tan^2 x \neq 0$

So A is invertible. Also,

$$\text{adj } A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

Now,

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

$\Rightarrow A^{-1} = \dfrac{1}{(1+\tan^2 x)}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$$= \begin{bmatrix} \dfrac{1}{1+\tan^2 x} & \dfrac{-\tan x}{1+\tan^2 x} \\ \dfrac{\tan x}{1+\tan^2} & \dfrac{1}{1+\tan^2 x} \end{bmatrix}$$

$\therefore A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1}{1+\tan^2 x} & \dfrac{-\tan x}{1+\tan^2 x} \\ \dfrac{\tan x}{1+\tan^2} & \dfrac{1}{1+\tan^2 x} \end{bmatrix}$

$$= \begin{bmatrix} \dfrac{1-\tan^2 x}{1+\tan^2 x} & \dfrac{-2\tan x}{1+\tan^2 x} \\ \dfrac{2\tan x}{1+\tan^2 x} & \dfrac{1-\tan^2 x}{1+\tan^2 x} \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

49. b. Since $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and given $A=\begin{bmatrix}0&\tan\alpha/2\\-\tan\alpha/2&0\end{bmatrix}$

$$\therefore \quad I-A=\begin{bmatrix}1&-\tan\alpha/2\\\tan\alpha/2&1\end{bmatrix} \qquad (1)$$

Now, $(I-A)\begin{bmatrix}\cos\alpha&-\sin\alpha\\\sin\alpha&\cos\alpha\end{bmatrix}$

$$=\begin{bmatrix}1&\tan\alpha/2\\-\tan\alpha/2&1\end{bmatrix}\begin{bmatrix}\cos\alpha&-\sin\alpha\\\sin\alpha&\cos\alpha\end{bmatrix}$$

$$=\begin{bmatrix}1&\tan\alpha/2\\-\tan\alpha/2&1\end{bmatrix}\begin{bmatrix}\dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2}&-\dfrac{2\tan\alpha/2}{1+\tan^2\alpha/2}\\\dfrac{2\tan\alpha/2}{1+\tan^2\alpha/2}&\dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2}\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2}+\dfrac{2\tan^2\alpha/2}{1+\tan^2\alpha/2}&-\dfrac{2\tan\alpha/2}{1+\tan^2\alpha/2}+\dfrac{\tan\alpha/2\,(1-\tan^2\alpha/2)}{1+\tan^2\alpha/2}\\\dfrac{-\tan\alpha/2\,(1-\tan^2\alpha/2)}{1+\tan^2\alpha/2}+\dfrac{2\tan\alpha/2}{1+\tan^2\alpha/2}&\dfrac{2\tan^2\alpha/2}{1+\tan^2\alpha/2}+\dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2}\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)}&-\dfrac{\tan\alpha/2\,(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)}\\\dfrac{\tan\alpha/2\,(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)}&\dfrac{(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)}\end{bmatrix}$$

$$=\begin{bmatrix}1&-\tan\alpha/2\\\tan\alpha/2&1\end{bmatrix}$$

$$=I-A \qquad \text{[Using (1)]}$$

50. d. Let, $A=\begin{bmatrix}1&-4\\3&-2\end{bmatrix}$ and $B=\begin{bmatrix}-16&-6\\7&2\end{bmatrix}$

Then the matrix equation is $AX=B$.

$$\therefore \quad |A|=\begin{vmatrix}1&-4\\3&-2\end{vmatrix}=-2+12\neq 0$$

So A is an invertible matrix. Also,

$$\text{adj } A=\begin{bmatrix}-2&-3\\4&1\end{bmatrix}^T=\begin{bmatrix}-2&4\\-3&1\end{bmatrix}$$

So,

$$A^{-1}=\frac{1}{|A|}\text{adj } A=\frac{1}{10}\begin{bmatrix}-2&4\\-3&1\end{bmatrix}$$

Now,

$AX=B$

$\Rightarrow A^{-1}(AX)=A^{-1}B$

$\Rightarrow (A^{-1}A)X=A^{-1}B$

$\Rightarrow IX=A^{-1}B$

$\Rightarrow X=A^{1}B$

$$\Rightarrow \quad X=\frac{1}{10}\begin{bmatrix}-2&4\\-3&1\end{bmatrix}\begin{bmatrix}-16&-6\\7&2\end{bmatrix}=\begin{bmatrix}6&2\\11/2&2\end{bmatrix}$$

51. c. $(A^{-1}BA)^2 = (A^{-1}BA)(A^{-1}BA)$

$= A^{-1}B(AA^{-1})BA$

$= A^{-1}BIBA = A^{-1}B^2A$

$\Rightarrow (A^{-1}BA)^3 = (A^{-1}B^2A)(A^{-1}BA)$

$= A^{-1}B^2(AA^{-1})BA$

$= A^{-1}B^2IBA$

$= A^{-1}B^3A$ and so on

$\Rightarrow (A^{-1}BA)^n = A^{-1}B^nA$

52. a. A adj $A = |A|\, I$

$\Rightarrow |A \text{ adj } A| = |A|^n$ [If A is of order $n\times n$]

$\Rightarrow |A|\,|\text{adj } A| = |A|^n$

$\Rightarrow |\text{adj } A| = |A|^{n-1}$

Now, A is singular,

$\therefore |A| = 0$

$\Rightarrow |\text{adj } A| = 0$

Hence adj A is singular.

53. a. $A = \text{diag}\,(d_1, d_2, d_3, \ldots, d_n)$

$\Rightarrow |A| = (d_1\times d_2\times d_3\times d_4 \cdots d_n)$

Now,

Cofactor of d_1 is $d_2d_3\cdots d_n$

Cofactor of d_2 is $d_1\times d_3\times d_4\cdots d_n$

Cofactor of d_3 is $d_1\times d_2\times d_4\cdots d_n$

$\vdots$

Cofactor of d_n is $d_1\times d_2\times d_3\cdots d_{n-1}$

$$\Rightarrow \quad A^{-1}=\frac{1}{|A|}\text{adj}(A)=\text{diag}\,(d_1^{-1}, d_2^{-1}, d_3^{-1}, \ldots, d_n^{-1})$$

Hence, A^{-1} is also a diagonal matrix.

54. a. We know that for any non-singular matrix A,

$$A^{-1}=\frac{1}{|A|}\text{adj}(A)$$

Now put $A = P^{-1}$. Then we have

$$(P^{-1})^{-1}=\frac{1}{|P^{-1}|}\text{adj}(P^{-1})$$

$\Rightarrow P = |P|\text{adj}(P^{-1})$

$$\Rightarrow \quad \text{adj}(P^{-1})=\frac{P}{|P|}$$

55. c. $\text{adj}(Q^{-1}BP^{-1}) = \text{adj}(P^{-1})\text{adj}(B)\text{adj}(Q^{-1})$

$$=\frac{P}{|P|}A\frac{Q}{|Q|}$$

$= PAQ$

56. b. We have,

$(A-2I)(A-4I) = O$

$\Rightarrow A^2-2A-4A+8I = O$

$\Rightarrow A^2-6A+8I = O$

$\Rightarrow A^{-1}(A^2-6A+8I) = A^{-1}O$

$\Rightarrow A-6I+8A^{-1} = O$

$\Rightarrow A+8A^{-1} = 6I$

$$\Rightarrow \quad \frac{1}{6}A+\frac{4}{3}A^{-1}=1$$

57. b. $|A(\alpha,\beta)| = \cos^2\alpha e^\beta+\sin^2\alpha e^\beta = e^\beta$

Now,

$$A(\alpha,\beta)^{-1}=\frac{1}{e^\beta}\text{adj}\,(A(\alpha,\beta))$$

$$= \frac{1}{e^{\beta}}\begin{bmatrix} e^{\beta}\cos\alpha & -\sin\alpha e^{\beta} & 0 \\ e^{\beta}\sin\alpha & \cos\alpha e^{\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & e^{-\beta} \end{bmatrix}$$

$= A(-\alpha, -\beta)$

58. a. As $B = -A^{1}BA$, we get

$AB = -BA$ or $AB + BA = O$

Now,

$(A + B)^2 = (A + B)(A + B)$

$= A^2 + BA + BA + B^2$

$= A^2 + O + B^2$

$= A^2 + B^2$

59. b. $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$\Rightarrow \quad A^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$

$\Rightarrow \quad A^3 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^3 & 0 \\ 0 & b^3 \end{pmatrix}$

$\Rightarrow \quad A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

$\Rightarrow \quad (A^n)^{-1} = \frac{1}{a^n b^n}\begin{pmatrix} b^n & 0 \\ 0 & a^n \end{pmatrix} = \begin{pmatrix} a^{-n} & 0 \\ 0 & b^{-n} \end{pmatrix}$

$\Rightarrow \quad \lim_{n\to\infty}(A^n)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ as $a > 1$ and $b > 1$

60. b. $(I - A)f(A) = I + A$

$\Rightarrow \quad f(A) = (I + A)(I - A)^{-1}$

$= (I + A)(I + A + A^2)$

$= I + A + A^2 + A + A^2 + A^3$

$= I + 2A + 2A^2$

61. d. If possible assume that A is non-singular, then A^{-1} exists. Thus,

$AB = O \Rightarrow A^{-1}(AB) = (A^{-1}A)B = O$

$\Rightarrow \quad IB = O$ or $B = O$ × (a contradiction)

Hence, both A and B must be singular.

62. a. $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}^{-1}$

$\Rightarrow \quad A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

63. c. $A^2 - A + I = 0$

$\Rightarrow \quad I = A - A^2$

$IA^{-1} = AA^{-1} - A^2A^{-1}$

$\Rightarrow \quad A^{-1} = I - A$

64. a. Let, $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow \quad X^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$

$\Rightarrow \quad a^2 + bc = 1$ and $ab + bd = 1 \Rightarrow b(a + d) = 1$

$ac + cd = 2 \Rightarrow c(a + d) = 2 \Rightarrow 2b = c$

Also,

$bc + d^2 = 3 \Rightarrow d^2 - a^2 = 2$

$\Rightarrow \quad (d - a)(a + d) = 2 \Rightarrow d - a = 2b \quad$ (using $bc = 1 - a^2$)

$a + d = 1/b$

$\Rightarrow \quad 2d = 2b + 1/b, \quad 2a = 1/b - 2b$

$d = b + 1/2b, \quad a = 1/(2b) - b$

$c = 2b$

$\Rightarrow \quad \left(b^2 + \frac{1}{4b^2} + 1\right) + 2b^2 = 3$

$\Rightarrow \quad 3b^2 + \frac{1}{4b^2} = 2$

$\Rightarrow \quad 3x + \frac{1}{4x} = 2$

$\Rightarrow \quad b = \pm\frac{1}{\sqrt{6}}$ or $b = \pm\frac{1}{\sqrt{2}}$

Therefore, matrices are

$$\begin{pmatrix} 0 & 1/\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 & -1/\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} \end{pmatrix}, \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} \\ 2/\sqrt{6} & 4\sqrt{6} \end{pmatrix}$$

65. c. Given that

$X = AB + BA \Rightarrow X = X^T$

and

$Y = AB - BA$

$\Rightarrow \quad Y = -Y^T$

Now, $(XY)^T = Y^TX^T = -YX$.

66. c. As A is a skew-symmetric matrix,

$A^T = -A$

$\Rightarrow \quad a_{ii} = 0, \forall\, i$

$\Rightarrow \quad \text{tr}(A) = 0$

Also,

$|A| = |A^T| = |-A| = (-1)^3|A|$

$\Rightarrow \quad 2|A| = 0$

$\Rightarrow \quad |A| = 0$

67. d. $\text{tr}(A) = \sum_{i=j} a_{ij}$

$= (a_{11} + a_{22} + a_{33} + \cdots + a_{10\times10})$

$= (w^2 + w^4 + w^6 + \cdots + w^{20})$

$= w^2(1 + w^2 + w^4 + \cdots + w^{18})$

$= w^2[(1 + w + w^2) + \cdots + (1 + w + w^2) + 1]$

$= w^2 \times 1$

$\Rightarrow \quad \text{tr}(A) = w^2$

68. b. We have,

$$F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{adj}(F(\alpha)) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Also,

$\det(F(\alpha)) = 1$

$$\Rightarrow \quad [F(\alpha)]^{-1} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(-\alpha)$$

69. b. We have,

$$[F(x)G(y)]^{-1} = [G(y)]^{-1}[F(x)]^{-1} = G(-y)F(-x)$$

70. a. Given A is skew-symmetric. Hence,

$$A^T = -A$$

$$\Rightarrow \quad A^n = (-A^T)^n = -(A^T)^n = -(A^n)^T \text{ (given } n \text{ is odd)}$$

Hence, A^n is skew-symmetric.

71. c. We have,

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + db \\ ac + cd & bc + d^2 \end{bmatrix}$$

As A satisfies $x^2 + k = 0$, therefore

$$A^2 + kI = O$$

$$\Rightarrow \quad \begin{bmatrix} a^2 + bc + k & (a + d)b \\ (a + d)c & bc + d^2 + k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad a^2 + bc + k = 0,\ bc + d^2 + k = 0$

and $(a + d)b = (a + d)c = 0$

As $bc \neq 0, b \neq 0, c \neq 0$, so

$a + d = 0$

$\Rightarrow \quad a = -d$

Also,

$$k = -(a^2 + bc) = -(d^2 + bc) = -((-ad) + bc) = |A|$$

72. b. Given $A, B, A + I, A + B$ are idempotent. Hence,

$A^2 = A,\ B^2 = B,\ (A + I)^2 = A + I$ and $(A + B)^2 = A + B$

$\Rightarrow \quad A^2 + B^2 + AB + BA = A + B$

$\Rightarrow \quad A + B + AB + BA = A + B$

$\Rightarrow \quad AB + BA = O$

73. b.

$$A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 - 24 + 0 & 40 - 40 + 0 & 0 + 0 + 0 \\ -15 + 15 + 0 & -24 + 25 + 0 & 0 + 0 + 0 \\ -5 + 6 - 1 & -8 + 10 - 2 & 0 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, the matrix A is involutory.

74. b. Let $A = [a_{ij}]$. Since A is skew-symmetric, therefore

$a_{ii} = 0$ and $a_{ij} = -a_{ij}\ (i \neq j)$

A is symmetric as well, so $a_{ij} = a_{ji}$ for all i and j.

$\therefore \quad a_{ij} = 0$ for all $i \neq j$

Hence, $a_{ij} = 0$ for all i and j, i.e., A is a null matrix.

75. c.

$$(A^{-1}BA)^2 = (A^{-1}BA)(A^{-1}BA) = A^{-1}B(AA^{-1})BA = A^{-1}BIBA = A^{-1}B^2A$$

$$(A^{-1}BA)^3 = (A^{-1}B^2A)(A^{-1}BA) = A^{-1}B^2(AA^{-1})BA = A^{-1}B^3A \text{ and so on}$$

$\therefore \quad (A^{-1}BA)^n = A^{-1}B^nA$

76. a. Matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is orthogonal if

$$\sum a_i^2 = \sum b_i^2 = \sum c_i^2 = 1;\ \sum a_ib_i = \sum b_ic_i = \sum c_ia_i = 0$$

77. b. $(kI_n)\text{ adj }(kI_n) = |kI_n|I_n$ [using $A(\text{adj } A) = |A|I$]

$\text{adj }(kI_n) = k^{n-1}I_n$

$|\text{adj }(kI_n)| = k^{n(n-1)}$

78. c. $A \text{ adj } A = |A|\, I$

$|A| = xyz - 8x - 3(z - 8) + 2(2 - 2y)$

$|A| = xyz - (8x + 3z + 4y) + 28$

$= 60 - 20 + 28$

$= 68$

$$\Rightarrow \quad A\,(\text{adj } A) = 68\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 68 & 0 & 0 \\ 0 & 68 & 0 \\ 0 & 0 & 68 \end{bmatrix}$$

79. c. $A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$ and $2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

$\Rightarrow \quad \text{tr}(A) + 2\text{tr}(B) = -1$

and $2\,\text{tr}(A) - \text{tr}(B) = 3$

Let $\text{tr}(A) = x$ and $\text{tr}(B) = y$. Then,

$x + 2y = -1$ and $2x - y = 3$

Solving, $x = 1$ and $y = -1$. Hence,

$\text{tr}(A) - \text{tr}(B) = x - y = 2$

80. b. $B = A_1 + 3A_3^3 + \cdots + (2n - 1)(A_{2n-1})^{2n-1}$

$B^T = -[A_1 + 3A_3^3 + \cdots + (2n - 1)(A_{2r-1})^{2r-1}]$

$= -B$

Hence, B is skew-symmetric.

81. a. $|A| = 1(0 - 10) - 2(2 - 6) + 3(4 - 0)$

$= -10 + 8 + 12 = 10$

$\Rightarrow \quad |A| \neq 0$

$\Rightarrow \quad$ Unique solution

82. a. $BC = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \quad \text{tr}(A) + \text{tr}\left(\frac{ABC}{2}\right) + \text{tr}\left(\frac{A(BC)^2}{4}\right) + \text{tr}\left(\frac{A(BC)^3}{8}\right) + \cdots + \infty$$

$$= \text{tr}(A) + \text{tr}\left(\frac{A}{2}\right) + \text{tr}\left(\frac{A}{2^2}\right) + \cdots$$

$$= \text{tr}(A) + \frac{1}{2}\text{tr}(A) + \frac{1}{2^2}\text{tr}(A) \cdots$$

$$= \frac{\text{tr}(A)}{1 - (1/2)}$$

$$= 2\text{tr}(A) = 2(2 + 1) = 6$$

Multiple Correct Answers Type

1. b, c.

$\det(-A) = (-1)^n \det(A)$

$\det(A^{-1}) = \dfrac{1}{\det(A)} = 1$

$\det(\text{adj } A) = |A|^{n-1} = 1$

Hence, $|\omega A| = \omega^n |A| = 1$ only when $n = 3k, k \in Z$.

2. a, d.

Given, $(A + B)^2 = A^2 + B^2 + 2AB$

$\Rightarrow \quad (A + B)(A + B) = A^2 + B^2 + 2AB$

$\Rightarrow \quad A^2 + AB + BA + B^2 = A^2 + B^2 + 2AB \Rightarrow BA = AB$

$$\Rightarrow \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+2 & -a+1 \\ b-2 & -b-1 \end{bmatrix} = \begin{bmatrix} a-b & 1+1 \\ 2a+b & 2-1 \end{bmatrix}$$

The corresponding elements of equal matrices are equal.

$a + 2 = a - b, -a + 1 = 2 \Rightarrow a = -1$

$b - 2 = 2a + b, -b - 1 = 1 \Rightarrow b = -2$

$\Rightarrow \quad a = -1, b = -2$

3. a, b, c.

Given, $AB = A$, $BA = B$

$\Rightarrow \quad B \times AB = B \times A$

$\Rightarrow \quad (BA)B = B$

$\Rightarrow \quad B^2 = B$

Also,

$A \times B \times A = AB$

$\Rightarrow \quad (AB)A = A$

$\Rightarrow \quad A^2 = A$

Now $(A^T)^2 = (A^T \times A^T) = (A \times A)^T = (A^2)^T = A^T$

Similarly, $(B^T)^2 = B^T$

$\Rightarrow \quad A^T$ and B^T are idempotent

4. a, c.

A is an orthogonal matrix.

$\therefore \quad AA^T = I$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\Rightarrow \quad a + 4 + 2b = 0, 2a + 2 - 2b = 0$ and $a^2 + 4 + b^2 = 9$

$\Rightarrow \quad a + 2b + 4 = 0, a - b + 1 = 0$ and $a^2 + b^2 = 5$

$\Rightarrow \quad a = -2, b = -1$

5. a, d.

$(B^TAB)^T = B^TA^T(B^T)^T = B^TA^TB = B^TAB$ if A is symmetric.

Therefore, B^TAB is symmetric if A is symmetric.

Also, $(B^TAB)^T = B^TA^TB = B^T(-A)B = -(B^TA^TB)$

Therefore, B^TAB if A is skew-symmetric if A is skew-symmetric.

6. a, b, c.

We have, $|A(\theta)| = 1$

Hence, A is invertible.

$$A(\pi + \theta) = \begin{bmatrix} \sin(\pi+\theta) & i\cos(\pi+\theta) \\ i\cos(\pi+\theta) & \sin(\pi+\theta) \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\theta & -i\cos\theta \\ -i\cos\theta & -\sin\theta \end{bmatrix} = -A(\theta)$$

$$\text{adj }(A(\theta)) = \begin{bmatrix} \sin\theta & -i\cos\theta \\ -i\cos\theta & \sin\theta \end{bmatrix}$$

$$\Rightarrow \quad A(\theta)^{-1} = \begin{bmatrix} \sin\theta & -i\cos\theta \\ -i\cos\theta & \sin\theta \end{bmatrix} = A(\pi - \theta)$$

7. a, c, d.

Given, $A^2 + A + 2I = O$

$\Rightarrow \quad A^2 + A = -2I$

$\Rightarrow \quad |A^2 + A| = |-2I|$

$\Rightarrow \quad |A||A + I| = (-2)^n$

$\Rightarrow \quad |A| \neq 0$

Therefore, A is non-singular, hence its inverse exists. Also, multiplying the given equation both sides with A^{-1}, we get

$A^{-1} = -\dfrac{1}{2}(A + I)$

8. a, b, c.

$$\because \quad |A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3(-3+4) + 3(2-0) + 4(-2-0) = 1$$

$\therefore \quad \text{adj}(\text{adj } A) = |A|^{3-2} A = A$ and $|\text{adj}(\text{adj } A)| = |A| = 1$

Also,

$|\text{adj } A| = |A|^{3-1} = |A|^2 = 1^2 = 1$

9. a, c.

We have,

$$\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \frac{1}{1+\tan^2\theta}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \cos^2\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$$

$$= \cos^2\theta \begin{bmatrix} 1-\tan^2\theta & -2\tan\theta \\ 2\tan\theta & 1-\tan^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

$\therefore \quad a = \cos 2\theta, b = \sin 2\theta$

10. a, b, c.

$|A^{-1}| = -1 \Rightarrow |A| = -1$

Now, use adj $A = |A|A^{-1}$ and $A = (A^{-1})^{-1}$

11. a, c, d.

B is an idempotent matrix

$\therefore \quad B^2 = B$

Now,

$$\begin{aligned}A^2 &= (I-B)^2\\ &= (I-B)(I-B)\\ &= I - IB - IB + B^2\\ &= I - B - B + B^2\\ &= I - 2B + B^2\\ &= I - 2B + B\\ &= I - B\\ &= A\end{aligned}$$

Therefore, A is idempotent. Again,

$AB = (I-B)B = IB - B^2 = B - B^2 = B^2 - B^2 = O$

Similarly, $BA = B(I-B) = BI - B^2 = B - B = O$.

12. b, c.

$$(-A)^{-1} = \frac{\text{adj}(-A)}{|-A|} = \frac{(-1)^{n-1}\text{adj}(A)}{(-1)^n\,|A|} = \frac{\text{adj}(A)}{-|A|} = -A^{-1}$$ (for any value of n)

Given, $A^n = O$

Now,

$(I-A)(I + A + A^2 + \cdots + A^{n-1}) = I - A^n = I$

$\Rightarrow \quad (I-A)^{-1} = I + A + A^2 + \cdots + A^{n-1}$

13. a, b.

Let $I = k = 1$ (say). Then,

$A_iA_k = A_kA_i = A_1A_1$

$$A_iA_k = A_1A_1 = \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\times\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = I$$

$$A_2A_2 = \begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix}\times\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = I$$

$\therefore \quad A_iA_k + A_kA_i = I + I = 2I$

If $i \neq k$ let $i = 3$ and $k = 2$, then

$$A_iA_k = A_1A_2 = \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\times\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix} = \begin{bmatrix}-i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&i\end{bmatrix}$$

$$A_2A_1 = \begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix}\times\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix} = \begin{bmatrix}-i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&-i\end{bmatrix}$$

$\Rightarrow \quad A_1A_2 + A_2A_1 = O$

14. a, c.

$$\sin A = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \text{ and } \cos A = \begin{bmatrix}\sin\theta & \cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$$

$\therefore \quad |\sin A| = \cos^2\theta + \sin^2\theta = 1.$

Hence $\sin A$ is invertible.

$$\text{Also, } (\sin A)\times(\sin A)^T = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\theta + \sin^2\theta & 0\\ 0 & \cos^2\theta + \sin^2\theta\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= I$$

Hence, $\sin A$ is orthogonal. Also,

$$2\sin A\cos A = 2\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\sin\theta & \cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$$

$$= 2\begin{bmatrix}2\sin\theta\cos\theta & \cos^2\theta + \sin^2\theta\\ \cos^2\theta - \sin^2\theta & 0\end{bmatrix}$$

$$= 2\begin{bmatrix}\sin 2\theta & 1\\ \cos 2\theta & 0\end{bmatrix}$$

$$\neq \sin 2A$$

15. a. c. Let,

$$A = \begin{bmatrix}0 & 2b & c\\ a & b & -c\\ a & -b & c\end{bmatrix}$$

Now,

$$A^T = \begin{bmatrix}0 & a & a\\ 2b & b & -b\\ c & -c & c\end{bmatrix}$$

Hence, A is orthogonal. Therefore,

$$AA^T = I \Rightarrow \begin{bmatrix}0 & 2b & c\\ a & b & -c\\ a & -b & c\end{bmatrix}\begin{bmatrix}0 & a & a\\ 2b & b & -b\\ c & -c & c\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Equating the corresponding elements, we get

$4b^2 + c^2 = 1$ (1)

$2b^2 - c^2 = 0$ (2)

$a^2 + b^2 + c^2 = 1$ (3)

Solving Eqs. (1), (2) and (3), we get

$$a = \pm\frac{1}{\sqrt{2}},\ b = \pm\frac{1}{\sqrt{6}},\ c = \pm\frac{1}{\sqrt{3}}$$

16. a, b, d. See theory.

17. a, b, c.

If $|A| \neq 0$, then

$AB = AC$

$\Rightarrow \quad A^{-1}AB = A^{-1}AC$

$\Rightarrow \quad B = C$

Also if A is orthogonal matrix, then, $AA^T = I$

$\Rightarrow \quad |AA^T| = 1 \Rightarrow |A|^2 = 1 \Rightarrow A$ is invertible

18. a, b, c.

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$|a| = 3\begin{vmatrix}a_1 & a_2 & a_3\\ d & d & d\\ d & d & d\end{vmatrix} = 0$$

where d is the common difference of the A.P.

Therefore, the given system of equations has infinite number of solutions. Also,

$|B| = a_1^{\,2} + a_2^{\,2} \neq 0$

19. a, b, d.

$$A=\begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ \cos\beta & \sin\beta & 0\\ \cos\gamma & \sin\gamma & 0\end{bmatrix}\begin{bmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \sin\alpha & \sin\beta & \sin\gamma\\ 0 & 0 & 0\end{bmatrix}$$

Thus, A is symmetric and $|A| = 0$, hence singular and not invertible. Also,

$$AA^T \neq I$$

20. b, c.

$$A^{-1}=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$$

$$\Rightarrow \quad (A^{-1})^2=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}$$

Similarly,

$$(A^{-1})^3=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}$$

and $(A^{-1})^n=\begin{bmatrix}1 & 0\\ -n & 1\end{bmatrix} \Rightarrow \lim\limits_{n\to\infty}\dfrac{1}{n}A^{-n}=\lim\limits_{n\to\infty}\begin{bmatrix}1/n & 0\\ -1 & 1/n\end{bmatrix}=\begin{bmatrix}0 & 0\\ -1 & 0\end{bmatrix}$

$$\lim_{n\to\infty}\frac{1}{n^2}A^{-n}=\lim_{n\to\infty}\begin{bmatrix}1/n^2 & 0\\ -1/n & 1/n^2\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

21. a, d.

Here X is a $n\times 1$ matrix, C is a $n\times n$ matrix and X^T is a $1\times n$ matrix. Hence X^TCX is a 1×1 matrix. Let $X^TCX = k$. Then,

$(X^TCX)^T = X^TC^T(X^T)^T = X^T(-C)X = -X^TCX = -k$

$\Rightarrow \quad k = -k$

$\Rightarrow \quad k = 0$

$\Rightarrow \quad X^TCX$ is null matrix

22. a, b, c.

$$S=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\Rightarrow S^{-1}=\frac{1}{2}\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$$

We have,

$$SA=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}b+c & c+a & b-c\\ c-b & c+a & a-b\\ b-c & a-c & a+b\end{bmatrix}$$

$$=\begin{bmatrix}0 & 2a & 2a\\ 2b & 0 & 2b\\ 2c & 2c & 0\end{bmatrix}$$

$$\therefore \quad SAS^{-1}=\begin{bmatrix}0 & 2a & 2a\\ 2b & 0 & 2b\\ 2c & 2c & 0\end{bmatrix}\frac{1}{2}\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}0 & a & a\\ b & 0 & b\\ c & c & 0\end{bmatrix}\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}2a & 0 & 0\\ 0 & 2b & 0\\ 0 & 0 & 2c\end{bmatrix}$$

$= \text{diag}\,(2a, 2b, 2c)$

23. a, b, c.

All are properties of diagonal matrix.

24. a, b, c.

Given that A and B commute, we have

$AB = BA \quad (\because A$ and B are symmetric) (1)

Also,

$A^T = A,\ B^T = B$ (2)

$(A^{-1}B)^T = B^T(A^{-1})^T = BA^{-1}$

($\because$ if A is symmetric, A^{-1} is also symmetric)

Also from Eq. (1),

$ABA^{-1} = B$ (3)

$\Rightarrow \quad A^{-1}ABA^{-1} = A^{-1}B$

$\Rightarrow \quad IBA^{-1} = A^{-1}B$

$\Rightarrow \quad BA^{-1} = A^{-1}B$

Hence, from Eq. (2),

$(A^{-1}B)^T = A^{-1}B$

Thus, $A^{-1}B$ is symmetric. Similarly, AB^{-1} is also symmetric. Also,

$BA = AB$

$\Rightarrow \quad (BA)^{-1} = (AB)^{-1}$

$\Rightarrow \quad A^{-1}B^{-1} = B^{-1}A^{-1}$

$\Rightarrow \quad (A^{-1}B^{-1})^T = (B^{-1}A^{-1})^T$

$= (A^{-1})^T(B^{-1})^T$

$= A^{-1}B^{-1}$

Hence, $A^{-1}B^{-1}$ is symmetric.

25. b, c.

Since A is skew-symmetric, $A^T = -A$. We have,

$A^2 + I = O$

$\Rightarrow \quad A^2 = -I$ or $AA = -I$

$\Rightarrow \quad A(-A) = I$

$\Rightarrow \quad AA^T = I$

Again, we know that

$|A| = |A^T|$ and $|kA| = k^n|A|$

where n is the order of A. Now,

$A^T = (-1)^n \times A$

$\Rightarrow \quad |A^T| = (1)^n\,|A|$

ΦΙ $\quad [1-(-1)^n]\,|A| = 0$

Hence either $|A| = 0$ or $1-(-1)^n = 0$, i.e., n is even. But

$A^2 = O - I = -I$

$\Rightarrow \quad |A|^2 = (-1)^n\,|I| = (-1)^n \neq 0$

Hence, the only possibility is that A is of even order.

26. a, b, c, d.

We have, $A^2B = A(AB) = AA = A^2$, $B^2A = B(BA) = BB = B^2$,

$ABA = A(BA) = AB = A$ and $BAB = B(AB) = BA = B$.

27. a, b, d.

$$A^2=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}=\begin{bmatrix}9 & 8 & 8\\ 8 & 9 & 8\\ 8 & 8 & 9\end{bmatrix}$$

We have,

$$A^2-4A-5I_3=\begin{bmatrix}9 & 8 & 8\\ 8 & 9 & 8\\ 8 & 8 & 9\end{bmatrix}-4\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}-5\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=O$$

$\Rightarrow \quad 5I_3 = A^2 - 4A = A(A - 4I_3)$

$$\Rightarrow \quad I_3 = \frac{1}{5}(A - 4I_3) \Rightarrow A^{-1} = \frac{1}{5}(A - 4I_3)$$

Note that $|A| = 5$. Since $|A^3| = |A|^3 = 5^3 \neq 0$, A^3 is invertible. Similarly, A^2 is invertible.

Reasoning Type

1. c. We know that $|\text{adj } A| = |A|^{n-1}$. Hence, statement 2 is false.

Now,

$$|\text{adj}(\text{adj } A)| = |\text{adj } A|^{n-1} = ||A|^{n-1}|^{n-1} = |A|^{(n-1)^2}$$

Then,

$$|\text{adj}(\text{adj}(\text{adj } A))| = |\text{adj}(\text{adj } A)|^{n-1}$$
$$= ||A|^{(n-1)^2}|^{n-1}$$
$$= |A|^{(n-1)^3}$$

Hence, statement 1 is true.

2. b. Both the statements are true as both are standard properties of diagonal matrix. But statement 2 does not explain statement 1.

3. d. Matrix $a_{ij} = \dfrac{i-j}{i+2j}$ is $A = \begin{bmatrix} 0 & -\frac{1}{5} & -\frac{2}{7} \\ \frac{1}{4} & 0 & -\frac{1}{8} \\ \frac{2}{5} & \frac{1}{7} & 0 \end{bmatrix}$, which is neither symmetric nor skew-symmetric. But this is not the reason for which A cannot be expressed as sum of symmetric and skew-symmetric matrix. In fact any matrix can be expressed as a sum of symmetric and skew-symmetric matrix. Hence, statement 1 is false but statement 2 is true.

4. a. Given,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \quad A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

Hence,

$$A^2 - (a+d)A + (ad-bc)I$$
$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} a^2+bc-(a^2+ad)+(ad-bc) & ab+bd-(ab+bd) \\ ac+cd-(ac+cd) & bc+d^2-(ad+d^2)+(ad-bc) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= O$$

Given,

$$A^3 = O$$
$$\Rightarrow \quad |A| = 0 \text{ or } ad - bc = 0$$
$$\Rightarrow \quad A^2 - (a+d)A = O \text{ or } A^2 = (a+d)A \qquad (1)$$

Case (i)

$$a + d = 0$$

From Eq. (1),

$$A^2 = O$$

Case (ii)

$$a + d \neq 0$$

Given,

$$A^3 = O$$
$$\Rightarrow \quad A^2A = O$$
$$\Rightarrow \quad (A+d)A \cdot A = O$$
$$\Rightarrow \quad A^2 = O$$

5. b. $\text{adj}(F(\alpha)) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Also,

$$|F(\alpha)| = 1$$

Then,

$$[F(\alpha)]^{-1} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos(-\alpha) & \sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= F(-\alpha)$$

Similarly, we can prove that $[G(\beta)]^{-1} = G(-\beta)$.

But again given matrices $F(\alpha)$ and $G(\beta)$ are special matrices for which this type of result holds.

In general, such result is not true. You can verify with any other matrix. Hence, both statements are true but statement 2 is not correct explanation of statement 1.

6. a. Statement 1 is true as $|A| = 0$. Since $|B| \neq 0$, statement 2 is also true and correct explanation of statement 1.

7. a. $A = -A^T \Rightarrow |A| = -|A^T| = -|A|$

$$\Rightarrow \quad 2|A| = 0$$
$$\Rightarrow \quad |A| = 0$$

8. d. Statement 1 is false.

$\because \quad A = [A_{ij}]_{n \times n}$ where $a_{ij} = 0, i \geq j$

Therefore, $|A| = 0$ and hence A is singular. So, inverse of A is not defined.

In statement 2, $|A| = 0$. Therefore, inverse of A is not defined.

9. d. A^{-1} exists only for non-singular matrix.

$\therefore \quad AB = AC \Rightarrow B = C$ if A^{-1} exists

10. d. ABC is not defined, as order of A, B and C are such that they are not conformable for multiplication.

11. c. $[A(A+B)^{-1}B]^{-1} = B^{-1}((A+B)^{-1})^{-1}A^{-1}$

$$= B^{-1}(A+B)A^{-1} = (B^{-1}A + I)A^{-1} = B^{-1}I + IA^{-1} = B^{-1} + A^{-1}$$

Hence, statement 1 is true. Statement 2 is false as $(A+B)^{-1} = A^{-1} + B^{-1}$ is not true.

12. b. Since $AB = BA$, we have

$$(A+B)^r = {}^rC_0A^r + {}^rC_1A^{r-1}B + {}^rC_2A^{r-2}B^2 + \cdots + {}^rC_sA^{r-s}B^s + \cdots + {}^rC_{r-1}AB^{r-1} + {}^rC_rB^r$$

If $r = m + n$, then

$$A^{r-p}B^p = A^mB^{r-p-m}B^p = O \text{ if } p \leq n$$

and $\quad A^{r-p}B^p = A^{r-p}B^nB^{p-n} = O$ if $p > n$

Then, $(A+B)^r = O$, for $r = m + n$

Thus, both the statements are correct but statement 2 is not correctly explaining statement 1.

13. b. Let, $A = \begin{bmatrix} d_1 & z_1 & z_2 \\ \bar{z}_1 & d_2 & z_3 \\ \bar{z}_2 & \bar{z}_3 & d_3 \end{bmatrix}$.

$A^2 = O$

$\Rightarrow \begin{bmatrix} d_1 & z_1 & z_2 \\ \bar{z}_1 & d_2 & z_3 \\ \bar{z}_2 & \bar{z}_3 & d_3 \end{bmatrix}\begin{bmatrix} d_1 & z_1 & z_2 \\ \bar{z}_1 & d_2 & z_3 \\ \bar{z}_2 & \bar{z}_3 & d_3 \end{bmatrix}$

$= \begin{bmatrix} d_1^2 + |z_1|^2 + |z_2|^2 & d_1 z_1 + d_2 z_1 + z_2\bar{z}_3 & d_1 z_2 + z_1 z_3 + z_2 d_3 \\ d_1\bar{z}_1 + d_2\bar{z}_1 + z_3\bar{z}_2 & d_2^2 + |z_1|^2 + |z_3|^2 & \bar{z}_1 z_2 + d_2 z_3 + z_3 d_3 \\ d_1\bar{z}_2 + \bar{z}_3\bar{z}_1 + d_3\bar{z}_2 & \bar{z}_1\bar{z}_2 + d_2\bar{z}_3 + d_3\bar{z}_3 & d_3^2 + |z_1|^2 + |z_2|^2 \end{bmatrix} = O$

$\Rightarrow$ Diagonal elements $d_1 = d_2 = d_3 = 0$ and $|z_1| = |z_2| = |z_3| = 0$

$\Rightarrow z_1 = z_2 = z_3 = 0$

$\Rightarrow A =$ Null matrix

Thus, statement 1 is true. Also,

$A^2 = O \Rightarrow |A|^2 = 0$ or $|A| = 0$

Thus, statement 2 is true but it does not explain statement 1.

14. a. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow a^2 + b^2 = 1$ (1)

$c^2 + d^2 = 1$ (2)

$ac + bd = 0$ (3)

$\Rightarrow \frac{a}{d} = \frac{-b}{c} = k$ (let)

$\Rightarrow c^2 + d^2 = 1/k^2$ or $k^2 = 1$ or $k = \pm 1$

$\Rightarrow \frac{a}{d} = \frac{-b}{c} = \pm 1$

Also, we must have $a, b, c, d \in [-1, 1]$ for Eqs. (1) and (2) to get defined. Hence, without loss of generality, we can assume $a = \cos\theta$ and $b = \sin\theta$.

So for $\frac{a}{d} = \frac{-b}{c} = 1$, we have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and

for $\frac{a}{d} = \frac{-b}{c} = -1$, we have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

Linked Comprehension Type

For Problems 1–3

1. b, 2. b, 3. d.

Sol.

1. Let,

$a = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow A^2 - (a+d)A + (ad-bc)I$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix}$

$+ \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$

$= O$

2. If $A = O$, tr$(A) = 0$. Suppose $A \neq O$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$|A| = 0$ and $A^2 - (A+d)A + (ad-bc)I = 0$

$\Rightarrow a + d = 0$

3. $(I+A)^{100} = {}^{100}C_0 I^{100} + {}^{100}C_1 I^{99}A + {}^{100}C_2 I^{98}A^2 + \cdots + {}^{100}C_{100}A^{100}$

$= I + 100A + O + O + \cdots + O$

$= I + 100A$

For Problems 4–6

4. c, 5. c, 6. c.

Sol.

4. $AB = A \Rightarrow |AB| = |A|$ (1)

$\Rightarrow |A| = 0$ or $|B| = 1$

$BA = B \Rightarrow |BA| = |B|$ (2)

$\Rightarrow |A| = 1$ or $|B| = 0$

If $|A| = 0$, then from Eq. (2), $|B| = 0$

If $|B| = 0$, then from Eq. (1), $|A| = 0$

5. $AB = A$, $BA = B$

$ABA = A^2 \Rightarrow A(BA) = A^2 \Rightarrow AB = A^2 \Rightarrow A = A^2$

Similarly, $B^2 = B$

$(A+B)^2 = A^2 + B^2 + AB + BA$

$= A + B + A + B = 2(A+B)$

$(A+B)^3 = (A+B)^2(A+B) = 2(A+B)^2 = 2^2(A+B)$

$\Rightarrow (A+B)^7 = 2^6(A+B) = 64\,(A+B)$

6. $(A+I)^5 = I + 5A + 10A^2 + 10A^3 + 5A^4 + A^5$

$= I + 5A + 10A + 10A + 5A + A$

$(\because A^2 = A \Rightarrow A^3 = A^4 = A^5 = \cdots = A)$

$= I + 31A$

For Problems 7–8

7. d, 8. c.

Sol.

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\Rightarrow B = \begin{bmatrix} a_{12}+a_{13} & a_{11}+a_{13} & a_{11}+a_{12} \\ a_{22}+a_{23} & a_{21}+a_{23} & a_{21}+a_{22} \\ a_{32}+a_{33} & a_{31}+a_{33} & a_{31}+a_{32} \end{bmatrix}$

$\Rightarrow X = A^{-1}B$

$= \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}\begin{bmatrix} a_{12}+a_{13} & a_{11}+a_{13} & a_{11}+a_{12} \\ a_{22}+a_{23} & a_{21}+a_{23} & a_{21}+a_{22} \\ a_{32}+a_{33} & a_{31}+a_{33} & a_{31}+a_{32} \end{bmatrix}$

$= \frac{1}{|A|}\begin{bmatrix} 0 & |A| & |A| \\ |A| & 0 & |A| \\ |A| & |A| & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$\Rightarrow |A^{-1}B| = 2$

$\Rightarrow |A^{-1}||B| = 2$

$\Rightarrow |B| = 2|A|$

For Problems 9–11

9. b, 10. d, 11. b.

Sol. $A^n - A^{n-2} = A^2 - I \Rightarrow A^{50} = A^{48} + A^2 - I$

Further,

$$\begin{aligned} A^{48} &= A^{46} + A^2 - I \\ A^{46} &= A^{44} + A^2 - I \\ &\vdots \\ A^4 &= A^2 + A^2 - I \\ \hline A^{50} &= 25A^2 - 24I \end{aligned}$$

Here,

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

$\therefore \quad |A^{50}| = 1$

Also, $\text{tr}(A^{50}) = 1 + 1 + 1 = 3$. Further,

$$\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cup_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly,

$$\cup_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \cup_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \cup \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ i.e., } |\cup| = 1$$

For Problems 12–14

12. c, 13. d, 14. b.

Sol.

12. $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$

$$\Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 4 \\ -1 & -1 & -2-\lambda \end{bmatrix}$$

$\Rightarrow$ det $(A - \lambda I) = -(\lambda - 1)(\lambda + 1)(\lambda - 3)$

Thus, the characteristic roots are –1, 1 and 3.

13. Option (a) is not correct since its characteristic determinant is

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

The characteristic equation is $\lambda^2 - 1 = 0$. Therefore, $\lambda = 1, -1$

Hence, eigenvalues are 1 and –1.

We similarly note that matrices given in options (b) and (c) have eigenvalues 1 and –1. Hence, they are not correct.

Option (d) has characteristic equation $(1 - \lambda)^2 = 0$. Hence, eigenvalues are not 1 and –1.

14. b. Let, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$\Rightarrow A - \lambda I = \begin{bmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_3-\lambda \end{bmatrix}$$

$\Rightarrow$ det $(A - \lambda I) = (a_1 - \lambda)[(b_2 - \lambda)(c_3 - \lambda) - b_3c_2] - b_1[a_2(c_3 - \lambda) - a_3c_2] + c_1[a_2 b_3 - a_3(b_2 - \lambda)]$

Now one of the eigen values is zero, so one root of equation should be zero. Therefore, constant term in the above polynomial is zero.

$\therefore \quad a_1b_2c_3 - a_1b_3c_2 - b_1a_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1a_3b_2 = 0$ (collecting constant terms)

But this value is value of determinant of A.

$\therefore \quad \det A = 0$

For Problems 15–17

15. c, 16. d, 17. c.

Sol.

15. As second row of all the options is same, we are to look at the elements of the first row. Let the left inverse be $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Then,

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad a + b + 2c = 1$

$-a + b + 3c = 1$, i.e., $b = \dfrac{1-5c}{2}$, $a = \dfrac{1+c}{2}$

Thus, matrices in the options (a), (b) and (d) are the inverses and matrix in option (c) is not the left inverse.

16. Let right inverse be $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Then,

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, $a - c + 2e = 1$

$b - d + 2f = 0$

$2a - c + e = 0$

$2b - d + f = 1$

This system of equations has infinite solutions.

17. By observation there cannot be any left inverse for options (b) and (d). So we will check for (a) and (c) only.

For option (a), let the left inverse be $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Then,

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ -3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $a - 3b = 1$, $2a + 2b = 0$ and $4a + b = 0$ which is not possible.

For option (c),

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $a + 2b + 5c = 1$, $4a - 3b + 4c = 0$, $d + 2e + 5f = 0$, $4d - 3e + 4f = 1$

Therefore, there are infinite number of left inverses.

$$\begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $a + 4d = 1$, $2a - 3d = 0$ and $5a + 4d = 0$

which is not possible. Therefore, there is no right inverse.

For Problems 18–20

18. a, 19. b, 20. c.

Sol. $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 2x^2 & 2x^2 \\ 2x^2 & 2x^2 \end{bmatrix}, A^3 = \begin{bmatrix} 2^2x^2 & 2^2x^2 \\ 2^2x^2 & 2^2x^2 \end{bmatrix}$

and so on. Then

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1+x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\cdots & x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\cdots \\ x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\cdots & 1+x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\cdots \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\cdots\right)+\frac{1}{2} & \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\cdots\right)-\frac{1}{2} \\ \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\cdots\right)-\frac{1}{2} & \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\cdots\right)+\frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{2x}+1 & e^{2x}-1 \\ e^{2x}-1 & e^{2x}+1 \end{bmatrix}$$

$\Rightarrow$ $f(x) = e^{2x} + 1$ and $g(x) = e^{2x} - 1$

18. $\displaystyle\int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \log_e(e^x + e^{-x}) + c$

19. $\displaystyle\int (g(x)+1)\sin x\,dx = \int e^{2x}\sin x\,dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + c$

20. $\displaystyle\int \frac{e^{2x}+1}{\sqrt{e^{2x}-1}}dx$

$$= \int \frac{e^{2x}}{\sqrt{e^{2x}-1}}dx + \int \frac{1}{\sqrt{e^{2x}-1}}dx$$

$$= \int \frac{e^{2x}}{\sqrt{e^{2x}-1}}dx + \int \frac{e^x}{e^x\sqrt{e^{2x}-1}}dx$$

$$= \frac{1}{2\sqrt{e^{2x}-1}} + \sec^{-1}(e^x) + c$$

Matrix-Match Type

1. a → s; b → p; c → q; d → r.

a. Since A is idempotent, hence,

$A^2 = A$

$\Rightarrow$ $A^3 = AA^2 = AA = A^2 = A$, $A^4 = AA^3 = AA = A^2 = A$

$\Rightarrow$ $A^n = A$

$\Rightarrow$ $(I - A)^n = {}^nC_0I - {}^nC_1A + {}^nC_2A^2 - {}^nC_3A^3 + \cdots$

$= I + (-{}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots)A$

$= I + [({}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots) - {}^nC_0]A = I - A$

b. A is involuntary. Hence,

$A^2 = I$

$\Rightarrow$ $A^3 = A^5 = \cdots = A$ and $A^2 = A^4 = A^6 = \cdots = I$

$\Rightarrow$ $(I - A)^n = {}^nC_0I - {}^nC_1A + {}^nC_2A^2 - {}^nC_3A^3 + \cdots$

$= {}^nC_0I - {}^nC_1A + {}^nC_2I - {}^nC_3A + {}^nC_4I - \cdots$

$= ({}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots)I - ({}^nC_1A + {}^nC_3 + {}^nC_5 + \cdots)$

$A = 2^{n-1}(I - A)$

$\Rightarrow$ $[(I - A)^n]A^{-1} = 2^{n-1}(I - a)A^{-1} = 2^{n-1}(A^{-1} - I)$

c. If A is nilpotent of index 2, then

$A^2 = A^3 = A^4 \cdots = A^n = O$

$\Rightarrow$ $(I - A)^n = {}^nC_0I - {}^nC_1A + {}^nC_2A^2 - {}^nC_3A^3 + \cdots$

$= I - nA + O + O + \cdots$

$= I - nA$

d. A is orthogonal. Hence,

$AA^T = I$

$\Rightarrow$ $(A^T)^{-1} = A$

2. a → r; b → s; c → p, r; d → p, q, r, s.

a. Since A is idempotent, $A^2 = A^3 = A^4 = \cdots = A$. Now,

$(A + I)^n = I + {}^nC_1A + {}^nC_2A^2 + \cdots + {}^nC_nA^n$

$= I + {}^nC_1A + {}^nC_2A + \cdots + {}^nC_nA$

$= I + {}^nC_1A + {}^nC_2A + \cdots + {}^nC_nA$

$= I + ({}^nC_1 + {}^nC_2 + \cdots + {}^nC_n)A$

$= I + (2^n - 1)A$

$\Rightarrow$ $2^n - 1 = 127$

$\Rightarrow$ $n = 7$

b. We have,

$(I - A)(I + A + A^2 + \cdots + A^7)$

$= I + A + A^2 + \cdots + A^7 + (-A - A^2 - A^3 - A^4 \cdots - A^8)$

$= I - A^8$

$= I$ (if $A^8 = O$)

c. Here matrix A is skew-symmetric and since $|A| = |A^T| = (-1)^n|A|$, so $|A|(1 - (-1)^n) = 0$. As n is odd, hence $|A| = 0$. Hence A is singular.

d. If A is symmetric, A^{-1} is also symmetric for matrix of any order.

3. a → q; b → p; c → s; d → r.

a. $|A| = 2 \Rightarrow |2A^{-1}| = 2^3/|A| = 4$

b. $|\text{adj}(\text{adj}(2A))| = |2A|^4 = 2^{12}|A|^4 = 2^{12}/2^{12} = 1$

c. $(A+B)^2 = A^2 + B^2$

$\Rightarrow AB + BA = O$

$\Rightarrow |AB| = |-BA| = -|BA| = -|AB|$

$\Rightarrow |AB| = 0$

$\Rightarrow |B| = 0$

d. Product ABC is not defined.

Integer Type

1.(0) $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$

$\Rightarrow A^2 = A.A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$\Rightarrow A^4 = A^2.A^2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & 3^2 \end{bmatrix}$

$\Rightarrow A^8 = \begin{bmatrix} 3^4 & 0 \\ 0 & 3^4 \end{bmatrix}$

and $A^6 = A^4.A^2 = \begin{bmatrix} 3^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^3 & 0 \\ 0 & 3^3 \end{bmatrix}$

Let $V = \begin{bmatrix} x \\ y \end{bmatrix}$

$A^8 + A^6 + A^4 + A^2 + I$

$\begin{bmatrix} 81 & 0 \\ 0 & 81 \end{bmatrix} + \begin{bmatrix} 27 & 0 \\ 0 & 27 \end{bmatrix} + \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 121 & 0 \\ 0 & 121 \end{bmatrix}$

$(A^8 + A^6 + A^4 + A^2 + I)\, V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 121 & 0 \\ 0 & 121 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 121x \\ 121y \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$

$\Rightarrow x = 0$ and $y = 1/11$

$\Rightarrow V = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1/11 \end{bmatrix}$

2.(4) $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix.

$\Rightarrow \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}^2 = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a^2+bc & ab+b-ab \\ ac+c-ac & bc+(1-a)^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a^2+bc & b \\ c & bc+(1-a)^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

$\Rightarrow a^2 + bc = a$

$a - a^2 = bc = 1/4$ (given)

$f(a) = 1/4$

3.(2) $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

$\Rightarrow A^2 = A.A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ 0-12+12 & 3+9-12 & -3-12+16 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow A^2 = I \Rightarrow A^4 = A^6 = A^8 = \cdots = I$

Now $A^x = I$

$\Rightarrow x = 2, 4, 6, 8, \ldots$

$\therefore \Sigma(\cos^x\theta + \sin^x\theta)$

$= (\cos^2\theta + \sin^2\theta) + (\cos^4\theta + \sin^4\theta) + (\cos^6\theta + \sin^6\theta) + \cdots$

$= (\cos^2\theta + \cos^4\theta + \cos^6\theta + \cdots) + (\sin^4\theta + \sin^4\theta + \sin^6\theta + \cdots)$

$= \dfrac{\cos^2\theta}{1-\cos^2\theta} + \dfrac{\sin^2\theta}{1-\sin^2\theta}$

$= \cot^2\theta + \tan^2\theta$

which has minimum value 2

4.(1) $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

Hence, det. $A = \sec^2 x$

$\therefore$ det $A^T = \sec^2 x$

Now $f(x) = \text{det.}(A^T A^{-1})$

$= (\text{det. } A^T)(\text{det. } A^{-1})$

$= (\text{det. } A^T)(\text{det. } A)^{-1}$

$= \dfrac{\text{det.}(A^T)}{\text{det.}(A)} = 1$

Hence, $f(x) = 1$.

5.(2) $\begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 3 & 4 & k \end{vmatrix} = 0$

$\Rightarrow 1(3k-16) - 2(k-12) + 2(4-9) = 0$

$\Rightarrow 3k - 16 - 2k + 24 - 10 = 0$

$\Rightarrow k = 2$

6.(9) Given $A^2 = A$

$\Rightarrow I = (I - 0.4A)(I - \alpha A)$

$= I - I\alpha A - 0.4AI + 0.4\alpha A^2$

$= I - A\alpha - 0.4A + 0.4\alpha A$

$= I - A(0.4 + \alpha) + 0.4\alpha A$

$\Rightarrow 0.4\alpha = 0.4 + \alpha$

$\Rightarrow \alpha = -2/3$

$\Rightarrow |9\alpha| = 6$

7.(7) We have $AB = \begin{bmatrix} 3ax^2 & 3bx^2 & 3cx^2 \\ a & b & c \\ 6ax & 6bx & 6cx \end{bmatrix}$

Now $\text{tr}(AB) = \text{tr}(C)$

$\Rightarrow \quad 3ax^2 + b + 6cx = (x+2)^2 + 2x + 5x^2 \quad \forall x \in R$ (Identity)

$\Rightarrow \quad 3ax^2 + 6cx + b = 6x^2 + 6x + 4$

$\Rightarrow \quad a = 2, c = 1, b = 4$

8.(8) In a skew symmetric matrix, diagonal elements are zero.

Also $a_{ij} + a_{ji} = 0$

Hence, number of matrices $= 2 \times 2 \times 2 = 8$

9.(4) Given that $AA^T = 4I$

$\Rightarrow \quad |A|^2 = 4$

$\Rightarrow \quad |A| = \pm 2,$

so $A^T = 4A^{-1} = 4\dfrac{adj\,A}{|A|}$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \frac{4}{|A|}\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

Now $a_{ij} = \dfrac{4}{|A|} c_{ij}$

$\Rightarrow \quad -2c_{ij} = \dfrac{4}{|A|} c_{ij} (\text{as } a_{ij} + 2c_{ij} = 0)$

$\Rightarrow \quad |A| = -2$

Now $|A + 4I| = |A + AA^T|$

$= |A|\,|I + A^T|$

$= -2\,|(I + A)^T|$

$= -2\,|I + A|$

$\Rightarrow \quad |A + 4I| + 2\,|A + I| = 0,$

so on comparing, we get $5\lambda = 2 \Rightarrow \lambda = \dfrac{2}{5}$

Hence, $10\lambda = 4$

10.(0) For idempotent matrix, $A^2 = A$

$\Rightarrow \quad A^{-1}A^2 = A^{-1}A \quad (\because\ A$ is non-singular)

$\Rightarrow \quad A = I$

Thus non-singular idempotent matrix is always a unit matrix.

$\therefore \quad l^2 - 3 = 1 \quad \Rightarrow \quad l = \pm 2$

$m^2 - 8 = 1 \quad \Rightarrow \quad m = \pm 3$

$n^2 - 15 = 1 \quad \Rightarrow \quad n = \pm 4$

and $p = q = r = 0$

$\Rightarrow$ required sum is 0.

11.(8) A diagonal matrix is commutative with every square matrix if it is scalar matrix so every diagonal element is 4.

$\therefore \quad |A| = 64$

12.(4) $|\text{adj } A^{-1}| = |A^{-1}|^2 = \dfrac{1}{|A|^2}$

$\Rightarrow \quad |(\text{adj } A^{-1})^{-1}| = \dfrac{1}{|adj\ A^{-1}|}$

$= |A|^2 = 2^2 = 4$

Archives

Subjective Type

1. $A^TA = I$

$$\Rightarrow \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad a^2 + b^2 + c^2 = 1 \quad (1)$

and

$ab + bc + ca = 0 \quad (2)$

Now,

$a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) + 3abc = (a + b + c) + 3 \quad (3)$

Now,

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) = 1 + 2 \times 0 = 1$

$\Rightarrow \quad a + b + c = 1$ (since a, b, c are real positive numbers)

Now from Eq. (3), $a^3 + b^3 + c^3 = 1 + 3 = 4$

Alternative solution:

$A^TA = I$

$\Rightarrow \quad |A^TA| = |I| \Rightarrow |A|^2 = 1$

$\Rightarrow \quad (a^3 + b^3 + c^3 - 3abc)^2 = 1$

$\Rightarrow \quad a^3 + b^3 + c^3 - 3abc = 1$

(since a, b, c are positive real numbers)

$\Rightarrow \quad a^3 + b^3 + c^3 \geq 3abc \quad (\because$ A.M. $\geq$ G.M.)

$\Rightarrow \quad a^3 + b^3 + c^3 = 4$

2. We are given that $MM^T = I$, where M is a square matrix of order 3 and det $M = 1$. Now,

$\det(M - I) = \det(M - MM^T) \quad [\because$ Given $MM^T = I]$

$= \det[M(I - M^T)]$

$= (\det M)[\det(I - M^T)] \quad [\because |AB| = |A|\,|B|]$

$= -(\det M)[\det(M^T - 1)]$

$= -1[\det(M^T - 1)] \quad [\because \det(M) = 1]$

$= -\det[(M - I)]^T$

$= -\det(M - I)$

$\Rightarrow \quad 2\det(M - I) = 0$

$\Rightarrow \quad \det(M - I) = 0$

3. Given that $A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$

and $AX = U$ has infinitely many solutions. Hence,

$|A| = 0$ and $|A_1| = |A_2| = |A_3| = 0$

Now, $|A| = 0 \Rightarrow \begin{vmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{vmatrix} = a(bc - bd) - 1(c - d) = 0$

$\Rightarrow \quad (ab - 1)(c - d) = 0$

$\Rightarrow \quad ab = 1$ or $c = d \quad (1)$

Also,

$$|A_1| = \begin{vmatrix} f & 1 & 0 \\ g & b & d \\ h & b & c \end{vmatrix} = 0$$

$\Rightarrow$ $f(bc - bd) - 1(gc - hd) = 0$

$\Rightarrow$ $fb(c - d) = gc - hd$ (2)

$$|A_2| = \begin{vmatrix} a & f & 0 \\ 1 & g & d \\ 1 & h & c \end{vmatrix} = 0$$

$\Rightarrow$ $a(gc - hd) - f(c - d) = 0$

$\Rightarrow$ $a(gc - hd) = f(c - d)$ (3)

$$|A_3| = \begin{vmatrix} a & 1 & f \\ 1 & b & g \\ 1 & b & h \end{vmatrix} = 0$$

$\Rightarrow$ $a(bh - bg) - 1(h - g) + f(b - b) = 0$

$\Rightarrow$ $ab(h - g) - (h - g) = 0$

$\Rightarrow$ $ab = 1$ or $h = g$ (4)

For $AX = U$ to have infinitely many solutions,

$c = d$ and $h = g$.

Now taking $BX = V$ where $B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}$, $V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$, we have

$$|B| = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix} = 0$$

(since in view of $c = d$ and $g = h$, C_2 and C_3 are identical)

$\Rightarrow$ $BX = V$ has no unique solution

Also,

$$|B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0 \quad (\because c = d, g = h)$$

$$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} = a^2cf = a^2df \quad (\because c = d)$$

$$|B_3| = \begin{vmatrix} a & 1 & a^2 \\ 0 & d & 0 \\ f & g & 0 \end{vmatrix} = -a^2df$$

Therefore, if $adf \neq 0$, then $|B_2| = |B_3| \neq 0$. Hence, no solution exists.

4. $(A + B)(A - B) = (A - B)(A + B)$

$\Rightarrow$ $AB = BA$

As A is symmetric and B is skew-symmetric,

$(AB)' = -AB$

$\Rightarrow$ k is an odd integer

Objective Type

Multiple choice questions with one correct answer

1. d. Given that $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ and $A^2 = B$. Hence,

$$\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$\Rightarrow$ $\begin{bmatrix} \alpha^2 & 0 \\ \alpha + 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

$\Rightarrow$ $\alpha^2 = 1, \alpha + 1 = 5$

$\Rightarrow$ $\alpha = \pm 1, \alpha = 4$

Hence, there is no real value.

2. c. $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$

Now,

$|A| = \alpha^2 - 4$

$\Rightarrow$ $(\alpha^2 - 4)^3 = 125 = 5^3$

$\Rightarrow$ $\alpha^2 - 4 = 5$

$\Rightarrow$ $\alpha = \pm 3$

3. c. We have,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}(A^2 + cA + dI)$$

$\Rightarrow$ $6AA^{-1} = A^3 + cA^2 + dAI$

$\Rightarrow$ $A^3 + cA^2 + dA - 6I = O$

We find that

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{bmatrix}$$

$\therefore$ $A^3 + cA^2 + dA - 6I = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{bmatrix} + c\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} + d\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 1 + c + d - 6 & 0 & 0 \\ 0 & -11 - c + d - 6 & 19 + 5c + d \\ 0 & -38 - 10c - 2d & 46 + 14c + 4d - 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $1 + c + d - 6 = 0$

$-11 - c + d - 6 = 0$

$\Rightarrow$ $c + d = 5$ and $-c + d = 17$

On solving, we get $c = -6$, $d = 11$. They also satisfy the equations

$-38 - 10c - 2d = 0$

$$46 + 14c + 4d - 6 = 0$$
$$19 + 5c + d = 0$$

4.a. Given that

$$P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$Q = PAP^T$ and
$X = P^T Q^{2005} P$

We have,

$$P^T P = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We observe that

$$Q = PAP^T$$
$$\Rightarrow \quad Q^2 = (PAP^T)(PAP^T)$$
$$= PA(P^TP)AP^T$$
$$= PA(IA)P^T$$
$$= PA^2P^T$$

Proceeding in the same way, we get

$$Q^{2005} = PA^{2005}P^T$$

Also,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

And proceeding in the same way,

$$A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

Now,

$$X = P^T Q^{2005} P$$
$$= P^T(PA^{2005}P^T)P$$
$$= (P^TP)A^{2005}(P^TP)$$
$$= IA^{2005}I$$
$$= A^{2005}$$
$$= \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

5. a. Three planes cannot intersect at two distinct points.

6. a. For being non–singular

$$\begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} \neq 0$$

$$\Rightarrow \quad ac\omega^2 - (a + c)\,\omega + 1 \neq 0$$

Hence number of possible triplets of (a, b, c) is 2.

i.e. $(\omega, \omega^2, \omega)$ and (ω, ω, ω).

Multiple choice questions with one or more correct answer

1. c. Given $MN = NM$

$$M^2N^2\,(M^TN)^{-1}\,(MN^{-1})^T$$
$$= M^2N^2\,N^{-1}\,(M^T)^{-1}\,(N^{-1})^T\,.\,M^T$$
$$= M^2N.\,(M^T)^{-1}(N^{-1})^T\,M^T$$
$$= -M^2.N(M)^{-1}\,(N^T)^{-1}\,M^T$$
$$= +\,M^2NM^{-1}N^{-1}\,M^T$$
$$= -M.NMM^{-1}\,N^{-1}\,M \qquad (\because MN = NM)$$
$$= -MNN^{-1}M = -M^2.$$

Comprehension

For Problems 1–3

1. a, 2. b, 3. a.

Sol.

1. Let $U_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ 2a + b \\ 3a + 2b + c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad a = 1,\ b = -2,\ c = 1$$

$$\therefore \quad U_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Similarly,

$$U_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}, U_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

$$\therefore \quad U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix} \Rightarrow |U| = 3$$

2. $$U^{-1} = \frac{1}{3}\begin{bmatrix} -1 & -2 & 0 \\ -7 & -5 & -3 \\ 9 & 6 & 3 \end{bmatrix}$$

Hence, sum of elements of U^{-1} is $\frac{1}{3}(0) = 0$.

3. $$\begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ -8 \\ -5 \end{bmatrix} = 5$$

For Problems 4–6

4. a, 5. b, 6. b.

Sol.

4. Let the matrix be

$$\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$$

We have five entries as 1 and remaining four entries as 0.

Since matrix is symmetric, we must have even number of zeros for $i \neq j$. We have two cases.

(i) Two entries in diagonal are zero. We can select two places from three (in diagonal) in ${}^3C_2$ ways. Now we have to select elements for upper triangle. For upper triangle, we have three places of which one entry is '0' and two are '1'. One place from three can be selected in ${}^3C_1$ ways. Hence, the number of matrices is ${}^3C_2 \times {}^3C_1 = 9$.

(ii) If all the entries in the principal diagonal are 1, we have two '0' and one '1' in upper triangle. Hence, the number of matrices is 3. Therefore, total number of matrices is 12.

5. $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 1 \end{bmatrix}$

Either $b = 0$ or $c = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ two matrices

$A = \begin{bmatrix} 0 & a & b \\ a & 1 & c \\ b & c & 0 \end{bmatrix}$

Either $a = 0$ or $c = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ two matrices

$A = \begin{bmatrix} 1 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$

Either $a = 0$ or $b = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ two matrices

$A = \begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix}$

$a = b = 0 \Rightarrow |A| = 0$

$a = c = 0 \Rightarrow |A| = 0$

$b = c = 0 \Rightarrow |A| = 0$

Therefore, there will be only six matrices.

6. The six matrices A for which $|A| = 0$ are:

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (inconsistent)

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (inconsistent)

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (infinite solutions)

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (inconsistent)

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (inconsistent)

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (inconsistent)

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (inconsistent)

For Problems 7–9

7. d, 8. c, 9. d.

Sol.

7. d We must have $a^2 - b^2 = kp$

$\Rightarrow$ $(a + b)(a - b) = kp$

$\Rightarrow$ either $a - b = 0$ or $a + b$ is multiple of p

when $a = b$; number of matrices is p

and when $a + b$ = multiple of $p \Rightarrow a, b$ has $p - 1$

$\therefore$ Total number of matrices $= p + p - 1$
$= 2p - 1.$

8. c.

9. d.

For Problems 10–12

10. d, 11. a, 12. b.

Sol.

10. $a + 8b + 7c = 0$

$9a + 2b + 3c = 0$

$a + b + c = 0$

Solving these, we get

$b = 6a \Rightarrow c = -7a$

now $2x + y + z = 0$

$\Rightarrow$ $2a + 6a + (-7a) = 1 \Rightarrow a = 1, b = 6, c = -7.$

11. $a = 2$, b and c satisfies (E)

$b = 12, c = -14$

$\dfrac{3}{\omega^a} + \dfrac{1}{\omega^b} + \dfrac{3}{\omega^c}$

$= \dfrac{3}{\omega^2} + \dfrac{1}{\omega^{12}} + \dfrac{3}{\omega^{-14}}$

$= 3\omega + 1 + 3\omega^2$

$= -2.$

12. $ax^2 + bx + c = 0$

$\Rightarrow$ $x^2 + 6x - 7 = 0$

$\Rightarrow$ $\alpha = 1, \beta = -7$

$\sum_{n=0}^{\infty}\left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)^n$

$= \sum_{n=0}^{\infty}\left(\dfrac{1}{1} - \dfrac{1}{7}\right)^n$

$= \sum_{n=0}^{\infty}\left(\dfrac{6}{7}\right)^n$

$= \dfrac{1}{1 - \dfrac{6}{7}} = 7$

Integer type

1.(4) $|A| = (2k+1)^3$, $|B| = 0$ (since B is a skew-symmetric matrix of order 3)

$\Rightarrow$ $\det(\text{adj } A) = |A|^{n-1} = ((2k+1)^3)^2 = 10^6$

$\Rightarrow$ $2k+1 = 10 \Rightarrow 2k = 9$

$\Rightarrow$ $[K] = 4.$

2.(9) let $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow b = -1, e = 2, h = 3$

$M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow a = 0, d = 3, g = 2$

$M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} = g + h + i = 12 \Rightarrow i = 7$

Therefore, sum of diagonal elements = 9.

CHAPTER 9

Probability

- Some Definitions
- Algebra of Events
- Different Types of Events
- Axiomatic Approach to Probability
- Mathematical or Classical Definition of Probability
- Addition Theorem of Probability
- Independent Events
- Compound and Conditional Probability
- Binomial Trials and Binomial Distribution
- Problems on Conditional Probability
- Bayes's Theorem
- Problems on Total Probability Theorem
- Problems on Bayes's Theorem

SOME DEFINITIONS

Experiment

An operation which results in some well-defined outcomes is called an experiment.

Random Experiment

An experiment whose outcome cannot be predicted with certainty is called a random experiment. In other words, if an experiment is performed many times under similar conditions and the outcome each time is not the same, then this experiment is called a random experiment.

An experiment whose outcome can be foretold before is not a random experiment. For example, when a stone is thrown upwards it is sure that the stone will fall downward. Therefore, throwing a stone upward is not a random experiment.

Examples:

(i) 'Tossing a fair coin' is a random experiment because if we toss a coin either a head or a tail will come up. But if we toss a coin again and again the outcome each time will not be the same.

(ii) 'Throwing an unbiased die' is a random experiment because when a die is thrown we cannot say with certainty which one of the numbers 1, 2, 3, 4, 5 and 6 will come up.

Sample Space

The set of all possible outcomes of a random experiment is called the sample space for that experiment. It is usually denoted by S.

Examples:

(i) When a coin is tossed either a head or a tail will come up. If H denotes the occurrence of head and T denotes the occurrence of tail, then sample space is $S = \{H, T\}$.

(ii) When two coins are tossed, sample Space is given by $S = \{(H, H), (H, T), (T, H), (T, T)\}$, where (H, H) denotes the occurrence of head on the first coin and occurrence of head on the second coin. Similarly, (H, T) denotes the occurrence of head on the first coin and occurrence of tail on the second coin.

(iii) When a die is thrown any one of the numbers 1, 2, 3, 4, 5 and 6 will come up. Therefore, sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Here, 1 denotes the occurrence of 1, 2 denotes the occurrence of 2 and so on.

(iv) When two balls are drawn from a bag containing three red and two black balls, the sample space is given by $S = \{(R_1, R_2), (R_1, R_3), (R_2, R_3), (B_1, B_2), (R_1, B_1), (R_1, B_2), (R_2, B_1)\ (R_2, B_2), (R_3, B_1), (R_3, B_2)\}$.

Event

Consider the experiment of tossing a coin two times. An associated sample space is $S = \{HH, HT, TH, TT\}$.

Now suppose that we are interested in those outcomes which correspond to the occurrence of exactly one head. We find that HT and TH are the only elements of S corresponding to the occurrence of this happening (event). These two elements form the set $E = \{HT, TH\}$.

We know that the set E is a subset of the sample space S. Similarly, we find the following correspondence between events and subsets of S.

Description of events	Corresponding subset of 'S'
Number of tails is exactly 2	$\{TT\}$
Number of tails is at least 1	$\{HT, TH, TT\}$
Number of heads is at most 1	$\{HT, TH, TT\}$
Second toss is not head	$\{HT, TT\}$
Number of tails is at most 2	$\{HH, HT, TH, TT\}$
Number of tails is more than 2	φ

When a die is thrown, sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{1, 3, 5\}$, where A is the event of occurrence of an odd number; $B = \{5, 6\}$, where B is the event of the occurrence of a number greater than 4; $C = \{2, 3, 5\}$, where C is the event of occurrence of a prime number.

Note:

- *Sample space S plays the same role as the universal set for all problems related to the particular experiment.*
- *ϕ is also a subset of S which is called an impossible event.*
- *S is also a subset of S which is called a sure event or a certain event.*

The above discussion suggests that a subset of sample space is associated with an event and an event is associated with a subset of sample space. In the light of this we define an event as follows.

Simple Event or Elementary Event

If an event E has only one sample point of a sample space, it is called a *simple* (or *elementary*) *event*.

Examples:

(i) When a coin is tossed, sample space is $S = \{H, T\}$. Let $A = \{H\}$ be the event of occurrence of head and $B = \{T\}$ be the event of occurrence of tail. Here, A and B are simple events.

(ii) When a die is thrown, sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{5\}$ be the event of occurrence of 5 and $B = \{2\}$ be the event of occurrence of 2. Here, A and B are simple events.

Mixed Event or Compound Event or Composite Event

A subset of the sample space S which contains more than one element is called a mixed event. For example, in the experiment of 'tossing a coin thrice,' the events

E: 'Exactly one head appeared'

F: 'At least one head appeared'

G: 'At most one head appeared', etc.

are all compound events.

The subsets of S associated with these events are

$$E = \{HTT, THT, TTH\}$$

$$F = \{HTT, THT, TTH, HHT, HTH, THH, HHH\}$$

$$G = \{TTT, THT, HTT, TTH\}$$

Each of the above subsets contains more than one sample point, hence they are all compound events.

Trial

When an experiment is repeated under similar conditions and it does not give the same result each time but may result in any one of the several possible outcomes, the experiment is called a trial and the outcomes are called cases. The number of times the experiment is repeated is called the number of trials. For example, one toss of a coin is a trial when the coin is tossed 5 times and one throw of a die is a trial when the die is thrown 4 times.

ALGEBRA OF EVENTS

Complementary Event

For every event A, there corresponds another event A' called the complementary event to A. It is also called the *event 'not A'*. For example, take the experiment 'of tossing three coins'. An associated sample space is $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$. Let $A = \{HTH, HHT, THH\}$ be the event 'only one tail appears'.

Clearly for the outcome HTT, the event A has not occurred. But we may say that the event 'not A' has occurred. Thus, with every outcome which is not in A, we say that 'not A' occurs.

Thus, the complementary event 'not A' to the event A is

$$A' = \{HHH, HTT, THT, TTH, TTT\}$$

or

$$A' = \{\omega : \omega \in S \text{ and } \omega \notin A\} = S - A$$

The Event 'A or B'

Recall that union of two sets A and B denoted by $A \cup B$ contains all those elements which are either in A or in B or in both. When the sets A and B are two events associated with a sample space, then '$A \cup B$' is the event 'either A or B or both'. This event '$A \cup B$' is also called 'A or B'.

The Event 'A and B'

We know that intersection of two sets $A \cap B$ is the set of those elements which are common to both A and B, i.e., which belong to both 'A and B'. If A and B are two events, then the set $A \cap B$ denotes the event 'A and B'.

The Event 'A but not B'

We know that $A - B$ is the set of all those elements which are in A but not in B. Therefore, the set $A - B$ may denote the event 'A but not B'. We know that $A - B = A - A \cap B$.

DIFFERENT TYPES OF EVENTS

Equally Likely Events

Cases (outcomes) are said to be equally likely when we have no reason to believe that one is more likely to occur than the other. Thus, when an unbiased die is thrown all the six faces 1, 2, 3, 4, 5 and 6 are equally likely to come up. Similarly, when an unbiased coin is tossed occurrence of head and tail are equally likely cases.

Exhaustive Cases (Events)

Consider the experiment of throwing a die. We have $S = \{1, 2, 3, 4, 5, 6\}$. Let us define the following events:

A: 'a number less than 4 appears'

B: 'a number greater than 2 but less than 5 appears',

C: 'a number greater than 4 appears'

Then $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{5, 6\}$. We observe that

$$A \cup B \cup C = \{1, 2, 3\} \cup \{3, 4\} \cup \{5, 6\} = S.$$

Such events A, B and C are called exhaustive events. In general, if $E_1, E_2, ..., E_n$ are n events of a sample space S and if $S = E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i$, then $E_1, E_2, ..., E_n$ are called *exhaustive events*. In other words, events $E_1, E_2, ..., E_n$ are said to be exhaustive if at least one of them necessarily occurs whenever the experiment is performed.

Mutually Exclusive Events

In the experiment of rolling a die, the sample space is given by $S = \{1, 2, 3, 4, 5, 6\}$.

Consider events, A 'an odd number appears' and B 'an even number appears'.

Clearly the event A excludes the event B and vice versa.

In other words, there is no outcome which ensures the occurrence of events A and B simultaneously.

Here, $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Clearly $A \cap B = \varphi$, i.e., A and B are disjoint sets.

In general, two events A and B are called *mutually exclusive* events if the occurrence of any one of them excludes the occurrence of the other event, i.e., if they can not occur simultaneously.

In this case, the sets A and B are disjoint.

Again in the experiment of rolling a die, consider the events A 'an odd number appears' and event B 'a number less than 4 appears'.

Obviously, $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$. Therefore, A and B are not mutually exclusive events.

Example 9.1 **A coin is tossed three times, consider the following events.**

A: 'no head appears'

B: 'exactly one head appears'

C: 'at least two heads appear'

Do they form a set of mutually exclusive and exhaustive events?

Sol. The sample space of the experiment is

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Events A, B and C are given by

$A = \{TTT\}$

$B = \{HTT, THT, TTH\}$

$C = \{HHT, HTH, THH, HHH\}$

Now,

$$A \cup B \cup C = \{TTT, HTT, THT, TTH, HHT, HTH, THH, HHH\} = S$$

Therefore, A, B and C are exhaustive events. Also, $A \cap B = \phi$, $A \cap C = \phi$ and $B \cap C = \phi$. Therefore, the events are pair-wise disjoint, i.e., they are mutually exclusive. Hence, A, B and C form a set of mutually exclusive and exhaustive events.

AXIOMATIC APPROACH TO PROBABILITY

Axiomatic approach is another way of describing probability of an event. In this approach, some axioms or rules are depicted to assign probabilities.

Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval $[0,1]$ satisfying the following axioms:

(i) For any event E, $P(E) \geq 0$

(ii) $P(S) = 1$

(iii) If E and F are mutually exclusive events, then

$$P(E \cup F) = P(E) + P(F)$$

It follows from (iii) that $P(\phi) = 0$.

Let S be a sample space containing outcomes $\omega_1, \omega_2, \ldots, \omega_n$, i.e., $S = \{\omega_1, \omega_2, \ldots, \omega_n\}$.

It follows from the axiomatic definition of probability that

(i) $0 \leq P(\omega_i) \leq 1$ for each $\omega_i \in S$

(ii) $P(\omega_1) + P(\omega_2) + \cdots + P(\omega_n) = 1$

(iii) For any event A, $P(A) = \Sigma P(\omega_i)$, $\omega_i \in A$.

MATHEMATICAL OR CLASSICAL DEFINITION OF PROBABILITY

Let S be the sample space. Then the probability of occurrence of an event E is denoted by $P(E)$ and is defined as

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of elements in } E}{\text{number of elements in } S}$$

$$= \frac{\text{number of cases favourable to event } E}{\text{total number of cases}}$$

Examples:

1. When a die is rolled, sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

Let A be the event of occurrence of an odd number, i.e., $\{1, 3, 5\}$ and B be the event of occurrence of a number greater than 4, i.e., $\{5, 6\}$. Then,

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

2. When one ball is drawn at random from a bag containing 3 black balls and 4 red balls (balls of the same colour being identical or different), then sample space is given by

$$S = \{B_1, B_2, B_3, R_1, R_2, R_3, R_4\}$$

$\therefore$ $n(S) = 7$

Here, the three black balls may be denoted by B_1, B_2 and B_3 even if they are identical because while finding probability only number of black and red balls are to be taken into account. Let E be the event of occurrence of a red ball. Then, $E = \{R_1, R_2, R_3, R_4\}$

$\therefore n(E) = 4$

Now,

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{7}$$

3. When two coins are tossed, sample space is given by

$S = \{HH, HT, TH, TT\}$. Let E be the event of occurrence of one head and one tail. Then $E = \{HT, TH\}$. Now,

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

Value of Probability of Occurrence of an Event

Let S be the sample space and E be an event. Then

$\phi \subseteq E \subseteq S$

$\therefore$ $n(\phi) \leq n(E) \leq n(S)$

$\Rightarrow$ $0 \leq n(E) \leq n(S)$

$$\Rightarrow \quad \frac{0}{n(S)} \le \frac{n(E)}{n(S)} \le \frac{n(S)}{n(S)} \quad \text{[Dividing by } n(S)\text{]}$$

$$\Rightarrow \quad 0 \le P(E) \le 1 \qquad \left[\because P(E) = \frac{n(E)}{n(S)}\right]$$

Thus, if ϕ is the impossible event, then

$$P(\phi) = \frac{n(\phi)}{n(S)} = \frac{0}{n(S)} = 0$$

and if S is the sure event, then

$$P(S) = \frac{n(S)}{n(S)} = 1$$

If E be Any Event and E' be the Complement of Event E, then $P(E) + P(E') = 1$.

Proof:

Let S be the sample space. Then

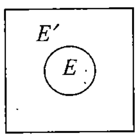


Fig. 9.1

$$n(E) + n(E') = n(S)$$

$$\therefore \quad \frac{n(E)}{n(S)} + \frac{n(E')}{n(S)} = 1 \text{ or } P(E) + P(E') = 1$$

Odds in Favour and Odds Against an Event

Let S be the sample space and E be an event. Let E' denote the complement of event E. Then

(i) odds in favour of event E is

$$\frac{n(E)}{n(E')} = \frac{\text{number of cases favourable to event } E}{\text{number of cases against event } E}$$

(ii) odds against an event E is

$$\frac{n(E')}{n(E)} = \frac{\text{number of cases against the event } E}{\text{number of cases favourable to event } E}$$

Note:

- *Odds in favour of event E is given by*

$$\frac{n(E)}{n(E')} = \frac{n(E)/n(S)}{n(E')/n(S)} = \frac{P(E)}{P(E')}$$

and odds against event E is

$$\frac{n(E')}{n(E)} = \frac{P(E')}{P(E)}$$

- *If any one of $P(E)$, odds in favour of E, and odds against E is given, then other two can be determined.*

Examples:

(i) If $P(E) = 2/7$, then odds in favour of E is 2/5 and odds against E is 5/2.

(ii) If odds against E is 3/11, then odds in favour of E is 11/3 and $P(E)$ is 11/4.

(iii) If odds in favour of E is 3/8, then odds against E is 8/3 and $P(E) = 3/11$.

Example 9.2 **Find the probability of getting more than 7 when two dice are rolled.**

Sol.

Sum of two dice	Result on two dice	Number of cases
8	(4,4), (3,5), (5,3), (2,6), (6,2)	5
9	(4,5), (5,4), (3,6), (6,3)	4
10	(5,5), (4,6), (6,4)	3
11	(5,6), (6,5)	2
12	(6,6)	1

Required probability is

$$P(8) + P(9) + P(10) + P(11) + P(12)$$

$$= \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{15}{36} = \frac{5}{12}$$

Example 9.3 **A card is drawn at random from a pack of cards. What is the probability that the drawn card is neither a heart nor a king.**

Sol. There are 13 heart cards and 4 king cards of which one is heart king. Therefore, the required probability is

$$1 - P(\text{drawn card is either a heart or a king})$$

$$1 - \frac{16}{52} = \frac{36}{52} = \frac{9}{13}$$

Example 9.4 **A determinant is chosen at random from the set of all determinant of order 2 with elements 0 or 1 only. Find the probability that the determinant chosen is non-zero.**

Sol. A determinant of order 2 is of the form

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

It is equal to $ad - bc$.

The total number of ways of choosing a, b, c and d is

$2 \times 2 \times 2 \times 2 = 16$.

Now, $\Delta \neq 0$ if and only if either $ad = 1$, $bc = 0$ or $ad = 0$, $bc = 1$. But $ad = 1$, $bc = 0$ if $a = d = 1$ and one of b, c is zero. Therefore, $ad = 1$, $bc = 0$ in three cases. Similarly, $ad = 0$, $bc = 1$ in three cases.

Therefore, the required probability is $6/16 = 3/8$.

Example 9.5 **A dice is rolled three times, find the probability of getting a larger number than the previous number each time.**

Sol. Exhaustive number of cases is $6^3 = 216$. Now if a larger number appears than the previous number each time, all the three numbers are distinct. Now three numbers can be selected from six numbers in ${}^6C_3$ ways and there is only one way in which three selected numbers can appear. Hence, probability is $20/216 = 5/54$.

Example 9.6 **An integer is chosen at random and squared. Find the probability that the last digit of the square is 1 or 5.**

Sol. The last digit of square will be 1 or 5 only when the integer is 1, 5 or 9. Therefore, required probability is 3/10.

Example 9.7 **If a coin be tossed n times then find the probability that the head comes odd times.**

Sol. Total number of cases is 2^n.

If head comes odd times (1, 3, 5, …, n times), then favourable number of ways is ${}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots) = 2^{n-1}$. Therefore, the required probability is

$$\frac{2^{n-1}}{2^n} = \frac{1}{2}$$

Example 9.8 **Find the probability that a leap year will have 53 Fridays or 53 Saturdays.**

Sol. There are 366 days in a leap year, in which there are 52 weeks and two days. The combination of 2 days can be

Sunday–Monday

Monday–Tuesday

Tuesday–Wednesday

Wednesday–Thursday

Thursday–Friday

Friday–Saturday

Saturday–Sunday

$$P(\text{53 Fridays}) = \frac{2}{7}; P(\text{53 Saturdays}) = \frac{2}{7}$$

$$P(\text{53 Fridays and 53 Saturdays}) = \frac{1}{7}$$

$\therefore$ P(53 Fridays or Saturdays)

$= P(\text{53 Fridays}) + P(\text{53 Saturdays})$

$-P(\text{53 Fridays and Saturdays})$

$$= \frac{2}{7} + \frac{2}{7} - \frac{1}{7} = \frac{3}{7}$$

Example 9.9 **A mapping is selected at random from the set of all the mappings of the set $A = \{1, 2, \ldots, n\}$ into itself. Find the probability that the mapping selected is an injection.**

Sol. Total number of functions from A to itself is n^n. Out of these functions, $n!$ functions are injective mappings (one–one and onto). So, required probability is

$$\frac{n!}{n^n} = \frac{(n-1)!}{n^{n-1}}$$

Example 9.10 **Two integers x and y are chosen with replacement out of the set $\{0, 1, 2, 3, \ldots, 10\}$. Then find the probability that $|x - y| > 5$.**

Sol. Since x and y each can take values from 0 to 10, so the total number of ways of selecting x and y is $11 \times 11 = 121$. Now,

$$|x - y| > 5 \Rightarrow x - y < -5 \text{ or } x - y > 5$$

When

$x - y > 5$, we have following cases:

Value of x	Value of y	Number of cases
6	0	1
7	0, 1	2
8	0, 1, 2	3
9	0, 1, 2, 3	4
10	0, 1, 2, 3, 4	5
	Total number of cases	15

Similarly, we have 15 cases for $x - y < -5$. There are 30 pairs of values of x and y satisfying these two inequalities. So, favourable number of ways is 30. Hence, required probability is 30/121.

Example 9.11 **Find the probability that the birthdays of six different persons will fall in exactly two calendar months.**

Sol. Since anyone's birthday can fall in one of 12 months, so total number of ways is 12^6.

Now, any two months can be chosen in ${}^{12}C_2$ ways. The six birthdays can fall in these two months in 2^6 ways. Out of these 2^6 ways there are two ways when all the six birthdays fall in one month. So, favourable number of ways is ${}^{12}C_2 \times (2^6 - 2)$. Hence, required probability is

$$\frac{{}^{12}C_2 \times (2^6 - 2)}{12^6} = \frac{12 \times 11 \times (2^5 - 1)}{12^6} = \frac{341}{12^5}$$

Example 9.12 **If 10 objects are distributed at random among 10 persons, then find the probability that at least one of them will not get anything.**

Sol. Since each object can be given to any one of 10 persons. So, 10 objects can be distributed among 10 persons in 10^{10} ways. Thus, the total number of ways is 10^{10}.

The number of ways of distribution in which each one gets only one thing is 10!.

So, the number of ways of distribution in which at least one of them does not get anything is $10^{10} - 10!$.

Hence, required probability is $10^{10} - 10!/10^{10}$.

Example 9.13 **Twelve balls are distributed among three boxes. Find the probability that the first box will contain three balls.**

Sol. Since each ball can be put into any one of the three boxes. So, the total number of ways in which 12 balls can be put into three boxes is 3^{12}. Out of 12 balls, 3 balls can be chosen in ${}^{12}C_3$ ways for first box. Now, remaining 9 balls can be put in the remaining 2 boxes in 2^9 ways. So, the total number of ways in which 3 balls are put in the first box and the remaining balls in other two boxes is ${}^{12}C_3 \times 2^9$. Hence, required probability is ${}^{12}C_3 \times 2^9/3^{12}$.

Example 9.14 **Two numbers a and b are chosen at random from the set of first 30 natural numbers. Find the probability that $a^2 - b^2$ is divisible by 3.**

Sol. The total number of ways of choosing two numbers out of 1, 2, 3, …, 30 is ${}^{30}C_2 = 435$. So, exhaustive number of cases is 435.

Since $a^2 - b^2$ is divisible by 3 if either a or b or both are divisible by 3 or none of a and b is divisible by 3. Thus, favourable number of cases is ${}^{10}C_2 + {}^{20}C_2 = 235$. Hence, the required probability is $235/435 = 47/87$.

Example 9.15 **Find the probability that the 3 N's come consecutively in the arrangement of the letters of the word 'CONSTANTINOPLE'.**

Sol. The total number of arrangement is of the letters of the word 'CONSTANTINOPLE' is (14)!/3! 2! 2!

Since 3 N's are consecutive, then considering all the 3 N's as single letter, the total number of arrangements is (12)!/2! 2!

Therefore, required probability is

$$\frac{(12)!/(2!\,2!)}{(14)!/(3!\,2!\,2!)} = \frac{3!}{14 \times 13} = \frac{3}{91}$$

Example 9.16 **Out of $3n$ consecutive integers, three are selected at random. Find the probability that their sum is divisible by 3.**

Sol. Let the $3n$ consecutive integers be $x, x + 1, x + 2, \ldots, x + 3n - 1$, where x is the starting integer. These $3n$ integers can be classified as

$$x, x + 3, x + 6, \ldots, x + 3n - 3$$
$$x + 1, x + 4, x + 7, \ldots, x + 3n - 2$$
$$x + 2, x + 5, x + 8, \ldots, x + 3n - 1$$

Each of these three rows contains 'n' numbers. If we take three numbers out of $3n$ numbers, obviously their sum shall be divisible by 3 only if either all the three numbers are from the same row or all the three numbers are from different rows.

The number of ways that the three numbers are from the same row is ${}^3C_1\,{}^nC_3 = 3\,{}^nC_3$ and the number of ways the three numbers are from different rows is $n \times n \times n = n^3$. Hence, favourable number of ways that the sum of the three numbers is divisible by 3 is $3\,{}^nC_3 + n^3$. Also the total number of ways of selecting three numbers out of $3n$ numbers is ${}^{3n}C_3$. Therefore, the required probability is

$$\frac{3 \times {}^nC_3 + n^3}{{}^{3n}C_3} = \frac{3n^2 - 3n + 2}{(3n - 1)(3n - 2)}$$

Example 9.17 **A die is loaded so that the probability of a face i is proportional to i, $i = 1, 2, \ldots, 6$. Then find the probability of an even number occurring when the die is rolled.**

Sol. Since the probabilities of the faces are proportional to the numbers on them, we can take the probabilities of faces 1, 2,…, 6 as $k, 2k, \ldots, 6k$, respectively. Since one of the faces must occur, we have

$$k + 2k + 3k + 4k + 5k + 6k = 1 \text{ or } k = 1/21$$

Therefore, the probability of occurence of an even number is $2k + 4k + 6k = 12k = 12/21 = 4/7$.

Example 9.18 **A card is drawn from a pack of 52 cards. A gambler bets that it is a spade or an ace. What are the odds against his winning this bet?**

Sol. Probability of the card being a spade or an ace is 16/52 = 4/13. Hence, odds in favour are 4:9. So odds against his winning are 9:4.

Concept Application Exercise 9.1

1. There are n letters and n addressed envelopes. Find the probability that all the letters are not kept in the right envelope.
2. Find the probability of getting a total of 5 or 6 in a single throw of two dice.
3. Two integers are chosen at random and multiplied. Find the probability that the product is an even integer.
4. If out of 20 consecutive whole numbers two are chosen at random, then find the probability that their sum is odd.
5. A bag contains 3 red, 7 white and 4 black balls. If three balls are drawn from the bag, then find the probability that all of them are of the same colour.
6. An ordinary cube has four blank faces, one face marked 2 and one face marked 3. Then find the probability of obtaining a total of exactly 12 in 5 throws.
7. If the letters of the word 'REGULATIONS' be arranged at random, find the probability that there will be exactly, four letters between the R and the E.
8. A five digit number is formed by the digits 1, 2, 3, 4, 5 without repetition. Find the probability that the number formed is divisible by 4.
9. Five persons entered the lift cabin on the ground floor of an 8-floor house. Suppose that each of them, independently and with equal probability, can leave the cabin at any floor beginning with the first. Find out the probability of all five persons leaving at different floors.
10. Two friends A and B have equal number of daughters. There are three cinema tickets which are to be distributed among the daughters of A and B. The probability that all the tickets go to the daughters of A is 1/20. Find the number of daughters each of them have.
11. A bag contains 12 pairs of socks. Four socks are picked up at random. Find the probability that there is at least one pair.
12. There are eight girls among whom two are sisters, all of them are to sit on a round table. Find the probability that the two sisters do not sit together.
13. A bag contains 50 tickets numbered 1, 2, 3, ..., 50 of which five are drawn at random and arranged in ascending order of magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$). Find the probability that $x_3 = 30$.
14. A pack of 52 cards is divided at random into two equal parts. Find the probability that both parts will have an equal number of black and red cards.

ADDITION THEOREM OF PROBABILITY

If A and B be any two events in a sample space S, then the probability of occurrence of at least one of the events A and B is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof:

From set theory, we know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Dividing both sides by $n(S)$, we get

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

or $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note:

- *If A and B are mutually exclusive events, then* $A \cap B = \phi$ *and hence* $P(A \cap B) = 0$.
  $\therefore P(A \cup B) = P(A) + P(B)$
- *Two events A and B are mutually exclusive if and only if* $P(A \cup B) = P(A) + P(B)$.
- $1 = P(S) = P(A \cup A') = P(A) + P(A')$ $\quad [\because A \cap A' = \phi]$
  or $P(A') = 1 - P(A)$
  If A, B and C are any three events in a sample space S, then
  $P(A \cup B \cup C) = P(A) + P(B) + P(C)$
  $-P(A \cap B) - P(B \cap C) - P(A \cap C)$
  $+ P(A \cap B \cap C)$
- *If A, B, C are mutually exclusive events, then*
  $A \cap B = \phi$, $B \cap C = \phi$, $A \cap C = \phi$, $A \cap B \cap C = \phi$
  $\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C)$
- *If A and B are any two events, then*
  $(A - B) \cap (A \cap B) = \phi$
  and $A = (A - B) \cap (A \cap B)$
  $\therefore P(A) = P(A - B) + P(A \cap B)$
  $= P(A \cap B') + P(A \cap B)$
  or $P(A) - P(A \cap B) = P(A - B) = P(A \cap B)$
  $[\because A - B = A \cap B']$
  Similarly, $P(B) - P(A \cap B) = P(B - A)$
  $= P(B \cap A')$

General Form of Addition Theorem of Probability

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

Note: *For any number of (finite or infinite) mutually exclusive events* $E_1, E_2, E_3, \ldots, E_n$,

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = P(E_1) + P(E_2) + \cdots + P(E_n)$$

Example 9.19 **The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then find $P(\bar{A})+P(\bar{B})$.**

Sol. It is given that $P(A \cup B)=0.6$ and $P(A \cap B)=0.2$. Therefore,

$$P(A \cup B)=P(A)+P(B)-P(A \cap B)$$
$$\Rightarrow \quad 0.6=P(A)+P(B)-0.2$$
$$\Rightarrow \quad P(A)+P(B)=0.8$$
$$\Rightarrow \quad 1-P(\bar{A})+1-P(\bar{B})=0.8$$
$$\Rightarrow \quad P(\bar{A})+P(\bar{B})=1.2$$

Example 9.20 **The probabilities of three events A, B and C are $P(A)=0.6$, $P(B)=0.4$ and $P(C)=0.5$. If $P(A \cup B)=0.8$, $P(A \cap C)=0.3$, $P(A \cap B \cap C)=0.2$ and $P(A \cup B \cup C) \geq 0.85$, then find the range of $P(B \cap C)$.**

Sol. We have,

$$P(A \cap B)=P(A)+P(B)-P(A \cup B)=0.6+0.4-0.8=0.2$$

Also,

$$P(A \cup B \cup C)=P(A)+P(B)+P(C)-P(A \cap C)+P(A \cap B \cap C)-P(A \cap B)-P(B \cap C)$$
$$\Rightarrow \quad P(B \cap C)=1.2-P(A \cup B \cup C) \quad (1)$$

Now,

$$0.85 \leq P(A \cup B \cup C) \leq 1$$
$$\therefore \quad \text{Eq. (1)} \Rightarrow 0.2 \leq P(B \cap C) \leq 0.35$$

Example 9.21 **Given two events A and B. If odds against A are as 2:1 and those in favour of $A \cup B$ are as 3:1, then find the range of $P(B)$.**

Sol. Clearly $P(A)=1/3$, $P(A \cup B)=3/4$. Now,

$$P(B) \leq P(A \cup B)$$
$$\Rightarrow \quad P(B) \leq 3/4 \quad (1)$$

Also,

$$P(B)=P(A \cup B)-P(A)+P(A \cap B)$$
$$\geq 3/4-1/3$$
$$=\frac{5}{12} \quad [P(A \cap B) \geq 0]$$

Example 9.22 **If A and B are two events, then which of the following does not represent the probability of at most one of A, B occurs**

a. $1-P(A \cap B)$
b. $P(\bar{A})+P(\bar{B})-P(\bar{A}+\bar{B})$
c. $P(\bar{A})+P(\bar{B})+P(A \cap B)-1$
d. $P(A \cap \bar{B})+P(\bar{A} \cap B)-P(\bar{A} \cap \bar{B})$

Sol. Required probability is

$$P(\bar{A} \cup \bar{B})=P(\overline{A \cap B})=1-P(A \cap B)$$

So, option (a) is correct. Again,

$$P(\bar{A} \cup \bar{B})=P\bar{A}+P\bar{B}-P(\bar{A} \cap \bar{B}) \quad \text{[by addition theorem]}$$

So, option (b) is correct. Again,

$$P(\bar{A} \cup \bar{B})=P\bar{A}+P\bar{B}-P(\bar{A} \cap \bar{B})$$
$$=P(\bar{A})+P(\bar{B})-P(\bar{A} \cap \bar{B})$$
$$=P(\bar{A})+P(\bar{B})-\{1-P(A \cup B)\}$$
$$=P(\bar{A})+P(\bar{B})+P(A \cup B)-1$$

So, option (c) is correct. Finally,

$$P(\bar{A} \cup \bar{B})=P[(A \cap \bar{B}) \cup (\bar{A} \cap B)+(\bar{A} \cap \bar{B})]$$
$$=P(A \cap \bar{B})+P(\bar{A} \cap B)+P(\bar{A} \cap \bar{B})$$

[$\because$ $A \cap \bar{B}$, $\bar{A} \cap B$ and $\bar{A} \cap \bar{B}$ are mutually exclusive events]

Example 9.23 **A box contains 6 nails and 10 nuts. Half of the nails and half of the nuts are rusted. If one item is chosen at random, then find the probability that it is rusted or is a nail.**

Sol. The total number of nails and nuts is $6+10=16$.

$$P(R)=\frac{1}{2} \quad (R \text{ stands for rusted})$$
$$P(N)=\frac{6}{16} \quad (N \text{ stands for nails})$$
$$P(R \cap N)=\frac{3}{16} \quad [\because 3 \text{ nails are rusted out of 6 nails}]$$
$$P(R \cup N)=P(R)+P(N)-P(R \cap N)$$
$$=\frac{1}{2}+\frac{6}{16}-\frac{3}{16}$$
$$=\frac{8+6-3}{16}=\frac{11}{16}$$

Example 9.24 **If $P(A \cup B)=3/4$ and $P(\bar{A})=2/3$, then find the value of $P(\bar{A} \cap B)$.**

Sol. Since $\bar{A} \cap B$ and A are mutually exclusive events such that

$$A \cup B=(\bar{A} \cap B) \cup A$$
$$\Rightarrow \quad P(A \cup B)=P(\bar{A} \cap B)+P(A)$$
$$\Rightarrow \quad \frac{3}{4}=P(\bar{A} \cap B)+1-\frac{2}{3}$$
$$\Rightarrow \quad P(\bar{A} \cap B)=\frac{5}{12}$$

Example 9.25 **Let A, B, C be three events. If the probability of occurring exactly one event out of A and B is $1-a$, out of B and C is $1-2a$, out of C and A is $1-a$ and that of occurring three events simultaneously is a^2, then prove that probability that at least one out of A, B, C will occur greater than 1/2.**

Sol. P(exactly one event out of A and B occurs)

$$=P[(A \cap B') \cup (A' \cap B)]$$
$$=P(A \cup B)-P(A \cap B)$$
$$=P(A)+P(B)-2P(A \cap B)$$
$$\therefore \quad P(A)+P(B)-2P(A \cap B)=1-a \quad (1)$$

Similarly,

$$P(B)+P(C)-2P(B \cap C)=1-2a \quad (2)$$
$$P(C)+P(A)-2P(C \cap A)=1-a \quad (3)$$
$$P(A \cap B \cap C)=a^2 \quad (4)$$

Now,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{1}{2}[P(A) + P(B) - 2P(B \cap C) + P(B) + P(C) - 2P(B \cap C) + P(C) + P(A) - 2P(C \cap A)] + P(A \cap B \cap C)$$

$$= \frac{1}{2}[1 - a + 1 - 2a + 1 - a] + a^2$$

[Using Eqs. (1), (2), (3) and (4)]

$$= \frac{3}{2} - 2a + a^2$$

$$= \frac{1}{2} + (a-1)^2 > \frac{1}{2}$$

Concept Application Exercise 9.2

1. Three students A and B and C are in a swimming race. A and B have the same probability of winning and each is twice as likely to win as C. Find the probability that B or C wins. Assume no two reach the winning point simultaneously.
2. If A and B are events such that $P(A \cup B) = 3/4$, $P(A \cap B) = 1/4$ and $P(A^c) = 2/3$, then find
   (a) $P(A)$
   (b) $P(B)$
   (c) $P(A \cap B^c)$
   (d) $P(A^c \cap B)$
3. If $P(A \cap B) = \frac{1}{2}, P(\bar{A} \cap \bar{B}) = \frac{1}{3}$, $P(A) = p$, $P(B) = 2p$, then find the value of p.
4. The probabilities of three mutually exclusive events are 2/3, 1/4 and 1/6. Is this statement correct?
5. The probability that at least one of A and B occurs is 0.6. If A and B occur simultaneously with probability 0.3, then find the value of $P(A') + P(B')$.
6. In a class of 125 students 70 passed in Mathematics, 55 in Statistics and 30 in both. Then find the probability that a student selected at random from the class has passed in only one subject.
7. In a certain population, 10% of the people are rich, 5% are famous and 3% are rich and famous. Then find the probability that a person picked at random from the population is either famous or rich but not both.

INDEPENDENT EVENTS

Two or more events are said to be independent if occurrence or non-occurrence of any of them does not affect the probability of occurrence or non-occurrence of other events.

In other words, two or more events are said to be independent if occurrence or non-occurrence of any of them does not influence the occurrence or non-occurrence of other events.

Examples:

(i) When a coin is tossed twice, the event of occurrence of head in the first throw and the event of occurrence of head in the second throw are independent events.

(ii) When two cards are drawn out of a full pack of 52 playing cards with replacement (the first card drawn is put back in the pack and then the second card is drawn), then the event of occurrence of a king in the first draw and the event of occurrence of a king in the second draw are independent events because the probability of drawing a king in the second draw is 4/52 whether a king is drawn in the first draw or not. But if the two cards are drawn without replacement, then the two events are not independent.

(iii) Let a bag contain 3 red and 2 black balls. Two balls are drawn one by one with replacement. Let A be the event of drawing a red ball in first draw and B be the event of drawing a black ball in the second draw. Then, $P(A) = 2/5$ when a red bal is drawn in the first draw and $P(B) = 2/5$ when a black ball is drawn in the first draw. Here probability of occurrence of event B is not affected by occurrence or non-occurrence of event A. Hence, A and B are independent events. But if the two balls are drawn one by one without replacement, then probability of occurrence of a black ball in the second draw when a red ball has been drawn in the first draw is 2/4. Probability of occurrence of a black ball in the second draw when a red ball is not drawn in the first draw is 1/4 (after a black ball is drawn there are only four balls in the bag out of which one is a black ball). Here, the events of drawing a ball in the first draw and the event of drawing a black ball in the second draw are not independent.

COMPOUND AND CONDITIONAL PROBABILITY

Compound Events

When two or more events occur together, their joint occurrence is called a compound event.

Examples:

Drawing a red and black ball from a bag containing 5 red and 6 black balls when two balls are drawn from the bag is a compound event. Compound events are of two types: (i) independent events and (ii) dependent events.

Conditional Probability

Let A and B be any two events and $B \neq \phi$. Then $P(A/B)$ denotes the conditional probability of occurrence of event A when B has already occurred.

Examples:

(i) Let a bag contain 2 red balls and 3 black balls. One ball is drawn from the bag and this ball is not replaced in the bag. Then, a second ball is drawn from the bag.

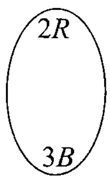


Fig. 9.2

Let B denote the events of occurrence of a red ball in the first draw and A denote the event of occurrence of a black ball in the second draw. When a red ball has been drawn in the first draw, the number of balls left is 4 and out of these four balls one is a red ball and three are black balls. Therefore, the probability of occurrence of a black ball in the second draw when a red ball has been drawn in the first draw is $P(A/B) = 3/4$.

(ii) When a die is thrown, sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

Let the event of occurrence of a number greater than 4 be $\{5, 6\}$. The event of occurrence of an odd number is $B = \{1, 3, 5\}$. Then $P(A/B)$ is the probability of occurrence of a number greater than 4, when an odd number has occurred. Here it is known that an odd number has occurred, i.e., one of 1, 3 and 5 has occurred. Out of these three numbers 1, 3 and 5, only 5 is greater than 4. Hence, here when an odd number has occurred, total number of cases is only 3 (not 6) and favourable number of cases is 1 (because out of 1, 3, 5, only 5 is greater than 4).

$$\therefore \quad P(A/B) = \frac{1}{2} = \frac{n(A \cap B)}{n(B)}$$

Clearly, while finding $P(A/B)$, B works as the sample space and $A \cap B$ works as the event.

In the example given above, the probability of occurrence of an odd number when a number greater than 4 has occurred is

$$P(B/A) \; \frac{n(B \cap A)}{n(A)} = \frac{1}{2}$$

Here, $A = \{5, 6\}$, $B \cap A = \{5\}$. Therefore, $n(A) = 2$ and $n(B \cap A) = 1$

Note: *$P(A/B)$ may or may not be equal to $P(B/A)$.*

Multiplication of Probability (Theorem of Compound Probability)

If A and B are any two events, then $P(A \cap B) = P(B).\ P\left(\frac{A}{B}\right)$, where $B \neq \phi$ and $P\left(\frac{A}{B}\right)$ denotes the probability of occurrence of event A when B has already occurred.

Proof:

Let S be the sample space. In case of occurrence of event A when B has already occurred, B works as the sample space and $A \cap B$ works as the event. Therefore,

$$P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}} = \frac{P(A \cap B)}{P(B)}$$

$$\therefore \quad P(A \cap B) = P(B)\, P(A/B) \qquad (1)$$

If A and B are independent events, then probability of occurrence of event A is not affected by occurrence or nor occurrence of event B. Therefore,

$$P\left(\frac{A}{B}\right) = P(A)$$

Hence from Eq. (1), $P(A \cap B) = P(B)\, P(A)$

Thus, $P(A \cap B) = P(A)\, P(B)$ where A and B are independent events.

Note:

1. *If A and B are two events associated with a random experiment, then $P(A \cap B = P(A).P(B/A)$, if $P(A) \neq 0$ or $P(A \cap B) = P(B).P(A/B)$, if $P(B) \neq 0$.*
2. *I Extension of multiplication theorem: If $A_1, A_2, \ldots, A_n$ are n events related to a random experiment, then $P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)P(A_2/A_1)P(A_3/A_1 \cap A_2) \ldots P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1})$*
   where $P(A_i/A_1 \cap A_2 \cap \ldots \cap A_{n-1})$ represents the conditional probability of the event A_i, given that the events $A_1, A_2, \ldots, A_{i-1}$ have already happened.
3. *Two events A and B are independent if and only if $P(A \cap B)$ or $P(AB) = P(A)\, P(B)$.*
4. *If A, B and C are any three independent events, then $P(A \cap B \cap C) = P([A \cap (B \cap C)] = P(A) \times P(B \cap C) = P(A)\, P(B)\, P(C)$.*
5. *General case: if $A_1, A_2, \ldots, A_n$ be any n events none of which is an impossible events, then*
   $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1A_2) \cdots P(A_n/A_1A_2\, A_{n-1})$.
   If $A_1, A_2, \ldots, A_n$ are independent events, then $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)\, P(A_2) \cdots P(A_n)$.
6. *Probability of at least one of the n independent events: If $p_1, p_2, p_3, \ldots, p_n$ be the probabilities of happening of n independent events $A_1, A_2, A_2, \ldots, A_n$ respectively, then*
   (i) Probability of happening none of them
   $$= P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \ldots \cap \bar{A}_n)$$
   $$= P(\bar{A}_1).P(\bar{A}_2).P(\bar{A}_3) \ldots P(\bar{A}_n)$$
   $$= (1-p_1)(1-p_2)(1-p_3) \ldots (1-p_n)$$
   (ii) Probability of happening at least one of them
   $$= P(A_1 \cup A_2 \cup A_3 \ldots \cup A_n)$$
   $$= 1 - P(\bar{A}_1)P(\bar{A}_2)P(\bar{A}_3) \ldots P(\bar{A}_n)$$
   $$= 1 - (1-p_1)(1-p_2)(1-p_3) \ldots (1-p_n)$$
   (iii) Probability of happening of first event and not happening of the remaining
   $$= P(A_1)P(\bar{A}_2)P(\bar{A}_3) \ldots P(\bar{A}_n)$$
   $$= p_1(1-p_2)(1-p_3) \ldots (1-p_n)$$

Complementation Rule

If A and B are two independent events, then

$P(A \cup B) = 1 - P(A')\, P(B')$

Proof:

$P(A \cup B) = 1 - P(A \cup B)'$

$= 1 - P(A' \cap B')$ [by De Morgan's law]

$= 1 - P(A') \times P(B)$

[Since A, B are independent events, therefore A' and B' are independent events. Hence, $P(A' \cap B') = P(A')\, P(B')$.]

Note:

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ *[always valid]*
2. $P(A \cup B) = 1 - P(A')\,P(B')$ *[valid only when A and B are independent]*
3. *If $A_1, A_2, \ldots, A_n$ are independent events, then* $P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - P(A'_1) \times P(A'_2) \cdots P(A'_n)$

Theorems on Independent Events

1. The events A and ϕ are independent.
2. The events A and S (sample space) are independent.
3. If A and B are independent events, then
   (i) A and B' are independent events
   (ii) A' and B are independent events
   (iii) A' and B' are independent events

Proof: Given, A and B are independent events, therefore

$$P(A \cap B) = P(A)\,P(B) \quad (1)$$

(i)

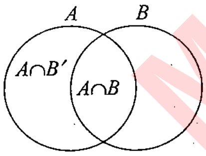


Fig. 9.3

Now, $A = (A \cap B) \cup (A \cap B')$

$\therefore \quad P(A) = P(A \cap B) + P(A \cap B')$ [$\because$ $A \cap B$ and $A \cap B'$ are mutually exclusive events]

$= P(A)\,P(B) + P(A \cap B')$

$\Rightarrow \quad P(A \cap B') = P(A) - P(A)P(B) = P(A)\,[1 - P(B)]$

$= P(A)\,P(B')$ [$\because$ $1 - P(B) = P(B')$]

Hence, A and B' are independent events.

(ii)

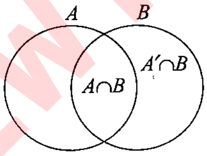


Fig. 9.4

$A' \cap B$ and $A \cap B$ are mutually exclusive events and $(A' \cap B) \cup (A \cap B) = B$.

$\therefore \quad P[(A' \cap B) \cup (A \cap B)] = P(B)$

$\Rightarrow \quad P(A' \cap B) + P(A \cap B) = P(B)$

$\Rightarrow \quad P(A' \cap B) = P(B) - P(A \cap B)$

$= P(B) - P(A)\,P(B)$ [From Eq. (1)]

$= P(B)(1 - P(A))$ [$\because$ $P(A') = 1 - P(A)$]

$= P(B)\,P(A')$

$= P(A')\,P(B)$

Hence, A and B are independent events.

(iii) $P(A' \cap B') = P(A \cup B)'$

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$ [By addition theorem of probability]

$= 1 - P(A) - P(B) + P(A)\,P(B)$ [From Eq. (1)]

$= 1 - P(A) - P(B)[1 - P(A)]$

$= [1 - P(A)][1 - P(B)]$

$= P(A')\,P(B')$

Hence, A' and B' are mutually exclusive events.

4. If A and B are two events such that $B \neq \phi$, then $P(A/B) + P(A/B) = 1$.

Proof: $P(A/B) + P(A'/B)$

$$= \frac{P(A \cap B)}{P(B)} + \frac{P(A' \cap B)}{P(B)}$$

$$= \frac{P(A \cap B) + P(A' \cap B)}{P(B)}$$

$$= \frac{P(B)}{P(B)} = 1$$ [$\because$ $(A \cap B) \cup (A' \cap B) = B$ and $A \cap B$ and $A' \cap B$ are mutually exclusive]

5. If A and B be two events such that $A \neq \phi$, then $P(B) = P(A)P(B/A) + P(A')P(B/A')$.

Proof: $P(A)P(B/A) + P(A')\,P(B/A')$

$= P(A \cap B) + P(A' \cap B)$

$= P[(A \cap B) \cup (A' \cap B)$

$= P[(A \cap B) \cup (A' \cap B)]$

[$\because$ $A \cap B$ and $A' \cap B$ are mutually exclusive]

$= P(B)$ [$\because (A \cap B) \cup (A' \cap B) = B$]

Example 9.26 **A fair coin is tossed repeatedly. If tail appears on first four tosses, then find the probability of head appearing on fifth toss.**

Sol. Since the trials are independent so the probability that head appears on the fifth toss does not depend upon previous results of the tosses. Hence, required probability is equal to probability of getting head, i.e., 1/2.

Example 9.27 **If a dice is thrown twice, then find the probability of getting 1 in the first throw only.**

Sol. Probability of getting 1 in first throw, $P(A) = 1/6$. Probability of not getting 1 in second throw, $P(B) = 5/6$.

Both are independent events, so the required probability is

$$P(A \cap B) = P(A)\,P(B) = \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$$

Example 9.28 **Find the probability of getting at least one tail in 4 tosses of a coin.**

Sol. Probability of getting head in each toss of coin is 1/2.

Then, probability of getting 4 heads in 4 tosses is

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^4$$

Therefore, required probability is

$$1 - P(\text{no coin shows tail}) = 1 - \left(\frac{1}{2}\right)^4 = \frac{15}{16}$$

Example 9.29 **Three persons work independently on a problem. If the respective probabilities that they will solve it are 1/3, 1/4 and 1/5, then find the probability that none can solve it.**

Sol. Required probability is

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)=\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{2}{5}$$

Example 9.30 **The probability of hitting a target by three marksmen are 1/2, 1/3 and 1/4. Then find the probability that one and only one of them will hit the target when they fire simultaneously.**

Sol. Here, $P(A) = 1/2$, $P(B) = 1/3$, $P(C) = 1/4$. Hence, required probability is

$$P(A)P(\bar{B})P(\bar{C})+P(\bar{A})P(B)P(\bar{C})+P(\bar{A})P(\bar{B})P(C)$$

$$=\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)+\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)$$

$$=\frac{11}{24}$$

Example 9.31 **An electrical system has open-closed switches S_1, S_2 and S_3 as shown in Fig. 9.5.**

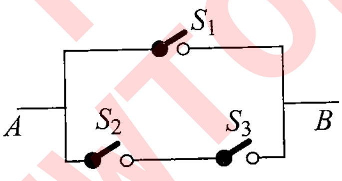


Fig 9.5

The switches operate independently of one another and the current will flow from A to B either if S_1 is closed or if both S_2 and S_3 are closed. If $P(S_1) = P(S_2) = P(S_3) = 1/2$, then find the probability that the circuit will work.

Sol. $P(S_1) = P(S_2) = P(S_3) = 1/2$

Let E be the event that the current will flow. Then,

$$P(E) = P((S_2 \cap S_3) \text{ or } S_1)$$
$$= P(S_2 \cap S_3) + P(S_1) - P(S_1 \cap S_2 \cap S_3)$$
$$= \frac{1}{4}+\frac{1}{2}-\frac{1}{8}=\frac{5}{8}$$

Example 9.32 **A bag contains 3 white, 3 black and 2 red balls. One by one three balls are drawn without replacing them, then find the probability that the third ball is red.**

Sol. Let R stand for drawing red ball, B for drawing black ball and W for drawing white ball. Then, the required probability is

$$P(WWR) + P(BBR) + P(WBR) + P(BWR) + P(WRR) + P(BRR) + P(RWR) + P(RBR)$$

$$=\frac{3\times2\times2}{8\times7\times6}+\frac{3\times2\times2}{8\times7\times6}+\frac{3\times3\times2}{8\times7\times6}+\frac{3\times3\times2}{8\times7\times6}$$
$$+\frac{3\times2\times1}{8\times7\times6}+\frac{3\times2\times1}{8\times7\times6}+\frac{2\times3\times1}{8\times7\times6}+\frac{2\times3\times1}{8\times7\times6}$$

$$=\frac{2}{56}+\frac{2}{56}+\frac{3}{56}+\frac{3}{56}+\frac{1}{56}+\frac{1}{56}+\frac{1}{56}+\frac{1}{56}$$

$$=\frac{1}{4}$$

Example 9.33 **The unbiased dice is tossed until a number greater than 4 appears. What is the probability that an even number of tosses is needed?**

Sol. Probability of success is $2/6 = 1/3 = p$. Probability of failure is $1 - 1/3 = 2/3 = q$. Probability that success occurs in even number of tosses is

$$P(FS) + P(FFFS) + P(FFFFFS) + \cdots$$
$$= pq + q^3p + q^5p + \cdots = \frac{pq}{1-q^2}=\frac{2}{5}$$

Example 9.34 **'X' speaks truth in 60% and 'Y' in 50% of the cases. Find the probability that they contradict each other narrating the same incident.**

Sol. Here, $P(X) = 3/5$, $P(Y) = 1/2$. Therefore, required probability is

$$P(X)P(\bar{Y})+P(\bar{X})P(Y)=\left(\frac{3}{5}\right)\left(1-\frac{1}{2}\right)+\left(1-\frac{3}{5}\right)\left(\frac{1}{2}\right)$$
$$=\frac{3}{5}\times\frac{1}{2}+\frac{2}{5}\times\frac{1}{2}=\frac{1}{2}$$

Example 9.35 **The odds against a certain event are 5:2 and the odds in favour of other event, independent of the former, are 6:5. Then find the probability that at least of the events will happen.**

Sol. $P\{\text{First event does not happen}\}=\frac{5}{5+2}=\frac{5}{7}$

$P\{\text{Second event does not happen}\}=\frac{5}{5+6}=\frac{5}{11}$

$\therefore$ $P\{\text{Both the events fail to happen}\}=\frac{5}{7}\times\frac{5}{11}=\frac{25}{77}$

Therefore, the probability that at least one of the events will happen is

$$1-P(\text{none of two happens})=1-\frac{25}{77}=\frac{52}{77}$$

Example 9.36 **If four whole numbers taken at random are multiplied together, then find the probability that the last digit in the product is 1, 3, 7 or 9.**

Sol. There are 10 digits 0, 1, 2, ..., 9 any of which can occur in any number at the last place, i.e., at the unit place. It is obvious that if the last digit in any of the four numbers is 0, 2, 4, 5, 6, 8, then the product of any of such four numbers will not give a number having its last digit as 1, 3, 7, 9. Hence, it is necessary that the last digit in each of the four numbers must be any of the four digits 1, 3, 7, 9.

Thus, for each of the four numbers, the number of ways of selection of the last digit is 10. Favourable number of ways of selection of the last digit is 4.

Therefore, the probability that the last digit be any of the four numbers 1, 3, 7, 9 is 4/10 = 2/5.

Hence, the required probability that the last digit in each of the four numbers is 1, 3, 7, 9 so that the last digit in their product is 1, 3, 7, 9 is $(2/5)^4 = 16/625$.

Example 9.37 **A rifleman is firing at a distance target and hence only 10% chance of hitting it. Find the number of rounds, he must fire in order to have more than 50% chance of hitting it at least once.**

Sol. We have,

$$p = \frac{10}{100} = \frac{1}{10}$$

$$\therefore q = \frac{9}{10}$$

By the given condition,

$$1 - q^n > \frac{1}{2}$$

$$\Rightarrow q^n < \frac{1}{2}$$

$$\Rightarrow \left(\frac{9}{10}\right)^n < \frac{1}{2}$$

which is possible if n is at least 7.

$\therefore \quad n = 7$

Example 9.38 **One of 10 keys open the door. If we try the keys one after another, then find the following:**

a. the probability that the door is opened on the first attempt.

b. the probability that the door is opened on the second attempt.

c. the probability that the door is opened on the 10th attempt.

Sol. Since out of 10 keys only one can open the door, so the door will open in first attempt with probability 1/10.

Now if he fails in first attempt which has the probability 9/10, then he will attempt next time with a different key. So, the probability that the door will open in second attempt is 9/10 × 1/9 = 1/10.

Probability that door will open only in 10th attempt is equal to the probability that the door will not open in first nine attempts which is equal to

$$\frac{9}{10}\times\frac{8}{9}\times\frac{7}{8}\times\cdots\times\frac{2}{3}\times\frac{1}{2}=\frac{1}{10}$$

as each time he tries with new key and keeps away the key which does not open the door.

Example 9.39 **A bag contains 'W' white and 3 black balls. Balls are drawn one by one without replacement till all the black balls are drawn. Then find the probability that this procedure for drawing the balls will come to an end at the r^{th} draw.**

Sol. Procedure of drawing the balls has to end at the r^{th} draw.

Exactly 2 black balls must have been drawn up to $(r-1)^{th}$ draw. Now probability of drawing exactly 2 black balls up to $(r-1)^{th}$ draw is

$$\frac{{}^3C_2\,{}^WC_{r-3}}{{}^{W+3}C_{r-1}} = \frac{\dfrac{3!}{2!1!}\,\dfrac{W!}{(r-3)!(W-r+3)!}}{\dfrac{(W+3)!}{(r-1)!(W-r+4)!}}$$

$$= \frac{3(r-1)(r-2)(W-r+4)}{(W+3)(W+2)(W+1)}$$

At the end of $(r-1)^{th}$ draw, we would be left with 1 black and $(W-r+3)$ white balls. Hence, the probability of drawing the black ball at the r^{th} draw is $1/(W-r+4)$. Therefore, the probability of required event is

$$\frac{3(r-1)(r-2)(W-r+4)}{(W+3)(W+2)(W+1)(W-r+4)}$$

$$= \frac{3(r-1)(r-2)}{(W+3)(W+2)(W+1)}$$

Example 9.40 **If A and B are two independent events, the probability that both A and B occur is 1/8 and the probability that neither of them occurs is 3/8. Find the probability of the occurrence of A.**

Sol. We have,

$$P(A\cap B) = \frac{1}{8} \text{ and } P(\bar{A}\cap\bar{B}) = \frac{3}{8}$$

$$\therefore \quad P(A)P(B) = \frac{1}{8} \text{ and } P(\bar{A})P(\bar{B}) = \frac{3}{8}$$

[$\because$ A and B are independent]

Now,

$$P(\bar{A}\cap\bar{B}) = \frac{3}{8} \Rightarrow 1 - P(A\cup B) = \frac{3}{8}$$

$$\Rightarrow \quad 1-(P(A)+P(B)-P(A\cap B)) = \frac{3}{8}$$

$$\Rightarrow \quad 1-(P(A)+P(B)) + \frac{1}{8} = \frac{3}{8}$$

$$\Rightarrow \quad P(A)+P(B) = \frac{3}{4}$$

The quadratic equation whose roots are $P(A)$ and $P(B)$ is

$$x^2 - x\{P(A)+P(B)\} + P(A)\,P(B) = 0$$

$$\Rightarrow \quad x^2 - \frac{3}{4}x + \frac{1}{8} = 0$$

$\Rightarrow \quad 8x^2 - 6x + 1 = 0$

$\Rightarrow \quad x = \frac{1}{2}, \frac{1}{4}$

BINOMIAL TRIALS AND BINOMIAL DISTRIBUTION

Consider a random experiment whose outcomes can be classified as success or failure. It means that experiment results in only two outcomes E_1(success) or E_2(failure). Further assume that experiment can be repeated several times, probability of success or failure in any trial are p and $q(p + q = 1)$ and do not vary from trial to trial and finally different trials are independent. Such a, experiment is called binomial experiment and trials are said to be binomial trials. For instance, tossing of a fair coin several times, each time outcome would be either a success (say occurrence of head) or failure (say occurrence of tail).

A probability distribution representing the binomial trials is said to binomial distribution. Let us consider a binomial experiment which has been repeated 'n' times. Let the probability of success and failure in any trial be p and q, respectively in these n-trials. Now number of ways of choosing 'r' success is in 'n' trials is ${}^nC_r$. Probability of 'r' successes and $(n - r)$ failures is $p^r q^{n-r}$. Thus, probability of having exactly r successes is ${}^nC_r p^r q^{n-r}$.

Let 'X' be a random variable representing the number of successes. Then,

$$P(X = r) = {}^nC_r p^r q^{n-r} \ (r = 0, 1, 2, \ldots, n)$$

Remark

- *Probability of utmost 'r' successes in n trials is* $\sum_{\lambda=0}^{r} {}^nC_\lambda p^\lambda q^{n-\lambda}$.
- *Probability of at least 'r' successes in n trials is* $\sum_{\lambda=r}^{n} {}^nC_\lambda p^\lambda q^{n-\lambda}$.
- *Probability of having first success at the r^{th} trial is* $p\,q^{r-1}$.

Example 9.41 **A die is thrown 7 times. What is the chance that an odd number turns up (i) exactly 4 times, (ii) at least 4 times?**

Sol. Probability of success is 3/6 = 1/2.

$\therefore \quad p = \frac{1}{2}$ and $q = \frac{1}{2}$

(i) For exactly 4 successes, the required probability is

$${}^7C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^3 = \frac{35}{128}$$

(ii) For at least 4 successes, the required probability is

$${}^7C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^3 + {}^7C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^2 + {}^7C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^1 + {}^7C_7\left(\frac{1}{2}\right)^7$$

$$= \frac{35}{128} + \frac{21}{128} + \frac{7}{128} + \frac{1}{128}$$

$$= \frac{64}{128}$$

$$= \frac{1}{2}$$

Example 9.42 ***A* and *B* play a series of games which cannot be drawn and *p*, *q* are their respective chances of winning a single game. What is the chance that *A* wins *m* games before *B* wins *n* games.**

Sol. For this to happen, A must win at least m out of the first $m + n - 1$ games. Therefore, the required probability is

$${}^{m+n-1}C_m p^m q^{n-1} + {}^{m+n-1}C_{m+1} p^{m+1} q^{n-2} + \cdots + {}^{m+n-1}C_{m+n-1} p^{m+n-1}$$

Example 9.43 **An experiment succeeds twice as often as it fails. Then find the probability that in the next 6 trials, there will be at least 4 successes.**

Sol. Let p be its probability of success and q that of failure. Then $p = 2q$. Also, $p + q = 1$. It gives $p = 2/3$ and $q = 1/3$.

P{4 successes in the 6 trials}

$$= {}^6C_4 p^4 q^2 = {}^6C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 \quad (1)$$

$$P\{5 \text{ successes in the 6 trials}\} = {}^6C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right) \quad (2)$$

$$P\{6 \text{ successes in the 6 trials}\} = {}^6C_6\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^0 \quad (3)$$

Therefore, the probability that there are at least 4 successes is

P {either 4 or 5 or 6 successes}

$$= {}^6C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 + {}^6C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right) + {}^6C_6\left(\frac{2}{3}\right)^6$$

$$= \frac{496}{729}$$

Example 9.44 **What is the probability of guessing correctly at least 8 out of 10 answers on a true–false examination?**

Sol. $p(X \geq 8) = {}^{10}C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + {}^{10}C_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1 + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$

$$= \left(\frac{1}{2}\right)^{10} [{}^{10}C_2 + {}^{10}C_1 + {}^{10}C_0]$$

$$= \left(\frac{1}{2}\right)^{10} [45 + 10 + 1]$$

$$= \frac{56}{8 \times 2^7}$$

$$= \frac{7}{128}$$

PROBLEMS ON CONDITIONAL PROBABILITY

Example 9.45 **Two dice are thrown. What is the probability that the sum of the numbers appearing on the two dice is 11, if 5 appears on the first?**

Sol. Since we are given that 5 appears on first die, so to get sum of 11, 6 must be one on the second and hence, the required probability is 1/6.

Example 9.46 **Three coins are tossed. If one of them shows tail, then find the probability that all three coins show tail.**

Sol. Let E be the event in which all three coins show tail and F be the event in which a coin shows tail. Therefore, $F = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$ and $E = \{TTT\}$. Hence, the required probability is

$$P(E/F) = \frac{P(E \cap F)}{P(E)} = \frac{1}{7}$$

Example 9.47 **One dice is thrown three times and the sum of the thrown numbers is 15. Find the probability for which number 4 appears in first throw.**

Sol. We have to find the bounded probability to get sum of 15 when 4 appears first. Let the event of getting a sum of 15 of three thrown number is A and the event of appearing 4 is B. So, we have to find $P(A/B)$. But

$$P(A/B) = \frac{n(A \cap B)}{n(B)}$$

Where $n(A \cap B)$ and $n(B)$, respectively, denote the number of digits in $A \cap B$ and B. Now $n(B) = 36$, because first throw is of 4. So another two throws stop by $6 \times 6 = 36$ types. Three dice have only two throws which starts from 4 and give a sum of 15, i.e., (4, 5, 6). So, $n(A \cap B) = 2$, $n(B) = 36$.

$$\therefore \quad P\left(\frac{A}{B}\right) = \frac{2}{36} = \frac{1}{18}$$

Example 9.48 **A box contains 10 mangoes out of which 4 are rotten. Two mangoes are taken out together. If one of them is found to be good, then find the probability that the other is also good.**

Sol. Let A be the event that the first mango is good and B be the event that the second one is good. Then, required probability is

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Now, probability that both mangoes are good is

$$P(A \cap B) = \frac{{}^6C_2}{{}^{10}C_2}$$

Probability that first mango is good is

$$P(A) = \frac{{}^6C_2}{{}^{10}C_2} + \frac{{}^6C_1 \times {}^4C_1}{{}^{10}C_2}$$

Hence,

$$P(B/A) = \frac{{}^6C_2}{{}^6C_2 + {}^6C_1 \times {}^4C_1} = \frac{15}{15+24} = \frac{5}{13}$$

Example 9.49 **If two events A and B are such that $P(A^c) = 0.3$, $P(B) = 0.4$ and $P(A \cap B^c) = 0.5$, then find the value of $P[B/(A \cup B^c)]$.**

Sol. $$P[B/(A \cup B^c)] = \frac{P(B \cap (A \cup B^c))}{P(A \cup B^c)}$$

$$= \frac{P(A \cap B)}{P(A) + P(B^c) - P(A \cap B^c)}$$

$$= \frac{P(A) - P(A \cap B^c)}{P(A) + P(B^c) - P(A \cap B^c)}$$

$$= \frac{0.7 - 0.5}{0.8} = \frac{1}{4}$$

Concept Application Exercise 9.3

1. A coin is tossed three times.
   Event A: two heads appear
   Event B: last should be head
   Then identify events A and B: independent or dependent.
2. Two cards are drawn one by one randomly from a pack of 52 cards. Then find the probability that both of them are king.
3. A coin is tossed and a dice is rolled. Find the probability that the coin shows the head and the dice shows 6.
4. The probability of happening an event A in one trial is 0.4. Find the probability that the event A happens at least once in three independent trials.
5. A fair coin is tossed n times. If the probability that head occurs 6 times is equal to the probability that head occurs 8 times, then find the value of n.
6. A, B, C in order cut a pack of cards replacing them after each cut on the condition that the first who cuts a spade shall win the prize. Find their respective chances.

7. In a bag, here are 6 balls of which 3 are white and 3 are black. They are drawn successively (i) without replacement, (ii) with replacement. What is the chance that the colours are alternate? It has been supposed that the number of balls drawn remains the same, i.e., six even with replacement.

8. The odds against a certain event is 5:2 and the odds in favour of another event is 6:5. If both the events are independent, then find the probability that at least one of the events will happen.

9. A pair of unbiased dice are rolled together till a sum of 'either 5 or 7' is obtained. Then find the probability that 5 comes before 7.

10. A man and a woman appear in an interview for two vacancies in the same post. The probability of man's selection is 1/4 and that of the woman's selection is 1/3. What is the probability that none of them will be selected?

11. A bag contains 5 white and 3 black balls. 4 balls are successively drawn out and not replaced. What is the probability that they are alternately of different colors.

12. The probability that Krishna will be alive 10 years hence is 7/15 and that Hari will be alive is 7/10. What is the probability that both Krishna and Hari will be dead 10 years hence?

13. A binary number is made up of 8 digits. Suppose that the probability of an incorrect digit appearing is p and that the errors in different digits are independent of each other. Then find the probability of forming an incorrect number.

14. A bag contains a white and b black balls. Two players, A and B alternately draw a ball from the bag, replacing the ball each time after the draw till one of them draws a white ball and wins the game. A begins the game. If the probability of A winning the game is three times that of B, then find the ratio $a:b$.

15. The probability of India winning a test match against West Indies is 1/2. Assuming independence from match to match find the probability that in a match series India's second win occurs at the third test.

16. In a single throw of two dice what is the probability of obtaining a number greater 7, if 4 appears on the first dice?

17. A coin is tossed three times in succession. If E is the event that there are at least two heads and F is the event in which first throw is a head, then find $P(E/F)$.

18. A die is thrown 4 times. Find the probability of getting at most two 6.

19. The probability that a student is not a swimmer is 1/5. Then find the probability that out of 5 students exactly 4 are swimmer.

20. A and B are two candidates seeking admission in IIT. The probability that A is selected is 0.5 and the probability that A and B are selected is at most 0.3. Is it possible that the probability of B getting selected is 0.9?

BAYES'S THEOREM

Partition of a Set

Consider a sample space 'S'. Let $A_1, A_2, \ldots, A_n$ be the set of mutually exclusive and exhaustive events of sample space S.

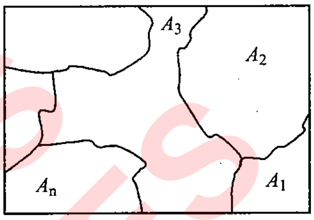


Fig. 9.6

These events $A_1, A_2, \ldots, A_n$ are said to partition the sample space S. We have,

$$A_i \cap A_j = \phi \text{ for } i \neq j,\ 1 \leq i, j \leq n$$

and $\sum_{i=1}^{n} P(A_i) = 1$

Bayes's Theorem

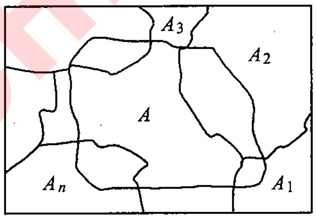


Fig. 9.7

If $A_1, A_2, A_3, \ldots, A_n$ be n mutually exclusive and exhaustive events and A is an event which occurs together (in conjunction) with either of A_i, i.e., if $A_1, A_2, \ldots, A_n$ form a partition of the sample space S and A be any event, then

$$p(A_k/A) = \frac{P(A_k)P(A/A_k)}{P(A_1)P(A/A_1) + P(A_2)P(A/A_2) + \cdots + P(A_n)P(A/A_m)}$$

Proof:

Since $A_1, A_2, \ldots, A_n$ form a partition of S, therefore

(i) $A_1, A_2, \ldots, A_n$ are non-empty

(ii) $A_i \cap A_j = \phi$ for $i \neq j$

(iii) $S = A_1 \cup A_2 \cup \cdots \cup A_n$

Now,

$$A = A \cap S = A \cap (A_1 \cup A_2 \cup \cdots \cup A_n)$$
$$= (A \cap A_1) \cup (A \cap A_2) \cup \cdots \cup (A \cap A_n) \quad (1)$$

Since $A_1, A_2, \ldots, A_n$ are disjoint sets, therefore $A \cap A_1, A \cap A_2, \ldots, A \cap A_n$ are also disjoint. Therefore, from (1), by addition theorem,

$$P(A) = P(A \cap A_1) + P(A \cap A_2) + \cdots + P(A \cap A_n) \quad (2)$$

Now,

$$P(A_k/A) = \frac{P(A_k \cap A)}{P(A)}$$

$$= \frac{P(A_k \cap A)}{P(A_1 \cap A) + P(A_2 \cap A) + \cdots + P(A_n \cap A)}$$

$$= \frac{P(A_k)P(A/A_k)}{P(A_1)P(A/A_1) + P(A_2)P(A/A_2) + \cdots + P(A_n)P(A/A_n)}$$

$[\because P(A \cap B) = P(A)P(B/A) = P(B)P(A/B)]$

Note:

1. *If $A_1, A_2, \ldots, A_n$ form a partition of S and A be any event then from Eq. (2),*

   $P(A) = P(A_1)\,P(A/A_1) + P(A_2)\,P(A/A_2) + \cdots + P(A_n)\,P(A/A_n)$

   $[\because P(A_i \cap A) = P(A_i)\,P(A/A_i)]$

2. *If $P(A_1) = P(A_2) = \cdots = P(A_n)$, then by Bayes's theorem,*

   $$P(A_k/A) = \frac{P(A/A_k)}{P(A/A_1) + P(A/A_2) + \cdots + P(A/A_n)}$$

3. *The probabilities $P(A_1), P(A_2), \ldots, P(A_n)$ which are known before the experiment takes place are called a priori probabilities and $P(A_i/A)$ are called a posteriori probabilities.*

4. ***Special case of Bayes's theorem:***

   If $A_1, A_2, \ldots, A_n$ form a partition of an event A, then

   $$P(A_k/A) = \frac{P(A_k)}{P(A_1) + P(A_2) + \cdots + P(A_n)}$$

   Proof:

   Since $A_1, A_2, \ldots, A_n$ form a partition of A, therefore

   (i) $A_1, A_2, \ldots, A_n$ are non-empty

   (ii) they are pairwise disjoint, i.e., no two of $A_1, A_2, \ldots, A_n$ have any common element

   (iii) $A = A_1 \cup A_2 \cup \cdots \cup A_n$

   From (i), (ii) and (iii), it is clear that $A \cap A_1, A \cap A_2, \ldots, A \cap A_n$ are non-empty pair-wise disjoint (they are mutually exclusive) and

   $A = (A \cap A_1) \cup (A \cap A_2) \cup \cdots \cup (A \cap A_n)$

   Hence by addition theorem of probability,

   $P(A) = P(A \cap A_1) + P(A \cap A_2) + \cdots + P(A \cap A_n)$ (3)

   Now,

   $$P(A_k/A) = \frac{P(A_k \cap A)}{P(A)} \qquad \left[\because P(A/B) = \frac{P(A \cap B)}{P(B)}\right]$$

   $$= \frac{P(A_k \cap A)}{P(A \cap A_1) + P(A \cap A_2) + \cdots + P(A \cap A_n)} \quad \text{[From Eq. (1)]}$$

   $$= \frac{P(A_k)P(A/A_k)}{P(A_1)P(A/A_1) + P(A_2)P(A/A_2) + \cdots + P(A_n)P(A/A_n)}$$

   $[\because P(A \cap B) = P(A)\,P(B/A)]$

   $$= \frac{P(A_k)}{P(A_1) + P(A_2) + \cdots + P(A_n)}$$

   $[\because A_1, A_2, \ldots, A_n$ *are subsets of A* $\therefore P(A/A_i) = 1$ *for* $i = 1, 2, 3, \ldots, n]$

PROBLEMS ON TOTAL PROBABILITY THEOREM

Example 9.50 **The probability that certain electronic component fails when first used is 0.10. If it does not fail immediately, the probability that it lasts for one year is 0.99. Find the probability that a new component will last for one year.**

Sol. Probability that the electronic component fails when first used is $P(F) = 0.10$. Therefore,

$P(F') = 1 - P(F) = 0.90$

Let E be the event that a new component will last for one year. Then,

$$P(E) = P(F)P\left(\frac{E}{F}\right) + P(F')P\left(\frac{E}{F'}\right)$$

[total probability theorem]

$= 0.10 \times 0 + 0.90 \times 0.99 = 0.891$

Example 9.51 **There are two bags, one of which contains 3 black and 4 white balls while the other contains 4 black and 3 white balls. A die is cast. If the face 1 or 3 turns up, a ball is taken from the first bag, and if any other face turn up, a ball is chosen from the second bag. Find the probability of choosing a black ball.**

Sol. Let E_1 be the event that a ball is drawn from first bag, E_2 the event that a ball is drawn from the second bag and E the event a black ball is drawn. Then we have

$P(E) = P(E_1)\,P(E/E_1) + P(E_2)\,P(E/E_2)$

$$= \frac{1}{3} \times \frac{3}{7} \times \frac{2}{3} \times \frac{4}{7} = \frac{11}{21}$$

Example 9.52 **A bag contains n + 1 coins. It is known that one of these coins shows heads on both sides, whereas the other coins are fair. One coin is selected at random and tossed. If the probability that toss results in heads is 7/12, then find the value of n.**

Sol. Let E_1 denote an event when a coin with two heads is selected and E_2 an event when a fair coin is selected. Let A be the event when the toss results in heads. Then, $P(E_1)$

$= 1/(n+1)$, $P(E_2) = 1/(n+1)$, $P(A/E_1) = 1$ and $P(A/E_2) = 1/2$.

$\therefore \quad P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2)$

$$\Rightarrow \quad \frac{7}{12} = \frac{1}{n\times 1}\times 1 + \frac{n}{n+1}\times\frac{1}{2}$$

$\Rightarrow \quad 12 + 6n = 7n + 7$

$\Rightarrow \quad n = 5$

Example 9.53 **An urn contains 6 white and 4 black balls. A fair die is rolled and that number of balls are chosen from the urn. Find the probability that the balls selected are white.**

Sol. Let A_i denote the event that the number i appears on the die and let E denote the event that only white balls are drawn. Then,

$$P(A_i) = \frac{1}{6} \text{ for } i = 1, 2, \ldots, 6$$

$$P(E/A_i) = \frac{{}^6C_i}{{}^{10}C_i}, \; i = 1, 2, \ldots, 6$$

Then, the required probability is

$$P(E) = P\left(\bigcup_{i=1}^{6}(E\cap A_i)\right)$$

$$= \sum_{i=1}^{6} P(E\cap A_i)$$

$$= \sum_{i=1}^{6} P(A_i)P(E/A_i)$$

$$= \frac{1}{6}\left[\frac{6}{10}+\frac{15}{45}+\frac{20}{120}+\frac{15}{210}+\frac{6}{252}+\frac{1}{210}\right] = \frac{1}{3}$$

PROBLEMS ON BAYES'S THEOREM

Example 9.54 **A pack of playing cards was found to contain only 51 cards. If the first 13 cards, which are examined, are all red, what is the probability that the missing card is black?**

Sol. Let A_1 be the event that black card is lost, A_2 be the event that red card is lost and let A denote occurrence of first 13 cards which are examined and are found to be all red. Then, we have to find $P(A_1/A)$. We have $P(A_1) = P(A_2) = 1/2$. Also, $P(A/A_1) = {}^{26}C_{13}/{}^{51}C_{13}$ and $P(A/A_2) = {}^{23}C_{13}/{}^{51}C_{13}$. Then by Bayes's rule,

$$P(A_1/A) = \frac{P(A_1)P(A/A_1)}{P(A_1)P(A/A_1)+P(A_2)P(A/A_2)}$$

$$= \frac{\frac{1}{2}\frac{{}^{26}C_{13}}{{}^{51}C_{13}}}{\frac{1}{2}\frac{{}^{26}C_{13}}{{}^{51}C_{13}}+\frac{1}{2}\frac{{}^{25}C_{13}}{{}^{51}C_{13}}}$$

$$= \frac{{}^{26}C_{13}}{{}^{26}C_{13}+{}^{25}C_{13}} = \frac{2}{2+1} = \frac{2}{3}$$

Example 9.55 **The chances of defective screws in three boxes A, B, C are 1/5, 1/6, 1/7, respectively. A box is selected at random and a screw drawn from it at random is found to be defective. Then, find the probability that it came from box A.**

Sol. Let E_1, E_2 and E_3 denote the events of selecting boxes A, B, C, respectively, and A be the event that a screw selected at random is defective. Then,

$$P(E_1) = P(E_2) = P(E_3) = 1/3$$

$$\therefore \quad P(A/E_1) = \frac{1}{5},\; P(A/E_2) = \frac{1}{6},\; P(A/E_3) = \frac{1}{7}$$

By Bayes's rule, the required probability is

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times\frac{1}{5}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{6}+\frac{1}{3}\times\frac{1}{7}} = \frac{42}{107}$$

Example 9.56 **In an entrance test, there are multiple choice questions. There are four possible answers to each question, of which one is correct. The probability that a student knows the answer to a question is 90%. If he gets the correct answer to a question, then find the probability that he was guessing.**

Sol. We define the following events:

A_1: He knows the answer

A_2: He does not know the answer

E: He gets the correct answer

Then, $P(A_1) = 9/10$, $P(A_2) = 1 - 9/10 = 1/10$, $P(E/A_1) = 1$, $P(E/A_2) = 1/4$.

Therefore, the required probability is

$$P(A_2/E) = \frac{P(A_2)P(E/A_2)}{P(A_1)P(E/A_1)+P(A_2)P(E/A_2)}$$

$$= \frac{\frac{1}{10}\times\frac{1}{4}}{\frac{9}{10}\times 1+\frac{1}{10}\times\frac{1}{4}} = \frac{1}{37}$$

Example 9.57 **Each of the 'n' urns contains 4 white and 6 black balls. The $(n+1)^{\text{th}}$ urn contains 5 white and 5 black balls. One of the $n+1$ urns is chosen at random and two balls are drawn from it without replacement. Both the balls turn out to be black. If the probability that the $(n+1)^{\text{th}}$ urn was chosen to draw the balls is 1/16, then find the value of n.**

Sol. Let E_1 denote the event that one of the first n urns is chosen and E_2 denote the event that $(n+1)^{\text{th}}$ urn is selected. A denotes the event that two balls drawn are black. Then,

$P(E_1) = n/(n+1)$, $p(E_2) = 1/(n+1)$, $P(A/E_1) = {}^6C_2/{}^{10}C_2$

$= 1/3$ and $P(A/E_2) = {}^5C_2/{}^{10}C_2 = 2/9$.

Using Bayes's theorem, the required probability is

$$P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$\Rightarrow \quad \frac{1}{16} = \frac{\left(\dfrac{1}{n+1}\right)\dfrac{2}{9}}{\left(\dfrac{n}{n+1}\right)\left(\dfrac{1}{3}\right)+\left(\dfrac{1}{n+1}\right)\left(\dfrac{2}{9}\right)} = \frac{2}{3n+2}$$

$\Rightarrow \quad n = 10$

Example 9.58 **Die A has 4 red and 2 white faces whereas die B has 2 red and 4 white faces. A coin is flipped once. If it shows a head, the game continues by throwing die A; if it shows tail, then die B is to be used. If the probability that die A is used is 32/33 when it is given that red turns up every time in first n throws, then find the value of n.**

Sol. Let R be the event that a red face appears in each of the first n throws.

E_1: Die A is used when head has already fallen

E_2: Die B is used when tail has already fallen

$$\therefore \quad P(R/E_1) = \left(\frac{2}{3}\right)^n \text{ and } P\left(\frac{R}{E_2}\right) = \left(\frac{1}{3}\right)^n$$

As per the given condition,

$$\frac{P(E_1)P(R/E_1)}{P(E_1)P(R/E_1)+P(E_2)P(R/E_2)} = \frac{32}{33}$$

$$\Rightarrow \quad \frac{1/2(2/3)^n}{\dfrac{1}{2}\left(\dfrac{2}{3}\right)^n+\dfrac{1}{2}\left(\dfrac{1}{3}\right)^n} = \frac{32}{33}$$

$$\Rightarrow \quad \frac{2^n}{2^n+1} = \frac{32}{33}$$

$\Rightarrow \quad n = 5$

Example 9.59 **A bag contain n balls out of which some balls are white. If probability that a bag contains exactly i white ball is proportional to i^2. A ball is drawn at random from the bag and found to be white, then find the probability that bag contains exactly 2 white balls.**

Sol. We have,

$P(A_i) = ki^2$

$\Rightarrow \quad 1 = k\Sigma n^2$

$$\Rightarrow \quad k = \frac{1}{\Sigma n^2}$$

$$\therefore \quad P(A_i) = \frac{6i^2}{n(n+1)(2n+1)}$$

Let event B denote that the ball drawn is white. Then,

$$\therefore \quad P(B) = \frac{6}{n(n+1)(2n+1)}\left[1^2\frac{1}{n}+2^2\frac{2}{n}+\cdots+n^2\frac{n}{n}\right]$$

$$= \frac{3(n+1)}{2(2n+1)}$$

$$P(A_2/B) = \frac{\left(\dfrac{6}{n(n+1)}\right)\left(\dfrac{2^3}{n}\right)}{\dfrac{2}{3(n+1)}}$$

Concept Application Exercise 9.4

1. An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well-shuffled pack of 11 cards numbered 2, 3, 4, ..., 12 is picked and the number on the card is noted. What is the probability that the noted number is either 7 or 8?
2. A number is selected at random from the first twenty-five natural numbers. If it is a composite number then it is divided by 5. But if it is not a composite number, it is divided by 2. Find the probability that there will be no remainder in the division.
3. A real estate man has eight master keys to open several new homes. Only one master key will open any given house. If 40% of these homes are usually left unlocked, what is the probability that the real estate man can get into a specific home if he selects three master keys at random before leaving the office?
4. A card from a pack of 52 cards is lost. From the remaining cards, two cards are drawn and are found to be spades. Find the probability that missing card is also a spade.
5. Consider a sample space 'S' representing the adults in a small town who have completed the requirements for a collage degree. They have been categorized according to sex and employment as follows:

	Employed	Unemployed
Male	460	40
Female	140	260

   An employed person is selected at random. Find the probability that chosen one is male.
6. A man has coins A, B, C. A is unbiased; the probability that a head will result when B is tossed is 2/3; and the probability that a head will result when C is tossed is 1/3. If one of the coins chosen at random, is tossed three times, giving two heads and one tail, then find the probability that the chosen coin was A.
7. A and B are two independent witnesses (i.e., there is no collusion between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statement. Find the probability that the statement is true.
8. An urn contains five balls. Two balls are drawn and are found to be white. Find the probability that all the balls are white.

EXERCISES

Subjective Type

Solutions on page 9.42

1. Let $A = \{0, 5, 10, 15, \ldots, 195\}$. Let B be any subset of A with atleast 15 elements. What is the probability that B has at least one pair of elements whose sum is divisible by 15?

2. A bag contains n white and n black balls, all of equal size. Balls are drawn at random. Find the probability that there are both white and black balls in the draw and that the number of white balls is greater than that of black balls by 1.

3. There are two bags each containing 10 books all having different titles but of the same size. A student draws out books from first bag as well as from the second bag. Find the probability that the different between the books drawn from the two bags does not exceed 2.

4. Suppose A and B shoot independently until each hits his target. They have probabilities 3.5 and 5/7 of hitting the targets at each shot. Find the probability that B will require more shots than A.

5. From an urn containing a white and b black balls, k balls are drawn and laid aside, their colour unnoted. Then one more ball is drawn. Find the probability that it is white assuming that $k < a, b$.

6. Of three independent events, the chance that only the first occurs is a, the chance that only the second occurs is b and the chance of only third is c. Show that the chances of three events are, respectively, $a/(a+x)$, $b/(b+x)$, $c/(c+x)$, where x is a root of the equation $(a+x)(b+x)(c+x) = x^2$.

7. Two natural numbers x and y are chosen at random. What is the probability that $x^2 + y^2$ is divisible by **(i)** 5 and **(ii)** 7.

8. If m things are distributed among a men and b women, show that the chance that the number of things received by man is

$$\frac{1}{2}\frac{(b+a)^m - (b-a)^m}{(b+a)^m}.$$

9. $8n$ players $P_1, P_2, P_3, \ldots, P_{8n}$ play a knock out tournament. It is known that all the players are of equal strength. The tournament is held in three rounds where the players are paired at random in each round. If it is given that P_1 wins in the third round. Find the probability that P_2 loses in the second round.

10. A tennis match of best of 5 sets is played by two players 'A' and 'B'. The probability that first set is won by A is 1/2 and if he loses the first, then probability of his winning the next set is 1/4 otherwise it remains same. Find the probability that A wins the match.

11. A and B participate in a tournament of 'best of 7 games'. It is equally likely that either A wins or B wins or the game ends in a draw. What is the probability that A wins the tournament.

12. Fourteen numbered balls (i.e., 1, 2, 3, …, 14) are divided in three groups randomly. Find the probability that sum of the numbers on the balls, in each group, is odd.

13. Let $P(x)$ denote the probability of the occurrence of event x. Plot all those point $(x, y) = (P(A), P(B))$ in a plane which satisfies the conditions,
$P(A \cup B) \geq 3/4$ and $1/8 \leq P(A \cap B) \leq 3/8$

14. Two players P_1 and P_2 are playing the final of a chess championship, which consists of a series of matches. Probability of P_1 winning a match is 2/3 and that of P_2 is 1/3. The winner will be the one who is ahead by 2 games as compared to the other player and wins at least 6 games. Now, if the player P_2 wins first four matches, find the probability of P_1 wining the championship.

15. Consider a game played by 10 people in which each flips a fair coin at the same time. If all but one of the coins comes up the same then the odd persons wins (e.g. if there are nine tails and one head then head wins). If such a situation does not occur, the players flips again. Find the probability that game is settled on or after n^{th} toss.

16. A bag contains 'n' balls, one of which is white. The probability that A and B speak truth are P_1 and P_2, respectively. One ball is drawn from the bag and A and B both assert that it is white. Find the probability that drawn ball is actually white.

17. A bag contains a total of 20 books on physics and mathematics. Any possible combination of books is equally likely. Ten books are chosen from the bag and it is found that it contains 6 books of mathematics. Find out the probability that the remaining books in the bag contains 2 books on mathematics.

18. A coin is tossed $(m+n)$ times $(m > n)$. Show that the probability of at least m consecutive heads is $n+2/2^{m+1}$.

Objective Type

Solutions on page 9.45

Each question has four choices a, b, c and d, out of which *only one* is correct. Find the correct answer.

1. Given two events A and B. If odds against A are as 2:1 and those in favour of $A \cup B$ are as 3:1, then

a. $1/2 \leq P(B) \leq 3/4$ **b.** $5/12 \leq P(B) \leq 3/4$
c. $1/4 \leq P(B) \leq 3/5$ **d.** none of these

2. The probability that a marksman will hit a target is given as $1/5$. Then the probability that at least once hit in 10 shots is

a. $1-(4/5)^{10}$ **b.** $1/5^{10}$
c. $1-(1/5^{10})$ **d.** $(4/5)^{10}$

3. A six-faced dice is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice, the probability that the sum of two numbers thrown is even is

a. 1/12 **b.** 1/6
c. 1/3 **d.** 5/9

4. A draws a card from a pack of n cards marked 1, 2, …, n. The card is replaced in the pack and B draws a card. Then the probability that A draws a higher card than B is

a. $(n+1)2n$ b. $1/2$
c. $(n-1)\,2n$ d. none of these

5. A student appears for tests I, II and III. The student is successful if he passes either in tests I and II or tests I and III. The probabilities of the student passing in tests I, II and III are, respectively, p, q and 1/2. If the probability that the student is successful is 1/2, then $p(1 + q) =$
a. 1/2 b. 1
c. 3/2 d. 3/4

6. A problem in mathematics is given to three students A, B, C and their respective probability of solving the problem is 1/2, 1/3 and 1/4. Probability that the problem is solved is
a. 3/4 b. 1/2
c. 2/3 d. 1/3

7. The probability that in a family of 5 members, exactly two members have birthday on Sunday is
a. $(12\times 5^3)\ 7^5$ b. $(10\times 6^2)\ 7^5$
c. 2/5 d. $(10\times 6^3)\ 7^5$

8. Three houses are available in a locality. Three persons apply for the houses. Each applies for one houses without consulting others. The probability that all three apply for the same houses is
a. 1/9 b. 2/9
c. 7/9 d. 8/9

9. The numbers 1, 2, 3, ..., n are arrange in random order. The probability that the digits 1, 2, 3, ..., k $(k < n)$ appear as neighbours in that order is
a. $1/n!$ b. $k!/n!$
c. $(n-k)!\,n!$ d. $(n-k+1)!\,n!$

10. A die is thrown a fixed number of times. If probability of getting even number 3 times is same as the probability of getting even number 4 times, then probability of getting even number exactly once is
a. 1/6 b. 1/9
c. 5/36 d. 7/128

11. A pair of four dice is thrown independently three times. The probability of getting a score of exactly 9 twice is
a. 8/9 b. 8/729
c. 8/243 d. 1/729

12. Five horses are in a race. Mr. A selects two of the horses at random and bets on them. The probability that Mr. A selected the winning horse is
a. 3/5 b. 1/5
c. 2/5 d. 4/5

13. Let A and B be two events such that $P(\overline{A\cup B})=1/6$, $P(A\cap B)=1/4$ and $P(\overline{A})=1/4$ where $\overline{A}$ stands for complment of event A. Then events A and B are
a. equally likely but not independent
b. equally likely and mutually exclusive
c. mutually exclusive and independent
d. independent but not equally likely

14. Words from the letters of the word PROBABILITY are formed by taking all letters at a time. The probability that both B's are not together and both I's are not together is
a. 52/55 b. 53/55
c. 54/55 d. none of these

15. A class consists of 80 students, 25 of them are girls and 55 are boys. If 10 of them are rich and the remaining are poor and also 20 of them are intelligent, then the probability of selecting an intelligent rich girl is
a. 5/128 b. 25/128
c. 5/512 d. none of these

16. Let A, B, C, D be independent events such that $P(A)=1/2$, $P(B) = 1/3$, $P(C) = 1/5$ and $P(D) = 1/6$. Then the probability that none of A, B, C and D occurs
a. 1/180 b. 1/45
c. 1/18 d. none of these

17. A sample space consists of 3 sample points with associated probabilities given as $2p$, p^2, $4p - 1$. Then the value of p is
a. $p=\sqrt{11}-3$ b. $\sqrt{10}-3$
c. $\frac{1}{4}<p<\frac{1}{2}$ d. none

18. South African cricket captain lost the toss of a coin 13 times out of 14. The chance of this happening was
a. $7/2^{13}$ b. $1/2^{13}$
c. $13/2^{14}$ d. $13/2^{13}$

19. Events A and C are independent. If the probabilities relating A, B and C are $P(A) = 1/5$, $P(B) = 1/6$; $P(A\cap C) = 1/20$; $P(B\cup C) = 3/8$. Then
a. events B and C are independent
b. events B and C are mutually exclusive
c. events B and C are neither independent nor mutually exclusive
d. events B and C are equiprobable

20. There are only two women among 20 persons taking part in a pleasure trip. The 20 persons are divided into two groups, each group consisting of 10 persons. Then the probability that the two women will be in the same group is
a. 9/19 b. 9/38
c. 9/35 d. none

21. Let A and B be two events. Suppose $P(A) = 0.4$, $P(B) = p$ and $P(A\cup B) = 0.7$. The value of p for which A and B are independent is
a. 1/3 b. 1/4
c. 1/2 d. 1/5

22. A man has 3 pairs of black socks and 2 pairs of brown socks kept together in a box. If he dressed hurriedly in the dark, the probability that after he has put on a black sock, he will then put on another black sock is

a. 1/3 b. 2/3
c. 3/5 d. 2/15

23. Five different games are to be distributed among 4 children randomly. The probability that each child get atleast one game is
a. 1/4 b. 15/64
c. 21/64 d. none of these

24. A drawer contains 5 brown socks and 4 blue socks well mixed. A man reaches the drawer and pulls out socks at random. What is the probability that they match?
a. 4/9 b. 5/8
c. 5/9 d. 7/12

25. Two dices are rolled one after the other. The probability that the number on the first is smaller than the number on the second is
a. 1/2 b. 7/18
c. 3/4 d. 5/12

26. A natural number is chosen at random from the first 100 natural numbers. The probability that $x+\dfrac{100}{x}>50$ is
a. 1/10 b. 11/50
c. 11/20 d. none of these

27. A four figure number is formed of the figures 1, 2, 3, 5 with no repetitions. The probability that the number is divisible by 5 is
a. 3/4 b. 1/4
c. 1/8 d. none of these

28. Twelve balls are distributed among three boxes. The probability that the first box contains three balls is
a. $\dfrac{110}{9}\left(\dfrac{2}{3}\right)^{10}$ b. $\dfrac{9}{110}\left(\dfrac{2}{3}\right)^{10}$
c. $\dfrac{{}^{12}C_3}{12^3}\times 2^9$ d. $\dfrac{{}^{12}C_3}{3^{12}}$

29. A box contains 2 black, 4 white and 3 red balls. One ball is drawn at random from the box and kept aside. From the remaining balls in the box, another ball is drawn at random and kept aside the first. This process is repeated till all the balls are drawn from the box. The probability that the balls drawn are in the sequence of 2 black, 4 white and 3 red is
a. 1/1260 b. 1/7560
c. 1/126 d. none of these

30. A cricket club has 15 members, of whom only 5 can bowl. If the names of 15 members are put into a box and 11 are drawn at random, then the probability of getting an eleven containing at least 3 bowlers is
a. 7/13 b. 6/13
c. 11/15 d. 12/13

31. *A* speaks truth in 60% cases and *B* speaks truth in 70% cases. The probability that they will say the same thing while describing a single event is
a. 0.56 b. 0.54
c. 0.38 d. 0.94

32. Three integers are chosen at random from the first 20 integers. The probability that their product is even is
a. 2/19 b. 3/29
c. 17/19 d. 4/29

33. There are 20 cards. Ten of these cards have the letter '*I*' printed on them and the other 10 have the letter '*T*' printed on them. If three cards are picked up at random and kept in the same order, the probability of making word IIT is
a. 4/27 b. 5/38
c. 1/8 d. 9/80

34. The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. The chances of the events are
a. 1/9, 1/3 b. 1/16, 1/4
c. 1/4, 1/2 d. none of these

35. If a party of n persons sit at a round table, then the odds against two specified individuals sitting next to each other are
a. $2:(n-3)$ b. $(n-3):2$
c. $(n-2):2$ d. $2:(n-2)$

36. The sum of two positive quantities is equal to $2n$. The probability that their product is not less than 3/4 times their greatest product is
a. 3/4 b. 1/2
c. 1/4 d. none of these

37. A bag contains an assortment of blue and red balls. If two balls are drawn at random, the probability of drawing two red balls is five times the probability of drawing two blue balls. Furthermore, the probability of drawing one ball of each colour is six times the probability of drawing two blue balls. The number of red and blue balls in the bag is
a. 6, 3 b. 3, 6
c. 2, 7 d. none of these

38. Dialing a telephone number an old man forgets the last two digits remembering only that these are different dialed at random. The probability that the number is dialed correctly is
a. 1/45 b. 1/90
c. 1/100 d. none of these

39. The probability that a teacher will give an unannounced test during any class meeting is 1/5. If a student is absent twice, then the probability that the student will miss at least one test is
a. 4/5 b. 2/5
c. 7/25 d. 9/25

40. A box contains tickets numbered from 1 to 20. Three tickets are drawn from the box with replacement. The probability that the largest number on the tickets is 7 is
a. 2/19 b. 7/20
c. $1-(7/20)^3$ d. none of these

41. An unbiased coin is tossed n times. If the probability that head occurs 6 times is equal to the probability that head occurs 8 times, then $n =$
   a. 7 b. 14
   c. 16 d. 19

42. One ticket is selected at random from 100 tickets numbered 00, 01, 02, ..., 98, 99. If x_1 and x_2 denotes the sum and product of the digits on the tickets, then $P(x_1 = 9/x_2 = 0)$ is equal to
   a. 2/19 b. 19/100
   c. 1/50 d. none of these

43. Let A and B be two events such that $P(A \cap B') = 0.20$, $P(A' \cap B) = 0.15$, $P(A' \cap B') = 0.1$, then $P(A/B)$ is equal to
   a. 11/14 b. 2/11
   c. 2/7 d. 1/7

44. A and B toss a fair coin each simultaneously 50 times. The probability that both of them will not get tail at the same toss is
   a. $(3/4)^{50}$ b. $(2/7)^{50}$
   c. $(1/8)^{50}$ d. $(7/8)^{50}$

45. Cards are drawn one-by-one at random from a well-shuffled pack of 52 playing cards until 2 aces are obtained from the first time. The probability that 18 draws are required for this is
   a. 3/34 b. 17/455
   c. 561/15925 d. none of these

46. A father has 3 children with at least one boy. The probability that he has 2 boys and 1 girl is
   a. 1/4 b. 1/3
   c. 2/3 d. None of these

47. Two players toss 4 coins each. The probability that they both obtain the same number of heads is
   a. 5/256 b. 1/16
   c. 35/128 d. none of these

48. In a game called 'odd man out' m $(m > 2)$ persons toss a coin to determine who will buy refreshments for the entire group. A person who gets an outcome different from that of the rest of the members of the group is called the odd man out. The probability that there is a loser in any game is
   a. $1/2m$ b. $m/2^{m-1}$
   c. $2/m$ d. none of these

49. If a is an integer lying in $[-5, 30]$, then the probability that the graph of $y = x^2 + 2(a + 4)x - 5a + 64$ is strictly above the x-axis is
   a. 1/6 b. 7/36
   c. 2/9 d. 3/5

50. $2n$ boys are randomly divided into two subgroups containing n boys each. The probability that the two tallest boys are in different groups is
   a. $n/(2n-1)$ b. $(n-1)/(2n-1)$
   c. $(n-1)/4n^2$ d. none of these

51. A bag contains some white and some black balls, all combinations of balls being equally likely. The total number of balls in the bag is 10. If three balls are drawn at random without replacement and all of them are found to be black, the probability that the bag contains 1 white and 9 black balls is
   a. 14/55 b. 12/55
   c. 2/11 d. 8/55

52. Three ships A, B and C sail from England to India. If the ratio of their arriving safely are 2:5, 3:7 and 6:11, respectively, then the probability of all the ships for arriving safely is
   a. 18/595 b. 6/17
   c. 3/10 d. 2/7

53. The probability of solving a question by three students are 1/2, 1/4, 1/6, respectively. Probability of question being solved will be
   a. 33/48 b. 35/48
   c. 31/48 d. 37/48

54. Let A, B, C be three mutually independent events. Consider the two statements S_1 and S_2.
   S_1: A and $B \cup C$ are independent.
   S_2: A and $B \cap C$ are independent.
   Then
   a. both S_1 and S_2 are true b. only S_1 is true
   c. only S_2 is true d. neither S_1 nor S_2 is true

55. If the papers of 4 students can be checked by any one of the 7 teachers, then the probability that all the 4 papers are checked by exactly 2 teachers is
   a. 2/7 b. 12/49
   c. 32/343 d. none of these

56. A and B play a game of tennis. The situation of the game is as follows: if one scores two consecutive points after a deuce, he wins, if loss of a point is followed by win of a point, it is deuce. The chance of a server to win a point is 2/3. The game is at deuce and A is serving. Probability that A will win the match is (serves are changed after each game)
   a. 3/5 b. 2/5
   c. 1/2 d. 4/5

57. A coin is tossed $2n$ times. The chance that the number of times one gets head is not equal to the number of times one gets tails is
   a. $\dfrac{(2n!)}{(n!)^2}\left(\dfrac{1}{2}\right)^{2n}$ b. $1 - \dfrac{(2n!)}{(n!)^2}$
   c. $1 - \dfrac{(2n!)}{(n!)^2}\dfrac{1}{4^n}$ d. none of these

58. The probability that a bulb produced by a factory will fuse after 150 days if used is 0.50. What is the probability that out of 5 such bulbs none will fuse after 150 days of use
   a. $1 - (19/20)^5$ b. $(19/20)^5$
   c. $(3/4)^5$ d. $90\,(1/4)^5$

59. If $\bar{E}$ and $\bar{F}$ are the complementary events of events E and F, respectively, and if $0 < P(F) < 1$, then

a. $P(\bar{E}/F) + P(\bar{E}/F) = 1/2$
b. $P(E/F) + P(E/\bar{F}) = 1$
c. $P(\bar{E}/F) + P(E/\bar{F}) = 1$
d. $P(E/\bar{F}) + P(\bar{E}/\bar{F}) = 1$

60. In a certain town, 40% of the people have brown hair, 25% have brown eyes and 15% have both brown hair and brown eyes. If a person selected at random from the town, has brown hair, the probability that he also has brown eyes is

a. 1/5
b. 3/8
c. 1/3
d. 2/3

61. A letter is known to have come either from LONDON or CLIFTON; on the postmark only the two consecutive letters ON are legible. The probability that it came from LONDON is

a. 1/17
b. 12/17
c. 17/30
d. 3/5

62. In a game a coin is tossed $2n + m$ times and a player wins if he does not get any two consecutive outcomes same for atleast $2n$ times in a row. The probability that player wins the game is

a. $\frac{m+2}{2^{2n}+1}$
b. $\frac{2n+2}{2^{2n}}$
c. $\frac{2n+2}{2^{2n+1}}$
d. $\frac{m+2}{2^{2n}}$

63. Words from the letters of the word PROBABILITY are formed by taking all letters at a time. The probability that both B's are not together and both I's are not together is

a. 52/55
b. 53/55
c. 54/55
d. none of these

64. The probabilities of winning a race by three persons A, B and C are 1/2, 1/4, and 1/4, respectively. They run two races. The probability of A winning the second race when B wins the first race is

a. 1/3
b. 1/2
c. 1/4
d. 2/3

65. A die is rolled 4 times. The probability of getting a larger number than the previous number each time is

a. 17/216
b. 5/432
c. 15/432
d. none of these

66. Four die are thrown simultaneously. The probability that 4 and 3 appear on two of the die given that 5 and 6 have appeared on other two die is

a. 1/6
b. 1/36
c. 12/151
d. none of these

67. A fair coin is tossed 5 times. then the probability that no two consecutive heads occur is

a. 11/32
b. 15/32
c. 13/32
d. none of these

68. A $2n$ digit number starts with 2 and all its digits are prime, then the probability that the sum of all 2 consecutive digits of the number is prime is

a. 4×2^{3n}
b. 4×2^{-3n}
c. 2^{3n}
d. none of these

69. The numbers (a, b, c) are selected by throwing a dice thrice, then the probability that (a, b, c) are in A.P. is

a. 1/12
b. 1/6
c. 1/4
d. none of these

70. In a n-sided regular polygon, the probability that the two diagonal chosen at random will intersect inside the polygon is

a. $\frac{2\,{}^nC_2}{{}^{({}^nC_2-n)}C_2}$
b. $\frac{{}^{n(n-1)}C_2}{{}^{({}^nC_2-n)}C_2}$
c. $\frac{{}^nC_4}{{}^{({}^nC_2-n)}C_2}$
d. none of these

71. A three-digit number is selected at random from the set of all three-digit numbers. The probability that the number selected has all the three digits same is

a. 1/9
b. 1/10
c. 1/50
d. 1/100

72. Two numbers a, b are chosen from the set of integers 1, 2, 3, …, 39. Then probability that the equation $7a - 9b = 0$ is satisfied is

a. 1/247
b. 2/247
c. 4/741
d. 5/741

73. Two numbers x and y are chosen at random (without replacement) from amongst the numbers 1, 2, 3, …, 2004. The probability that $x^3 + y^3$ is divisible by 3 is

a. 1/3
b. 2/3
c. 1/6
d. 1/4

74. One mapping is selected at random from all mappings of the set $S = \{1, 2, 3, \ldots, n\}$ into itself. If the probability that the mapping is one-one is 3/32, then the value of n is

a. 2
b. 3
c. 4
d. none of these

75. A bag contains 20 coins. If the probability that thebag contains exactly 4 biased coin is 1/3 and that of exactly 5 biased coin is 2/3, then the probability that all the biased coin are sorted out from the bag in exactly 10 draws is

a. $\frac{5}{10}\frac{{}^{16}C_6}{{}^{20}C_9} + \frac{1}{11}\frac{{}^{15}C_5}{{}^{20}C_9}$
b. $\frac{2}{33}\left[\frac{{}^{16}C_6 + 5\,{}^{15}C_5}{{}^{20}C_9}\right]$
c. $\frac{5}{33}\frac{{}^{16}C_7}{{}^{20}C_9} + \frac{1}{11}\frac{{}^{15}C_6}{{}^{20}C_9}$
d. none of these

76. Cards are drawn one by one without replacement from a pack of 52 cards. The probability that 10 cards will precede the first ace is

a. 241/1456
b. 164/4165
c. 451/884
d. none of these

77. If any four numbers are selected and they are multiplied, then the probability that the last digit will be 1, 3, 5 or 7 is

a. 4/625
b. 18/625
c. 16/625
d. none of these

78. Four numbers are multiplied together. Then the probability that the product will be divisible by 5 or 10 is

a. 369/625 **b.** 399/625
c. 123/625 **d.** 133/625

79. A coin is tossed 7 times. Then the probability that at least 4 consecutive heads appear is

a. 3/16 **b.** 5/32
c. 3/16 **d.** 1/8

80. If odds against solving a question by three students are 2:1, 5:2 and 5:3, respectively, then probability that the question is solved only by one student is

a. 31/56 **b.** 24/56
c. 25/56 **d.** none of these

81. An unbiased coin is tossed 6 times. The probability that third head appears on the sixth trial is

a. 5/16 **b.** 5/32
c. 5/8 **d.** 5/64

82. There are two urns A and B. Urn A contains 5 red, 3 blue and 2 white balls, urn B contains 4 red, 3 blue and 3 white balls. An urn is choosen at random and a ball is drawn. Probability that the ball drawn is red is

a. 9/10 **b.** 1/2
c. 11/20 **d.** 9/20

83. Three critics review a book. Odds in favour of the book are 5:2, 4:3 and 3:4, respectively, for the three critics. The probability that majority are in favour of the book is

a. 35/49 **b.** 125/343
c. 164/343 **d.** 209/343

84. Let A and B are events of an experiment and $P(A) = 1/4$, $P(A \cup B) = 1/2$, then value of $P(B/A^c)$ is

a. 2/3 **b.** 1/3
c. 5/6 **d.** 1/2

85. The probability that an automobile will be stolen and found within one week is 0.0006. The probability that an automobile will be stolen is 0.0015. The probability that a stolen automobile will be found in one week is

a. 0.3 **b.** 0.4
c. 0.5 **d.** 0.6

86. A pair of numbers is picked up randomly (without replacement) from the set {1, 2, 3, 5, 7, 11, 12, 13, 17, 19}. The probability that the number 11 was picked given that the sum of the numbers was even is nearly

a. 0.1 **b.** 0.125
c. 0.24 **d.** 0.18

87. An unbiased cubic die marked with 1, 2, 2, 3, 3, 3 is rolled 3 times. The probability of getting a total score of 4 or 6 is

a. 16/216 **b.** 50/216
c. 60/216 **d.** none of these

88. A bag contains 3 red and 3 green balls and a person draws out 3 at random. He then drops 3 blue balls into the bag and again draws out 3 at random. The chance that the 3 later balls being all of different colours is

a. 15% **b.** 20%
c. 27% **d.** 40%

89. A bag contains 20 coins. If the probability that bag contains exactly 4 biased coin is 1/3 and that of exactly 5 biased coin is 2/3, then the probability that all the biased coin are sorted out from the bag in exactly 10 draws is

a. $\frac{5}{33}\frac{{}^{16}C_6}{{}^{20}C_9}+\frac{1}{11}\frac{{}^{15}C_5}{{}^{20}C_9}$ **b.** $\frac{2}{33}\left[\frac{{}^{16}C_6+5\ {}^{15}C_5}{{}^{20}C_9}\right]$

c. $\frac{5}{33}\frac{{}^{16}C_7}{{}^{20}C_9}+\frac{1}{11}\frac{{}^{15}C_6}{{}^{20}C_9}$ **d.** none of these

90. A doctor is called to see a sick child. The doctor knows (prior to the visit) that 90% of the sick children in that neighbourhood are sick with the flu, denoted by F, while 10% are sick with the measles, denoted by M. A well-known symptom of measles is a rash, denoted by R. The probability of having a rash for a child sick with the measles is 0.95. However, occasionally children with the flu also develop a rash, with conditional probability 0.08. Upon examination the child, the doctor finds a rash. Then what is the probability that the child has the measles?

a. 91/165 **b.** 90/163
c. 82/161 **d.** 95/167

91. A dice is thrown six times, it being known that each time a different digit is shown. The probability that a sum of 12 will be obtained in the first three throws is

a. 5/24 **b.** 25/216
c. 3/20 **d.** 1/12

92. A composite number is selected at random from the first 30 natural numbers and it is divided by 5. The probability that there will be a remainder is

a. 14/19 **b.** 5/19
c. 5/6 **d.** 7/15

93. Let E be an event which is neither a certainty nor an impossibility. If probability is such that $P(E) = 1 + \lambda + \lambda^2$ and $P(E') = (1 + \lambda)^2$ in terms of an unknown λ. Then $P(E)$ is equal to

a. 1 **b.** 3/4
c. 1/4 **d.** none of these

94. A student can solve 2 out of 4 problems of mathematics, 3 out of 5 problem of physics and 4 out of 5 problems of chemistry. There are equal number of books of math, physics and chemistry in his shelf. He selects one book randomly and attempts 10 problems from it. If he solves the first problem, then the probability that he will be able to solve the second problem is

a. 2/3 **b.** 25/38
c. 13/21 **d.** 14/23

95. A fair die is tossed repeatedly. A wins if it is 1 or 2 on two consecutive tosses and B wins if it is 3, 4, 5 or 6 on two consecutive tosses. The probability that A wins if the die is tossed indefinitely is

a. 1/3 **b.** 5/21

c. 1/4 **d.** 2/5

96. Whenever horses a, b, c race together, their respective probabilities of winning the race are 0.3, 0.5 and 0.2, respectively. If they race three times the probability that 'the same horse wins all the three races, and the probability that a, b, c each wins one race are, respectively

a. 8/50, 9/50 **b.** 16/100, 3/100

c. 12/50, 15/50 **d.** 10/50, 8/50

97. Five different games are to be distributed among 4 children randomly. The probability that each child get atleast one game is

a. 1/4 **b.** 15/64

c. 21/64 **d.** none of these

98. Forty teams play a tournament. Each team plays every other team just once. Each game results in a win for one team. If each team has a 50% chance of winning each game, the probability that at the end of the tournament, every team has won a different number of games is

a. 1/780 **b.** $40!/2^{780}$

c. $40!/3^{780}$ **d.** none of these

99. If three squares are selected at random from chessboard, then the probability that they form the letter 'L' is

a. $196/{}^{64}C_3$ **b.** $49/{}^{64}C_3$

c. $36/{}^{64}C_3$ **d.** $98/{}^{64}C_3$

100. A bag has 10 balls. Six balls are drawn in an attempt and replaced. Then another draw of 5 balls is made from the bag. The probability that exactly two balls are common to both the draw is

a. 5/21 **b.** 2/21

c. 7/21 **d.** 3/21

101. There are 3 bags. Bag 1 contains 2 red and $a^2 - 4a + 8$ black balls, bag 2 contains 1 red and $a^2 - 4a + 9$ black balls and bag 3 containa 3 red and $a^2 - 4a + 7$ black balls. A ball is drawn at random from at random chosen bag. Then the maximum value of probability that it is a red ball is

a. 1/3 **b.** 1/2

c. 2/9 **d.** 4/9

102. The probability that a random chosen three-digit number has exactly 3 factors is

a. 2/225 **b.** 7/900

c. 1/800 **d.** none of these

103. Let p, q be chosen one by one from the set $\{1, \sqrt{2}, \sqrt{3}, 2, e, \pi\}$ with replacement. Now a circle is drawn taking (p, q) as its centre. Then the probability that at the most two rational points exist on the circle is (rational points are those points whose both the coordinates are rational)

a. 2/3 **b.** 7/8

c. 8/9 **d.** none of these

104. An event X can take place in conjuction with any one of the mutually exclusive and exhaustive events A, B and C. If A, B, C are equiprobable and the probability of X is 5/12, and the probability of X taking place when A has happened is 3/8, while it is 1/4 when B has taken place, then the probability of X taking place in conjuction with C is

a. 5/8 **b.** 3/8

c. 5/24 **d.** none of these

105. On a Saturday night, 20% of all drivers in U.S.A. are under the influence of alcohol. The probability that a driver under the influence of alcohol will have an accident is 0.001. The probability that a sober driver will have an accident is 0.0001. If a car on a Saturday night smashed into a tree, the probability that the driver was under the influence of alcohol is

a. 3/7 **b.** 4/7

c. 5/7 **d.** 6/7

106. A purse contains 2 six-sided dice. One is a normal fair die, while the other has two 1's, two 3's, and two 5's. A die is picked up and rolled. Because of some secret magnetic attraction of the unfair die, there is 75% chance of picking the unfair die and a 25% chance of picking a fair die. The die is rolled and shows up the face 3. The probability that a fair die was picked up is

a. 1/7 **b.** 1/4

c. 1/6 **d.** 1/24

107. Five different marbles are placed in 5 different boxes randomly. Then the probability that exactly two boxes remain empty is (each box can hold any number of marbles)

a. 2/5 **b.** 12/25

c. 3/5 **d.** none of these

108. There are 10 prizes, five A's, three B's and two C's, placed in identical sealed envelopes for the top 10 contestants in a mathematics contest. The prizes are awarded by allowing winners to select an envelope at random from those remaining. When the 8th contestant goes to select the prize, the probability that the remaining three prizes are one A, one B and one C is

a. 1/4 **b.** 1/3

c. 1/12 **d.** 1/10

109. Thirty-two players ranked 1 to 32 are playing in a knockout tournament. Assume that in every match between any two players, the better-ranked player wins, the probability that ranked 1 and ranked 2 players are winner and runner up, respectively, is

a. 16/31 **b.** 1/2

c. 17/31 **d.** none of these

110. Three integers are chosen at random from the set of first 20 natural numbers. The chance that their product is a multiple of 3 is

a. 194/285 **b.** 1/57

c. 13/19 **d.** 3/4

111. A car is parked among N cars standing in a row, but not at either end. On his return, the owner finds that exactly 'r' of the N places are still occupied. The probability that the places neighouring his car are empty is

a. $\dfrac{(r-1)!}{(N-1)!}$
b. $\dfrac{(r-1)!\ (N-r)!}{(N-1)!}$
c. $\dfrac{(N-r)(N-r-1)}{(N+1)(N+2)}$
d. $\dfrac{{}^{N-r}C_2}{{}^{N-1}C_2}$

112. One ticket is selected at random from 100 tickets numbered 00, 01, 02, ..., 99. Suppose A and B are the sum and product of the digit found on the ticket. Then $P((A = 7)/(B = 0))$ is given by
a. 2/13
b. 2/19
c. 1/50
d. none of these

113. A fair coin is tossed 100 times. The probability of getting tails 1, 3, ..., 49 times is
a. 1/2
b. 1/4
c. 1/8
d. 1/16

114. A pair of unbiased dice is rolled together till a sum of either 5 or 7 is obtained. The probability that 5 comes before 7 is
a. 2/5
b. 3/5
c. 4/5
d. none of these

115. If n integers taken at random are multiplied together, then the probability that the last digit of the product is 1, 3, 7 or 9 is
a. $2^n/5^n$
b. $4^n - 2^n/5^n$
c. $4^n/5^n$
d. none of these

116. If A and B each toss three coins. The probability that both get the same number of heads is
a. 1/9
b. 3/16
c. 5/16
d. 3/8

117. Let A be a set containing n elements. A subset P of the set A is chosen at random. The set A is reconstructed by replacing the elements of P, and another subset Q of A is chosen at random. The probability that $P \cap Q$ contains exactly $m(m < n)$ elements is
a. $3^{n-m}/4^n$
b. ${}^nC_m \times 3^m/4^n$
c. ${}^nC_m \times 3^{n-m}/4^n$
d. none of these

118. A fair die is thrown 20 times. The probability that on the 10th throw, the fourth six appears is
a. ${}^{20}C_{10} \times 5^6/6^{20}$
b. $120 \times 5^7/6^{10}$
c. $84 \times 5^6/6^{10}$
d. none of these

119. If p is the probability that a man aged x will die in a year, then the probability that out of n men $A_1, A_2, \ldots, A_n$ each aged x, A_1 will die in an year and be the first to die is
a. $1-(1-p)^n$
b. $(1-p)^n$
c. $1/n\,[1-(1-p)^n]$
d. $1/n\,(1-p)^n$

120. A bag contains n white and n black balls. Pairs of balls are drawn without replacement until the bag is empty. The probability that each pair consists of one white and one black ball is
a. $1/{}^{2n}C_n$
b. $2n/{}^{2n}C_n$
c. $2n/n!$
d. $2n/(2n!)$

121. A man alternately tosses a coin and throws a die beginning with the coin. The probability that he gets a head in the coin before he gets a 5 or 6 in the dice is
a. 3/4
b. 1/2
c. 1/3
d. none of these

122. There are 3 bags which are known to contain 2 white and 3 black, 4 white and 1 black, and 3 white and 7 black balls, respectively. A ball is drawn at random from one of the bags and found to be a black ball. Then the probability that it was drawn from the bag containing the most black ball is
a. 7/15
b. 5/19
c. 3/4
d. None of these

123. Consider $f(x) = x^3 + ax^2 + bx + c$. Parameters a, b, c are chosen, respectively, by throwing a die three times. Then the probability that $f(x)$ is an increasing function is
a. 5/36
b. 8/36
c. 4/9
d. 1/3

124. A fair coin is tossed 10 times. Then the probability that two heads do not occur consecutively is
a. 7/64
b. 1/8
c. 9/16
d. 9/64

125. If a and b are chosen randomly from the set consisting of numbers 1, 2, 3, 4, 5, 6 with replacement. Then the probability that $\lim_{x\to 0} [(a^x + b^x)/2]^{2/x} = 6$ is
a. 1/3
c. 1/4
c. 1/9
d. 2/9

126. An artillery target may be either at point I with probability 8/9 or at point II with probability 1/9. We have 55 shells, each of which can be fired either rat point I or II. Each shell may hit the target, independent of the other shells, with probability 1/2. Maximum number of shells must be fired at point I to have maximum probability is
a. 20
b. 25
c. 29
d. 35

127. An urn contains 3 red balls and n white balls. Mr. A draws two balls together from the urn. The probability that they have the same colour is 1/2. Mr. B draws one balls form the urn, notes its colour and replaces it. He then draws a second ball from the urn and finds that both balls have the same colour is 5/8. The possible value of n is
a. 9
b. 6
c. 5
d. 1

128. A hat contains a number of cards with 30% white on both sides, 50% black on one side and white on the other side, 20% black on both sides. The cards are mixed up, and a single card is drawn at random and placed on the table. Its upper side shows up black. The probability that its other side is also black is
a. 2/9
b. 4/9
c. 2/3
d. 2/7

129. All the jacks, queens, kings and aces of a regular 52 cards deck are taken out. The 16 cards are thoroughly shuffled and my

opponent, a person who always tells the truth, simultaneously draws two cards at random and says, 'I hold at least one ace'. The probability that he holds two aces is

a. 2/8 b. 4/9
c. 2/3 d. 1/9

130. Mr. A lives at origin on the Cartesian plane and has his office at (4, 5). His friend lives at (2, 3) on the same plane. Mr. A can go to his office travelling one block at a time either in the $+y$ or $+x$ direction. If all possible paths are equally likely then the probability that Mr. A passed his friends house is (shortest path for any event must be considered)

a. 1/2 b. 10/21
c. 1/4 d. 11/21

Multiple Correct Answers Type

Solutions on page 9.60

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. If A and B are two independent events such that $P(A)=1/2$ and $P(B)=1/5$, then

a. $P(A\cup B)=3/5$ b. $P(A/B)=1/4$
c. $P(A/A\cup B)=5/6$ d. $P(A\cap B/\bar{A}\cup\bar{B})=0$

2. Let A and B be two events such that $P(A\cap B')=0.20$, $P(A'\cap B)=0.15$ and $P(A$ and B both fail$)=0.10$. Then

a. $P(A/B)=2/7$ b. $P(A)=0.3$
c. $P(A\cup B)=0.55$ d. $P(B/A)=1/2$

3. If A and B are two events such that $P(A)=3/4$ and $P(B)=5/8$, then

a. $P(A\cup B)\geq 3/4$ b. $P(A'\cap B)\leq 1/4$
c. $3/8\leq P(A\cap B)\leq 5/8$ d. $3/8\leq P(A\cap B)\leq 5/8$

4. If A and B are two mutually exclusive events, then

a. $P(A)\leq P(\bar{B})$ b. $P(A)>P(B)$
c. $P(B)\leq P(\bar{A})$ d. $P(A)>P(B)$

5. The probability that a married man watches a certain TV show is 0.4 and the probability that a married woman watches the show is 0.5. The probability that a man watches the show, given that his wife does, is 0.7. Then

a. the probability that married couple watchs the show is 0.35
b. the probability that a wife watches the show given that her husband does is 7/8
c. the probability that atleat one person of a married couple will watch the show is 0.55
d. none of these

6. A and B are two events defined as follows:

A: It rains today with $P(A)=40\%$
B: It rains tomorrow with $P(B)=50\%$
Also, P(it rains today and tomorrow) $=30\%$
Also, E_1: $P((A\cap B)/(A\cup B))$ and E_2: $P(\{(A\cap\bar{B})$or $(B\cup\bar{A})\}/(A\cup B))$. Then which of the following is/are true?

a. A and B are independent
b. $P(A/B)<P(B/A)$
c. E_1 and E_2 are equiprobable
d. $P(A/(A\cup B))=P(B/(A\cup B))$

7. Two numbers are randomly selected and multiplied. Consider two events E_1 and E_2 defined as

E_1: Their product is divisible by 5
E_2: Unit's places in their product is 5

Which of the following statement is/are correct?

a. E_1 is twice as likely to occur as E_2
b. E_1 and E_2 are disjoint
c. $P(E_2/E_1)=1/4$
d. $P(E_1/E_2)=1$

8. Probability if n heads in $2n$ tosses of a fair coin can be given by

a. $\prod_{r=1}^{n}\left(\frac{2r-1}{2r}\right)$ b. $\prod_{r=1}^{n}\left(\frac{n+r}{2r}\right)$

c. $\sum_{r=0}^{n}\left(\frac{{}^nC_r}{2^n}\right)^2$ d. $\dfrac{\sum_{r=0}^{n}({}^nC_r)^2}{\left(\sum_{r=0}^{2n}{}^{2n}C_r\right)}$

9. The probability that a 50-year-old man will be alive at 60 is 0.83 and the probability that a 45-year-old woman will be alive at 55 is 0.87. Then

a. the probability that both will be alive is 0.7221
b. at least one of them will alive is 0.9779
c. at least one of them will alive is 0.8230
d. the probability that both will be alive is 0.6320

10. Which of the following statement is/are correct?

a. Three coins are tossed once. At least two of them must land the same way. No mater whether they land heads or tails, the third coin is equally likely to land either the same way or oppositely. So, the chance that all the three coins land the same way is 1/2.
b. Let $0<P(B)<1$ and $P(A/B)=P(A/B^C)$. Then A and B are independent.
c. Suppose an urn contains 'w' white and 'b' black balls and a ball is drawn from it and is replaced along with 'd' additional balls of the same colour. Now a second ball is drawn from it. The probability that the second drawn ball is white is independent of the value of 'd'.
d. A, B, C simultaneously satisfy

$P(ABC)=P(A)\,P(B)\,P(C)$
$P(AB\bar{C})=P(A)\,P(B)\,P(\bar{C})$
$P(A\bar{B}C)=P(A)\,P(\bar{B})\,P(C)$
$P(A\text{–}BC)=P(\bar{A})\,P(B)\,P(C)$

Then A, B, C are independent.

11. A bag initially contains 1 red and 2 blue balls. An experiment consisting of selecting a ball at random, noting its colour and replacing it together with an additional ball of the same colour. If three such trials are made, then

a. probability that at least one blue ball is drawn is 0.9

b. probability that exactly one blue ball is drawn is 0.2

c. probability that all the drawn balls are red given that all the drawn balls are of same colour is 0.2

d. probability that atleast one red ball is drawn is 0.6

12. $P(A) = 3/8$; $P(B) = 1/2$; $P(A \cup B) = 5/8$, which of the following do/does hold good?

a. $P(A^C/B) = 2P(A/B^C)$ b. $P(B) = P(A/B)$

c. $15\ P(A^C/B^C) = 8P(B/A^C)$ d. $P(A/B^C) = (A \cap B)$

13. The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. The chances of the events are

a. $p_1 = 1/9$ b. $p_1 = 1/16$

c. $p_2 = 1/3$ d. $p_2 = 1/4$

14. A bag contains b blue balls and r red balls. If two balls are drawn at random, the probability of drawing two red balls is five times the probability of drawing two blue balls. Furthermore, the probability of drawing one ball of each colour is six times the probability of drawing two blue balls. Then

a. $b + r = 9$ b. $br = 18$

c. $|b - r| = 4$ d. $b/r = 2$

15. In a precision bombing attack, there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. The number of bombs which should be dropped to give a 99% chance or better of completely destroying the target can be

a. 12 b. 11

c. 10 d. 13

16. If A and B are two events, the probability that exactly one of them occurs is given by

a. $P(A) + P(B) - 2P(A \cap B)$ b. $P(A \cap \bar{B}) + P(\bar{A} \cap B)$

c. $P(A \cup B) - P(A \cap B)$ d. $P(\bar{A}) + P(\bar{B}) - 2P(\bar{A} \cap \bar{B})$

17. If A and B are two events, then which one of the following is/are always true?

a. $P(A \cap B) \geq P(A) + P(B) - 1$

b. $P(A \cap B) \leq P(A)$

c. $P(A' \cap B') \geq P(A') + P(B') - 1$

d. $P(A \cap B) = P(A)\ P(B)$

18. If A and B are two independent events such that $P(A) = 1/2$, $P(B) = 1/5$, then

a. $P(A/B) = 1/2$ b. $P\left(\dfrac{A}{A \cup B}\right) = \dfrac{5}{6}$

c. $P\left(\dfrac{A \cap B}{A' \cup B'}\right) = 0$ d. none of these

19. If A and B are two independent events such that $P(\bar{A} \cap B) = 2/15$ and $P(A \cap \bar{B}) = 1/6$, then $P(B)$ is

a. 1/5 b. 1/6

c. 4/5 d. 5/6

20. Two buses A and B are scheduled to arrive at a town central bus station at noon. The probability that bus A will be late is 1/5. The probability that bus B will be late is 7/25. The probability that the bus B is late given that bus A is late is 9/10. Then,

a. probability that neither bus will be late on a particular day is 7/10

b. probability that bus A is late given that bus B is late is 18/28

c. probability that at least one bus is late is 3/10

d. probability that at least one bus is in time is 4/5

21. If p and q are chosen randomly from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} with replacement, then the probability that the roots of the equation $x^2 + px + q = 0$

a. are real is 33/50 b. are imaginary is 19/50

c. are real and equal is 3/50 d. are real and distinct is 3/5

22. Two numbers are chosen from {1, 2, 3, 4, 5, 6, 7, 8} one after another without replacement. Then the probability that

a. the smaller value of two is less than 3 is 13/28

b. the bigger value of two is more than 5 is 9/14

c. product of two number is even is 11/14

d. none of these

23. A fair coin is tossed 99 times. Let X be the number of times heads occurs. Then $P(X = r)$ is maximum when r is

a. 49 b. 52

c. 51 d. 50

24. If the probability of choosing an integer 'k' out of $2m$ integers 1, 2, 3, ..., $2m$ is inversely proportional to k^4 $(1 \leq k \leq m)$. If x_1 is the probability that chosen number is odd and x_2 is the probability that chosen number is even, then

a. $x_1 > 1/2$ b. $x_1 > 2/3$

c. $x_2 < 1/2$ d. $x_2 < 2/3$

Reasoning Type

Solutions on page 9.64

Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.

b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.

c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** For events A and B of sample space if $P\left(\dfrac{A}{B}\right) \geq P(A)$, then $P\left(\dfrac{B}{A}\right) \geq P(B)$.

Statement 2: $P\left(\dfrac{A}{B}\right) = \dfrac{P(A \cap B)}{P(B)}\ (P(B) \neq 0)$.

2. A fair die is rolled once.

Statement 1: The probability of getting a composite number is 1/3.

Statement 2: There are three possibilities for the obtained number: (i) the number is a prime number, (ii) the number is a composite number and (iii) the number is 1. Hence probability of getting a prime number is 1/3.

3. Statement 1: If $P(A) = 0.25$, $P(B) = 0.50$ and $P(A \cap B) = 0.14$, then the probability that neither A nor B occurs is 0.39.

Statement 2: $\overline{(A \cup B)} = \bar{A} \cup \bar{B}$.

4. Let A and B be two events such that $P(A) = 3/5$ and $P(B) = 2/3$. Then

Statement 1: $4/15 \le P(A \cap B) \le 3/5$.

Statement 2: $2/5 \le P(A/B)) \le 9/10$.

5. Statement 1: If A, B, C be three mutually independent events, then A and $B \cup C$ are also independent events.

Statement 2: Two events A and B are independent if and only if $P(A \cap B) = P(A)\,P(B)$.

6. Statement 1: Out of 5 tickets consecutively numbered, three are drawn at random. The chance that the numbers on them are in A.P. is 2/15.

Statement 2: Out of $2n + 1$ tickets consecutively numbered, three are drawn at random, the chance that the numbers on them are in A.P. is $3n/(4n^2 - 1)$.

7. Let A and B be two event such that $P(A \cup B) \ge 3/4$ and $1/8 \le P(A \cap B) \le 3/8$.

Statement 1: $P(A) + P(B) \ge 7/8$.

Statement 2: $P(A) + P(B) \le 11/8$.

8. Statement 1: The probability of drawing either an ace or a king from a pack of card in a single draw is 2/13.

Statement 2: For two events A and B which are not mutually exclusive, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

9. Let A and B be two independent events.

Statement 1: If $P(A) = 0.4$ and then $P(A \cup \bar{B}) = 0.9$, then $P(B)$ is 1/6.

Statement 2: If A and B are independent, then $P(A \cap B) = P(A)P(B)$.

10. Consider an event for which probability of success is 1/2.

Statement 1: Probability that in n trials, there are r successes where $r = 4k$ and k is an integer is

$$\frac{1}{4} + \frac{1}{2^{n/2+1}} \cos\left(\frac{n\pi}{4}\right)$$

Statement 2: ${}^nC_0 + {}^nC_4 + {}^nC_8 + \cdots = 2^{n/2} \sin\left(\frac{n\pi}{4}\right)$.

11. Statement 1: If A and B are two events such that $0 < P(A)$, $P(B) < 1$, then $P(A/\bar{B}) + P(\bar{A}/\bar{B}) = 3/2$.

Statement 2: If A and B are two events such that $0 < P(A)$, $P(B) < 1$, then

$P(A/B) = P(A \cap B)/P(B)$ and $P(\bar{B}) = P(A \cap \bar{B}) + P(\bar{A} \cap \bar{B})$

12. Statement 1: If a fair coin is tossed 15 times, then the probability of getting head as many times in the first ten throws as in the last five is 3003/32768.

Statement 2: Sum of the series ${}^mC_r\,{}^nC_0 + {}^mC_{r-1}\,{}^nC_1 + \cdots + {}^mC_0\,{}^nC_r = {}^{m+n}C_r$.

13. Statement 1: If $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$ where A and B are the events of numbers occurring on a dice, then $P(A) + P(B) = 1$.

Statement 2: If $A_1, A_2, A_3, \ldots, A_n$ are all mutually exclusive events, then $P(A_1) + P(A_2) + \cdots + P(A_n) = 1$.

14. Statement 1: There are 4 addressed envelopes and 4 letters for each one of them. The probability that no letter is mailed in its correct envelops is 3/8.

Statement 2: The probability that all letters are not mailed in their correct envelope is 23/24.

15. Let A and B be two independent events.

Statement 1: If $P(A) = 0.3$ and $P(A \cup \bar{B}) = 0.8$, then $P(B)$ is 2/7.

Statement 2: $P(\bar{E}) = 1 - P(E)$, where E is any event.

Linked Comprehension Type

Solutions on page 9.66

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

In a class of 10 students, probability of exactly i students passing an examination is directly proportional to i^2. Then answer the following questions:

1. The probability that exactly 5 students passing an examination is

a. 1/11 **b.** 5/77
c. 25/77 **d.** 10/77

2. If a student is selected at random, then the probability that he has passed the examination is

a. 1/7 **b.** 11/35
c. 11/14 **d.** none of these

3. If a students selected at random is found to have passed the examination, then the probability that he was the only student who has passed the examination is

a. 1/3025 **b.** 1/605
c. 1/275 **d.** 1/121

For Problems 4–6

A shoping mall is running a scheme: 'Each packet of detergent "SURF" contains a coupon which bears letter of the word "SURF", if a person buys at least four packets of detergent "SURF", and produce all the letters of the word "SURF", then he gets one free packet of detergent.

4. If a person buys 8 such packets at a time, then number of different combinations of coupon he has

a. 4^8 **b.** 8^4
c. ${}^{11}C_3$ **d.** ${}^{12}C_4$

5. If person buys 8 such packets, then the probability that he gets exactly one free packets is

a. 7/33 b. 102/495
c. 13/55 d. none of these

6. If a person buys 8 such packets, then the probability that he gets two free packets is

a. 1/7 b. 1/5
c. 1/42 d. 1/165

For Problems 7–9

In an objective paper, there are two sections of 10 questions each. For 'section 1', each question has 5 options and only one option is correct and 'section 2' has 4 options with multiple answers and marks for a question in this section is awarded only if he ticks all correct answers. Marks for each question in 'section 1' is 1 and in 'section 2' is 3. (There is no negative marking.)

7. If a candidate attempts only two questions by guessing, one from 'section 1' and one from 'section 2', the probability that he scores in both questions is

a. 74/75 b. 1/25
c. 1/15 d. 1/75

8. If a candidate in total attempts 4 questions all by guessing, then the probability of scoring 10 marks is

a. $1/15(1/15)^3$ b. $4/5(1/15)^3$
c. $1/5(14/15)^3$ d. none of these

9. The probability of getting a score less than 40 by answering all the questions by guessing in this paper is

a. $(1/75)^{10}$ b. $1-(1/75)^{10}$
c. $(74/75)^{10}$ d. none of these

For Problems 10–12

There are two die A and B both having six faces. Die A has three faces marked with 1, two faces marked with 2 and one face marked with 3. Die B has one face marked with 1, two faces marked with 2 and three faces marked with 3. Both dices are thrown randomly once. If E be the event of getting sum of the numbers appearing on top faces equal to x and let $P(E)$ be the probability of even E, then

10. $P(E)$ is maximum when x equal to

a. 5 b. 3
c. 4 d. 6

11. $P(E)$ is minimum when x equals to

a. 3 b. 4
c. 5 d. 6

12. When $x = 4$, then $P(E)$ is equal to

a. 5/9 b. 6/7
c. 7/18 d. 8/19

For Problems 13–15

A JEE aspirant estimates that she will be successful with an 80% chance if she studies 10 hours per day, with a 60% chance if she studies 7 hours per day and with a 40% chance if she studies 4 hours per day. She further believes that she will study 10 hours, 7 hours and 4 hours per day with probabilities 0.1, 0.2 and 0.7, respectively.

13. The chance she will be successful is

a. 0.28 b. 0.38
c. 0.48 d. 0.58

14. Given that she is successful, the chance that she studied for 4 hours is

a. 6/12 b. 7/12
c. 8/12 d. 9/12

15. Given that she does not achieve success, the chance she studied for 4 hour is

a. 18/26 b. 19/26
c. 20/26 d. 21/26

For Problems 16–18

Let S and T are two events defined on a sample space with probabilities $P(S) = 0.5$, $P(T) = 0.69$, $P(S/T) = 0.5$

16. Events S and T are

a. mutually exclusive
b. independent
c. mutually exclusive and independent
d. neither mutually exclusive nor independent

17. The value of $P(S$ and $T)$ is

a. 0.3450 b. 0.2500
c. 0.6900 d. 0.350

18. The value of $P(S$ or $T)$ is

a. 0.6900 b. 1.19
c. 0.8450 d. 0

For Problems 19–21

An amoeba either splits into two or remains the same or eventually dies out immediately after completion of every second with probabilities, respectively, 1/2, 1/4 and 1/4. Let the initial amoeba be called as mother amoeba and after every second, the amoeba, if it is distinct from the previous one, be called as 2nd, 3rd, … generations.

19. The probability that immediately after completion of 2 s all the amoeba population dies out is

a. 9/32 b. 11/32
c. 1/2 d. 3/32

20. The probability that after 2 s exactly 4 amoeba are alive is

a. 1/16 b. 1/8
c. 1/4 d. 1/2

21. The probability that amoeba population will be maximum after completion of 3 s is

a. $1/2^7$ b. $1/2^6$
c. $1/2^8$ d. none of these

For Problems 22–24

A cube having all of its sides painted is cut by two horizontal, two vertical and other two planes so as to form 27 cubes all having the same dimensions. Of these cubes, a cube is selected at random.

22. The probability that the cube selected has none of its sides painted is

a. 1/9 **b.** 1/27
c. 1/18 **d.** 5/54

23. The probability that the cube selected has two sides painted is

a. 1/9 **b.** 4/9
c. 8/27 **d.** none of these

24. The total number of cubes having at least one of its sides painted is

a. 8 **b.** 53
c. 49 **d.** 26

For Problems 25–27

There are some experiments in which the outcomes cannot be identified discretely. For example, an ellipse of eccentricity $2\sqrt{2}/3$ is inscribed in a circle and a point within the circle is chosen at random. Now, we want to find the probability that this point lies outside the ellipse. Then, the point must lie in the shaded region shown in Fig. 9.8. Let the radius of the circle be a and length of minor axis of the ellipse be $2b$. Given that

$$1-\frac{b^2}{a^2}=\frac{8}{9} \Rightarrow \frac{b^2}{a^2}=\frac{1}{9}$$

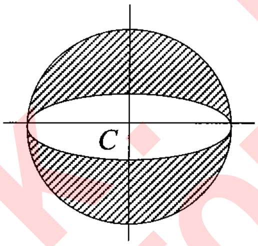


Fig. 9.8

Then, the area of circle serves as sample space and area of the shaded region represents the area for favourable cases. Then, required probability is

$$p=\frac{\text{area of shaded region}}{\text{area of circle}}$$
$$=\frac{\pi a^2-\pi ab}{\pi a^2}$$
$$=1-\frac{b}{a}$$
$$=1-\frac{b}{a}$$
$$=1-\frac{1}{3}$$
$$=\frac{2}{3}$$

Now answer the following questions.

25. A point is selected at random inside a circle. The probability that the point is closer to the centre of the circle than to its circumference is

a. 1/4 **b.** 1/2
c. 1/3 **d.** $1/\sqrt{2}$

26. Two persons A and B agree to meet at a place between 5 and 6 pm. The first one to arrive waits for 20 min and then leave. If the time of their arrival be independent and at random, then the probability that A and B meet is

a. 1/3 **b.** 1/3
c. 2/3 **d.** 5/9

27. If points x, y are chosen randomly from the intervals [0, 2] and [0, 1], respectively, then the probability that $y \le x^2$ is

a. 1/2 **b.** 2/3
c. 3/4 **d.** 1/4

For Problems 28–30

If the squares of a 8 × 8 chessboard are painted either red or black at random.

28. The probability that not all the squares in any column are alternating in colour is

a. $(1-1/2^7)^8$ **b.** $1/2^{56}$
c. $1-1/2^7$ **d.** none of these

29. The probability that the chessboard contains equal number of red and black squares is

a. $\frac{^{64}C_{32}}{2^{64}}$ **b.** $\frac{64!}{32^{264}!}$
c. $\frac{2^{32}-1}{2^{64}}$ **d.** none of these

30. The probability that all the squares in any column are of same colour and that of a row are of alternating colour is

a. $1/2^{64}$ **b.** $1/2^{63}$
c. 1/2 **d.** none of these

For Problems 31–33

Two fair dice are rolled. Let $P(A_i) > 0$ denote the event that the sum of the faces of the dice is divisible by i.

31. Which one of the following events is most probable?

a. A_3 **b.** A_4
c. A_5 **d.** A_6

32. For which one of the following pairs (i, j) are the events A_i and A_j independent?

a. (3, 4) **b.** (4, 6)
c. (2, 3) **d.** (4, 2)

33. The number of all possible ordered pairs (i, j) for which the events A_i and A_j are independent is

a. 6 **b.** 12
c. 13 **d.** 25

For Problems 34–36

A player tosses a coin and scores one point for every head and two points for every tail that turns up. He plays on until his score reaches or passes n. P_n denotes the probability of getting a score of exactly n.

34. The value of P_n is equal to

a. $(1/2)[P_{n-1}+P_{n-2}]$ **b.** $(1/2)[2P_{n-1}+P_{n-2}]$
c. $(1/2)[P_{n-1}+2P_{n-2}]$ **d.** none of these

35. The value of $P_n + (1/2)\, P_{n-1}$ is equal to

a. 1/2 b. 2/3
c. 1 d. none of these

36. Which of the following is not true?

a. $P_{100} > 2/3$ b. $P_{101} < 2/3$
c. $P_{100}, P_{101} > 2/3$ d. none of these

For Problems 37–39

The probability that a family has exactly n children is αp^n, $n \geq 1$. All sex distributions of n children in a family have the same probability.

37. The probability that a family contains exactly k boys is (where $k \geq 1$)

a. $\alpha p^k(1-p)^{-k-1}$ b. $2\alpha p^k(2-p)^{-k-1}$
c. $2\alpha p^k(2-p)^{-k}$ d. $2\alpha p^{k-1}(2-p)^{-k-1}$

38. The probability that a family includes at least one boy is

a. $\dfrac{\alpha^2 p}{(2-p)(1-p)}$ b. $\dfrac{\alpha p^2}{(2-p)(1-p)}$

c. $\dfrac{\alpha p}{(2-p)(1-p)}$ d. $\dfrac{2\alpha p}{(2-p)(1-p)}$

39. Given that a family includes at least one boy, show that the probability that there are two or more boys is

a. $p/(2-p)$ b. $p/(1-p)$
c. $p/(2-p)^2$ d. $p/(1-p)^2$

Matrix-Match Type

Solutions on page 9.69

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a→p, a→s, b→q, b→r, c→p, c→q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1. An urn contains four black and eight white balls. Three balls are drawn from the urn without replacement. Three events are defined on this experiment.

A: Exactly one black ball is drawn.

B: All balls are drawn are of the same colour.

C: Third drawn ball is black.

Match the entries of column I with none, one or more entries of column II.

Column I	Column II
a. The events A and B are	p. mutually exclusive
b. The events B and C are	q. independent
c. The events C and A are	r. neither independent nor mutually exclusive
d. The events A, B and C are	s. exhaustive

2.

Column I	Column II
a. If the probability of getting at least one head is at least 0.8 is n trials then value of n can be	p. 2
b. One mapping is selected at random from all mappings of the set $s = \{1, 2, 3, \ldots, n\}$ into itself. If the probability that the mapping being one-one is 3/32, then find the value of n is	q. 3
c. If m is selected at random from set $\{1, 2, \ldots, 10\}$ and the probability that the quadratic equation $2x^2 + 2mx + m + 1 = 0$ has real roots is k, then value of $5k$ is more than	r. 4
d. A man firing at a distant target as 20% chance of hitting the target in one shoot. If P be the probability of hitting the target in 'n' attempts, where $20P^2 - 13P + 2 \leq 0$, then the ratio of maximum and minimum value of n is less than	s. 5

3.

Column I	Column II
a. The probability of a bomb hitting a bridge is 1/2. Two direct hits are needed to destroy it. The number of bombs required so that the probability of the bridge being destroyed is greater than 0.9 can be	p. 4
b. A bag contains 2 red, 3 white and 5 black balls, a ball is drawn its colour is noted and replaced. The number of times, a ball can be drawn so that the probability of getting a red ball for the first time is at least 1/2	q. 6
c. A drawer contains a mixture of red socks and blue socks, at most 17 in all. It so happens that when two socks are selected randomly without replacement, there is a probability of exactly 1/2 that both are red or both are blue. Then number of red socks in the drawer can be	r. 7
d. There are two red, two blue, two white and certain number (greater than 0) of green socks in a drawer. If two socks are taken at random from the drawer without replacement, the probability that they are of the same colour is 1/5, then the number of green socks are	s. 10

4. Let A and B are two independent events such that $P(A) = 1/3$ and $P(B) = 1/4$.

Column I	Column II
a. $P(A \cup B)$ is equal to	p. 1/12
b. $P(A/A \cup B)$ is equal to	q. 1/2
c. $P(B/A' \cap B')$ is equal to	r. 2/3
d. $P(A'/B)$ is equal to	s. 0

5. A bag contains some white and some black balls, all combinations being equally likely. The total number of balls in the bag is 12. Four balls are drawn at random from the bag at random without replacement.

Column I	Column II
a. Probability that all the four balls are black is equal to	**p.** 14/33
b. If the bag contains 10 black and 2 white balls then the probability that all four balls are black is equal to	**q.** 1/3
c. If all the four balls are black, then the probability that the bag contains 10 black balls is equal to	**r.** 70/429
d. Probability that two balls are black and two are white is	**s.** 13/165

6.

Column I	Column II
a. Six different balls are put in three different boxes, none being empty. The probability of putting the balls equal number is	**p.** 20/27
b. Six letters are posted in three letter boxes. The probability that no letter box remains empty is	**q.** 1/6
c. Two persons A and B throw two dice each. If A throw a sum of 9, then the probability of B throwing a sum greater than A is	**r.** 1/3
d. If A and B are independent and $P(A) = 0.3$ and $P(A \cup \bar{B}) = 0.8$, then $P(B)$ is equal to	**s.** 2/7

7. An urn contains r red balls and b black balls.

Column I	Column II
a. If the probability of getting two red balls in first two draws (without replacement) is 1/2, then value of r can be	**p.** 10
b. If the probability of getting two red balls in first two draws (without replacement) is 1/2 and b is an even number, then r can be	**q.** 3
c. If the probability of getting exactly two red balls in four draws (with replacement) from the urn is 3/8 and $b = 10$, then r can be	**r.** 8
d. If the probability of getting exactly n red balls in $2n$ draw (with replacement) is equal to probability of getting exactly n black balls in $2n$ draws (with replacement), then the ratio r/b can be	**s.** 2

8. 'n' whole numbers are randomly chosen and multiplied,

Column I	Column II
a. The probability that the last digit is 1, 3, 7 or 9 is	**p.** $\frac{8^n - 4^n}{10^n}$
b. The probability that the last digit is 2, 4, 6, 8 is	**q.** $\frac{5^n - 4^n}{10^n}$
c. The probability that the last digit is 5 is	**r.** $\frac{4^n}{10^n}$
d. The probability that the last digit is zero is	**s.** $\frac{10^n - 8^n - 5^n + 4^n}{10^n}$

Integer Type

Solutions on page 9.72

1. If the probability of a six-digit number N whose six digits are 1, 2, 3, 4, 5, 6 written as random order is divisible by 6 is p, then the value of $1/p$ is.
2. If the probability that the product of the outcomes of three rolls of a fair dice is a prime number is p, then the value of $1/(4p)$ is.
3. If two loaded dice each have the property that 2 or 4 is three times as likely to appear as 1, 3, 5 or 6 on each roll. When two such dice are rolled, the probability of obtaining a total of 7 is p, then the value of $[1/p]$ is, where $[x]$ represents the greatest integer less than or equal to x.
4. An urn contains three red balls and n white balls. Mr. A draws two balls together from the urn. The probability that they have the same colour is 1/2. Mr. B draws one balls from the urn, notes its colour and replaces it. He then draws a second ball from the urn and finds that both balls have the same colour is. 5/8. The possible value of n is.
5. Suppose A and B are two events with $P(A) = 0.5$ and $P(A \cup B) = 0.8$. Let $P(B) = p$ if A and B are mutually exclusive and $P(B) = q$ if A and B are independent events, then the value of q/p is.
6. There are two red, two blue, two white and certain number (greater than 0) of green socks in a drawer. If two socks are taken at random from the drawer without replacement, the probability that they are of the same colour is 1/5, then the number of green socks are.
7. A drawer contains a mixture of red socks and blue socks, at most 17 in all. It so happens that when two socks are selected randomly without replacement, there is a probability of exactly 1/2 that both are red or both are blue. The largest possible number of red socks in the drawer that is consistent with this data is.
8. Two different numbers are taken from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. The probability that their sum and positive difference are both multiple of 4 is $x/55$, then x equals.
9. Thirty-two players ranked 1 to 32 are playing in a knockout tournament. Assume that in every match between any two players the better ranked player wins, the probability that ranked 1 and ranked 2 players are winner and runner up

respectively is p, then the value of $[2/p]$ is, where [.] represents the greatest integer function

10. Five different games are to be distributed among four children randomly. The probability that each child get at least one game is p, then the value of $[1/p]$ is, where [.] represents the greatest integer function

11 A die is weighted such that the probability of rolling the face numbered n is proportional to n^2 ($n = 1, 2, 3, 4, 5, 6$). The die is rolled twice, yielding the numbers a and b. The probability that $a < b$ is p, then the value of $[2/p]$ is, where [.] represents the greatest integer function.

12. A bag contains 10 different balls. Five balls are drawn simultaneously and then replaced and then seven balls are drawn. The probability that exactly three balls are common to the two drawn is p, then the value of $12p$ is.

13. If A and B are two events such that $P(A) = 0.6$ and $P(B) = 0.8$, if the greatest value that $P(A/B)$ can have is p, then the value of $8p$ is.

14. A die is thrown three times. The chance that the highest number shown on the die is 4 is p, then the value of $[1/p]$ is where [.] represents greatest integer function is.

15. Two cards are drawn from a well shuffled pack of 52 cards. The probability that one is a heart card and the other is a king is p, then the value of $104p$ is.

Archives

Solutions on page 9.73

Subjective Type

1. Balls are drawn one-by-one without replacement from a box containing 2 black, 4 white and 3 red balls till all the balls are drawn. Find the probability that the balls drawn are in the order 2 black, 4 white and 3 red. **(IIT-JEE, 1978)**

2. Six boys and six girls sit in a row randomly. Find the probability that (i) the six girls sit together, (ii) the boys and girls sit alternately. **(IIT-JEE, 1979)**

3. An anti-aircraft gun can take a maximum of four shots at an enemy plane moving away from it. The probabilities of hitting the plane at the first, second, third and fourth shot are 0.4, 0.3, 0.2 and 0.1, respectively. What is the probability that the gun hits the plane? **(IIT-JEE, 1981)**

4. A and B are two candidates seeking admission in IIT. The probability that A is selected is 0.5 and the probability that A and B are selected is at most 0.3. Is it possible that the probability of B getting selected is 0.9? **(IIT-JEE, 1982)**

5. Cards are drawn one at random from a well-shuffled full pack of 52 playing cards until 2 aces are obtained for the first time. If N is the number of cards required to the drawn, then show that

$$P_r\{N = n\} = \frac{(n-1)(52-n)(51-n)}{50 \times 49 \times 17 \times 13}, \text{ where } 2 < n < 50$$

(IIT-JEE, 1983)

6. Let A, B, C be three events such that $P(A) = 0.3$, $P(B) = 0.4$, $P(C) = 0.8$, $P(A \cap B) = 0.88$, $P(A \cap C) = 0.28$, $P(A \cap B \cap C) = 0.09$. If $P(A \cup B \cup C) \geq 0.75$, then show that $0.23 \leq P(B \cap C) \leq 0.48$.

7. A and B are two independent events. The probability that both A and B occur is 1/6 and the probability that neither of them occurs is 1/3. Find the probability of the occurrence of A. **(IIT-JEE, 1984)**

8. In a certain city, only 2 newspapers A and B are published. It is known that 25% of the city population read A and 20% read B while 8% reads both A and B. It is also known that 30% of those who read A but not B look into advertisement and 40% of those who read B but not A look into advertisements while 50% of those who read both A and B look into advertisements. What is the percentage of the population who read an advertisement? **(IIT-JEE, 1984)**

9. In a multiple choice question, there are four alternative answers of which one or more than one is correct. A candidate will get marks on the question only if he ticks the correct answer. The candidate decides to tick answers at random. If he is allowed up to three chances to answer the question, then find the probability that he will get marks on it. **(IIT-JEE, 1985)**

10. A lot contains 20 articles. The probability that the lot contains exactly two defective articles is 0.4 and the probability that it contains exactly 3 defective articles is 0.6. Articles are drawn from the lot at random one by one without replacement and are tested till all defective articles are found. Then find the probability that the testing procedure ends at the twelfth testing. **(IIT-JEE, 1985)**

11. A man takes a step forward with probability 0.4 and backward with probability 0.6. Then find the probability that at the end of eleven steps he is one step away from the starting point. **(IIT-JEE, 1987)**

12. An urn contains 2 white and 2 black balls. A ball is drawn at random. If it is white, it is not replaced into urn otherwise it is replaced along with another ball of the same colour. The process is repeated. Then find the probability that the third ball drawn is black. **(IIT-JEE, 1987)**

13. A box contains two 50-paise coins, five 25-paise coins and a certain fixed number $N(\geq 2)$ of 10 and 5-paise coins. Five coins are taken out of the box at random. Find the probability that the total value of these 5 coins is less than 1 rupee and 50 paise. **(IIT-JEE, 1988)**

14. Suppose the probability for A to win a game against B is 0.4. If A has an option of playing either a 'best of 3 games' or a 'best of 5 games' match against B, which option should be chosen so that the probability of his winning the match is higher? (No game ends in a draw.) **(IIT-JEE, 1989)**

15. A is a set containing n elements. A subset P of A is chosen at random. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen at random. Find the probability that P and Q have no common elements. **(IIT-JEE, 1990)**

16. In a test, an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is 1/3 and the probability that he copies the answer is 1/6. The probability that his answer is correct given that he copied it is 1/8. Find the probability that he knew the answer to the question, given that he correctly answered it.

17. A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as follows:

A: the first bulb is detective

B: the second bulb is non-defective

C: the two bulbs are both defective or both non-defective

Determine whether

(i) A, B, C are pair-wise independent

(ii) A, B, C are independent **(IIT-JEE, 1992)**

18. Numbers are selected at random, one at a time, from the two-digit numbers 00, 01, 02 ..., 99 with replacement. An event E occurs if the only product of the two digits of a selected number is 18. If four numbers are selected, find the probability that the event E occurs at least 3 times. **(IIT-JEE, 1993)**

19. An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well-shuffled pack of 11 cards numbered 2, 3, 4, ...,12 is picked and the number on the card is noted. What is the probability that the noted number is either 7 or 8? **(IIT-JEE, 1994)**

20. In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls should sit together in a back row on adjacent seats? Now, if all the seating arrangements are equally likely, what is the probability of 3 girls sitting together in a back row on adjacent seats? **(IIT-JEE, 1996)**

21. Sixteen players $S_1, S_2, \ldots, S_{16}$ play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of a game played between the two players of the pair. Assume that all the players are of equal strength. **(IIT-JEE, 1997)**

a. Find the probability that the player S_1 is among the eight winners.

b. Find the probability that exactly one of the two players S_1 and S_2 is among the eight winners.

22. If p and q are chosen randomly from the set {1, 2, 3, 4,5, 6, 7, 8, 9, 10} with replacement, determine the probability that the roots of the equation $x^2 + px + q = 0$ are real. **(IIT-JEE, 1997)**

23. Three players, A, B and C, toss a coin cyclically in that order (i.e., is $A, B, C, A, B, C, A, B, \ldots$) till a head shows. Let p be the probability that the coin shows a head. Let α, β and γ be, respectively, the probabilities that A, B and C gets the first head. Prove that $\beta = (1-p)\alpha$. Determine α, β and γ (in terms of p). **(IIT-JEE, 1998)**

24. Eight players $P_1, P_2, \ldots, P_8$ play a knock-out tournament. It is known that whenever the players P_i and P_j play, the player P_i will win if $i < j$. Assuming that the players are paired at random in each round, what is the probability that the player P_4 reaches the final? **(IIT-JEE, 1999)**

25. A coin has probability p of showing head when tossed. It is tossed n times. Let P_n denote the probability that no two (or more) consecutive heads occur. Prove that $p_1 = 1$, $p_2 = 1 - p^2$ and $p_n = (1-p)\,p_{n-1} + p(1-p)\,p_{n-2}$ for all $n \geq 3$. **(IIT-JEE, 2000)**

26. An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. What is the probability that the ball drawn now is white? **(IIT-JEE, 2001)**

27. An unbiased die, with faces numbered 1, 2, 3, 4, 5, 6 is thrown n times and the list of n numbers showing up is noted. What is the probability that, among the numbers 1, 2, 3, 4, 5, 6, only three numbers appear in this list? **(IIT-JEE, 2001)**

28. A box contains N coins, m of which are fair and the rest are biased. The probability of getting a head when a fair coin is tossed is 1/2, while it is 2/3 when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. The first time it shows head and the second time it shows tail. What is the probability that the coin drawn is fair? **(IIT-JEE, 2002)**

29. For a student to qualify, he must pass at least two out of three exams. The probability that he will pass the first exam is p. If he fails in one of the exams then the probability of his passing in the next exam is $p/2$, otherwise it remains the same. Find the probability that he will qualify. **(IIT-JEE, 2003)**

30. A is targeting to B, B and C are targeting to A. Probability of hitting the target by A, B and C are 2/3, 1.2 and 1/3, respectively. If A is hit, then find the probability that B hits the target and C does not. **(IIT-JEE, 2003)**

31. A and B are two independent events. C is an event in which exactly one of A or B occurs. Prove that $P(C) \geq P(A \cup B)$ $P(A \cap B)$. **(IIT-JEE, 2004)**

32. A box contains 12 red and 6 white balls. Balls are drawn from the bag one at a time without replacement. If in 6 draws, there are at least 4 white balls, find the probability that exactly one white ball is drawn in the next two draws. (Binomial coefficients can be left as such.) **(IIT-JEE, 2004)**

33. A person goes to office either by car, scooter, bus or train, the probability of which being 1/7, 3/7, 2/7 and 1/7, respectively. Probability that he reaches office late, if he takes car, scooter, bus or train is 2/9, 1/9, 4/9 and 1/9 respectively. Given that he reached office in time, then what is the probability that he travelled by a car. **(IIT-JEE, 2005)**

Objective Type

Fill in the blanks

1. For a biased die, the probability for the different face to turn up are given below:

Face	1	2	3	4	5	6
Probability	0.1	0.32	0.21	0.15	0.05	0.17

This die is tossed and you are told that either face 1 or face 2 has turned up. Then the probability that it is face 1 is ______. **(IIT-JEE, 1981)**

2. $P(A \cup B) = P(A \cap B)$ if and only if the relation between $P(A)$ and $P(B)$ is ______. **(IIT-JEE, 1985)**

3. A box contains 100 tickets numbered 1, 2, ..., 100. Two tickets are chosen at random. It is given that the maximum number on the two chosen tickets is not more than 10. The minimum number on them is 5 with probability ______. **(IIT-JEE, 1985)**

4. If $(1 + 3p)/3$, $(1 - p)/4$ and $(1 - 2p)/2$ are the probabilities of three mutually exclusive events, then the set of all values of p is ______. **(IIT-JEE, 1986)**

5. Urn A contains 6 red and 4 black balls and urn B contains 4 red and 6 black balls. One ball is drawn at random from urn P and placed in urn Q. Then one ball is drawn at random from urn Q and placed in urn A. If one ball is now drawn at random from urn P, the probability that it is found to be red is ______. **(IIT-JEE, 1988)**

6. A pair of fair dice is rolled together till a sum of either 5 or 7 is obtained. Then the probability that 5 comes before 7 is ______. **(IIT-JEE, 1989)**

7. Let A and B be two events such that $P(A) = 0.3$ and $P(A \cup B) = 0.8$. If A and B are independent events, then $P(B) =$ ______. **(IIT-JEE, 1990)**

8. Three faces of a fair die are yellow, two faces red and one blue. The die is tossed three times. The probability that the colours, yellow, red and blue, appear in the first, second and the third tosses, respectively, is ______. **(IIT-JEE, 1992)**

9. If two events A and B are such that $P(A^C) = 0.3$, $P(B) = 0.4$ and $P(A \cap B^C) = 0.5$, then $P(B/(A \cup B^C)] =$ ______. **(IIT-JEE, 1994)**

10. Three numbers are chosen at random without replacement from {1, 2, ..., 10}. The probability that the minimum of the chosen numbers is 3, or their maximum is 7, is ______. **(IIT-JEE, 1997)**

True or false

1. If the letters of the word 'ASSASSIN' are written down at random in a row, the probability that no two S's occur together is 1/35. **(IIT-JEE, 1983)**

2. If the probability for A to fail in an examination is 0.2 and that for B is 0.3, then the probability that either A or B fails is 0.5. **(IIT-JEE, 1989)**

Multiple choice questions with one correct answer

1. Two fair dice are tossed. Let x be the event that the first die shows an even number and y be the event that the second die shows an odd number. The two events x and y are
   a. mutually exclusive
   b. independent and mutually exclusive
   c. dependent
   d. none of these **(IIT-JEE, 1979)**

2. Two events A and B have probabilities 0.25 and 0.50, respectively. The probability that both A and B occur simultaneously is 0.14. Then the probability that neither A nor B occurs is
   a. 0.39 **b.** 0.25
   c. 0.11 **d.** none of these
   (IIT-JEE, 1980)

3. The probability that an event A happens in one trial of an experiment is 0.4. Three independent trials of the experiment are performed. The probability that the event A happens at least once is
   a. 0.936 **b.** 0.784
   c. 0.904 **d.** none of these
   (IIT-JEE, 1980)

4. If A and B are two events such that $P(A) > 0$ and $P(B) \neq 1$, then $P(\bar{A}/\bar{B})$ is equal to

   (Here $\bar{A}$ and $\bar{B}$ are complements of A and B, respectively.)
   a. $1 - P\left(\frac{A}{B}\right)$ **b.** $1 - P\left(\frac{\bar{A}}{B}\right)$
   c. $\frac{1 - P(A \cup B)}{P(\bar{B})}$ **d.** $\frac{P(\bar{A})}{P(B)}$
   (IIT-JEE, 1982)

5. Fifteen coupons are numbered 1, 2, 3, ..., 15. Seven coupons are selected at random one at a time with replacement. The probability that the largest number appearing on selected coupon is 9 is
   a. $(9/16)^6$ **b.** $(8/15)^7$
   c. $(3/5)^7$ **d.** none of these
   (IIT-JEE, 1983)

6. One-hundred identical coins, each with probability, p, of showing up heads are tossed once. If $0 < p < 1$ and the probability of heads showing on 50 coins is equal to that of heads showing on 51 coins, then the value of p is
   a. 1/2 **b.** 49/101
   c. 50/101 **d.** 51/101
   (IIT-JEE, 1988)

7. India plays two matches each with West Indies and Australia. In any match, the probabilities of India getting points 0, 1 and 2 are 0.45, 0.05 and 0.50, respectively. Assuming that the outcomes are independent, the probability of India getting at least 7 points is
   a. 0.8750 **b.** 0.0875
   c. 0.0625 **d.** 0.0250 **(IIT-JEE, 1982)**

8. An unbiased die with faced marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained, the probability that the minimum face value is not less than 2 and the maximum face value is not greater than five is then
   a. 16/81 b. 1/81
   c. 80/81 d. 65/81 **(IIT-JEE, 1993)**

9. The probability of India winning a test match against West Indies is 1/2. Assuming independence from match to match, the probability that in a five match series India's second win occurs at third test is
   a. 1/8 b. 1/4
   c. 1/2 d. 2/3 **(IIT-JEE, 1995)**

10. Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle with three vertices is equilateral is
   a. 1/2 b. 1/5
   c. 1/10 d. 1/20 **(IIT-JEE, 1995)**

11. For the three events A, B and C, P(exactly one of the events A or B occurs) = P(exactly one of the two events B or C occurs) = P(exactly one of the events C or A occurs) = p and P(all the three events occur simultaneously) = p^2, where $0 < p < 1/2$. Then the probability of at least one of the three events A, B and C occurring is
   a. $\frac{3p+2p^2}{2}$ b. $\frac{p+3p^2}{4}$
   c. $\frac{p+3p^2}{2}$ d. $\frac{3p+2p^2}{4}$
   (IIT-JEE, 1996)

12. If the integers m and n are chosen at random between 1 and 100, then the probability that a number of the form $7^m + 7^n$ is divisible by 5 equals
   a. 1/4 b. 1/7
   c. 1/8 d. 1/49 **(IIT-JEE, 1999)**

13. Two numbers are selected randomly from the set $S = \{1, 2, 3, 4, 5, 6\}$ without replacement one by one. The probability that minimum of the two numbers is less than 4 is
   a. 1/15 b. 14/15
   c. 1/5 d. 4/5 **(IIT-JEE, 2003)**

14. If $P(B) = 3/4$, $P(A \cap B \cap \bar{C}) = 1/3$ and $P(\bar{A} \cap B \cap \bar{C}) = 1/3$, then $P(B \cap C)$ is
   a. 1/12 b. 1/6
   c. 1/15 d. 1/9 **(IIT-JEE, 2003)**

15. If three distinct numbers are chosen randomly from the first 100 natural numbers, then the probability that all three of them are divisible by both 2 and 3 is
   a. 4/25 b. 4/35
   c. 4/33 d. 4/1155
   (IIT-JEE, 2004)

16. A six-faced fair dice is shown until 1 comes. Then the probability that 1 comes in even number of trials is
   a. 5/11 b. 5/6
   c. 6/11 d. 1/6 **(IIT-JEE, 2005)**

17. Three identical dice are rolled. The probability that the same number will appear on each of them is
   a. 1/6 b. 1/36
   c. 1/18 d. 3/28 **(IIT-JEE, 1984)**

18. A box contains 24 identical balls of which 12 are white and 12 are black. The balls are drawn at random from the box one at a time with replacement. The probability that a white ball is drawn for the 4th time on the 7th draw is
   a. 5/64 b. 27/32
   c. 5/32 d. 1/2 **(IIT-JEE, 1984)**

19. Let A, B, C be three mutually independent events. Consider the two statements S_1 and S_2.
   S_1: A and $B \cup C$ are independent
   S_2: A and $B \cap C$ are independent
   Then,
   a. both S_1 and S_2 are true b. only S_1 is true
   c. only S_2 is true d. neither S_1 nor S_2 is true
   (IIT-JEE, 1994)

20. One Indian and four American men and their wives are to be seated randomly around a circular table. Then, the conditional probability that the Indian man is seated adjacent to his wife given that each American man is seated adjacent to his wife is
   a. 1/2 b. 1/3
   c. 2/5 d. 1/5 **(IIT-JEE, 2007)**

21. Let E^c denote the complement of an event E. Let E, F, G be pairwise independent events with $P(G) > 0$ and $P(E \cap F \cap G) = 0$. Then $P(E^c \cap F^c/G)$ equals
   a. $P(E^c) + P(F^c)$ b. $P(E^c) - P(F^c)$
   c. $P(E^c) - P(F)$ d. $P(E) - P(F^c)$
   (IIT-JEE, 2007)

22. A signal which can be green or red with probability $\frac{2}{3}$ and $\frac{1}{5}$ respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is $\frac{3}{4}$. If the signal received at station B is green, then the probability that the original signal was green is
   (a) $\frac{3}{5}$ (b) $\frac{6}{7}$
   (c) $\frac{20}{23}$ (d) $\frac{9}{20}$ **(IIT-JEE, 2010)**

23. Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is

a. 1/18
b. 1/9
c. 2/9
d. 1/36 (IIT-JEE, 2010)

Multiple choice questions with one or more than one correct answer

1. If M and N are any two events, the probability that exactly one of them occurs is
   a. $P(M) + P(N) - 2P(M \cap N)$
   b. $P(M) + P(N) - P(M \cap N)$
   c. $P(M^C) + P(N^C) - 2P(M^C \cap N^C)$
   d. $P(M \cap N^C) + P(M^C \cap N)$ (IIT-JEE, 1984)

2. A student appears for tests I, II and III. The student is successful if he passes either in tests I and II or tests I and III. The probabilities of the student passing in tests I, II and III are p, q and 1/2, respectively. The probability that the student is successful is then
   a. $p = q = 1$
   b. $p = q = 1/2$
   c. $p = 1, q = 0$
   d. none of these
   (IIT-JEE, 1986)

3. The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then $P(\bar{A}) + P(\bar{B})$ is
   (Here $\bar{A}$ and $\bar{B}$ are complements of A and B, respectively.)
   a. 0.4
   b. 0.8
   c. 1.2
   d. none
   (IIT-JEE, 1987)

4. For two given events A and B, $P(A \cap B)$ is
   a. not less than $P(A) + P(B) - 1$
   b. not greater than $P(A) + P(B)$
   c. equal to $P(A) + P(B) - P(A \cup B)$
   d. equal to $P(A) + P(B) + P(A \cup B)$
   (IIT-JEE, 1988)

5. If E and F are independent events such that $0 < P(E) < 1$ and $0 < P(F) < 1$, then
   a. B and F are mutually exclusive
   b. E and F^C (the complement of the event F) are independent
   c. E^C and F^C are independent
   d. $P(E|F) + P(E^C|F) = 1$ (IIT-JEE, 1989)

6. For any two events A and B in a sample space,
   a. $P(A/B) \geq \dfrac{P(A) + P(B) - 1}{P(B)}$ $(P(B) \neq 0)$ is always true
   b. $P(A \cap \bar{B}) = P(A) - P(A \cap B)$ does not hold
   c. $P(A \cup B) = 1 - P(\bar{A})\, P(\bar{B})$, if A and B are independent
   d. $P(A \cup B) = 1 - P(\bar{A})\, P(\bar{B})$, if A and B are disjoint
   (IIT-JEE, 1997)

7. E and F are two independent events. The probability that both E and F happen is 1/12 and the probability that neither E nor F happens is 1/2. Then,
   a. $P(E) = 1/3, P(F) = 1/4$
   b. $P(E) = 1/4, P(F) = 1/3$
   c. $P(E) = 1/6, P(F) = 1/2$
   d. $P(E) = 1/2, P(F) = 1/6$ (IIT-JEE, 1993)

8. If from each of the three boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, one ball is drawn at random, then the probability that 2 white and 1 black ball will be drawn is
   a. 13/32
   b. 1/4
   c. 1/32
   d. 3/16 (IIT-JEE, 1998)

9. If $\bar{E}$ and $\bar{F}$ are the complementary events of events E and F, respectively, and if $0 < P(F) < 1$, then
   a. $P(E/F) + P(\bar{E}/F) = 1$
   b. $P(E/F) + P(E/\bar{F}) = 1$
   c. $P(\bar{E}/F) + P(E/\bar{F}) = 1$
   d. $P(E/\bar{F}) + P(\bar{E}/\bar{F}) = 1$ (IIT-JEE, 1998)

10. There are four machines and it is known that exactly two of them are faulty. They are tested, one by one, in a random order till both the faulty machines are identified. Then the probability that only two tests are needed is
    a. 1/3
    b. 1/6
    c. 1/2
    d. 1/4 (IIT-JEE, 1998)

11. If E and F are events with $P(E) \leq P(F)$ and $P(E \cap F) > 0$, then
    a. occurrence of $E \Rightarrow$ occurrence of F
    b. occurrence of $F \Rightarrow$ occurrence of E
    c. non-occurrence of $E \Rightarrow$ non-occurrence of F
    d. none of the above implications holds (IIT-JEE, 1998)

12. A fair coin is tossed repeatedly. If the tail appears on first four tosses, then the probability of the head appearing on the fifth toss equals
    a. 1/2
    b. 1/32
    c. 31/32
    d. 1/5 (IIT-JEE, 1998)

13. The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c, respectively. Of these subjects, the student has a 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two. Which of the following relations are true?
    a. $p + m + c = 19/20$
    b. $p + m + c = 27/20$
    c. $pmc = 1/10$
    d. $pmc = 1/4$
    (IIT-JEE, 1999)

14. Seven white balls and three black balls are randomly placed in a row. The probability that no two black balls are placed adjacently equals

a. 1/2 **b.** 7/15
c. 2/15 **d.** 1/3 **(IIT-JEE, 1998)**

15. Let $0 < P(A) < 1$, $0 < P(B) < 1$ and $P(A \cup B) = P(A) + P(B) - P(A)\,P(B)$. Then

a. $P(B/A) = P(B) - P(A)$
b. $P(A' - B') = P(A') - P(B')$
c. $P(A \cup B)' = P(A')\,P(B')$
d. $P(A/B) = P(A)$ **(IIT-JEE, 1995)**

16. Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If $P(T)$ denotes the probability of occurrence of the event T, then

(a) $P(E) = \frac{4}{5}, P(F) = \frac{3}{5}$
(b) $P(E) = \frac{1}{5}, P(F) = \frac{2}{5}$
(c) $P(E) = \frac{2}{5}, P(F) = \frac{1}{5}$
(d) $P(E) = \frac{3}{5}, P(F) = \frac{4}{5}$ **(IIT-JEE, 2011)**

Comprehension

Read the passage given below and answer the questions that follow.

For Problems 1–3

There are n urns, each of these contain $n + 1$ balls. The i^{th} urn contains i white balls and $n + 1 - i$ red balls. Let u_i be the event of selecting i^{th} urn, $i = 1, 2, 3, \ldots, n$ and w be the event of getting a white ball.

1. If $P(u_i) \propto i$, where $i = 1, 2, 3, \ldots, n$, then $\lim_{n\to\infty} P(w)$ equals to

a. 1 **b.** 2/3
c. 3/4 **d.** 1/4 **(IIT-JEE, 2005)**

2. If $P(u_i) = c$ (a constant), then $P(u_n/w)$ equals to

a. $2/(n + 1)$ **b.** $1/(n + 1)$
c. $n/(n + 1)$ **d.** 1/2 **(IIT-JEE, 2006)**

3. Let $P(u_i) = 1/n$. If n is even and E denotes the event of choosing even numbered urn then the value of $P(w/E)$ is

a. $\frac{n+2}{2n+1}$ **b.** $\frac{n+2}{2(n+1)}$
c. $\frac{2}{n+1}$ **d.** $\frac{1}{n+1}$ **(IIT-JEE, 2006)**

For Problems 4–6

A fair die is tossed repeatedly until a 6 is obtained. Let X denote the number of tosses required.

4. The probability that $X = 3$ equals

a. 25/216 **b.** 25/36
c. 5/36 **d.** 125/216 **(IIT-JEE, 2009)**

5. The probability that $X \geq 3$ equals

a. 125/216 **b.** 25/36
c. 5/36 **d.** 25/216

6. The conditional probability that $X \geq 6$ given $X > 3$ equals

a. 125/216 **b.** 25/36
c. 5/36 **d.** 25/216

For Problems 7–8

Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls, and U_2 contains only 1 white ball. A fair coin is tossed. If head appears then 1 ball is drawn at random for U_1 and put into U_2. However, if tail appears then 2 balls are drawn at random from U_1 and put into U_2. Now I ball is drawn at random from U_2 **(IIT-JEE, 2011)**

7. The probability of the drawn ball from U_2 being white is

a. $\frac{13}{30}$ **b.** $\frac{23}{30}$
c. $\frac{19}{30}$ **d.** $\frac{11}{30}$

8. Given that the drawn ball from U_2 is white, the probability that head appeared on the coin is

a. $\frac{17}{23}$ **b.** $\frac{11}{23}$
c. $\frac{15}{23}$ **d.** $\frac{12}{23}$

Assertion and reason

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and Statement 2 is the correct explanation of Statement 1.
b. Both the statements are TRUE but Statement 2 is NOT the correct explanation of Statement 1.
c. Statements 1 is TRUE and Statement 2 is FALSE.
d. Statements 1 is FALSE and Statement 2 is TRUE.

1. Let $H_1, H_2, \ldots, H_n$ be mutually exclusive and exhaustive events with $P(H_i) > 0$, $i = 1, 2, \ldots, n$. Let E be any other event with $0 < P(E) < 1$.

Statement 1:
$P(H_i/E) > P(E/H_i) \times P(H_i)$ for $i = 1, 2, \ldots, n$.

Statement 2 : $\sum_{i=1}^{n} P(H_i) = 1$. **(IIT-JEE, 2007)**

2. Consider the system of equations $ax + by = 0$, $cx + dy = 0$, where $a, b, c, d \in \{0, 1\}$.

Statement 1: The probability that the system of equations has a unique solution is 3/8.

Statement 2: The probability that the system of equations has a solution is 1. **(IIT-JEE, 2008)**

ANSWERS AND SOLUTIONS

Subjective Type

1. The elements of A are all multiples of 5. Sum of every pair of elements of A is divisible by 5. Therefore, we have to find the probability that B has two distinct elements whose sum is divisible by 3.

Let A_0 be the set of elements of A of the form $3k$, i.e., $\{0, 15, 30, ..., 195\}$; A_1 be the set of elements of A of the form $3k + 1$, i.e., $\{10, 25, ..., 190\}$; A_2 be the set of elements of A of the form $3k + 2$, i.e., $\{5, 20, 35, 185\}$. Then $n(A_0) = 14$, $n(A_1) = n(A_2) = 13$.

If B has at least two elements from A_0, then we are done.

If B contains at most one element of A_0, then it must have at least one element from each of A_1 and A_2 for which the sum of these two elements will be divisible by 3. So, the required probability is 1.

2. Let S be the sample space consisting of elements representing balls that can be drawn from the bag containing $2n$ balls (n white + n black). Let E_{ij} be the event representing drawing balls such that number of white balls is greater than that of black balls by one. Then,

$$E = E_{21} \cup E_{32} \cup \cdots \cup E_{n,n-1}$$

$$m(S) = {}^{2n}C_1 + {}^{2n}C_2 + \cdots + {}^{2n}C_{2n} = 2^{2n} - 1$$

$$m(E) = m(E_{21}) + m(E_{32}) + \cdots + m(E_{n,n-1})$$

$$= {}^{n}C_2\,{}^{n}C_1 + {}^{n}C_3\,{}^{n}C_2 + \cdots + {}^{n}C_n\,{}^{n}C_{n-1}$$

$$= {}^{2n}C_{n-1} - {}^{n}C_1\,{}^{n}C_0$$

$$= {}^{2n}C_{n-1} - n$$

Hence,

$$P(E) = \frac{{}^{2n}C_{n-1} - n}{2^{2n} - 1}$$

3. Let 'S' be the sample space, A_0 be the event that books drawn from two bags are equal in number, A_1 be the event that number of books drawn from one bag exceed those drawn from another bag by one, and A_2 be the event that number of books drawn from one bag exceed those drawn from other bag by two. Total number of ways is $({}^{10}C_1 + {}^{10}C_2 + \cdots + {}^{10}C_{10})^2 = (2^{10} - 1)^2$. Favourable number of ways for A_0 is $({}^{10}C_1)^2 + ({}^{10}C_2)^2 + \cdots + ({}^{10}C_{10})^2 = {}^{20}C_{10} - 1$.

Favourable number of ways for A_1 is

$$2({}^{10}C_1\,{}^{10}C_2 + {}^{10}C_2\,{}^{10}C_3 + \cdots + {}^{10}C_9\,{}^{10}C_{10})$$

$$= 2({}^{20}C_9 - {}^{10}C_0\,{}^{10}C_1)$$

$$= 2({}^{20}C_9 - 10)$$

Favourable number of ways for A_2 is

$$2({}^{10}C_1\,{}^{10}C_3 + {}^{10}C_2\,{}^{10}C_4 + \cdots + {}^{10}C_8\,{}^{10}C_{10})$$

$$= 2({}^{20}C_8 - {}^{10}C_0\,{}^{10}C_2)$$

$$= 2({}^{20}C_8 - 45)$$

Therefore, the required probability is

$$\frac{2\,{}^{20}C_8 + 2\,{}^{20}C_9 + {}^{20}C_{10} - 111}{(2^{10} - 1)^2}$$

4. Let X be the number of times A shoots at the target to hit it for the first time and Y be the number of times B shoots at the target to hit for the first time. Then,

$$P(X = m) = \left(\frac{2}{5}\right)^{m-1}\left(\frac{3}{5}\right) \text{ and } P(Y = n) = \left(\frac{2}{7}\right)^{n-1}\left(\frac{5}{7}\right)$$

We have,

$$P(Y > X) = \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} P(X = m)\,P(Y = n)$$

[$\because X$ and Y are independent]

$$= \sum_{m=1}^{\infty}\left[\left\{\left(\frac{2}{5}\right)^{m-1}\left(\frac{3}{5}\right)\right\}\sum_{n=m+1}^{\infty}\left\{\left(\frac{2}{7}\right)^{n-1}\left(\frac{5}{7}\right)\right\}\right]$$

$$= \sum_{m=1}^{\infty}\left(\frac{2}{5}\right)^{m-1}\left(\frac{3}{5}\right)\left\{\frac{5}{7}\cdot\frac{\left(\frac{2}{7}\right)^m}{1 - \frac{2}{7}}\right\}$$

$$= \sum_{m=1}^{\infty}\left(\frac{2}{5}\right)^{m-1}\left(\frac{3}{5}\right)\left(\frac{2}{7}\right)^m$$

$$= \frac{6}{35}\sum_{m=1}^{\infty}\left(\frac{4}{35}\right)^{m-1}$$

$$= \frac{6}{35}\,\frac{1}{1 - \frac{4}{35}} = \frac{6}{31}$$

5. Let E_i denote the event that out of the first k balls drawn, i balls are white and A be the event that $(k + 1)^{\text{th}}$ ball drawn is also white. We have to find $P(A)$. Now ways of selecting i white balls from a white balls and $k - i$ black balls from b black balls is ${}^{a}C_i\,{}^{b}C_{k-i}$ $(0 \le i \le k)$. Ways to select k balls from $a + b$ balls is ${}^{a+b}C_k$.

$\therefore \quad P(E_i) = ({}^{a}C_i\,{}^{b}C_{k-i})/({}^{a+b}C_k),\ 0 \le i \le k$

Also,

$$P(A/E_i) = \frac{{}^{a-i}C_1}{{}^{a+b-k}C_1} = \frac{a-i}{a+b-k} \quad (0 \le i \le k)$$

By the theorem of total probability, we have

$$P(A) = \sum_{i=0}^{k} P(E_i)\,P(A/E_i)$$

$$= \sum_{i=0}^{k} \frac{{}^{a}C_i\,{}^{b}C_{k-i}}{{}^{a+b}C_k}\,\frac{a-i}{a+b-k}$$

$$= \sum_{i=0}^{k} \frac{[(a-i)\,{}^{a}C_{a-i}]\,{}^{b}C_{k-i}}{(a+b-k)\,{}^{a+b}C_{a+b-k}}$$

$$= \frac{a}{a+b}\sum_{i=0}^{k} \frac{{}^{a-1}C_{a-1-i}\,{}^{b}C_{k-i}}{{}^{a+b-1}C_{a+b-k-1}}$$

$$= \frac{a}{a+b}\sum_{i=0}^{k} \frac{{}^{a-1}C_i\,{}^{b}C_{k-i}}{{}^{a+b-1}C_k}$$

$$= \frac{a}{a + b({}^{a+b-1}C_k)} \sum_{i=0}^{k} {}^{a-1}C_i \, {}^{b}C_{k-i}$$

$$= \frac{a}{a + b({}^{a+b-1}C_k)} {}^{a+b-1}C_k$$

$$= \frac{a}{a+b}$$

6. Let A, B, C be the three independent events having probability p, q, and r, respectively. Then according to the hypothesis, we have

$$p(1-q)(1-r) = a, \ (1-p)q(1-r) = b \text{ and } (1-p)(1-q)r = c$$

$$\therefore \quad pqr\,[(1-p)(1-q)(1-r)]^2 = abc \text{ or}$$

$$\frac{abc}{pqr}[(1-p)(1-q)(1-r)]^2 - 1 = x^2 \text{ (say)} \qquad (1)$$

Then,

$$\frac{a}{x} = \frac{p}{1-p}$$

$$\Rightarrow \quad a - ap = px$$

$$\Rightarrow \quad p = a/(a+x)$$

Similarly, $q = b/(b+x)$ and $r = c/(c+x)$. Then from Eq. (1), clearly $(a+x)(b+x)(c+x) = x^2$, i.e., x is a root of the equation $(a+x)(b+x)(c+x) = x^2$.

7. (i) Let r and s be the remainders when x and y are divided by 5, so

$$x = 5p + r, \ y = 5q + s, \ p, q \in N, \ 0 \le r \le 4, \ 0 \le s \le 4$$

$$\therefore \quad x^2 + y^2 = 25(p^2 + q^2) + 10(pr + qs) + r^2 + s^2 = 5k + (r^2 + s^2), \ k \in N$$

Thus, $x^2 + y^2$ will be divisible by 5 if $r^2 + s^2$ is divisible by 5. Therefore, total number of ways is equal to the number of ways of selecting r and s, which is $5^2 = 25$.

For favourable ways, we should have $r^2 + s^2$ is divisible by 5 and $0 \le r^2 + s^2 \le 32$ so $r^2 + s^2 = 0, 5, 10, 20, 25$ or 30. Thus

r	0	1	2	3	4
s	0	2, 3	1, 4	1, 4	2, 3

So number of favourable ways is 9. Hence, the probability is 9/25.

(ii) Similarly, following the above steps, we can find the required result.

8. Suppose that p and q, respectively, denote the probability that a thing goes to a man and to a woman, respectively. Then,

$$p = \frac{a}{a+b} \text{ and } q = \frac{b}{a+b}$$

Now, the probabilities of 0, 1, 2, 3, … things going to man are the first, second, third terms, etc., in the following binomial expansion.

$$(q+p)^m = q^m + {}^mC_1 a^{m-1} p + {}^mC_2 a^{m-2} p^2 + \cdots + p^m \qquad (1)$$

But men are to receive an odd number of things. Hence, the required probability is the sum of even terms in Eq. (1). To obtain the sum of even terms, we write the expansion

$$(q+p)^m = q^m + {}^mC_1 a^{m-1} p + {}^mC_2 a^{m-2} p^2 + \cdots + (-1)^m p^m \qquad (2)$$

Taking the difference of Eq. (2) from Eq. (1), we obtain

$(q+p)^m - (q-p)^m$ = sum of even terms in Eq. (1). Hence, the required probability is

$$\frac{1}{2}[(q+p)^m - (q-p)^m]$$

$$= \frac{1}{2}\left[1 - \left(\frac{b-a}{b+a}\right)^m\right]$$

$$= \frac{1}{2}\frac{(b+a)^m (b-a)^m}{(b+a)^m}$$

9. Let A be the event of P_1 winning in third round and B be the event of P_2 winning in first round but loosing in second round. We have

$$P(A) = \frac{{}^{8n-1}C_{n-1}}{{}^{8n}C_n} = \frac{1}{8}$$

$P(B \cap A)$

= Probability of both P_1 and P_2 winning in first round

× probability of P_1 winning and P_2 losing in second round

× probability of P_1 winning in third round

$$= \frac{{}^{8n-2}C_{4n-2}}{{}^{8n}C_{4n}} \times \frac{{}^{4n-2}C_{2n-1}}{{}^{4n}C_{2n}} \times \frac{{}^{2n-1}C_{n-1}}{{}^{2n}C_n}$$

$$= \frac{n}{4(8n-1)}$$

Hence,

$$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}$$

$$= \frac{2n}{8n-1}$$

Alternative Solution:

Probability that P_2 wins in first round given P_1 wins is

$$\frac{{}^{8n-2}C_{4n-2}}{{}^{8n-1}C_{4n-1}} = \frac{4n-1}{8n-1}$$

In second round, probability that P_2 loses in second round given P_1 wins is

$$1 - \frac{2n-1}{4n-1} = \frac{2n}{4n-1}$$

Hence, probability that P_2 loses in second round, given P_1 wins in third round is $2n/(8n-1)$.

10. In the tennis match of best of 5 sets, A can win the match, if score of A against the score of B is (3, 0), (3, 1) or (3, 2).

The probability of A's doing the score of (3, 0) is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

The probability of A's winning by the score of (3, 1) is

$P(A$ wins the first III sets)

$+ P(A$ wins I, loses II and wins III, IV sets)

$+ P(A$ wins sets I, II, loses III and wins set IV) 2

$$= \frac{1}{4}\left(\frac{1}{2}\right)^3 + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{4} = \frac{3}{32}$$

The probability of A's winning by the score of (3, 2) is

$P(A$ looses I and II sets)
$+ P(A$ looses I and III sets)
$+ P(A$ looses I and IV sets)
$+ P(A$ looses II and III sets)
$+ P(A$ looses II and IV sets)
$+ P(A$ looses III and IV sets)

$$=\frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^2+\frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)$$
$$+\frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)$$
$$+\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)+\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\frac{1}{4}$$
$$=\frac{3}{128}+\frac{1}{128}+\frac{1}{128}+\frac{3}{128}+\frac{1}{128}+\frac{3}{128}$$
$$=\frac{12}{128}$$

The probability that A wins the match is

$$\frac{1}{8}+\frac{3}{32}+\frac{12}{128}=\frac{16+12+12}{128}=\frac{40}{128}=\frac{5}{16}$$

11. Let the probability that A wins the tournament be x and the probability that B wins the tournament be x. Also let the probability that the game ends in a draw be y so that

$$2x+y=1 \Rightarrow x=\frac{1-y}{2}$$

Case I: The probability that the game ends in 1 draw, 3 wins and 3 losses is

$$\frac{7!}{3!3!}\left(\frac{1}{3}\right)^7$$

Case II: The probability that the game ends in 3 draws, 2 wins and 2 losses is

$$\frac{7!}{3!2!2!}\left(\frac{1}{3}\right)^7$$

Case III: The probability that the game ends in 5 draws, 1 win and 1 losses is

$$\frac{7}{5!}\left(\frac{1}{3}\right)^7$$

Case IV: The probability that the game ends in all draws is

$$\left(\frac{1}{3}\right)^7$$

$$\therefore \quad y=\frac{1}{3^7}\left(\frac{7!}{3!3!}+\frac{7!}{3!2!2!}+\frac{7!}{5!}+1\right)=\frac{131}{729}$$

$$\Rightarrow \quad x=\frac{1}{2}\left(1-\frac{131}{729}\right)=\frac{299}{729}$$

12. Each group should have odd number of odd numbered balls.

Case I: Two groups have three odd-numbered balls and the third group has only one odd-numbered ball. The number of such cases is

$$\frac{7!}{(3!)^2 2!}\times 3^7 \text{ (each even numbered ball has three possibilities)}$$

Case II: Two groups have one odd-numbered ball and the third group has five odd-numbered balls.

The number of such cases is

$$\frac{7!}{5!\times 2!}\times 3^7$$

The total number of cases for dividing 14 balls into three non-empty groups is $(3^{14}-{}^3C_1 2^{14}+{}^3C_2)/3!$ Hence, the required probability is

$$\frac{\left(\frac{7!\times 3!}{(3!)^2 2!}+\frac{7!\times 3!}{5!\times 2!}\right)}{(3^{14}-{}^3C_1 2^{14}+{}^3C_2)}\times 3^7$$

13.

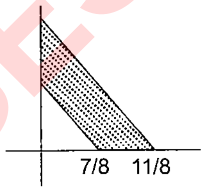


Fig. 9.9

$$P(A\cup B)\geq\frac{3}{4} \text{ and } \frac{1}{8}\leq P(A\cap B)\leq\frac{3}{8}$$

Hence,

$$P(A)+P(B)-P(A\cap B)\geq\frac{3}{4}$$

$$\Rightarrow \quad P(A)+P(B)\geq + P(A\cap B)\geq\frac{3}{4}+\frac{1}{8}=\frac{7}{8}$$

$$\Rightarrow \quad x+y\geq\frac{7}{8}$$

We know that

$$P(A\cap B)\leq 1$$

$$P(A)+P(B)\leq 1+P(A\cap B)\leq 1+\frac{3}{8}=\frac{11}{8}$$

$$\Rightarrow \quad x+y\leq\frac{11}{8}$$

The shaded part in the figure is the required region.

14. P_1 can win in the following mutually exclusive ways:

a. P_1 wins the next six matches.

b. P_1 wins five out of next six matches, so that after next six matches scores of P_1 and P_2 are tied up. This tie continues up to next '$2n$' matches ($n\geq 0$) and finally P_1 wins 2 consecutive matches. Now, for case (a), probability is given by $(2/3)^6$ and probability of tie after 6 matches [in case (b)] is

$${}^6C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)=6\times\frac{2^5}{3^6}=\frac{2^6}{3^5}$$

Now probability that scores are still tied up after another next two matches is

$$\frac{2}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{3}=\frac{4}{9}$$

[First match won by P_1 and second by P_2 or first by P_2 and second by P_1.]

Similarly, probability that scores are still tied up after another $2n$ matches is $(4/9)^n$.

Therefore, the total probability of P_1 winning the championship is

$$\left(\frac{2}{3}\right)^6+\frac{2^6}{3^5}\left(\sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^n\left(\frac{2}{3}\right)^2\right)$$

$$=\left(\frac{2}{3}\right)^6+\frac{2^5}{3^5}\left(\frac{2}{3}\right)^2\left(\frac{1}{1-\frac{4}{9}}\right)$$

$$=\frac{17}{5}\left(\frac{2}{3}\right)^6=\frac{1088}{3645}$$

15. (i) The probability that one gets tail and nine get head is

$$^{10}C_1\frac{1}{2}\left(\frac{1}{2}\right)^9$$

(ii) The probability that one gets head and nine get tail is

$$^{10}C_1\frac{1}{2}\left(\frac{1}{2}\right)^9$$

Hence, probability that the game is settled is

$$2\times{}^{10}C_1\frac{1}{2}\left(\frac{1}{2}\right)^9=\frac{5}{2^8}$$

If the game is not settled in First toss, its probability is $1-5/2^8$. If the game is not settled in Second toss, its probability is $(1-5/2^8)^2$. Similarly, the probability that game is not settled in first $n-1$ toss is $(1-5/2^8)^{n-1}$, which is equal to the probability that the game is settled on or after the n^{th} toss.

16. Let A_1 (A_2) be the event that drawn ball is white (non-white) and 'E' be the event that A and B claim that drawn ball is white. Clearly,

$$P(A_1)=\frac{1}{n},\ P(A_2)=\frac{n-1}{n}$$
$$P(E/A_1)=P_1P_2$$
$$P(E/A_2)=(1-P_1)(1-P_2)(n-1)^{-2}$$

[As $n-1$ balls remain in the bag and one of them is white, the chance that 'A' should choose this ball and wrongly assert that it was drawn from the bag is $(1-P_1)/(n-1)$.]

$$\therefore\ P(E)=P(A_1)P(E/A_1)+P(A_2)P(E/A_2)$$

$$=\frac{P_1P_2}{n}+\frac{(n-1)}{n}\frac{(1-P_1)(1-P_2)}{(n-1)^2}$$

$$=\frac{(n-1)P_1P_2+(1-P_1)(1-P_2)}{n(n-1)}$$

$$\Rightarrow\ P(A_1/E)=\frac{P(A_1)\,P(E/A_1)}{P(E)}$$

$$=\frac{(n-1)P_1P_2}{(n-1)P_1P_2+(1-P_1)(1-P_2)}$$

17. Let E_i ($i=0,1,2,\ldots,20$) be the event that the bag contains i books on mathematics. Since all these events are equally likely and mutually exclusive and exhaustive, so $P(E_i)=1/21$ ($i=0,1,2,\ldots,20$) and let A be the event that a draw of 10 books contains 6 books on mathematics. Then,

$$P(A)=\sum_{i=0}^{20}P(E_i)\cdot P(A/E_i)$$

$$=\frac{1}{21}\left[\sum_{i=0}^{20}P(A/E_i)\right]$$

$$=\frac{1}{21}\left[\sum_{i=6}^{16}\frac{^iC_6\times{}^{20-i}C_4}{^{20}C_{10}}\right]$$

Now, we want that the bag should contain 2 more books on mathematics, i.e., E_8 must occur.

$$P(E_8/A)=\frac{P(E_8)P(A/E_8)}{P(A)}$$

$$=\frac{\dfrac{^8C_6\times{}^{12}C_4}{^{20}C_{10}}}{\sum_{i=6}^{16}\left(\dfrac{^iC_6\times{}^{20-i}C_4}{^{20}C_{10}}\right)}$$

$$=\frac{^8C_6\times{}^{12}C_4}{\sum_{i=6}^{16}(^iC_6\times{}^{20-i}C_4)}$$

18. Let H and T denote turning up of the head and tail and X denote the turning of head or tail. Then

$$P(H)=P(T)=\frac{1}{2}\text{ and }P(X)=1$$

$P(HH\ldots m\text{ times})(XXX\ldots n\text{ times})$

$$=\frac{1}{2}\times\frac{1}{2}\cdots m\text{ times}$$

$$=\frac{1}{2^m}$$

$P[T\{HHH\ldots m\text{ times}\}\{XX\ldots n-1\text{ times}\}]$

$=P(T)\,P(H)+\cdots+m$ times.

$$P(XX\ldots n-1\text{ times})=\frac{1}{2^{m+1}}$$

If the sequence of heads starts with $(r+1)^{\text{th}}$ throw, then the first $r-1$ throws may be head or tail but r^{th} throw must be tail and we have

$(XX\ldots(r-1)\text{ times})\,T\,(HH\ldots m\text{ times})\,(XX\ldots$

$$(n-m-r)\text{ times})=\frac{1}{2^{m+1}}$$

Since all the above cases are mutually exclusive, the required probability is

$$\frac{1}{2^m}+\left[\frac{1}{2^{m+1}}+\frac{1}{2^{m+1}}+\cdots+n\text{ times}\right]$$

$$=\frac{1}{2^m}+\frac{n}{2^{m+1}}=\frac{n+2}{2^{m+1}}$$

Objective Type

1. b. $P(A)=\dfrac{1}{3},\ P(A\cup B)=\dfrac{3}{4}$

Now,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B)$$

$$\Rightarrow \quad \frac{3}{4} \le \frac{1}{3} + P(B)$$

$$\Rightarrow \quad \frac{5}{12} \le P(B)$$

Again we have $B \subseteq A \cup B$.

$$\therefore \quad P(B) \le P(A \cup B) = \frac{3}{4}$$

Hence, $5/12 \le P(B) \le 3/4$.

2. a. The probability of hitting a target is $p = 1/5$. Therefore, the probability of not hitting a target is $q = 1 - 1/5 = 4/5$. Hence, the required probability is $1 - (4/5)^{10}$.

3. d. Let the probability for getting an odd number be p. Therefore, the probability for getting an even number is $2p$.

$$\therefore \quad p + 2p = 1 \Rightarrow 3p = 1 \Rightarrow p = \frac{1}{3}$$

Sum of two nunbers is even means either both are odd or both are even. Therefore, the required probability is

$$\frac{1}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{2}{3} = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}$$

4. c. If A draws card higher than B, then number of favourable cases is $(n-1) + (n-2) + \cdots + 3 + 2 + 1$ (as when B draws card number 1, then A can draw any card from 2 to n and so on). Therefore, the required probability is

$$\frac{\frac{n(n-1)}{2}}{n^2} = \frac{n-1}{2n}$$

5. b. If A, B, C represent events that the student is successful in tests I, II, III, respectively, Then the probability tha the student is successful is

$$P[(A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A \cap B \cap C)]$$

$$= P(A \cap B \cap C') + P(A \cap B' \cap C) + P(A \cap B \cap C)$$

$$= P(A)\, P(B)\, P(C') + P(A)\, P(B')\, P(C) + P(A)\, P(B)\, P(C)$$

[$\therefore$ A, B, C are independent events]

$$= pq\left(1 - \frac{1}{2}\right) + p(1-q)\,\frac{1}{2} + pq\frac{1}{2}$$

$$= pq + \frac{1}{2}p - \frac{1}{2}pq$$

$$= \frac{1}{2}(pq + p)$$

$$\therefore \quad \frac{1}{2}p(1+q) = \frac{1}{2}$$

$$\Rightarrow \quad p(1+q) = 1$$

6. a. $P(A) = \frac{1}{2},\ P(B) = \frac{1}{3},\ P(C) = \frac{1}{4}$

$$\therefore \quad P(\bar{A}) = 1 - \frac{1}{2} = \frac{1}{2},\ P(\bar{B}) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(\bar{C}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Therefore, the required probability is

$$1 - P(\bar{A})\, P(\bar{B})\, P(\bar{C}) = 1 - \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

7. d. A person can have his/her birthday on any one of the seven days of the week. So 5 persons can have their birthdays in 7^5 ways. Out of 5, three persons can have their birthdays on days other than Sundays in 6^3 ways and other 2 on Sundays. Hence, the required probability is

$$\frac{{}^5C_2 \times 6^3}{7^5} = \frac{10 \times 6^3}{7^5}$$

(Note that 2 persons can be selected out of 5 in ${}^5C_2$ ways.)

8. a. Required probability $= \dfrac{\text{No. of favourable cases}}{\text{Total no. of exhaustive cases}}$

$$= \frac{3}{3 \times 3 \times 3} = \frac{1}{9}$$

9. d. The number of ways of arranging n numbers is $n!$ In each order obtained, we must now arrange the digits $1, 2, \ldots, k$ as group and the $n - k$ remaining digits. This can be done in $(n - k + 1)!$ ways. Therefore, the probability for the required event is $(n - k + 1)!/n!$

10. d. According to the given condition

$${}^nC_3\left(\frac{1}{2}\right)^n = {}^nC_4\left(\frac{1}{2}\right)^n$$

where n is the number of times die is thrown.

$$\therefore \quad {}^nC_3 = {}^nC_4 \Rightarrow n = 7$$

Thus, the required probability is

$${}^7C_1\left(\frac{1}{2}\right)^7 = \frac{7}{2^7} = \frac{7}{128}$$

11. c. Possibilities of getting 9 are (5, 4), (4, 5), (6, 3), (3, 6).

$$\therefore \quad p = \frac{4}{36} = \frac{1}{9} \text{ and } q = 1 - \frac{1}{9} = \frac{8}{9}$$

Therefore, the required probability is

$${}^3C_2 q^1 p^2 = (3)\left(\frac{8}{9}\right)\left(\frac{1}{9}\right)^2 = \frac{8}{243}$$

12. c. Out of 5 horses, only one is the wining horse. The probability that Mr. A selected that losing horse is $4/5 \times 3/4$. Therefore, the required probability is

$$1 - \frac{4}{5} \times \frac{3}{4} = 1 - \frac{3}{5} = \frac{2}{5}$$

13. d. We have,

$$P(\overline{A \cup B}) = \frac{1}{4}, P(A \cap B) = \frac{1}{4}$$

$$P(\bar{A}) = \frac{1}{4} \Rightarrow P(A) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\therefore \quad P(\overline{A \cup B}) = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$\Rightarrow \quad \frac{1}{6} = 1 - \frac{3}{4} - P(B) + \frac{1}{4}$$

$$\Rightarrow \quad P(B) = 1 - \frac{1}{2} - \frac{1}{6} = \frac{6-3-1}{6} = \frac{2}{6} = \frac{1}{3}$$

Since $P(A \cap B) = P(A)P(B)$ and $P(A) \neq P(B)$, therefore A and B are independent but not equally likely.

14. b. The total number of cases is $11!/2! \times 2!$. The number of favourable cases is $[11!/(2! \times 2!)] - 9!$ Therefore, the required probability is

$$1 - \frac{9! \times 4}{11!} = \frac{53}{55}$$

15. c. Total number of the students is 80. Total number of girls is 25. Total number of boys is 55. There are 10 rich, 70 poor, 20 intelligent students in the class. Therefore, required probability is

$$\underset{(I)}{\frac{1}{4}} \times \underset{(R)}{\frac{1}{8}} \times \underset{(G)}{\frac{25}{80}} = \frac{5}{512}$$

16. b. $P(A' \cap B \cap C' \cap D) = P(A')\, P(B)\, P(C')\, P(D)$

$$= \left(1 - \frac{1}{2}\right)\frac{1}{3}\left(1 - \frac{1}{5}\right)\left(\frac{1}{6}\right)$$

$$= \frac{1}{2} \times \frac{1}{3} \times \frac{4}{5} \times \frac{1}{6} = \frac{1}{45}$$

17. a. $p^2 + 2p + 4p - 1 = 1$ (Exhaustive)

$$p^2 + 6p - 2 = 0$$

$$\Rightarrow \quad p = -3 \pm \sqrt{11}$$

$$\Rightarrow \quad p = \sqrt{11} - 3$$

18. a. L and W can be filled at 14 places in 2^{14} ways.

$\therefore \quad n(S) = 2^{14}$

Now 13 L's and 1 W can be arranged at 14 places in 14 ways.

Hence, $n(A) = 14$.

$$\therefore \quad p = \frac{14}{2^{14}} = \frac{7}{2^{13}}$$

19. a. $P(A \cap C) = P(A)\, P(C)$

$$\Rightarrow \quad \frac{1}{20} = \frac{1}{5} P(C)$$

$$\Rightarrow \quad P(C) = \frac{1}{4}$$

Now,

$$P(B \cup C) = \frac{1}{6} + \frac{1}{4} - P(B \cap C)$$

$$\Rightarrow \quad P(B \cup) = \frac{3}{8} - \frac{1}{3} = \frac{1}{24} = \frac{1}{24} = P(B)\, P(C)$$

Therefore, B and C are independent.

20. a. The number of ways in which 20 people can be divided into two equal groups is

$$n(s) = \frac{20!}{10!\ 10!\ 2!}$$

The number of ways in which 18 people can be divided into groups of 10 and 8 is

$$n(A) = \frac{18!}{10!\ 8!}$$

$$\therefore \quad P(E) = \frac{18!}{10!\,8!} \frac{10!\,10!\,2}{20!} = \frac{10 \times 9 \times 2}{20 \times 19} = \frac{9}{19}$$

21. c. $P(A \cup B) = P(A) + P(B) - P(A) \times P(B)$

$$\Rightarrow \quad 0.7 = 0.4 + p - 0.4p$$

$$\therefore \quad 0.6p = 0.3 \Rightarrow p = \frac{1}{2}$$

22. a. $P(B_1) = \dfrac{{}^6C_1}{{}^{10}C_1} = \dfrac{6}{10} = \dfrac{3}{5}$

$$P(B_2 / B_1) = \frac{5}{9}\ (B_2 = \text{black})$$

$$\therefore \quad P(B_1 \cap B_2) = P(B_1)\, P(B_2 / B_1) = \frac{3}{5} \times \frac{5}{9} = \frac{1}{3}$$

23. b. Total number of ways of distribution is 4^5.

$\therefore \quad n(S) = 4^5$

Total number of ways of distribution so that each child gets at least one game is

$$4^5 - {}^4C_1\, 3^5 + {}^4C_2\, 2^5 - {}^4C_3 = 1024 - 4 \times 243 + 6 \times 32 - 4$$
$$= 240$$

$\therefore \quad n(E) = 240$

Therefore, the required probability is

$$\frac{n(E)}{n(S)} = \frac{240}{4^5} = \frac{15}{64}$$

24. a. Out of 9 socks, 2 can be drawn in ${}^9C_2$ ways. Therefore, the total number of cases is ${}^9C_2$. Two socks drawn from the drawer will match if either both are brown or both are blue. Therefore, favourable number of cases is ${}^5C_2 + {}^4C_2$. Hence, the required probability is

$$\frac{{}^5C_2 + {}^4C_2}{{}^9C_2} = \frac{4}{9}$$

25. d. Consider two events as follows:

A_i: getting number i on first die

B_i: getting a number more than i on second die

The required probability is

$$P(A_1 \cap B_1) + P(A_2 \cap B_2) + P(A_3 \cap B_3) + P(A_4 \cap B_4)$$
$$+ P(A_5 \cap B_5) = \sum_{i=1}^{5} P(A_i \cap B_i) = \sum_{i=1}^{5} P(A_i)\, P(B_i)$$

[$\because$ A_i, B_i are independent]

$$= \frac{1}{6}[P(B_1) + P(B_2) + \cdots + P(B_5)]$$

$$= \frac{1}{6}\left(\frac{5}{6} + \frac{4}{6} + \frac{3}{6} + \frac{2}{6} + \frac{1}{6}\right) = \frac{5}{12}$$

26. d. We have,

$$x + \frac{100}{x} > 50$$

$\Rightarrow$ $x^2 + 100 > 50x$

$\Rightarrow$ $(x-25)^2 > 525$

$\Rightarrow$ $x - 25 < \sqrt{525}$ or $x - 25 > \sqrt{525}$

$\Rightarrow$ $x < 25 - \sqrt{525}$ or $25 + \sqrt{525}$

As x is a positive integer and $\sqrt{525} = 22.91$, we must have $x \le 2$ or $x \ge 48$. Thus, the favourable number of cases is 2 + 53 = 55. Hence, the required probability is 55/100 = 11/20.

27. b. The total number of ways in which four-figure numbers can be formed is 4! = 24. A number is divisible by 5 if at unit's place we have 5. Therefore, unit's place can be filled in one way and the remaining 3 places can be filled with the other digits in 3! ways. Hence, total number of numbers divisible by 5 is 3! = 6. So, the required probability is 6/24 = 1/4.

28. a. Since each ball can be placed in any one of the 3 boxes, therefore there are 3 ways in which a ball can be placed in any one of the three boxes. Thus, there are 3^{12} ways in which 12 balls can be placed in 3 boxes. The number of ways in which 3 balls out of 12 can be put in the box is ${}^{12}C_3$. The remaining 9 balls can be placed in 2 boxes in 2^9 ways. So, required probability is

$$\frac{{}^{12}C_3}{3^{12}} 2^9 = \frac{110}{9}\left(\frac{2}{3}\right)^{10}$$

29. a. The required probability is

$$\frac{2}{9} \times \frac{1}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} \times 1 \times 1 \times 1 = \frac{1}{1260}$$

30. d. The total number of ways of choosing 11 players out of 15 is ${}^{15}C_{11}$. A team of 11 players containing at least 3 bowlers can be chosen in the following mutually exclusive ways:

(I) Three bowlers out of 5 bowlers and 8 other players out of the remaining 10 players.

(II) Four bowlers out of 5 bowlers and 7 other players out of the remaining 10 players.

(III) Five bowlers out of 5 bowlers and 6 other players out of the remaining 10 players.

So, required probability is

$$P(\text{I}) + P(\text{II}) + P(\text{III}) = \frac{{}^5C_3 \times {}^{10}C_8}{{}^{15}C_{11}} + \frac{{}^5C_4 \times {}^{10}C_7}{{}^{15}C_{11}} + \frac{{}^5C_5 \times {}^{10}C_6}{{}^{15}C_{11}}$$

$$= \frac{1260}{1365} = \frac{12}{13}$$

31. b. Consider the following events:

A_1: A speaks truth

A_2: B speaks truth

Then, $P(A_1) = 60/100 = 3/5$, $P(A_2) = 70/100 = 7/10$.

For the required event, either both of them should speak the truth or both of them should tell a lie. Thus, the required probability is

$$P((A_1 \cap A_2) \cup (\bar{A}_1 \cap \bar{A}_2)) = P(A_1 \cap A_2) + P(\bar{A}_1 \cap \bar{A}_2)$$

$$= P(A_1)\, P(A_2) + P(\bar{A}_1)\, (\bar{A}_2)$$

$$= \frac{3}{5} \times \frac{7}{10} + \left(1 - \frac{3}{5}\right)\left(1 - \frac{7}{10}\right) = 0.54$$

32. c. The total number of ways in which 3 integers can be chosen from first 20 integers is ${}^{20}C_3$. The product of three integers will be even if at least one of the integers is even. Therefore, the required probability is

1 – Probability that none of the three integers is even

$$= 1 - \frac{{}^{10}C_3}{{}^{20}C_3} = 1 - \frac{2}{19} = \frac{17}{19}$$

33. b. Consider the following events:

A: getting a card with mark I in first draw

B: getting a card with mark I in second draw

C: getting a card with mark T in this draw

Then, the required probability is

$$P(A \cap B \cap C) = P(A)\, P(B/A)\, P(C/A \cap B)$$
$$= \frac{10}{20} \times \frac{9}{19} \times \frac{10}{18} = \frac{5}{38}$$

34. a. Let p_1 and p_2 be the chances of happening of the first and second events, respectively, then according to the given conditions, we have

$$p_1 = p_2^2 \text{ and } \frac{1-p_1}{p_1} = \left(\frac{1-p_2}{p_2}\right)^3$$

$$\Rightarrow \frac{1-p_2^2}{p_2^2} = \left(\frac{1-p_2}{p_2}\right)^3$$

$$\Rightarrow p_2(1 + p_2) = (1 - p_2)^2$$

$$\Rightarrow \quad p_2 = \frac{1}{3}$$

and so

$$p_1 = \frac{1}{9}$$

35. b. The total number of ways in which n persons can sit at a round table is $(n-1)!$ So, total number of cases is $(n-1)!$.

Let A and B be two specified persons. Considering these two as one person, the total number of ways in which $n-1$ persons, $n-2$ other persons and one AB can sit at a round table is $(n-2)!$ So, favourable number of cases is $2!(n-2)!$ Thus, the required probability is

$$p = \frac{2!\,(n-2)!}{(n-1)!} = \frac{2}{n-1}$$

Hence, the required odds are $(1-p):p$ or $(n-3):2$.

36. b. Let one of the quantities be x. Then the other is $2n-x$. Their product will be greatest when they are equal, i.e., each is n in which case the product is n^2. According to the proposition,

$$x(2n-x) \geq \frac{3}{4}n^2$$

$$\Rightarrow \quad 4x^2 - 8nx + 3n^2 \leq 0$$

$$\Rightarrow \quad (2x-3n)(2x-n) \leq 0$$

$$\Rightarrow \quad \frac{n}{2} \leq x \leq \frac{3}{2}n$$

So, favourable number of cases is $3/2n - n - 2 = n$. Hence, the required probability is $n/2n = 1/2$.

37. a. Let the number of red and blue balls be r and b, respectively.

Then, the probability of drawing two red balls is

$$p_1 = \frac{{}^rC_2}{{}^{r+b}C_2} = \frac{r(r-1)}{(r+b)(r+b-1)}$$

The probability of drawing two blue balls is

$$p_2 = \frac{{}^bC_2}{{}^{r+b}C_2} = \frac{b(b-1)}{(r+b)(r+b-1)}$$

The probability of drawing one red and one blue ball is

$$p_3 = \frac{{}^rC_1\,{}^bC_1}{{}^{r+b}C_2} = \frac{2br}{(r+b)(r+b-1)}$$

By hypothesis, $p_1 = 5p_2$ and $p_3 = 6p_2$.

$\therefore \quad r(r-1) = 5b(b-1)$ and $2br = 6b(b-1)$

$\Rightarrow \quad r = 6, b = 3$

38. b. There are 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The last two digits can be dialled in ${}^{10}P_2 = 90$ ways out of which only one way is favourable, thus, the required probability is 1/90.

39. d. The probability that one test is held is $2 \times (1 \times 5) \times (4 \times 5) = 8/25$. The probability that test is held on both days is $(1 \times 5) \times (1 \times 5) = 1/25$.

Therefore, the probability that the student misses at least one test is $8/25 + 1/25 = 9/25$.

40. d. Let X denote the largest number on the 3 tickets drawn.

Then, $P(X \leq 7) = (7/20)^3$ and $P(X \leq 6) = (6/20)^3$. Then, the required probability is

$$P(X=7) = \left(\frac{7}{10}\right)^3 - \left(\frac{6}{20}\right)^3$$

41. b. Let X denote the number of heads in n trials. Then X is a binomial variant with $p = q = 1/2$. Therefore,

$$P\,(X=r) = {}^nC_r\left(\frac{1}{2}\right)^n$$

Now,

$$P(X=6) = P(X=8)$$

$$\Rightarrow \quad {}^nC_6\left(\frac{1}{2}\right)^n = {}^nC_8\left(\frac{1}{2}\right)^n$$

$$\Rightarrow \quad {}^nC_6 = {}^nC_8 \Rightarrow n = 14$$

42. a. Let the number selected be xy. Then

$$x + y = 9,\ 0 \leq x, y \leq 9$$

and

$$xy = 0 \Rightarrow x = 0, y = 9$$

or

$$y = 0, x = 9$$

$$P(x_1 = 9/x_2 = 0) = \frac{P\,(x_1 = 9 \cap x_2 = 0)}{P\,(x_2 = 0)}$$

Now,

$$P(x_2 = 0) = \frac{19}{100}$$

and

$$P(x_1 = 9 \cap x_2 = 0) = \frac{2}{100}$$

$$\Rightarrow \quad P(x_1 = 9/x_2 = 0) = \frac{2/100}{19/100} = \frac{2}{19}$$

43. a. $P(A \cap B') = P(A) - P(A \cap B) = 0.20$

Also,

$$P(A' \cap B) = P(B) - P(A \cap B) = 0.15$$

$$\Rightarrow \quad P(A) + P(B) - 2P(A - B) = 0.35$$

Now,

$$P(A' \cap B') = 1 - P(A \cup B)$$

$$\Rightarrow \quad 0.1 = 1 - P(A) - P(B) + P(A \cap B)$$

$$\Rightarrow \quad P(A) + P(B) - P\,(A \cap B) = 0.9$$

$$\Rightarrow \quad P(A \cap B) = 0.9 - 0.35 = 0.55$$

and

$$P(A) = 0.75, P(B) = 0.70$$

Now,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.55}{0.70}$$

44. a. For each toss, there are four choices:

(i) A gets head, B gets head

(ii) A gets tail, B gets head

(iii) A gets head, B gets tail

(iv) A gets tail, B gets tail

Thus, exhaustive number of ways is 4^{50}. Out of the four choices listed above, (iv) is not favourable to the required event in a toss. Therefore, favourable number of cases is 3^{50}. Hence, the required probability is $(3/4)^{50}$.

45. c. 18 draws are required for 2 aces means in the first 17 draws, there is one ace and 16 other cards and 18th draw produces an ace. So, the required probability is

$$\frac{^{48}C_{16} \times {}^4C_1}{^{52}C_{17}} \times \frac{3}{35} = \frac{561}{15925}$$

46. b. Consider the following events:

A: Father has at least one boy

B: Father has 2 boys and one girl

Then,

A = one boy and 2 girls, 2 boys and one girl, 3 boys and no girl

$A \cap B$ = 2 boys and one girl

Now, the required probability is

$$P(A/B) = \frac{P(A \cap B)}{P(A)} = \frac{1}{3}$$

47. c. The required probability is

$$\left[{}^4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4\right]^2 + \left[{}^4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3\right]^2$$
$$+\left[{}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2\right]^2 + \left[{}^4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1\right]^2$$
$$+\left[{}^4C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0\right]^2 = \frac{35}{128}$$

48. b. Let A denote the event that there is an odd man out in a game. The total number of possible cases is 2^m. A person is odd man out if he is alone in getting a head or a tail.

The number of ways in which there is exactly one tail (head) and the rest are heads (tails) is ${}^mC_1 = m$. Thus, the number of favourable ways is $m + m = 2m$. Therefore,

$$P(A) = \frac{2m}{2^m} = \frac{m}{2^{m-1}}$$

49. c. $x^2 + 2(a+4)x - 5a + 64 \geq 0$

If $D \leq 0$, then

$$(a+4)^2 - (-5a+64) < 0$$
$$\Rightarrow a^2 + 13a - 48 < 0$$
$$\Rightarrow (a+16)(a-3) < 0$$
$$\Rightarrow -16 < a < 3 \Leftrightarrow -5 \leq a \leq 2$$

Then, the favorable cases is equal to the number of integers in the interval [–5, 2], i.e., 8.

Total number of cases is equal to the number of integers in the interval [–5, 30], i.e., 36.

Hence, the required probability is 8/36 = 2/9.

50. a. The total number of ways in which $2n$ boys can be divided into two equal groups is

$$\frac{(2n)!}{(n!)^2 2!}$$

Now, the number of ways in which $2n - 2$ boys other than the two tallest boys can be divided into two equal groups is

$$\frac{(2n-2)!}{((n-1)!)^2 2!}$$

Two tallest boys can be put in different groups in ${}^2C_1$ ways. Hence, the required probability is

$$\frac{2\dfrac{(2n-2)!}{((n-1)!)^2 2!}}{\dfrac{(2n)!}{(n!)^2 2!}} = \frac{n}{2n-1}$$

51. a. Let E_i denote the event that the bag contains i black and $(10 - i)$ white balls $(i = 0, 1, 2, \ldots, 10)$. Let A denote the event that the three balls drawn at random from the bag are black. We have,

$$P(E_i) = \frac{1}{11} \; (i = 0, 1, 2, \ldots, 10)$$

$$P(A/E_i) = 0 \text{ for } i = 0, 1, 2 \text{ and } P(A/E_i) = {}^iC_3/{}^{10}C_3 \text{ for } i \geq 3$$

$$\Rightarrow P(A) = \frac{1}{11} \times \frac{1}{^{10}C_3}\left[{}^3C_3 + {}^4C_3 + \cdots + {}^{10}C_3\right]$$

But

$${}^3C_3 + {}^4C_3 + {}^5C_3 + \cdots + {}^{10}C_3 = {}^4C_4 + {}^4C_3 + {}^5C_3 + \cdots + {}^{10}C_3$$
$$= {}^5C_4 + {}^5C_3 + {}^6C_3 + \cdots + {}^{10}C_3$$
$$\vdots$$
$$= {}^{11}C_4$$

$$\Rightarrow p(A) = \frac{1}{11} \times \frac{1}{^{10}C_3} \times {}^{11}C_4$$

$$= \frac{\dfrac{11 \times 10 \times 9 \times 8}{4!}}{11 \times \dfrac{10 \times 9 \times 8}{3!}} = \frac{1}{4}$$

$$\therefore \quad P(E_9/A) = \frac{P(E_9)\,P(A/E_9)}{P(A)}$$

$$= \frac{\frac{1}{11} \times \frac{{}^9C_3}{{}^{10}C_3}}{\frac{1}{4}}$$

$$= \frac{14}{55}$$

52. a. We have ratio of the ships A, B and C for arriving safely are 2:5, 3:7 and 6:11, respectively. Therefore, the probability of ship A for arriving safely is $2/(2 + 5) = 2/7$.

Similarly, for B the probability is $3/(3 + 7) = 3/10$ and for C the probability is $C = 6/(6 + 11) = 6/17$.

Therefore, the probability of all the ships for arriving safely is $(2/7) \times (3/10) \times (6/17)$ 18/595.

53. a. (i) This question can also be solved by one student.

(ii) This question can be solved by two students simultaneously.

(iii) This question can be solved by three students all together.

$$P(A) = \frac{1}{2},\ P(B) = \frac{1}{4},\ P(C) = \frac{1}{6}$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - [P(A)\,P(B) + P(B)\,P(C) + P(C)\,P(A)] + [P(A)\,P(B)\,P(C)]$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} - \left[\frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{2}\right] + \left[\frac{1}{2} \times \frac{1}{4} \times \frac{1}{6}\right]$$

$$= \frac{33}{48}$$

Alternative solution:

We have,

$$P(\bar{A}) = \frac{1}{2},\ P(\bar{B}) = \frac{3}{4},\ P(\bar{C}) = \frac{5}{6}$$

Then the probability that the problem is not solved is

$$P(\bar{A})\,P(\bar{B})\,P(\bar{C}) = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) = \frac{5}{16}$$

Hence probability that problem is solved is $1 - 5/16 = 11/16$.

54. a. We are given that

$P(A \cap B) = P(A)\,P(B)$

$P(B \cap C) = P(B)P(C)$

$P(C \cap A) = P(C)P(A)$

$P(A \cap B \cap C) = P(A)(B)P(C)$

We have,

$$P(A \cap (B \cap C)) = P(A \cap B \cap C) = P(A)P(B)P(C) = P(A)P(B \cap C)$$

$\Rightarrow$ A and $B \cap C$ are indepedent

Therefore, S_2 is true. Also,

$$P[(A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$$
$$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$
$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$
$$= P(A)P(B) + P(A)\,P(C) - P(A)\,P(B)\,P(C)$$
$$= P(A)[P(B) + P(C) - P(B)P(C)]$$
$$= P(A)[P(B) + P(C) - P(B \cap C)]$$
$$= P(A)P(B \cup C)$$

Therefore, A and $B \cup C$ are independent.

55. d. The total number of ways in which papers of 4 students can be checked by seven teachers is 7^4. The number of ways of choosing two teachers out of 7 is ${}^7C_2$. The number of ways in which they can check four papers is 2^4. But this includes two ways in which all the papers will be checked by a single teacher. Therefore, the number of ways in which 4 papers can be checked by exactly two teachers is $2^4 - 2 = 14$. Therefore, the number of favourable ways is $({}^7C_2)$ (14) = (21) (14). Thus, the required probability is $(21)(14)/7^4 = 6/49$.

56. c. Let us assume that A wins after n deuces, n = 0, 1, 2, 3, ... The probability of a deuce is $(2/3) \times (2/3) + (1/3) \times (1/3) = (5/9)$. [A wins his serve, then B wins his serve or A loses his serve.] So, the probability that 'A' wins game after n deuces is $(5/9)^n \times (2/3) \times (1/3)$. [After n^{th} deuce, A serves and wins, then B serves and loses.] Therefore, the required probability of 'A' winning the game is

$$\sum_{n=0}^{\infty} \left(\frac{5}{9}\right)^n \times \frac{2}{3} \times \frac{1}{3} = \frac{1}{1 - \frac{5}{9}} \times \frac{2}{9} = \frac{1}{2}$$

57. c. The required probability is

1 – Probability of getting equal number of heads and tails

$$= 1 - {}^{2n}C_n \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{2n-n}$$

$$= 1 - \frac{(2n)!}{n!n!}\left(\frac{1}{4}\right)^n$$

$$= 1 - \frac{(2n)!}{(n!)^2} \times \frac{1}{4^n}$$

58. b. Here $p = 19/20$, $q = 1/20$, $n = 5$, $r = 5$. The required probability is

$${}^5C_5 \left(\frac{19}{20}\right)^5 \left(\frac{1}{20}\right)^0 = \left(\frac{19}{20}\right)^5$$

59. d. $$P(E/F) + P(\bar{E}/F) = \frac{P(E \cap F) + P(\bar{E} \cap F)}{P(F)}$$

$$= \frac{P\{(E \cap F) \cup (\bar{E} \cap F)\}}{P(F)}$$

[$\because$ $E \cap F$ and $\bar{E} \cap F$ are disjoint]

$$= \frac{P\{(E \cup \bar{E}) \cap F\}}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Similarly, we can show that (b) and (c) are not true while (d) is true.

$$P\left(\frac{E}{\bar{F}}\right)+P\left(\frac{\bar{E}}{\bar{F}}\right)=\frac{P(E\cap\bar{F})}{P(\bar{F})}+\frac{P(\bar{E}\cap\bar{F})}{P(\bar{F})}=\frac{P(\bar{F})}{P(\bar{F})}=1$$

60. b. We have,

$$P(A)=\frac{40}{100},\ P(B)=\frac{25}{100}\text{ and }P(A\cap B)=\frac{15}{100}$$

So,

$$P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)}=\frac{15/100}{40/100}=\frac{3}{8}$$

61. b. We define the following events:

A_1: Selecting a pair of consecutive letters from the word LONDON

A_2: Selecting a pair of consecutive letters from the word CLIFTON

E: Selecting a pair of letters 'ON'

Then, $P(A_1\cap E)=2/5$ as there are 5 pairs of consecutive letters out of which 2 are ON and $P(A_2\cap E)=1/6$ as there are 6 pairs of consecutive letters of which 1 is ON. Therefore, the required probability is

$$P\left(\frac{A_1}{E}\right)=\frac{P(A_1\cap E)}{P(A_1\cap E)+P(A_2\cap E)}=\frac{\frac{2}{5}}{\frac{2}{5}+\frac{1}{6}}=\frac{12}{17}$$

62. d. Player should get (HT, HT, HT, ...) or (TH, TH, ...) at least $2n$ times. If the sequence starts from first place, then the probability is $1/2^{2n}$ and if starts from any other place, then the probability is $1/2^{2n+1}$. Hence, required probability is

$$2\left(\frac{1}{2^{2n}}+\frac{m}{2^{2n+1}}\right)=\frac{m+2}{2^{2n}}$$

63. b. The total number of cases is $11!/2!\times 2!$ The number of favourable cases is $11!/(2!\times 2!)-9!$ Therefore, required probability is

$$1-\frac{9!\times 4}{11!}=\frac{53}{55}$$

64. b. The probability of winning of A the second race is 1/2 (since both events are independent).

65. b. Given that $n(S)=6\times 6\times 6\times 6=6^4$. The number of favourable ways is ${}^6C_4=6\times 5/2=15$. Therefore, the required probability is

$$\frac{15}{6\times 216}=\frac{5}{2\times 216}=\frac{5}{432}$$

66. c. Given that 5 and 6 have appeared on two of the dice, the sample space reduces to $6^4-2\times 5^4+4^4$ (inclusion–exclusion principle). Also, the number of favourable cases are $4!=24$. So, the required probability is $24/302=12/151$.

67. c. Let a_n be the number of strings of H and T of length n with no two adjacent H's. Then $a_1=2$, $a_2=3$. Also,

$$a_{n+2}=a_{n+1}+a_n \quad \text{(since the string must begin with } T \text{ or } HT)$$

So,

$$a_3=5,\ a_4=8,\ a_5=8+5=13$$

Therefore, the required probability is $13/2^5=13/32$.

68. b.

2							

The prime digits are 2, 3, 5, 7. If we fix 2 at first place, then other $2n-1$ places are filled by all four digits. So the total number of cases is 4^{2n-1}.

Now, sum of 2 consecutive digits is prime when consecutive digits are (2, 3) or (2, 5). Then 2 will be fixed at all alternative places.

2		2		2		2	

So favourable number of cases is 2^n. Therefore, probability is

$$\frac{2^n}{4^{2n-1}}=2^n\,2^{-4n+2}=2^2\,2^{-3n}=\frac{4}{2^{3n}}$$

69. d. Since a, b, c are in A.P., therefore, $2b=a+c$. The possible cases are tabulated as follows.

b	a	c	Number of ways
1	1	1	1
2	2	2	1
2	1	3	6
3	3	3	1
3	1	5	6
3	2	4	6

Total number of ways is 21. So, required probability is 21/216 = 7/72.

70. c. When 4 points are selected, we get one intersecting point. So, probability is

$$\frac{{}^nC_4}{{}^{({}^nC_2-n)}C_2}$$

71. d. Three-digit numbers are 100, 101, ..., 999. Total number of such numbers is 900. The three-digit numbers (which have all same digits) are 111, 222, 333, ..., 999. Favorable number of cases is 9. Therefore, the required probability is 9/900 = 1/100.

72. c. Given,

$$7a-9b=0\Rightarrow b=\frac{7}{9}a$$

Hence, number of pairs (a, b) can be (9, 7); (18, 14); (27, 21); (36, 28). Hence, the required probability is $4/{}^{39}C_2=4/741$.

73. a. Let $E_1=1, 4, 7, \ldots$ (n each)

$E_2=2, 5, 8, \ldots$ (n each)

$E_3 = 3, 6, 9, \ldots$ (n each)

x and y belong to (E_1, E_2), (E_2, E_1) or (E_3, E_3). So, the required probability is

$$\frac{n^2 + {}^nC_2}{{}^{3n}C_2} = \frac{1}{3}$$

74. c. The total number of mapping is n^n. The number of one-one mapping is ${}^nC_1\,{}^{n-1}C_1 \cdots {}^1C_1 = n!$ Hence, the probability is

$$\frac{n!}{n^n} = \frac{3}{32} = \frac{4!}{4^4}$$

Comparing, we get $n = 4$.

75. b. $P(4 \text{ biased coin}) = \frac{1}{3}$

$P(5 \text{ biased coin}) = \frac{1}{4}$

Hence, the required probability is

$$\frac{1}{3}\frac{{}^4C_3\,{}^{16}C_6}{{}^{20}C_9} + \frac{2}{3}\frac{{}^5C_4\,{}^{15}C_5}{{}^{20}C_9}\frac{1}{{}^{11}C_1}$$

$$= \frac{2}{33}\left[\frac{{}^{16}C_6 + 5\,{}^{15}C_5}{{}^{20}C_9}\right]$$

76. b. Let event A be drawing 9 cards which are not ace and B be drawing an ace card. Therefore, the required probability is $P(A \cap B) = P(A) \times P(B)$

Now, there are four aces and 48 other cards. Hence,

$$P(A) = \frac{{}^{48}C_9}{{}^{52}C_9}$$

After having drawn 9 non-ace cards, the 10th card must be ace. Hence,

$$P(B) = \frac{{}^4C_1}{{}^{42}C_1} = \frac{4}{42}$$

Hence,

$$P(A \cap B) = \frac{{}^{48}C_9}{{}^{52}C_9}\frac{4}{42}$$

77. c. The total number of digits in any number at the unit's place is 10.

$\therefore \quad n(S) = 10$

If the last digit in product is 1, 3, 5 or 7, then it is necessary that the last digit in each number must be 1, 3, 5 or 7.

$\therefore \quad n(A) = 4$

$\therefore \quad P(A) = \frac{4}{10} = \frac{2}{5}$

Hence, the required probability is $(2/5)^4 = 16/625$.

78. a. The divisibility of the product of four numbers depends upon the value of the last digit of each number. The last digit of a number can be any one of the 10 digits 0, 1, 2, …, 9. So, the total number of ways of selecting last digits of four numbers is $10 \times 10 \times 10 \times 10 = 10^4$. If the product of the 4 numbers is not divisible by 5 or 10, then the number of choices for the last digit of each number is 8 (excluding 0 or 5). So, favourable number of ways is 8^4. Therefore, the probability that the product is not divisible by 5 or 10 is $(8/10)^4$. Hence, the required probability is $1 - (8/10)^4 = 369/625$.

79. b. Let H denote the head, T the tail and * any of the head or tail. Then, $P(H) = 1/2$, $P(T) = 1/2$ and $P(*) = 1$. For at least four consecutive heads, we should have any of the following patterns:

	Probability
(i) $HHHH***$	$(1/2)^4 \times 1 = 1/16$
(ii) $THHHH**$	$(1/2)^5 = 1/32$
(iii) $*THHHH*$	$(1/2)^5 = 1/32$
(iv) $**THHHH$	$(1/2)^5 = 1/32$

Since all the above cases are mutually exclusive, the probability of getting at least four consecutive heads (on adding) is $1/16 + 3/32 = 5/32$.

80. c. The probabilities of solving the question by these three students are 1/3, 2/7 and 3/8, respectively.

$$\therefore \quad P(A) = \frac{1}{3};\ P(B) = \frac{2}{7};\ P(C) = \frac{3}{8}$$

Then probability of question solved by only one student is

$$P((A\bar{B}\bar{C} \text{ or } \bar{A}\,B\bar{C} \text{ or } \bar{A}\,\bar{B}\,C)) = P(A)\,P(\bar{B})\,P(\bar{C}) + P(\bar{A})\,P(B)\,P(\bar{C}) + P(\bar{A})\,P(\bar{B})\,P(C)$$

$$= \frac{1}{3}\times\frac{5}{7}\times\frac{5}{8} + \frac{2}{3}\times\frac{2}{7}\times\frac{5}{8} + \frac{2}{3}\times\frac{5}{7}\times\frac{3}{8}$$

$$= \frac{25+20+30}{168} = \frac{25}{56}$$

81. b. Probability of getting 2 heads in the first 5 trials is

$${}^5C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{10}{32} = \frac{5}{16}$$

Therefore, the probability that third head appears on the sixth trial is $5/16 \times 1/2 = 5/32$.

82. d. Let A and B, respectively, be the events that urn A and urn B are selected. Let R be the event that the selected ball is red. Since the urn is chosen at random, Therefore,

$$P(A) = P(B) = \frac{1}{2}$$

and

$$P(R) = P(A)P(R/A) + P(B)P(R/B)$$

$$= \frac{1}{2}\times\frac{5}{10} + \frac{1}{2}\times\frac{4}{10}$$

$$= \frac{9}{20}$$

83. d. The probability that the first critic favours the book is

$$P(E_1) = \frac{5}{5+2} = \frac{5}{7}$$

The probability that the second critic favours the book is

$$P(E_2) = \frac{4}{4+3} = \frac{4}{7}$$

The probability that the third critic favours the book is

$$P(E_3) = \frac{3}{3+4} = \frac{3}{7}$$

Majority will be in favour of the book if at least two critics favour the book. Hence, the probability is

$$P(E_1 \cap E_2 \cap \bar{E}_3) + P(E_1 \cap \bar{E}_2 \cap E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

$$= P(E_1)P(E_2)P(\bar{E}_3) + P(E_1)P(\bar{E}_2)\,P(E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1)P(E_2)P(E_3)$$

$$= \frac{5}{7}\times\frac{4}{7}\times\left(1-\frac{3}{7}\right)+\frac{5}{7}\times\left(1-\frac{4}{7}\right)\times\frac{3}{7} + \left(1-\frac{5}{7}\right)\times\frac{4}{7}\times\frac{3}{7}+\frac{5}{7}\times\frac{4}{7}\times\frac{3}{7}$$

$$= \frac{209}{343}$$

84. b.

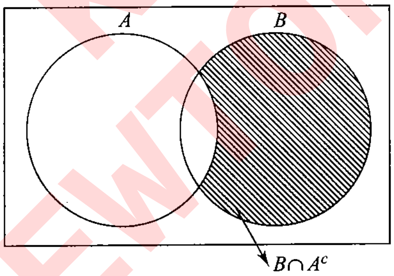


Fig. 9.10

$$P(A) = \frac{1}{4},\ P(A \cup B) = \frac{1}{2}$$

$$P\left(\frac{B}{A^C}\right) = \frac{P(B \cap A^C)}{P(A^C)}$$

$$= \frac{P(A \cup B) - P(A)}{1 - P(A)} \quad [\because P(A \cup B) + P(B) - P(A \cap B)]$$

$$= \frac{\frac{1}{2}-\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{4}\times\frac{4}{3} = \frac{1}{3}$$

85. b. $P(S \cap F) = 0.0006$, where S is the event that the motor cycle is stolen and F is the event that the moter cycle is found. Therefore,

$$P(S) = 0.0015$$

$$P(F/S) = \frac{P(F \cap S)}{P(S)} = \frac{6\times 10^{-4}}{15\times 10^{-4}} = \frac{2}{5}$$

86. c. Let A be the event that 11 is picked and B be the event that sum is even. The number of ways of selecting 11 along with one more odd number is $n(A \cap B) = {}^7C_1$.

The number of ways of selecting either two even numbers or selecting two odd numbers is $n(B) = 1 + {}^8C_2$.

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{7}{29} = 0.24$$

87. b. Die marked with 1, 2, 2, 3, 3, 3 is thrown 3 times.

$$P(1) = \frac{1}{6},\ P(2) = \frac{2}{6},\ P(3) = \frac{3}{6}$$

$$P(S) = P(4 \text{ or } 6)$$

$$= P(112(3 \text{ cases}) \text{ or } 123\ (6 \text{ cases}) \text{ or } 222)$$

$$= 3\times\frac{1}{6}\times\frac{1}{6}\times\frac{2}{6}+6\frac{1}{6}\times\frac{2}{6}\times\frac{3}{6}+\frac{2}{6}\times\frac{2}{6}\times\frac{2}{6}$$

$$= \frac{6+36+8}{216} = \frac{50}{216} = \frac{25}{108}$$

88. c.

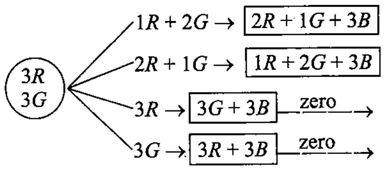


Fig. 9.12

The required probability is

$$\frac{{}^3C_1\,{}^3C_2}{{}^6C_3}\frac{{}^2C_1\,{}^1C_1\,{}^3C_1}{{}^6C_3} + \frac{{}^3C_2\,{}^3C_1}{{}^6C_3}\frac{{}^1C_1\,{}^2C_1\,{}^3C_1}{{}^6C_3}$$

$$= 2\frac{9}{20}\times\frac{6}{20}$$

$$= \frac{27}{100}$$

89. c. $P(4 \text{ biased coins}) = \frac{1}{3}$

$P(5 \text{ biased coins}) = \frac{1}{4}$

The required probability is

$$\frac{1}{3}\frac{{}^4C_3\,{}^{16}C_6}{{}^{20}C_9}\frac{1}{{}^{11}C_1}+\frac{2}{3}\frac{{}^5C_4\,{}^{15}C_5}{{}^{20}C_9}\frac{1}{{}^{11}C_1}$$

$$=\frac{2}{33}\left[\frac{{}^{16}C_6+5\,{}^{15}C_5}{{}^{20}C_9}\right]$$

90. d. A: Doctor finds a rash

B_1: Child has measles

S: Sick children

$P(S/F)=0.9$

B_2: Child has flu $\Rightarrow P(B_2)=9/10$

$P(S/M)=0.10$

$P(A/B_1)=0.95$

$P(R/M)=0.95$

$P(A/B_2)=0.08$

$P(R/F)=0.08$

$$P(B_1/A)=\frac{0.1\times 0.95}{0.1\times 0.95+0.9\times 0.08}$$

$$=\frac{0.095}{0.095+0.072}$$

$$=\frac{0.095}{0.167}=\frac{95}{167}$$

91. c. The sum is 12 in the first three throws if they are (1, 5, 6) in any order or (2, 4, 6) in any order or (3, 4, 5) in any order. Therefore, the required probability is

$$\frac{1}{6}\times\frac{1}{5}\times\frac{1}{4}\times 3!+\frac{1}{6}\times\frac{1}{5}\times\frac{1}{4}\times 3!+\frac{1}{6}\times\frac{1}{5}\times\frac{1}{4}\times 3!=\frac{3}{20}$$

(because after throwing 1, in the next throw 1 cannot come, etc.)

92. a. The number of composite numbers in 1 to 30 is $n(S)=19$.

The number of composite number when divided by 5 leaves a remainder is $n(E)=14$. Therefore, the required probability is 14/19.

93. b. $P(E)+P(E')=1=1+\lambda+\lambda^2+(1+\lambda)^2$

$\Rightarrow\ 2\lambda^2+3\lambda+1=0$

$\Rightarrow\ (2\lambda+1)(\lambda+1)=0$

$\Rightarrow\ \lambda=-1,\ -\frac{1}{2}$

Then,

$P(E)=1+(-1)+(-1)^2=1$ (not possible)

$$\Rightarrow\ P(E)=1+\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)^2=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

94. b. Let $P(m)$, $P(p)$, $P(c)$ be the probability of selecting a book of maths, physics and chemistry, respectively. Clearly,

$$P(m)=P(p)=P(c)=\frac{1}{3}$$

Again let $P(s_1)$ and $P(s_2)$ be the probability that he solves the first as well as second problem, respectively. Then,

$$P(s_1)=P(m)\times P\left(\frac{s_1}{m}\right)+P(p)\times P\left(\frac{s_1}{p}\right)+P(c)\times P\left(\frac{s_1}{c}\right)$$

$$\Rightarrow\ P(s_1)=\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{3}{5}+\frac{1}{3}\times\frac{4}{5}=\frac{19}{30}$$

Similarly,

$$P(S_2)=\frac{1}{3}\times\left(\frac{1}{2}\right)^2+\frac{1}{3}\times\left(\frac{3}{5}\right)^2+\frac{1}{3}\times\left(\frac{4}{5}\right)^2=\frac{125}{300}$$

$$\Rightarrow\ P\left(\frac{S_2}{S_1}\right)=\frac{\frac{125}{300}}{\frac{19}{30}}=\frac{25}{38}$$

95. b. Let,

$P(S)=P(1 \text{ or } 2)=1/3$

$P(F)=P(3 \text{ or } 4 \text{ or } 5 \text{ or } 6)=2/3$

$P(A \text{ wins})=P[(S\,S \text{ or } S\,F\,S\,S \text{ or } S\,F\,S\,F\,S\,S \text{ or } \ldots)$

or $(F\,S\,S \text{ or } F\,S\,F\,S\,S \text{ or } \ldots)]$

$$=\frac{\frac{1}{9}}{1-\frac{2}{9}}+\frac{\frac{2}{27}}{1-\frac{2}{9}}$$

$$=\frac{1}{9}\times\frac{9}{7}+\frac{2}{27}\times\frac{9}{7}$$

$$=\frac{1}{7}+\frac{2}{21}=\frac{3+2}{21}=\frac{5}{21}$$

$$P(A \text{ winning})=\frac{5}{21},\ P(B \text{ winning})=\frac{16}{21}$$

96. a. $P(a)=0.3$, $P(b)=0.5$, $P(c)=0.2$. Hence, a, b, c are exhaustive.

$P(\text{same horse wins all the three races})=P(aaa \text{ or } bbb \text{ or } ccc)$

$$=(0.3)^3+(0.5)^3+(0.2)^3$$

$$=\frac{27+125+8}{1000}=\frac{160}{1000}$$

$$=\frac{4}{25}$$

$P(\text{each horse wins exactly one race})$

$=P(abc \text{ or } acb \text{ or } bca \text{ or } bac \text{ or } cab \text{ or } cba)$

$$=0.3\times 0.5\times 0.2\times 6=0.18=\frac{9}{50}$$

97. b. The total number of ways of distribution is 4^5.

$\therefore\ n(S)=4^5$

The total number of ways of distribution so that each child gets at least one game is $4^5 - {}^4C_1\,3^5 + {}^4C_2\,2^5 - {}^4C_3$.

$\therefore \quad n(E) = 240$

Hence, the required probability is

$$\frac{n(E)}{n(S)} = \frac{240}{4^5} = \frac{15}{64}$$

98. b. Team totals must be 0, 1, 2, …, 39. Let the teams be $T_1T_2, \ldots, T_{40}$, so that T_i loses to T_j for $i < j$. In other words, this order uniquely determines the result of every game. There are 40! such orders and 780 games, so 2^{780} possible outcomes for the games. Hence, the probability is $40!/2^{780}$.

99. a. We have,

$n(S) = {}^{64}C_3$

Let 'E' be the event of selecting 3 squares which form the letter 'L'.

The number of ways of selecting squares consisting of 4 unit squares is $7 \times 7 = 49$.

Each square with 4 unit squares form 4 L-shapes consisting of 3 unit squares.

$\therefore \quad n(E) = 7 \times 7 \times 4 = 196$

$$\therefore \quad P(E) = \frac{196}{{}^{64}C_3}$$

100.a. The total number of ways of making the second draw is ${}^{10}C_5$.

The number of draws of 5 balls containing 2 balls common with first draw of 6 balls is ${}^6C_2\,{}^4C_3$. Therefore, the probability is

$$\frac{{}^6C_2\,{}^4C_3}{{}^{10}C_5} = \frac{5}{21}$$

101. a. The required probability is

$$P(A) = \frac{1}{3}\,\frac{6}{a^2 - 4a + 10}$$

$$(P(A))_{\max} = \frac{1}{3} \times \frac{6}{6} = \frac{1}{3}$$

102. b. A number has exactly 3 factors if the number is squares of a prime number. Squares of 11, 13, 17, 19, 23, 29, 31 are 3-digit numbers. Hence, the required probability is 7/900.

103. c. Suppose, there exist three rational points or more on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Therefore, if (x_1, y_1), (x_2, y_2) and (x_3, y_3) be those three points, then

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad (1)$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_1 + c = 0 \quad (2)$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \quad (3)$$

Solving Eqs. (1), (2) and (3), we will get g, f, c as rational. Thus, centre of the circle $(-g, -f)$ is a rational point. Therefore, both the coordinates of the centre are rational numbers. Obviously, the possible values of p are 1, 2. Similarly, the possible values of q are 1, 2. Thus for this case, (p, q) may be chosen in 2×2, i.e., 4 ways. Now, (p, q) can be, without restriction, chosen in 6×6, i.e., 36 ways.

Hence, the probability that at the most two rational points exist on the circle is $(36 - 4)/36 = 32/36 = 8/9$.

104. a. $P(A) = P(B) = P(C)$ and $P(A) + P(B) + P(C) = 1$

$$\therefore \quad P(A) = P(B) = P(C) = \frac{1}{3}$$

Also,

$$P(X) = \frac{5}{12}, P(X/A) = \frac{3}{8}, P(X/B) = \frac{1}{4}$$

We have,

$$P(X) = P(A)\,P(X/A) + P(B)\,P(X/B) + P(C)\,P(X/C)$$

$$\therefore \quad \frac{5}{12} = \frac{1}{3}\left\{\frac{3}{8} + \frac{1}{4} + P(X/C)\right\}$$

$$\Rightarrow \quad P(X/C) = \frac{5}{8}$$

105. c. A: car met with an accident

B_1: driver was alcoholic, $P(B_1) = 1/5$

B_2: driver was sober, $P(B_2) = 4/5$

$P(A/B_1) = 0.001$; $P(A/B_2) = 0.0001$

$$P(B_1/A) = \frac{(0.2)\,(0.001)}{(0.2)\,(0.001) + (0.8)(0.0001)} = 5/7$$

106. a.

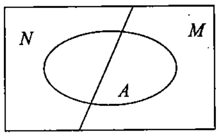


Fig. 9.12

Let N be the event of picking up a normal die; $P(N) = 1/4$. Let M be the event of picking up a magnetic die; $P(M) = 3/4$. Let A be the event that die shows up 3.

$$\therefore \quad P(A) = P(A \cap N) + P(A \cap M)$$

$$= P(N)P(A/N) + P(M)\,P(A/M)$$

$$= \frac{1}{4} \times \frac{1}{6} + \frac{3}{4} \times \frac{2}{3} = \frac{7}{24}$$

$$P(N/A) = \frac{P(N \cap A)}{P(A)} = \frac{(1/4)(1/6)}{7/24} = \frac{1}{7}$$

107. c. We have,

$n(S) = 5^5$

For computing favourable outcomes, 2 boxes which are to remain empty, can be selected in ${}^5C_2$ ways and 5 marbles can be placed in the remaining 3 boxes in groups of 221 or 311 in

$$3!\left[\frac{5!}{2!\,2!\,2!} + \frac{5!}{3!\,2!}\right] = 150 \text{ ways} \Rightarrow n(A) = {}^5C_2 \times 150$$

Hence,

$$P(E) = {}^5C_2 \times \frac{150}{5^5} = \frac{60}{125} = \frac{12}{25}$$

108. a. $n(S) = {}^{10}C_7 = 120$

$$n(A) = {}^5C_4 \times {}^3C_2 \times {}^2C_1$$

$$P(E) = \frac{5\times 3\times 2}{120} = \frac{1}{4}$$

109. a. For ranked 1 and 2 players to be winners and runners up, respectively, they should not be paired with each other in any round. Therefore, the required probability is 30/31 × 14/15 × 6/7 × 2/3 = 16/31.

110. a. The total number of ways of selecting 3 integers from 20 natural numbers is ${}^{20}C_3 = 1140$. Their product is a multiple of 3 means at least one number is divisible by 3. The numbers which are divisible by 3 are 3, 6, 9, 12, 15, 18 and the number of ways of selecting at least one of them is ${}^6C_1 \times {}^{14}C_2 + {}^6C_2 \times {}^{14}C_1 + {}^6C_3 = 776$. Hence, the required probability is 776/1140 = 194/285.

111. d. Since there are r cars in N places, total number of selection of places out of $N-1$ places for $r-1$ cars (excepting the owner's car) is

$${}^{N-1}C_{r-1} = \frac{(N-1)!}{(r-1)!\,(N-r)!}$$

If neighbouring places are empty, then $r-1$ cars must be parked in $N-3$ places. So, the favourable number of cases is

$${}^{N-3}C_{r-1} = \frac{(n-3)!}{(r-1)!\,(N-r-2)!}$$

Therefore, the required probability is

$$\frac{(N-3)!}{(r-1)!\,(N-r-2)!} \times \frac{(r-1)!\,(N-r)!}{(N-1)!}$$

$$= \frac{(N-r)(N-r-1)}{(N-1)(N-2)} = \frac{{}^{N-r}C_2}{{}^{N-1}C_2}$$

112. b. The sum of the digits can be 7 in the following ways: 07, 16, 25, 34, 43, 52, 61, 70.

$\therefore \quad (A = 7) = \{07, 16, 25, 34, 43, 52, 61, 70\}$

Similarly,

$(B = 0) = \{00, 01, 02, \ldots, 10, 20, 30, \ldots, 90\}$

Thus,

$(A = 7) \cap (B = 0) = \{09, 70\}$

$$\therefore \quad P((A=7)\cap(B=0)) = \frac{2}{100},\ P((B=0)) = \frac{19}{100}$$

Hence,

$$P(A=7 \mid B=0) = \frac{P((A=7)\cap(B=0))}{P(B=0)}$$

$$= \frac{\frac{2}{100}}{\frac{19}{100}} = \frac{2}{19}$$

113. b. Let the probability of getting a tail in a single trial be $p = 1/2$. The number of trials be $n = 100$ and the number of trials in 100 trials be X. We have,

$$P(X = r) = {}^{100}C_r p^r q^{n-r}$$

$$= {}^{100}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{100-r}$$

$$= {}^{100}C_r \left(\frac{1}{2}\right)^{100}$$

Now,

$$P(X=1) + P(X=3) + \cdots + P(X=49)$$

$$= {}^{100}C_1\left(\frac{1}{2}\right)^{100} + {}^{100}C_3\left(\frac{1}{2}\right)^{100} + \cdots + {}^{100}C_{49}\left(\frac{1}{2}\right)^{100}$$

$$= ({}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{49})\left(\frac{1}{2}\right)^{100}$$

But

$${}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{99} = 2^{99}$$

Also,

$${}^{100}C_{99} = {}^{100}C_1$$

$${}^{100}C_{97} = {}^{100}C_3, \ldots\ {}^{100}C_{51} = {}^{100}C_{49}$$

Thus,

$$2({}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{49}) = 2^{99}$$

$$\Rightarrow \quad {}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{49} = 2^{98}$$

Therefore, probability of required event is

$$\frac{2^{98}}{2^{100}} = \frac{1}{4}\ 2^{98}/2^{100} = 1/4$$

114. a. Let A denote the event that a sum of 5 occurs, B the event that a sum of 7 occurs and C the event that neither a sum of 5 nor a sum of 7 occurs. We have,

$$P(A) = \frac{4}{36} = \frac{1}{9},\ P(B) = \frac{6}{36} = \frac{1}{6}$$

$$P(C) = \frac{26}{36} = \frac{13}{18}$$

Thus, probability that A occurs before B is

$$P[A \text{ or } (C\cap A) \text{ or } (C\cap C\cap A) \text{ or } \ldots]$$

$$= P(A) + P(C\cap A) + P(C\cap C\cap A) + \cdots$$

$$= P(A) + P(C)P(A) + P(C)^2P(A) + \cdots$$

$$= \frac{1}{9} + \left(\frac{13}{18}\right)\times\frac{1}{9} + \left(\frac{13}{18}\right)^2\frac{1}{9} + \cdots$$

$$= \frac{1/9}{1-13/18} = \frac{2}{5}$$

115. a. In any number the last digits can be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Therefore, last digit of each number can be chosen in 10 ways. Thus, exhaustive number of ways is 10^n. If the last digit be 1, 3, 7 or 9, then none of the numbers can be even or end in 0 or 5. Thus, we have a choice of 4 digits, viz., 1, 3, 7 or 9 with which each of n numbers should end. So favourable number of ways is 4^n. Hence, the required probability is

$$\frac{4^n}{10^n}=\left(\frac{2}{5}\right)^n$$

116. c. The probability that A gets r heads in three tosses of a coin is

$$P(X=r)={}^3C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{3-r}={}^3C_r\left(\frac{1}{2}\right)^3$$

The probability that A and B both get r heads in three tosses of a coin is

$${}^3C_r\left(\frac{1}{2}\right)^3 {}^3C_r\left(\frac{1}{2}\right)^3=({}^3C_r)^2\left(\frac{1}{2}\right)^6$$

Hence, the required probability is

$$\sum_{r=0}^{3}({}^3C_r)^2\left(\frac{1}{2}\right)^6=\left(\frac{1}{2}\right)^6\{1+9+9+1\}=\frac{20}{64}=\frac{5}{16}$$

117. c. We know that the number of subsets of a set containing n elements is 2^n. Therefore, the number of ways of choosing P and Q is

$${}^{2^n}C_1\times{}^{2^n}C_1=2^n\times2^n=4^n$$

Out of n elements, m elements are chosen and then from the remaining $n-m$ elements either an element belongs to P or Q. But not both P and Q. Suppose P contains r elements from the remaining $n-m$ elements. Then, Q may contain any number of elements from the remaining $(n-m)-r$ elements Therefore, P and Q can be chosen in ${}^{n-m}C_r 2^{(n-m)-r}$ ways.

But r can vary from 0 to $n-m$. So, in general the number of ways in which P and Q can be chosen is

$$\left(\sum_{r=0}^{n-m}{}^{n-m}C_r 2^{(n-m)-r}\right){}^nC_m=(1+2)^{n-m}\,{}^nC_m={}^nC_m 3^{n-m}$$

Hence, the required probability is ${}^nC_m 3^{n-m}/4^n$.

118. c. In the first 9 throws, we should have three sixes and six non-sixes; and a six in the 10th throw, and thereafter it does not matter whatever face appears. So, the required probability is

$${}^9C_3\left(\frac{1}{6}\right)^3\times\left(\frac{5}{6}\right)^6\times\frac{1}{6}\times\underbrace{1\times1\times1\times\cdots\times1}_{10\text{ times}}$$

$$=\frac{84\times5^6}{6^{10}}$$

119. c. Let the probability that a man aged x dies in a year p. Thus the probability that a man aged x does not die in a year $=1-p$. The probability that all n men aged x do not die in a year is $(1-p)^n$. Therefore, the probability that at least one man dies in a year is $1-(1-p)^n$. The probability that out of n men, A_1 dies first is $1/n$. Since this event is independent of the event that at least one man dies in a year, hence, the probability that A_1 dies in the year and he is first one to die is $1/n\,[1-(1-p)^n]$.

120. b. The required probability is

$$\frac{n^2}{{}^{2n}C_2}\frac{(n-1)^2}{{}^{2n-2}C_2}\frac{(n-2)^2}{{}^{2n-4}C_2}\cdots\frac{2^2}{{}^4C_2}\frac{1^2}{{}^2C_2}$$

$$=\frac{(1\times2\times3\times4\times\cdots\times(n-1)n)^2}{\dfrac{(2n)!}{2^n}}=\frac{2^n(n!)^2}{(2n)!}=\frac{2^n}{{}^{2n}C_n}$$

121. a. The probability of getting a head in a single toss of a coin is $p=1/2$ (say). The probability of getting 5 or 6 in a single throw of a die is $q=2/6=1/3$ (say). Therefore, the required probability is

$$p+(1-p)(1-q)p+(1-p)(1-q)(1-p)(1-q)p+\cdots$$
$$=p+(1-p)(1-q)p+(1-p)^2(1-q)^2p+\cdots$$
$$=\frac{p}{1-(1-p)(1-q)}$$
$$=\frac{1/2}{1-1/2\times2/3}=\frac{3}{4}$$

122. a. Consider the following events.

A: ball drawn is black

E_1: bag I is chosen

E_2: bag II is chosen

E_3: bag III is chosen

Then,

$$P(E_1)=(E_2)=P(E_3)=\frac{1}{3}$$

$$P(A/E_1)=\frac{3}{5},\ P(A/E_2)=\frac{1}{5},\ P(A/E_3)=\frac{7}{10}$$

Therefore, the required probability is

$$P(E_3/A)$$
$$=\frac{P(E_3)P(A/E_3)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}=\frac{7}{15}$$

123. c. $f'(x)=3x^2+2ax+9$

$y=f(x)$ is increasing

$\Rightarrow$ $f'(x)\ge0,\ \forall\ x$ and for $f'(x)=0$ should not form an interval

$\Rightarrow$ $(2a)^2-4\times3\times b\le0\Rightarrow a^2-3b\le0$

This is true for exactly 16 ordered pairs (a, b), $1\le a, b\le6$, namely (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 6) and (4, 6). Thus, the required probability is $16/36=4/9$

124. d. Let p_i denote the probability that out of 10 tosses, head occurs i times and no two heads occur consecutively. It is clear that $i>5$.

For $i=0$, i.e., no head, $p_0=1/2^{10}$.

For $i=1$, i.e., one head, $p_1={}^{10}C_1\ (1/2)^1\,(1/2)^9=10/2^{10}$.

Now for $i=2$, we have 2 heads and 8 tails. Then, we have 9 possible places for heads. For example, see the construction:

$$x\,T\,x\,T\,x\,T\,x\,T\,x\,T\,x\,T\,x\,T\,x\,T\,x$$

Here x represents possible places for heads.

$$\therefore\quad p_2={}^9C_2\left(\frac{1}{2}\right)^2(1/2)^8=36/2^{10}$$

Similarly,

$$p_3={}^8C_3/2^{10}=56/2^{10}$$

$$p_4 = {}^7C_2/2^{10} = 35/2^{10}$$

$$p_5 = {}^6C_5/2^{10} = 6/2^{10}$$

$$\therefore \quad p = p_0 + p_1 + p_2 + p_3 + p_4 + p_5$$

$$= \frac{1+10+36+56+35+6}{2^{10}} = \frac{144}{2^{10}} = \frac{9}{64}$$

125. c. Given limit,

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{2}{x}}$$

$$= \lim_{x\to 0}\left(1+\frac{a^x+b^x-2}{2}\right)^{\frac{2}{a^x+b^x-2}\lim\limits_{x\to 0}\left(\frac{a^x-1+b^x-1}{x}\right)}$$

$$= e^{\log ab} = ab = 6$$

Total number of possible ways in which a, b can take values is $6 \times 6 = 36$. Total possible ways are (1, 6), (6, 1), (2, 3), (3, 2). The total number of possible ways is 4. Hence, the required probability is $4/36 = 1/9$.

126. c. Let A denote the event that target is hit when x shells are fired at point I. Let P_1 and P_2 denote the event that the target is at point I and II, respectively. We have $P(P_1) = 8/9$, $P(P_2) = 1/9$, $P(A/P_1) = 1 - (1/2)^x$, $P(A/P_2) = 1 - (1/2)^{55-x}$.

Now from total probability theorem,

$$P(A) = P(P_1)\, P(A/P) + P(P_2)\, P(A/P_2)$$

$$= \frac{1}{9}\left(8 - 8\left(\frac{1}{2}\right)^x + 1 - \left(\frac{1}{2}\right)^{55-x}\right)$$

$$= \frac{1}{9}\left(9 - 8\left(\frac{1}{2}\right)^x - \left(\frac{1}{2}\right)^{55-x}\right)$$

Now,

$$\frac{dP(A)}{dx} = \frac{1}{9}\left(-8\left(\frac{1}{2}\right)^x \ln\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{55-x} \ln\left(\frac{1}{2}\right)\right)$$

(Note that in this step, it is being assumed that $x \in R^+$)

$$= \frac{1}{9}\ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{55-x}\left(1 - \left(\frac{1}{2}\right)^{2x-58}\right)$$

> 0 if $x < 29$

< 0 if $x > 29$

Therefore, $P(A)$ is maximum at $x = 29$. Thus, '29' shells must be fired at point I.

127. d. In the first case, the urn constains 3 red and n white balls. The probability that colour of both the balls matches is

$$\frac{{}^3C_2 + {}^nC_2}{{}^{n+3}C_2} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{6 + n(n-1)}{(n+3)(n+2)} = \frac{1}{2}$$

$$\Rightarrow \quad 2(n^2 - n + 6) = n^2 + 5n + 6$$

$$\Rightarrow \quad n^2 - 7n + 6 = 0$$

$$\Rightarrow \quad n = 1 \text{ or } 6 \qquad (1)$$

In the second case,

$$\frac{3}{n+3}\,\frac{3}{n+3} + \frac{n}{n+3}\,\frac{n}{n+3} = \frac{5}{8}$$

Solving, we get

$$n^2 - 10n + 9 = 0$$

$$\Rightarrow \quad n = 9 \text{ or } 1. \qquad (2)$$

From Eqs. (1) and (2), we have $n = 1$.

128. b. Let us consider the following events.

A: card shows up black

B_1: card with both sides black

B_2: card with both sides white

B_3: card with one side white and one black

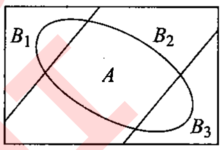


Fig. 9.13

$$P(B_1) = \frac{2}{10},\ P(B_2) = \frac{3}{10},\ P(B_3) = \frac{5}{10}$$

$$P(A/B_1) = 1,\ P(A/B_2) = 0,\ P(A/B_3) = \frac{1}{2}$$

$$P(B_1/A) = \frac{\frac{2}{10}\times 1}{\frac{2}{10}\times 1 + \frac{3}{10}\times(0) + \frac{5}{10}\times\frac{1}{2}} = \frac{4}{4+5} = \frac{4}{9}$$

129.d. A: exactly one ace

B: both aces

E: $A \cup B$

$$P(B/A \cup B)) = \frac{{}^4C_2}{{}^4C_1\,{}^{12}C_1 + {}^4C_2} = \frac{6}{54} = \frac{1}{9}$$

130. b. $n(S) = \dfrac{9!}{4!\,.\,5!} = 126$

$n(A) = 0$ to F and F to P

$$= \frac{5!}{2!\times 3!} \times \frac{4}{2!\times 2!} = 60$$

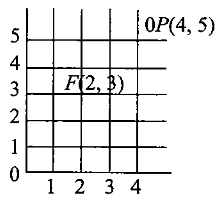


Fig. 9.14

$$\Rightarrow \quad P(A) = \frac{60}{126} = \frac{10}{21}$$

Multiple Correct Answers Type

1. a, c, d.

Since A and B are independent events, therefore,

$$P(A \cap B) = P(A)\, P(B) = \frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$$

$$P(A/B) = P(A) = \frac{1}{2}$$

Now,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{2} + \frac{1}{5} - \frac{1}{10} = \frac{3}{5}$$

Now,

$$P(A/A \cup B) = \frac{P[A \cap (A \cup B)]}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{1/2}{3/5} = \frac{5}{6}$$

$$P[(A \cap B)/(\bar{A} \cup \bar{B})] = P(A \cap B/(\overline{A \cap B})) = 0$$

2. a, b, c.

We are given that

$$P(A \cap B') = 0.20,\ P(A' \cap B) = 0.15,\ P(A \cap B) = 0.10$$

Now,

$$P(B) = P(A' \cap B) + P(A \cap B) = 0.15 + 0.20 = 0.35$$

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.10}{0.35} = \frac{2}{7}$$

$$P(A) = P(A \cap B') + P(A \cap B) = 0.3$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.55$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{1}{3}$$

3. a, b, c, d.

$$A \subseteq A \cup B$$

$$\Rightarrow \quad P(A) \le P(A \cup B) \Rightarrow P(A \cup B) \ge \frac{3}{4}$$

Also,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$\ge P(A) + P(B) - 1$$

$$= \frac{3}{4} + \frac{5}{8} - 1 = \frac{3}{8}$$

Now,

$$A \cap B \subseteq B$$

$$\Rightarrow \quad P(A \cap B) \le P(B) = \frac{5}{8}$$

$$\therefore \quad \frac{3}{8} \le P(A \cap B) \le \frac{5}{8} \qquad (1)$$

and

$$P(A \cap B') = P(A) - P(A \cap B)$$

$$\Rightarrow \quad \frac{3}{4} - \frac{5}{8} \le P(A \cap B') \le \frac{3}{4} - \frac{3}{8}$$

$$\Rightarrow \quad \frac{1}{8} \le P(A \cap B') \le \frac{3}{8}$$

$$\therefore \quad P(A \cap B) = P(B) - P(A' \cap B) \qquad \text{[Using Eq. (1)]}$$

$$\Rightarrow \quad \frac{3}{8} \le P(B) - P(A' \cap B) \le \frac{5}{8}$$

$$\Rightarrow \quad 0 \le P(A' \cap B) \le \frac{1}{4}$$

4. a, c.

Given that A and B are mutually exclusive events.

$$\therefore \quad A \cap B = \phi$$

$$\Rightarrow \quad A \subseteq \bar{B} \text{ and } B \subseteq \bar{A}$$

$$\Rightarrow \quad P(A) \le P(\bar{B}) \text{ and } P(B) \le P(\bar{A})$$

5. a, b, c.

Let 'H' be the event that married man watches the show and 'W' be the probability that married woman watches the show.

$$\therefore \quad P(H) = 0.4,\ P(W) = 0.5,\ P(H/W) = 0.7$$

a. $P(H \cap W) = P(W)\, P(H/W) = 0.5 \times 0.7 = 0.35$

b. $P(W/H) = \dfrac{P(H \cap W)}{P(H)} = \dfrac{0.35}{0.4} = \dfrac{7}{8}$

c. $P(H \cup W) = P(H) + P(W) - P(H \cap W)$

$$= 0.4 + 0.5 - 0.35 = 0.55$$

6. b, c.

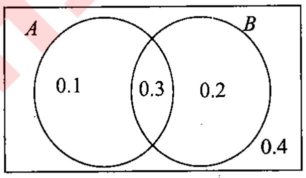


Fig. 9.15

$$P(A) = 0.4,\ P(B) = 0.5,\ P(A \cap B) = 0.3$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$$

Now,

$$P(E_1) = \frac{P(A \cap B)}{P(A \cup B)} = \frac{0.3}{0.6} = \frac{1}{2}$$

$$P(E_2) = \frac{P(A \cap \bar{B}) + P(\bar{A} \cap B)}{P(A \cup B)} = \frac{0.1 + 0.2}{0.6} = \frac{1}{2}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.5} = \frac{3}{5} = 0.60;$$

$$P(B/A) = \frac{0.3}{0.4} = \frac{3}{4} = 0.75 = 0.75$$

$$P(A/(A \cup B)) = \frac{P(A)}{P(A \cup B)} = \frac{0.4}{0.6} = \frac{2}{3}$$

$$\frac{P(B)}{P(A \cup B)} = \frac{0.5}{0.6} = \frac{5}{6}$$

7. c, d.

$P(E_1) = 1 - P$ (unit's place in both is 1, 2, 3, 4, 6, 7, 8, 9)

$$P(E_1 = 0 \text{ or } 5) = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}$$

$$P(E_2: 5) = P(1\ 3\ 5\ 7\ 9) - P(1\ 3\ 7\ 9)$$

$$= \frac{1}{4} - \frac{4}{25}$$

$$= \frac{25 - 16}{100} = \frac{9}{100}$$

$$\frac{P(E_2)}{P(E_1)} = \frac{9}{100} \times \frac{25}{9} = \frac{1}{4}$$

$$P(E_1) = 4P(E_2)$$

$$P(E_2/E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{P(E_2)}{P(E_1)} = \frac{1}{4}$$

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(E_2)}{P(E_2)} = 1$$

8. a, c, d.

$$P(E) = \frac{{}^{2n}C_n}{2^{2n}} = \frac{(2n)!}{n!\,n!\,2^n\,2^n}$$

$$= \frac{1 \times 2 \times 3 \times \cdots \times (2n)}{n!\,n!\,2^n\,2^n}$$

$$= \frac{1 \times 3 \times 5 \cdots \times (2n-1)}{n!\;2^n}$$

Now,

$$\prod_{r=1}^{n}\left(\frac{2r-1}{2r}\right) = \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times (2n)}$$

$$= \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{(1 \times 2 \times 3 \times \cdots \times n)2^n}$$

$$\sum_{r=0}^{n}\left(\frac{{}^nC_r}{2^n}\right)^2 = \frac{1}{2^n 2^n}\sum_{r=0}^{n}({}^nC_r)^2$$

$$= \frac{1}{2^n 2^n}\,{}^{2n}C_n$$

Also,

$$\frac{\sum_{r=0}^{n}({}^nC_r)^2}{\left(\sum_{r=0}^{2n}{}^{2n}C_r\right)} = \frac{{}^{2n}C_n}{2^{2n}}$$

9. a, b. The probability that both will be alive for 10 years, hence, i.e., the probability that the man and his wife both will be alive 10 years hence is $0.83 \times 0.87 = 0.7221$. The probability that at least one of them will be alive is

$1 - P$ {That none of them remains alive 10 years hence}

$= 1 - (1 - 0.83)(1 - 0.87) = 1 - 0.17 \times 0.13$

$= 0.9779$

10. b, c, d.

a. False.

$$P(TTT \text{ or } HHH) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

b. $$\frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B^c)}{1 - P(B)} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$P(A \cap B)\,[1 - P(B)] = P(B)\,P(A) - P(B)\,P(A \cap B)$$

$$P(A \cap B) = P(A)\,P(B)$$

Hence, the given statement is true.

c. Let E_1 be the white ball is drawn in first draw; E_2 be the event that black ball is drawn in second draw; E be the event that white ball is drawn in second draw.

$$\therefore\ P(E) = P(E/E_1)\,P(E_1) + P(E/E_2)\,P(E_2)$$

$$= \frac{d+w}{w+b+d}\left(\frac{w}{w+b}\right) + \frac{w}{w+b+d}\left(\frac{b}{w+b}\right)$$

$$= \left(\frac{w}{w+b}\right)\left(\frac{d+w}{w+b+d} + \frac{b}{w+b+d}\right)$$

$$= \left(\frac{w}{w+b}\right)$$

which is independent of d.

d. To prove that A, B, C are pairwise independent only. Now,

$$P(A \cap B) = P((A \cap B \cap C) \cup (A \cap B \cap \bar{C}))$$

$$= P(A \cap B \cap C) + P(A \cap B \cap \bar{C})$$

$$= P(A)\,P(B)\,P(\bar{C}) + P(A)\,P(B)\,P(C) \qquad \text{(given)}$$

$$= P(A) \times P(B)\,[P(C) + P(\bar{C})]$$

$$= P(A) \times P(B)$$

Similarly, for the other two. Hence, this statement is correct.

11. a, b, c, d.

a. $$P(E_1) = 1 - P(RRR)$$

$$= 1 - \left[\frac{1}{3} \times \frac{2}{4} \times \frac{3}{5}\right] = 0.9$$

b. $$P(E_2) = 3P(BRR) = 3 \times \frac{2}{3} \times \frac{1}{4} \times \frac{2}{5} = 0.2$$

c. $$P(E_3) = P(RRR/(RRR \cup BBB))$$

$$= \frac{P(RRR)}{P(RRR) + P(BBB)}$$

$$= \frac{0.1}{0.1 + \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5}}$$

$$= \frac{0.1}{0.1 + 0.4} = 0.2$$

d. $$P(E_4) = 1 - P(BBB) = 1 - \frac{2}{5} = 0.6$$

12. a, b, c, d.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \frac{5}{8} = \frac{3}{8} + \frac{4}{8} - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = \frac{2}{8} = \frac{1}{4}$$

Now,

$$P(A^c/B) = \frac{P(A^c \cap B)}{P(B)}$$

$$= \frac{P(B) - P(A \cap B)}{P(B)}$$

$$= 1 - 2\left(\frac{1}{4}\right)$$

$$= \frac{1}{2}$$

$$2P(A/B^c) = \frac{2P(A \cap B^C)}{P(B^C)}$$

$$= \frac{2(P(A) - P(A \cap B))}{1 - P(B)}$$

$$= 4\left(\frac{3}{8} - \frac{2}{8}\right) = \frac{1}{2}$$

Hence option (a) is correct.

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2} = P(B)$$

Hence (b) is correct. Again,

$$P(A^c/B^c) = \frac{P(A^C \cap B^C)}{P(B^C)}$$

$$= \frac{1 - P(A \cup B)}{1 - P(B)}$$

$$= 2\left(1 - \frac{5}{8}\right) = \frac{3}{4}$$

$$P(B/A^c) = \frac{P(B \cap A^C)}{1 - P(A)}$$

$$= \frac{P(B) - P(A \cap B)}{5/8}$$

$$= \frac{1/2 - 1/4}{5/8}$$

$$= \frac{1}{4} \times \frac{8}{5}$$

$$= \frac{2}{5}$$

Hence,

$$8P(A^c/B^c) = 15P(B/A^c)$$

Hence, (c) is not correct. Again,

$$2P(A/B^c) = \frac{1}{2}$$

$$\Rightarrow \quad P(A/B^c) = \frac{1}{4} = P(A \cap B)$$

Hence (d) is correct.

13. a, c.

Let p_1, p_2 be the chances of happening of the first and second events, respectively. Then according to the given conditions, we have

$$p_1 = p_2^2$$

and

$$\frac{1-p_1}{p_1} = \left(\frac{1-p_2}{p_2}\right)^3$$

Hence,

$$\frac{1-p_2^2}{p_2^2} = \left(\frac{1-p_2}{p_2}\right)^3 \Rightarrow p_2(1+p_2) = (1-p_2)^2$$

$$\Rightarrow \quad 3p_2 = 1 \Rightarrow p_2 = \frac{1}{3}$$

and so

$$p_1 = \frac{1}{9}$$

14. a, b.

Let the number of red and blue balls be r and b, respectively.

Then, the probability of drawing two red balls is

$$p_1 = \frac{{}^rC_2}{{}^{r+b}C_2} = \frac{r(r-1)}{(r+b)(r+b-1)}$$

The probability of drawing two blue balls is

$$p_2 = \frac{{}^bC_2}{{}^{r+b}C_2} = \frac{b(b-1)}{(r+b)(r+b-1)}$$

The probability of drawing one red and one blue ball is

$$p_3 = \frac{{}^rC_1 \times {}^bC_1}{{}^{r+b}C_2} = \frac{2br}{(r+b)(r+b-1)}$$

By hypothesis, $p_1 = 5p_2$ and $p_3 = 6p_2$.

$\therefore \quad r(r-1) = 5b(b-1)$ and $2br = 6b(b-1)$

$\Rightarrow \quad r = 6, b = 3$

15. a, b, d.

We have, the probability that the bomb strikes the target is $p = 1/2$. Let n be the number of bombs which should be dropped to ensure 99% chance or better of completely destroying the target. Then, the probability that out of n bombs at least two bombs strike the target is greater than 0.99. Let X denote the number of bombs striking the target. Then

$$P(X = r) = {}^nC_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{n-r} = {}^nC_r\left(\frac{1}{2}\right)^n, r = 0, 1, 2, \ldots, n$$

We should have

$$P(X \geq 2) \geq 0.99$$

$$\Rightarrow \quad \{1 - P(X < 2)\} \geq 0.99$$

$$\Rightarrow \quad 1 - \{P(X = 0) + P(X = 1)\} \geq 0.99$$

$$\Rightarrow \quad 1 - \left\{(1+n)\frac{1}{2^n}\right\} \geq 0.99$$

$$\Rightarrow \quad 0.001 \geq \frac{1+n}{2^n}$$

$$\Rightarrow \quad 2^n > 100 + 100n \Rightarrow n \geq 11$$

Thus, the minimum number of bombs is 11.

16. a, b, c, d.

We have,

P(exactly one of A, B occurs)

$= P[(A \cap \bar{B}) \cup \bar{A} \cap B)]$

$= P(A \cap \bar{B}) + P(\bar{A} \cap B)$

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$= P(A) + P(B) - 2P(A \cap B)$

$= P(A \cup B) - P(A \cap B)$

Also,

P(exactly one of A, B occurs)

$= [1 - P(\bar{A} \cap \bar{B})] - [1 - P(\bar{A} \cup \bar{B})]$

$= P(\bar{A} \cup \bar{B}) - P(\bar{A} \cap \bar{B})] = P(\bar{A}) + P(\bar{B}) - 2P(\bar{A} \cap \bar{B})$

17. a, b, c.

Option (d) is true if and only if A and B are independent.

18. a, b, c.

$$P(A/B) = P(A) = \frac{1}{2}$$

$$P\{A/(A \cup B)\} = \frac{P[A \cap (A \cup B)]}{P(A \cup B)}$$

$$[\because A \cap (A \cup B) = A \cap (A - B - A \cap B)] = A - A \cap B - A \cap B = a\}$$

$$\Rightarrow \quad P\left(\frac{A}{A \cup B}\right) = \frac{P(A)}{P(A \cup B)}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2} - \frac{1}{5} - \frac{1}{10}} = \frac{\frac{1}{2}}{\frac{6}{10}} = \frac{5}{6}$$

and similarly,

$$P\left(\frac{A \cap B}{A' \cup B'}\right) = 0$$

19. b, c.

Let $P(A) = x$ and $P(B) = y$. Since A and B are independent events, therefore,

$$P(\bar{A} \cap B) = 2/15 \quad \Rightarrow P(\bar{A})P(B) = 2/15$$

$$\Rightarrow (1 - P(A))\, P(B) = 2/15$$

$$\Rightarrow (1 - x)y = 2/15 \qquad (1)$$

and

$$P(A \cap \bar{B}) = \frac{1}{6} \Rightarrow P(A)\, P(\bar{B}) = \frac{1}{6}$$

$$\Rightarrow x(1 - y) = \frac{1}{6}$$

$$\Rightarrow x - xy = \frac{1}{6} \qquad (2)$$

Subtracting Eq. (1) from Eq. (2), we get

$$x - y = \frac{1}{30} \quad \Rightarrow x = \frac{1}{30} + y$$

Putting this value of x in Eq. (1), we get

$$y - y\left(\frac{1}{30} + y\right) = \frac{2}{15}$$

$\Rightarrow \quad 30y - y - 30y^2 = 2/5$

$\Rightarrow \quad 30y^2 - 29y + 4 = 0$

$\Rightarrow \quad (6y - 1)(5y - 4) = 0$

$\Rightarrow \quad y = 1/6$ or $y = 4/5$

$\Rightarrow \quad P(B) = 1/6$ or $P(B) = 4/5$

20. a, b, c.

a. $P(A) = \frac{1}{5}, P(B) = \frac{7}{25}, P(B/A) = \frac{9}{10}$

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \left[\frac{1}{5} + \frac{7}{25} - P(A)\, P(B/A)\right]$$

$$= 1 - \left[\frac{1}{5} + \frac{7}{25} - \frac{1}{5} \times \frac{7}{25}\right] = \frac{7}{10}$$

b. $$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(A)\, P(B/A)}{P(B)}$$

$$= \frac{\frac{1}{5} \times \frac{9}{10}}{\frac{7}{25}}$$

$$= \frac{9}{50} \times \frac{25}{7} = \frac{9}{14} = \frac{18}{28}$$

c. $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)P(B/A)$$

$$= \frac{1}{5} + \frac{7}{25} - \frac{1}{5} \times \frac{9}{10}$$

$$= \frac{10 + 14 - 9}{50}$$

$$= \frac{3}{10}$$

d. $$P(A' \cup B') = 1 - P(A \cap B)$$

$$= 1 - P(A)P(B/A)$$

$$= 1 - \frac{1}{5} \times \frac{9}{10}$$

$$= \frac{41}{50}$$

21. b, c, d.

Roots of $x^2 + px + q = 0$ will be real if $p^2 \geq 4q$.

The possible selections are as follows:

p	q
1	–
2	1
3	1, 2
4	1, 2, 3, 4
5	1, 2, 3, 4, 5, 6
6	1, 2, ..., 9
7	1, 2, ..., 10
8	1, 2, ..., 10
9	1, 2, ..., 10
10	1, 2, ..., 10
Total	62

Therefore, number of favourable ways is 62 and total number of ways is $10^2 = 100$. Hence, the required probability is 62/100 = 31/50.

The probability that the roots are imaginary is $1 - 31/50 = 19/50$.

Roots are equal when $(p, q) \equiv (2, 1), (4, 4), (9, 6)$. The probability that the roots are real and equal is 3/50. Hence, probability that the roots are real and distinct is 3/5.

22. a, b, c.

Here total number of cases is ${}^8C_2 = 28$.

a. Favourable number of case is 13.

For 2 $\rightarrow$ 6 choices
For 1 $\rightarrow$ 7 choices

b. For 6 $\rightarrow$ 5 choices
For 7 $\rightarrow$ 6 choices
For 8 $\rightarrow$ 7 choices

c. For 1 $\rightarrow$ 4 choices (2, 4, 6, 8)
For 2 $\rightarrow$ 6 choices (3, 4, 5, 6, 7, 8)
For 3 $\rightarrow$ 3 choices (4, 6, 8)
For 4 $\rightarrow$ 4 choices (5, 6, 7, 8)
For 5 $\rightarrow$ 2 choices (6, 8)
For 6 $\rightarrow$ 2 choices (7, 8)
For 7 $\rightarrow$ 1 choice (8)

Alternative solution:

a. $$\frac{{}^8C_2 - {}^6C_2}{{}^8C_2} = \frac{13}{28}$$

b. $$\frac{{}^8C_2 - {}^5C_2}{{}^8C_2} = \frac{9}{14}$$

c. $$\frac{{}^8C_2 - {}^4C_2}{{}^8C_2} = \frac{11}{14}$$

23. a, d.

The probability that head appears r times is

$$ {}^{99}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{99-r}$$

which is maximum when $r = 49$ or 50.

24. a, c.

Let one probability of choosing one integer k be $P(k) = \lambda/k^4$. (λ is one constant of proportionality). Then,

$$\sum_{k=1}^{2m} \frac{\lambda}{k^4} = 1$$

$$\Rightarrow \lambda \sum_{k=1}^{2m} \frac{1}{k^4} = 1$$

Let x_1 be the probability of choosing the odd number. Then,

$$x_1 = \sum_{k=1}^{m} P(2k-1) = \lambda \sum_{k=1}^{m} \frac{1}{(2k-1)^4}$$

Also,

$$1 - x_1 = \sum_{k=1}^{m} P(2k)$$

$$= \lambda \sum_{k=1}^{m} \frac{1}{(2k)^4}$$

$$< \lambda \sum_{k=1}^{m} \frac{1}{(2k-1)^4}$$

$\Rightarrow 1 - x_1 < x_1$
$\Rightarrow x_1 > 1/2$
$\Rightarrow x_2 < 1/2$

Reasoning Type

1. a. $P(A/B) \geq P(A)$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} \geq P(A)$$

$$\Rightarrow \frac{P(A \cap B)}{P(A)} \geq P(B)$$

$$\Rightarrow P(B/A) \geq P(B)$$

2. c. Statement 1 is true as there are six equally likely possibilities of which only two are favourable (4 and 6). Hence, probability that the obtained number is composite) is $2/6 = 1/3$.

Statement 2 is not true, as the three possibilities are not equally likely.

3. c. The required probability is

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(A \cap B)]$$
$$= 0.39$$

4. a. $\because P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$\geq P(A) + P(B) - 1$$

$$\therefore P(A \cap B) \geq \frac{3}{5} + \frac{2}{3} - 1$$

$$\Rightarrow P(A \cap B) \geq \frac{4}{15} \quad (1)$$

$$P(A \cap B) \leq P(A)$$

$$\Rightarrow P(A \cap B) \leq \frac{3}{5} \quad (2)$$

From Eqs. (1) and (2),

$$\frac{4}{15} \leq P A \cap B) \leq \frac{3}{5} \quad (3)$$

From (3),

$$\frac{4}{15\,P(B)} \leq \frac{P(A \cap B)}{P(B)} \leq \frac{3}{5\,P(B)}$$

$$\Rightarrow \frac{2}{5} \leq P\left(\frac{A}{B}\right) \leq \frac{9}{10}$$

5. a. $P\{A \cap (B \cap C)\} = P(A \cap B \cap C) = P(A)P(B)P(C)$

$\therefore P[A \cap (B \cup C)]$
$= P[(A \cap B) \cup (A \cap C)]$
$= P[(A \cap B) + (A \cap C) - P[(A \cap B) \cap (A \cap C)]$
$= p(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$
$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)$
$= P(A)[P(B) + P(C) - P(B)P(C)]$
$= P(A)\, P(B \cup C)$

Therefore, A and $B \cup C$ are independent events.

6. a. $2n + 1 = 5, n = 2$

$$P(E) = \frac{3n}{4n^2 - 1} = \frac{6}{15} = \frac{2}{5}$$

As a, b, c are in A.P., so

$a + c = 2b$

$\Rightarrow$ $a + c$ is even

Therefore, a and c are both even or both odd. So, the number of ways of choosing a and c is ${}^nC_2 + {}^{n+1}C_2 = n^2$.

$$\therefore \quad P(E) = \frac{n^2}{{}^{2n+1}C_3} = \frac{3n}{4n^2 - 1}$$

7. d. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore$ $1 \geq P(A) + P(B) - P(A \cap B) \geq 3/4$

$\Rightarrow$ $P(A) + P(B) - 1/8 \geq 3/4$

[since minimum value of $P(A \cap B)$ is 1/8)

$\Rightarrow$ $P(A) + P(B) \leq 1/8 + 3/4 = 7/8$

As the maximum value of $P(A \cap B)$ is 3/8, we get

$1 \geq P(A) + P(B) - 3/8$

$\Rightarrow$ $P(A) + P(B) \leq 1 + 3/8 = 11/8$

8. b. Clearly both are correct but statement 2 is not the correct explanation for statement 1.

9. b. $P(A \cup \bar{B}) = 1 - (\overline{A \cup \bar{B}}) = 1 - (\bar{A} \cap B) = 1 - P(\bar{A})P(B)$

$\Rightarrow$ $0.9 = 1 - 0.6 \times P(B)$

$\Rightarrow$ $P(B) = \dfrac{1}{6}$

Clearly, statement 2 is not correct explanation of statement 1.

10. c. According to statement 1, the required probability is

$${}^nC_0\left(\frac{1}{2}\right)^n + {}^nC_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{n-4} + {}^nC_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{n-8} + \cdots$$

$$= ({}^nC_0 + {}^nC_4 + {}^nC_8 + \cdots)\left(\frac{1}{2}\right)^n$$

Now consider the binomial expansion,

$$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots$$

Putting $x + i$, where $i = \sqrt{-1}$, we get

$$(1 + i)^n = ({}^nC_0 - {}^nC_2 + {}^nC_4 - \cdots) + i({}^nC_1 - {}^nC_3 + {}^nC_5 - \cdots)$$

$$\Rightarrow \quad \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n = ({}^nC_0 - {}^nC_2 + {}^nC_4 - \cdots) + i({}^nC_1 - {}^nC_3 + {}^nC_5 - \cdots)$$

$$\Rightarrow \quad {}^nC_0 - {}^nC_2 + {}^nC_4 - \cdots = 2^{n/2}\cos\frac{n\pi}{4}$$

Also we know that

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = 2^{n-1}$$

$$\Rightarrow \quad 2({}^nC_0 + {}^nC_4 + {}^nC_8 + \cdots) = 2^{n-1} + 2^{n/2}\cos\frac{n\pi}{4}$$

Hence, the required probability is

$$\frac{1}{4} + \frac{1}{2^{n/2+1}}\cos\left(\frac{n\pi}{4}\right)$$

11. d. $P\left(\dfrac{A}{B}\right) = \dfrac{P(A \cap B)}{P(B)}$ (by definition)

$\Rightarrow$ $P(\bar{B}) = P((A \cup \bar{A}) \cap \bar{B}) = P((A \cap \bar{B}) \cup (\bar{A} \cap \bar{B}))$

Hence, statement 2 is true. Now,

$$P(A/\bar{B}) + P(\bar{A}/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} + \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$$

$$= \frac{P(A \cap \bar{B}) + P(\bar{A} \cap \bar{B})}{P(\bar{B})}$$

$$= \frac{P(\bar{B})}{P(\bar{B})} = 1$$

Therefore, statement 1 is false.

12. a. In binomial theorem, we have proved statement 2. Now, there may be 0, 1, 2, 3, 4 or 5 heads in the last five throws and the same can be for the first 10 throws. The number of cases thus may be given by

$$m = {}^5C_0{}^{10}C_0 + {}^5C_1{}^{10}C_1 + {}^5C_2{}^{10}C_2 + {}^5C_3{}^{10}C_3 + {}^5C_4{}^{10}C_4 + {}^5C_5{}^{10}C_5$$

$$= {}^5C_0{}^{10}C_{10} + {}^5C_1{}^{10}C_9 + {}^5C_2{}^{10}C_8 + {}^5C_3{}^{10}C_7 + {}^5C_4{}^{10}C_6 + {}^5C_5{}^{10}C_5$$

$$= {}^{10+5}C_{10} = {}^{15}C_{10}$$

$$= 3003$$

The total number of ways (N) is $2^{15} = 32768$. Hence, the required probability is $m/N = 3003/32768$.

13. c. $P(A) + P(B) = 1$ is true as A and B are mutually exclusive and exhaustive events, but statement 2 is false as it is not given that the events are exhaustive.

14. b. The total number of cases, $n(S) = 4!$. Let E be the event that no letter is mailed in its correct envelope. Then the favourable number of cases is

$$4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$$

Hence, the required probability is

$$P(E) = \frac{9}{24} = \frac{3}{8}$$

Also, the probability that all the letters are placed in the correct envelope is 1/24.

Hence, the probability that all the letters are not placed in the correct envelope is 23/24.

Hence, statement 2 is correct but does not explain statement 1.

15. a. We have,

$$P(A \cup \bar{B}) = 1 - (\overline{A \cup \bar{B}}) = 1 - (\bar{A} \cap B) = 1 - P(\bar{A})\,P(B)$$

$$0.8 = 1 - 0.7 \times P(B)$$

$$\Rightarrow \quad P(B) = \frac{2}{7}$$

Linked Comprehension Type

For Problems 1–3

1. b, 2. c, 3. a.

Sol. Let $P(i)$ be the probability that exactly i students are passing an examination. Now given that

$P(A_i) = \lambda i^2$ (where λ is constant)

$$\Rightarrow \sum_{i=1}^{10} P(A_i) = \sum_{i=1}^{10} \lambda i^2 = \lambda \frac{10\times 11\times 21}{6} = \lambda 385 = 1 \Rightarrow \lambda = 1/385$$

Now, $P(5) = 25/385 = 5/77$.

Let A represent the event that selected students have passed the examination.

$$\therefore \quad P(A) = \sum_{i=1}^{10} P(A/A_i)P(A_i)$$

$$= \sum_{i=1}^{10} \frac{i}{10}\frac{i^2}{385}$$

$$= \frac{1}{3850}\sum_{i=1}^{10} i^3$$

$$= \frac{10^2 11^2}{4\times 3850} = \frac{11}{14}$$

Now,

$$P(A_1/A) = \frac{P(A/A_1)P(A_1)}{P(A)}$$

$$= \frac{\frac{1}{385}\frac{1}{10}}{\frac{11}{14}}$$

$$= \frac{1}{11\times 55}\frac{1}{5}$$

$$= \frac{1}{3025}$$

For Problems 4–6

4. c, 5. b, 6. d.

Sol. Let in 8 coupons S, U, R, F appears x_1, x_2, x_3, x_4 times. Then $x_1 + x_2 + x_3 + x_4 = 8$, where $x_1, x_2, x_3, x_4 \geq 0$.

We have to find non-negative integral solutions of the equation. The total number of such solutions is ${}^{8+4-1}C_{4-1} = {}^{11}C_3 = 165$.

If a person gets at least one free packet, then he must get each coupon at least once, which is equal to number of positive integral solutions of the equation. The number of such solutions is ${}^{8-1}C_{4-1} = {}^{7}C_3 = 35$. Then, the probability that he gets exactly one free packet is

$(35 - 1)/165 = 102/495$.

The probability that he gets two free packets is $1/{}^{11}C_3 = 1/165$.

For Problems 7–9

7. d, 8. d, 9. b.

Sol. 7. Let p_1 be the probability of being an answer correct from section 1. Then $p_1 = 1/5$. Let p_2 be the probability of being an answer correct from section 2. Then $p_2 = 1/15$. Hence, the required probability is $1/5 \times 1/15 = 1/75$.

8. Scoring 10 marks from four questions can be done in 3 + 3 + 3 + 1 = 10 ways so as to answer 3 questions from section 2 and 1 question from section 1 correctly. Hence the required probability is

$$\frac{{}^{10}C_3\,{}^{10}C_1}{{}^{20}C_4}\frac{1}{5}\left(\frac{1}{15}\right)^3$$

9. To get 40 marks, he has to answer all questions correctly and its probability is $(1/5)^{10}$ $(1/15)^{10}$. Hence, probability of getting a score less than 40 is

$$1 - \left(\frac{1}{5}\right)^{10}\left(\frac{1}{15}\right)^{10} = 1 - \left(\frac{1}{75}\right)^{10}$$

For Problems 10–12

10. c., 11. d., 12.c.

Sol. 10. x can be 2, 3, 4, 5, 6. The number of ways in which sum of 2, 3, 4, 5, 6 can occur is given by the coefficients of x^2, x^3, x^4, x^5, x^6 in

$$(3x + 2x^2 + x^3)(x + 2x^2 + 3x^3) = 3x^2 + 8x^3 + 14x^4 + 8x^5 + 3x^6$$

This shows that sum that occurs most often is 4.

11. Sum that occurs for minimum times is 2 or 6.

12. The number of ways in which different sums can occur is (3 + 2 + 1) (1 + 2 + 3) = 36. The probability of 4 is 14/36 = 7/18.

For Problems 13–15

13. c, 14. b, 15. d.

Sol. A: She gets a success

T: She studies 10 h: $P(T) = 0.1$

S: She studies 7 h: $P(S) = 0.2$

F: She studies 4 h: $P(F) = 0.7$

$P(A/T) = 0.8$, $P(A/S) = 0.6$, $P(A/F) = 0.4$

$$P(A) = P(A\cap T) + P(A\cap S) + P(A\cap F)$$

$$= P(T)\,P(A/T) + P(S)\,P(A/S) + P(F)\,P(A/F)$$

$$= (0.1)(0.8) + (0.2)(0.6) + (0.7)(0.4)$$

$$= 0.08 + 0.12 + 0.28 = 0.48$$

$$= P(F/A) = \frac{P(F\cap A)}{P(A)}$$

$$= \frac{(0.7)(0.4)}{0.48}$$

$$= \frac{0.28}{0.48} = \frac{7}{12}$$

$$P(F/\bar{A}) = \frac{P(F \cap \bar{A})}{P(\bar{A})}$$

$$= \frac{P(F) - P(F \cap A)}{0.52}$$

$$= \frac{(0.7) - 0.28}{0.52}$$

$$= \frac{0.42}{0.52} = \frac{21}{26}$$

For Problems 16–18

16. b, 17. a, 18. c.

Sol. $P(S/T) = \dfrac{P(S \cap T)}{P(T)}$

$\Rightarrow \quad 0.5 = \dfrac{P(S \cap T)}{0.69}$

$\Rightarrow \quad P(S \cap T) = 0.5 \times 0.69 = P(S)\,P(T)$

Therefore, S and T are independent.

$\therefore \quad P(S \text{ and } T) = P(S)P(T)$

$= 0.69 \times 0.5 = 0.345$

$P(S \text{ or } T) = P(S) + P(T) - P(S \cap T)$

$= 0.5 + 0.69 - 0.345$

$= 0.8450$

For Problems 19–21

19. d, 20. b, 21. a.

Sol. 19. Let E be the event that all the amoeba population dies out;

E_1 be the event that after first second amoeba splits into two;

E_2 be the event that after first second amoeba remains the same. Then,

$P(E) = P(E_1)P(E/E_1) + P(E_2)P(E/E_2)$

$$= \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4}$$

$$= \frac{3}{32}$$

20. After 2s, exactly 4 amoeba are alive, i.e. initially amoeba must split into two and in 2nd second again amoeba must split into two. Hence, the required probability is $(1/2) \times (1/2) \times (1/2) = 1/8$.

21. In first second, mother amoeba splits into two with the probability 1/2.

In 2nd second, two amoeba split into two with probability $(1/2) \times (1/2) = 1/4$.

In 3rd second, four amoeba split into two with probability $(1/2) \times (1/2) \times (1/2) \times (1/2) = 1/16$.

Hence, the probability that the population is maximum after 3s is $(1/2) \times (1/4) \times (1/16) = 1/2^7$.

For Problems 22–24

22. b, 23. b, 24. d.

Sol. The number of cubes having at least one side painted is $9 + 9 + 3 + 3 + 1 + 1 = 26$. The number of cubes having two sides painted is $4 + 4 + 1 + 1 + 1 = 12$.

For Problems 25–27

25. a, 26. d, 27. b.

Sol.

25. For the favourable cases, the points should lie inside the concentric circle of radius $r/2$. So the desired probability is given by

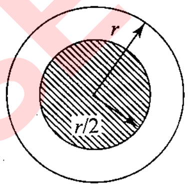


Fig. 9.16

$$\frac{\text{Area of smaller circle}}{\text{Area of larger circle}} = \frac{\pi\left(\frac{r}{2}\right)^2}{\pi r^2} = \frac{1}{4}$$

26.

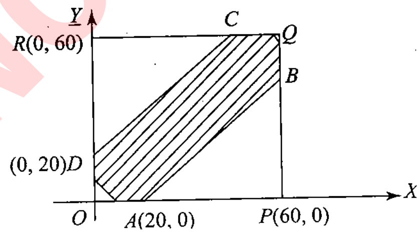


Fig. 9.17

Let A and B arrive at the place of their meeting 'a' minutes and 'b' minutes after 5 pm. Their meeting is possible only if

$|a - b| \le 20$ (1)

Clearly, $0 \le a \le 60$ and $0 \le b \le 60$. Therefore, a and b can be selected as an ordered pair (a, b) from the set $[0, 60] \times [0, 60]$.

Alternatively, it is equivalent to select a point (a, b) from the square $OPQR$, where P is $(60, 0)$ and R is $(0, 60)$ in the Cartesian plane. Now,

$$|a - b| \le 20 \Rightarrow -20 \le a - b \le 20$$

Therefore, points (a, b) satisfy the equation $-20 \le x - y \le 20$.

Hence, favourable condition is equivalent to selecting a point from the region bounded by $y \le x + 20$ and $y \ge x - 20$. Therefore, the required probability is

$$\frac{\text{Area of } OABQCDO}{\text{Area of square } OPQR} = \frac{[\text{Ar}(OPQR) - 2\text{Ar}(\Delta APB)]}{\text{Ar}(OPQR)}$$

$$= \frac{60 \times 60 - \frac{2}{2} \times 40 \times 40}{60 \times 60} = \frac{5}{9}$$

27. Picking two points x and y randomly from the intervals [0, 2] and [0, 1] is equivalent to picking a single point (x, y) randomly from the rectangle S shown in the following figure, which has vertices at (0, 0) (2, 0), (2, 1) and (0, 1). So, we take S as our sample space. Now, the condition $y \le x^2$ is satisfied if and only if the point (x, y) lies in the shaded region. It is the portion of the rectangle lying below the parabola $y = x^2$. Therefore, the required probability is given by

$$\frac{\text{Area of the shaded region}}{\text{Area of the rectangle } S}$$

Area of rectangle $S = 2 \times 1 = 2$. Area of shaded region is

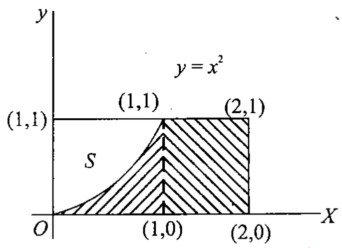


Fig. 9.18

$$\int_0^1 x^2\,dx + 1\times 1 = \frac{1}{3} + \frac{4}{3}$$

Hence, required probability is

$$\frac{\frac{4}{3}}{2} = \frac{2}{3}$$

For Problems 28–30

28. a, **29.** a, **30.** b.

Sol. 28. The total number of ways of painting first column when colours are not alternating is $2^8 - 2$. The total number of ways when no column has alternating colours is $(2^8 - 2)^8/2^{64}$.

29. The number of ways the square has equal number of red and black squares is ${}^{64}C_{32}$. Hence, the required probability is ${}^{64}C_{32}/2^{64}$.

30. This is possible only when the column are alternating red and black. Hence, the required probability is $2/2^{64} = 1/2^{63}$.

For Problems 31–33

31. a, **32.** c, **33.** d.

31. $P(A_2) = \frac{18}{36}$

$P(A_3) = \frac{12}{36} = \frac{1}{3}$

$P(A_4) = \frac{9}{36} = \frac{1}{4}$

$P(A_5) = \frac{7}{36}$

$P(A_6) = \frac{6}{36} = \frac{1}{6}$

Hence, A_3 is most probable.

32. $P(A_2) = \frac{1}{2}$, $P(A_3) = \frac{1}{3}$, $P(A_6) = \frac{1}{6}$

$\therefore\ P(A_2 \cap A_3) = P(A_2)\,P(A_3)$

$\Rightarrow\ P(A_6) = P(A_2)\,P(A_3)$

$\frac{6}{36} = \frac{1}{2} \times \frac{1}{3}$

Hence, A_2 and A_3 are independent.

33. Note that A_1 is independent with all events $A_1, A_2, A_3, A_4, \ldots, A_{12}$. Now, total number of ordered pairs is 23.

$$\underbrace{(1, 1), (1, 2), (1, 3), \ldots, (1, 11)}_{22} + (1, 12)$$

Also, A_2, A_3 and A_3, A_2 are independent. Hence, there are 25 ordered pairs.

For Problems 34–36

34. a, **35.** c, **36.** c.

Sol. The scores of n can be reached in the following two mutually exlcusive events:

(i) by throwing a head when the score is $(n - 1)$;

(ii) by throwing a tail when the score is $(n - 2)$

Hence,

$$P_n = P_{n-1} \times \frac{1}{2} + P_{n-2} \times \frac{1}{2} \qquad [\because P(\text{head}) = P(\text{tail}) = 1/2]$$

$$\Rightarrow P_n = \frac{1}{2}[P_{n-1} + P_{n-2}] \qquad (1)$$

$$\Rightarrow P_0 + \frac{1}{2}P_{n-1} = P_{n-1} + \frac{1}{2}P_{n-2}$$

(adding $(1/2)\,P_{n-1}$ on both sides)

$$= P_{n-2} + \frac{1}{2}P_{n-3}$$

$$\vdots$$

$$= P_2 + \frac{1}{2}P_1 \qquad (2)$$

Now, a score of 1 can be obtained by throwing a head at a single toss.

$$\therefore\ P_1 = \frac{1}{2}$$

And a score of 2 can be obtained by throwing either a tail at a single toss or a head at the first toss as well as second toss.

$$\therefore\ P_2 = \frac{1}{2} + \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{3}{4}$$

From Eq. (2), we have

$$P_n + \frac{1}{2}P_{n-1} = \frac{3}{4} + \frac{1}{2}\left(\frac{1}{2}\right) = 1$$

$$\Rightarrow P_n = 1 - \frac{1}{2}P_{n-1}$$

$$\Rightarrow P_n - \frac{2}{3} = 1 - \frac{1}{2}P_{n-1} - \frac{2}{3}$$

$$\Rightarrow P_n - \frac{2}{3} = -\frac{1}{2}\left(P_{n-1} - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^2 \left(P_{n-2} - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^3 \left(P_{n-3} - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^{n-1} \left(P_1 - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^{n-1} \left(\frac{1}{2} - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^{n-1} \left(-\frac{1}{6}\right)$$

$$= \left(-\frac{1}{2}\right)^{n} \frac{1}{3}$$

$$\Rightarrow \quad P_n = \frac{2}{3} + \frac{(-1)^n}{2^n}\frac{1}{3} = \frac{1}{3}\left\{2 + \frac{(-1)^n}{2^n}\right\}$$

Now,

$$P_{100} = \frac{2}{3} + \frac{1}{3 \times 2^{100}} > \frac{2}{3} \text{ and } P_{101} = \frac{2}{3} - \frac{1}{3 \times 2^{101}} < \frac{2}{3}$$

$$\Rightarrow \quad P_{101} < \frac{2}{3} < P_{100}$$

For Problems 37–39

37. b, 38. c, 39. a.

Sol. 37. If a family of n children contains exactly k boys, then, by binomial distribution, its probability is

$${}^nC_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$$

Hence, by total probability law, the probability of a family of n children having exactly k boys is given by

$$\alpha p^n \, {}^nC_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \quad (\text{where } n \geq k)$$

Therefore, the required probability is

$$= \sum_{n=k}^{\infty} \alpha p^n \, {}^nC_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$$

$$= \alpha \left(\frac{1}{2}\right)^k p^k \sum_{n=k}^{\infty} {}^nC_k \left(\frac{1}{2}\right)^{n-k} (p^{n-k})$$

$$= \alpha \left(\frac{1}{2}\right)^k p^k \left[1 + {}^{(k+1)}C_1 \left(\frac{p}{2}\right) + {}^{(k+2)}C_2 \left(\frac{p}{2}\right)^2 + \cdots\right]$$

$$= \alpha \left(\frac{1}{2}\right)^k p^k \left(1 - \frac{p}{2}\right)^{-(k+1)}$$

$$= \alpha p^k (2-p)^{-(k+1)}$$

$$= \frac{2\alpha}{2-p}\left(\frac{p}{2-p}\right)^k, k \geq 1$$

38. Let A denote the event of a family including at least one boy. Then,

$$P(A) = \frac{2\alpha}{2-p} \sum_{k=1}^{\infty} \left(\frac{p}{2-p}\right)^k$$

$$= \frac{2\alpha}{2-p} \frac{\left(\frac{p}{2-p}\right)}{1 - \left(\frac{p}{2-p}\right)} \quad \text{(sum of infinite terms of G.P.)}$$

$$= \frac{\alpha p}{(2-p)(1-p)} \qquad (1)$$

39. Let B denote the event of a family including at least two or more boys. Then,

$$P(B) = \frac{2\alpha}{2-p} \sum_{k=2}^{\infty} \left(\frac{p}{2-p}\right)^k$$

$$= \frac{\alpha p^2}{(2-p)^2(1-p)} \qquad (2)$$

$$\therefore \quad P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)} \qquad (\text{Since } B \subset A)$$

$$= \frac{p}{2-p} \qquad \text{[From Eqs. (1) and (2)]}$$

Matrix-Match Type

1. a → p; b → r; c → q; d → p.

$$P(A) = \frac{{}^4C_1\, {}^8C_2}{{}^{12}C_3} = \frac{4\,28}{220} = \frac{112}{220} = \frac{28}{55}$$

$$P(B) = \frac{{}^4C_3 + {}^8C_3}{{}^{12}C_3} = \frac{4+56}{220} = \frac{60}{220} = \frac{3}{11}$$

$$P(C) = P(WBB \text{ or } BWB \text{ or } WWB \text{ or } BBB)$$

$$= \frac{8}{12}\times\frac{4}{11}\times\frac{3}{10}\times\frac{4}{12}\times\frac{8}{11}\times\frac{3}{10}\times\frac{8}{12}\times\frac{7}{11}\times\frac{4}{10}\times\frac{4}{12}\times\frac{3}{11}\times\frac{2}{10}$$

$$= \frac{96+96+224+24}{12 \times 110} = \frac{440}{12 \times 110} = \frac{1}{3}$$

$A \cap B = \phi \Rightarrow A$ and B are mutually exclusive

$$P(B \cap C) = P(BBB) = \frac{4 \times 3 \times 2}{12 \times 110} = \frac{1}{55}$$

Also,

$$P(B)\,P(C) = \frac{3}{11} \times \frac{1}{3} = \frac{1}{11}$$

Hence, B and C are neither independent nor mutually exclusive.

$$P(C \cap A) = P(WWB)\,\frac{8 \times 7 \times 4}{12 \times 11 \times 10} = \frac{28}{3 \times 55}$$

$$P(C)\,P(A) = \frac{1}{3} \times \frac{112}{220} = \frac{28}{3 \times 55} \Rightarrow C \text{ and } A \text{ are independent}$$

Also, A, B, C are mutually exclusive as A and B are mutually exclusive.

2. a → q, r, s; b → r; c → p, q; d → p, q, r, s.

a. Suppose the coin is tossed n times. The probability of getting head or tail is 1/2. The probability of not getting any head in n tosses is $(1/2)^n$. The probability of getting at least one head is $1-(1/2)^n$. Now given that

$$1-(1/2)^n \geq 0.8$$

$$\Rightarrow \left(\frac{1}{2}\right)^n \leq 0.2$$

$$\Rightarrow 2^n \geq 5$$

Therefore, the least value of n is 3.

b. The total number of mappings is n^n. The number of one-one mappings is $n!$ Hence the probability is

$$\frac{n!}{n^n}=\frac{3}{32}=\frac{6}{64}=\frac{3!}{4^3}=\frac{4!}{4^4}$$

Comparing, we get $n = 4$.

c. Given equation is

$$2x^2+2mx+m+1=0$$

$$D=4m^2-8(m+1)\geq 0$$

$$m^2-2m-2\geq 0$$

$$(m-1)^2-3\geq 0$$

$$\Rightarrow m=3, 4, 5, 6, 7, 8, 9, 10$$

Also, the number of ways of choosing m is 10. Therefore, the required probability is 4/5.

$$\therefore \quad 5k=4$$

d. $$20P^2-13P+2\leq 0$$

$$\Rightarrow (4P-1)(5P-2)\leq 0$$

$$\Rightarrow \frac{1}{4}\leq P\leq \frac{2}{5}$$

$$\Rightarrow \frac{1}{4}\leq \frac{1}{5}+\frac{1}{5}\left(\frac{4}{5}\right)+\frac{1}{5}\left(\frac{4}{5}\right)^2+\cdots+\frac{1}{5}\left(\frac{4}{5}\right)^{n-1}\leq \frac{2}{5}$$

$$\Rightarrow n=2$$

Hence, maximum as well as minimum value of n is 2.

3. a → r, s; b → p, q, r, s; c → p, q, r, s; d → p.

a. $P(\text{success}) = 1/2$; $P(\text{failure}) = 1/2$

Suppose 'n' bombs are to dropped. Let E be the event that the bridge is destroyed. Then,

$$P(E)=1-P(0 \text{ or } 1 \text{ success})$$

$$=1-\left(\left(\frac{1}{2}\right)^n+{}^nC_1\frac{1}{2}\left(\frac{1}{2}\right)^{n-1}\right)=1-\left(\frac{1}{2^n}+\frac{n}{2^n}\right)\geq 0.9$$

$$\Rightarrow \frac{1}{10}\geq\frac{n+1}{2^n} \text{ or } \frac{2^n}{10(n+1)}\geq 1$$

b. The bag contains 2 red, 3 white and 5 black balls. Hence $P(S)=1/5$; $P(F)=4/5$; Let E be the even of getting a red ball.

$$P(E)=P(S \text{ or } FS \text{ or } FFS \text{ or } \ldots]\geq\frac{1}{2}$$

$$\therefore \quad P(F)^n\leq\frac{1}{2};\left(\frac{4}{5}\right)^n\leq\frac{1}{2}$$

The value of n consistent is 4.

c. Let there be x red socks and y blue socks and $x > y$. Then

$$\frac{{}^xC_2+{}^yC_2}{{}^{x+y}C_2}=\frac{1}{2}$$

or

$$\frac{x(x-1)+y(y-1)}{(x+y)(x+y-1)}=\frac{1}{2}$$

Multiplying both sides by $2(x+y)(x+y-1)$ and expanding, we get

$$2x^2-2x+2y^2-2y=x^2+2xy+y^2-x-y$$

Rearranging, we have

$$x^2-2xy+y^2=x+y$$

$$\Rightarrow (x-y)^2=x+y$$

$$\Rightarrow |x-y|=x+y$$

Now,

$$x+y\leq 17$$

$$x-y\leq\sqrt{17}$$

As $x - y$ must be an integer, so

$$x-y=4$$

$$\therefore \quad x+y=16$$

Adding both together and dividing by 2 yields $x \leq 10$.

d. Let the number of green socks be $x > 0$. Let E: be the event that two socks drawn are of the same colour.

$$P(E)=P(RR \text{ or } BB \text{ or } WW \text{ or } GG)$$

$$=\frac{3}{{}^{6+x}C_2}+\frac{{}^xC_2}{{}^{6+x}C_2}$$

$$=\frac{6}{(x+6)(x+5)}+\frac{x(x-1)}{(x+6)(x+5)}$$

$$=\frac{1}{5}$$

$$\Rightarrow 5(x^2-x+6)=x^2+11x+30$$

$$\Rightarrow 4x^2-16x \quad =0$$

$$\Rightarrow x=4$$

4. a → q; b → r; c → s; d → r.

We have,

$$P(A\cap B)=P(A)\,P(B)=\frac{1}{12}$$

a. $$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{1}{3}+\frac{1}{4}-\frac{1}{12}=\frac{1}{2}$$

b. $$P\left(\frac{A}{A\cup B}\right)=\frac{P(A)}{P(A\cup B)}=\frac{2}{3}$$

c. $$(C)\,P\left(\frac{B}{A'\cap B'}\right)=\frac{P(B\cap(A'\cap B'))}{P(A'\cap B')}=\frac{P(\phi)}{P(A'\cap B')}=0$$

d. $P\left(\frac{A'}{B}\right) = \frac{P(A' \cap B)}{P(B)} = P(A') = \frac{2}{3}$

5. a → q; b → p; c → r; d → q.

Let E_i denote the event that the bag contains i black and $(12-i)$ white balls ($i = 0, 1, 2, \ldots, 12$) and A denote the event that the four balls drawn are all black. Then

$$P(E_i) = \frac{1}{13}\ (i = 0, 1, 2, \ldots, 12)$$

$$P\left(\frac{A}{E_i}\right) = 0 \text{ for } i = 0, 1, 2, 3$$

$$P\left(\frac{A}{E_i}\right) = \frac{{}^iC_4}{{}^{12}C_4} \text{ for } i \ge 4$$

a. $P(A) = \sum_{i=0}^{12} P(E_i)\, P\left(\frac{A}{E_i}\right)$

$$= \frac{1}{13} \times \frac{1}{{}^{12}C_4}\left[{}^4C_4 + {}^5C_4 + \cdots + {}^{12}C_4\right]$$

$$= \frac{{}^{13}C_5}{13 \times {}^{12}C_4} = \frac{1}{5}$$

b. Clearly,

$$P\left(\frac{A}{E_{10}}\right) = \frac{{}^{10}C_4}{{}^{12}C_4} = \frac{14}{33}$$

c. By Bayer's theorem,

$$P\left(\frac{E_{10}}{A}\right) = \frac{P(E_{10})\left(\frac{A}{E_{10}}\right)}{P(A)}$$

$$= \frac{\frac{1}{13} \times \frac{14}{33}}{\frac{1}{5}} = \frac{70}{429}$$

b. Let B denote the probability of drawing 2 white and 2 black balls. Then

$$P\left(\frac{B}{E_i}\right) = 0 \text{ if } i = 0, 1 \text{ or } 11, 12$$

$$P\left(\frac{B}{E_i}\right) = \frac{{}^iC_2 \times {}^{12-i}C_2}{{}^{12}C_4} \text{ for } i = 2, 3, \ldots, 10$$

$$\therefore\quad P(B) = \sum_{i=0}^{12} P(E_i)\, P\left(\frac{B}{E_i}\right)$$

$$= \frac{1}{13} \times \frac{1}{{}^{12}C_4}[2\{{}^2C_2 \times {}^{10}C_2 + {}^3C_2 + \cdots + {}^{10}C_2 \times {}^2C_2]$$

$$= \frac{1}{13} \times \frac{1}{{}^{12}C_4}[2\{{}^2C_2\ {}^{10}C_2 + {}^3C_2\ {}^9C_2 + \cdots + {}^5C_2 \times {}^7C_2\} + {}^6C_2 \times {}^6C_2]$$

$$= \frac{1}{13} \times \frac{1}{495}(1287)$$

$$= \frac{1}{5}$$

6. a → q; b → p; c → q; d → s.

a. When no box remain empty, then

$$n(S) = 3^6 - {}^3C_1 2^6 + {}^3C_2 = 3(243 - 64 + 1)$$
$$= 540$$

When each box contains equal number of balls, then

$$n(E) = \frac{6!}{2!^3\, 3!}\, 3! = 90$$

Therefore, the required probability is 90/540 = 1/6.

b. The required probability is

$$\frac{3^6 - {}^3C_1 2^6 + {}^3C_2}{3^6} = \frac{20}{27}$$

c. Let A be the event that A is throwing sum of 9 and B be the event that B throws a number greater than that thrown by A. We have to find $P(B/A) = P(A \cap B)/P(A) = P(B)$ (as A and B are independent). The probability that is throwing dice so that sum is higher than 9 is

$$P(B) = P((4, 6) \text{ or } (6, 4) \text{ or } (5, 5) \text{ or } (6, 5) \text{ or } (5, 6) \text{ or } (6, 6))$$
$$= \frac{6}{36} = \frac{1}{6}$$

d. $P(A \cup \bar{B}) = P(A) + P(\bar{B}) - P(A \cap \bar{B})$

$$= P(A) + P(\bar{B}) - P(A)P(\bar{B})$$

$\Rightarrow\ 0.8 = 0.3 + P(\bar{B}) - 0.3\, P(\bar{B})$

$\Rightarrow\ 0.5 = 0.7\, P(\bar{B})$

$\Rightarrow\ P(\bar{B}) = \frac{5}{7}$

$\Rightarrow\ P(B) = 1 - \frac{5}{7} = \frac{2}{7}$

7. a → q, r; b → r; c → p; d → a, b, c, d.

a. $\frac{{}^rC_2}{{}^{r+b}C_2} = \frac{1}{2}\ 2r(r-1) = (r-b)(r+b-1)$

$$= 2r(r-1) = (r+b)(r+b-1)$$

$\Rightarrow\ 2r^2 - 2r = r^2 + (2b-1)r + b^2 - 1$

$\Rightarrow\ r^2 - (1 + 2b)r + 1 - b^2 = 0$

$\Rightarrow\ b^2 + 2br + r - r^2 - 1 = 0$

$\Rightarrow\ b = \frac{-2r \pm \sqrt{4r^2 - 4(r - r^2 - 1)}}{2}$

$$= -r \pm \sqrt{2r^2 - r + 1}$$

Since b is integer, possible values of r are 3 and 8.

b. $${}^4C_2\left(\frac{r}{r+10}\right)^2\left(\frac{10}{r+10}\right)^2=\frac{3}{8}$$

c. $$\left(\frac{r}{r+10}\right)^2\left(\frac{10}{r+10}\right)^2=\frac{1}{16}$$

$\Rightarrow$ $r=10$

d. Probability of getting exactly n red balls in $2n$ draws is always equal to probability of getting exactly n black balls in $2n$ draws for any value of r and b, hence the ratio r/b can be 10, 3, 8, 2.

8. a $\to$ r; b $\to$ p; c. $\to$ q; d $\to$ s.

a. The required event will occur if last digit in all the chosen numbers is 1, 3, 7 or 9. Therefore, the required probability is $(4/10)^n$.

b. The required probability is equal to the probability that the last digit is 2, 4, 6, 8 and is given by P (last digit is 1, 2, 3, 4, 6, 7, 8, 9) $-P$ (last digit is 1, 3, 7, 9) $=\dfrac{8^n-4^n}{10^n}$

c. $P(1, 3, 5, 7, 9)-P(1, 3, 7, 9)=\dfrac{5^n-4^n}{10^n}$

d. The required probability is

$$P(0, 5)-P(5)=\frac{(10^n-8^n)-(5^n-4^n)}{10^n}$$

$$=\frac{10^n-8^n-5^n+4^n}{10^n}$$

Integer Type

1.(2) Total number of cases $=n(S)=6!$

Now sum the given digits is $1+2+3+4+5+6=21$, which is divisible by 3.

Now we have to form the number which is divisible by 6, then we have to ensure that the digit in unit place is even.

$\Rightarrow$ Favorable cases $=n(A)=3\cdot 5!$

Hence, $P(A)=\dfrac{3\cdot 5!}{6!}=\dfrac{1}{2}$.

2.(6) Total number of cases $n(S)=6^3=216$

Product is prime only when two outcomes are 1 and the third is prime i.e. 2, 3, 5.

If it is 2, 1, 1, then the number of cases is 3.

Similarly, 3 cases for 3, 1, 1 and 5, 1, 1 each

Hence, favorable cases = 9.

Hence, required probability $p=\dfrac{1}{24}$

$\Rightarrow$ $\dfrac{1}{4p}=6$

3.(7) Let the probability of the faces 1, 3, 5 or 6 be p for each face.

Hence, probability of each of the faces 2 or 4 is $3p$

Now according to the question $4p+6p=1$

$\Rightarrow$ $p=\dfrac{1}{10}$

$\therefore$ $P(1)=P(3)=P(5)=P(6)=\dfrac{1}{10}$

and $P(2)=P(4)=\dfrac{3}{10}$

$\Rightarrow$ Required probability

$p=P$(total of 7 with a draw of dice)

$=P(16, 61, 25, 52, 43, 34)$

$$=2\left(\frac{1}{10}\cdot\frac{1}{10}\right)+2\left(\frac{3}{10}\cdot\frac{1}{10}\right)+2\left(\frac{3}{10}\cdot\frac{1}{10}\right)$$

$$=\frac{2+6+6}{100}=\frac{14}{100}=\frac{7}{50}$$

4.(1) There are n white balls in the turn.

$\Rightarrow$ Probability of Mr. A to draw two balls of same color is

$$\frac{{}^3C_2+{}^nC_2}{{}^{n+3}C_2}=\frac{1}{2}\text{ (given)}$$

$$\Rightarrow \frac{6+n(n-1)}{(n+3)(n+2)}=\frac{1}{2}$$

$\Rightarrow$ $n^2-7n+6=0$

$\Rightarrow$ $n=1$ or 6 (1)

Also required probability for Mr. B according to the question is

$$\frac{3}{n+3}\frac{3}{n+3}+\frac{n}{n+3}\frac{n}{n+3}=\frac{5}{8}\text{ (given)}$$

Solving, we get $n^2-10n+9=0$; $n=1$ or 9 (2)

From (1) and (2), $n=1$

5.(2) When A and B are mutually exclusive, $P(A\cap B)=0$

$\therefore$ $P(A\cup B)=P(A)+P(B)$ (1)

$\Rightarrow$ $0.8=0.5+p$

$\Rightarrow$ $p=0.3$ (2)

$P(A\cup B)=P(A)+P(B)$

$=P(A)+P(B)-P(A\cap B)$

$=P(A)+P(B)-P(A)P(B)$

$\Rightarrow$ $0.8=0.5+q-(0.5)q$

$\Rightarrow$ $0.3=q/2$

$\Rightarrow$ $q=0.6$ (3)

$\Rightarrow$ $q/p=2$

6.(4) Let the number of green socks be $x>0$.

E: Two socks drawn are of the same color

$\Rightarrow$ $P(E)=P(R\,R$ or $B\,B$ or $W\,W$ or $G\,G)$

$$=\frac{3}{{}^{6+x}C_2}+\frac{{}^xC_2}{{}^{6+x}C_2}$$

$$=\frac{6}{(x+6)(x+5)}+\frac{x(x-1)}{(x+6)(x+5)}=\frac{1}{5}\text{ (given)}$$

$\Rightarrow$ $5(x^2-x+6)=x^2+11x+30$

$\Rightarrow$ $4x^2-16x=0$

$\Rightarrow$ $x=4$

7.(7) Let there be x red socks and y blue socks. Let $x>y$

Then $$\frac{{}^xC_2+{}^yC_2}{{}^{x+y}C_2}=\frac{1}{2}$$

$$\Rightarrow \frac{x(x-1)+y(y-1)}{(x+y)(x+y-1)}=\frac{1}{2}$$

$\Rightarrow$ $2x^2-2x+2y^2-2y=x^2+2xy+y^2-x-y$.

$\Rightarrow$ $x^2-2xy+y^2=x+y$

$\Rightarrow$ $(x-y)^2=x+y$

$\Rightarrow \quad |x-y| = (x+y)^{1/2}$

Since $x+y \le 17$, $x-y \le \sqrt{17}$.

As $x-y$ must be an integer $\Rightarrow x-y=4$

$\therefore \quad x+y=16$

Solving, we get $x=7$.

8.(6) Let the two numbers are 'a' and 'b'

According to the question $a+b=4p$ and $a-b=4q$ where $p, q \in I$

$\Rightarrow \quad 2a = 4(p+q)$ and $2b = 4(p-q)$

$\Rightarrow \quad a = 2I_1$ and $b = 2I_2$

Hence, both a and b must be even.

Also if $(a-b)$ is a multiple of 4 then $(a+b)$ will also be a multiple of 4.

Hence, $n(S) = {}^{11}C_2$

$n(A) = (0, 4), (0, 8), (2, 6), (2, 10), (4, 8), (6, 10) = 6$

$$\therefore \quad P(A) = \frac{6}{{}^{11}C_2} = \frac{6}{55}$$

9.(3) For ranked 1 and 2 players to be winners and runners up, respectively, they should not be paired with each other in any round

$$\Rightarrow p = \frac{30}{31} \times \frac{14}{15} \times \frac{6}{7} \times \frac{2}{3} = \frac{16}{31}$$

10.(4) Total ways of distribution $= n(S) = 4^5$

Total ways of distribution so that each child get at least one game

$$n(E) = 4^5 - {}^4C_1\, 3^5 + {}^4C_2\, 2^5 - {}^4C_3$$
$$= 1024 - 4 \times 243 + 6 \times 32 - 4 = 240$$

Required probability $p = \dfrac{n(E)}{n(S)} = \dfrac{240}{4^5} = \dfrac{15}{64}$

11.(5) $P(n) = Kn^2$

Given $P(1) = K$; $P(2) = 2^2K$; $P(3) = 3^2K$; $P(4) = 4^2K$; $P(5) = 5^2K$; $P(6) = 6^2K$

$\therefore \quad$ Total $= 91\,K = 1$

$$\Rightarrow \quad K = \frac{1}{91}$$

$$\therefore \quad P(1) = \frac{1}{91};\ P(2) = \frac{4}{91} \text{ and so on}$$

Let three events A, B, C are defined as

$A : a < b$

$B : a = b$

$C : a > b$

By symmetry, $P(A) = P(C)$. Also $P(A) + P(B) + P(C) = 1$ (1)

$$\text{Since} \quad P(B) = \sum_{i=1}^{6} [P(i)]^2$$
$$= \left[\frac{1+16+81+256+625+1296}{91 \times 91}\right]$$
$$= \frac{2275}{91 \cdot 91} = \frac{25}{91}$$

Now $2P(A) + P(B) = 1$

$$\Rightarrow \quad P(A) = \frac{1}{2}[1 - P(B)] = \frac{33}{91}$$

12.(5) The number of ways of drawing 7 balls (second draws) $= {}^{10}C_7$.

For each set of 7 balls of the second draw, 3 must be common to the set of 5 balls of the first draw, i.e., 2 other balls can be drawn in ${}^3C_2$ ways.

Thus, for each set of 7 balls of the second draw, there are ${}^7C_3 \times {}^3C_2$ ways of making the first draw so that there are 3 balls common.

Hence, the probability of having three balls in common is

$$\frac{{}^7C_3 \times {}^3C_2}{{}^{10}C_7} = \frac{5}{12}$$

13.(6) $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.6}{0.8} = \dfrac{3}{4}$

(Maximum value of $P(A \cap B) = P(A) = 0.6$)

14.(5) Highest number in three throws 4

$\Rightarrow$ At least one of the throws must be equal to 4.

Number of ways when three blocks are filled from $\{1,2,3,4\} = 4^3$

$\therefore$ number of ways when filled from $\{1,2,3\} = 3^3$

$\therefore$ required number of ways $= 4^3 - 3^3$

$$\therefore \quad \text{Probability } p = \frac{4^3 - 3^3}{6^3} = \frac{37}{216}$$

15.(4) Let event A: Card is of heart but not king (12 cards)

Event B : King but not heart (3 cards)

Event C: Heart and king (1 card)

$\therefore$ required probability

$$p = P(E) = \frac{{}^{12}C_1 . {}^3C_1 + {}^3C_1 . {}^1C_1 + {}^1C_1 . {}^{12}C_1}{{}^{52}C_2} = \frac{2}{52}$$

$\therefore \quad 104\,p = 4$

Archives

Subjective Type

1. To draw 2 black, 4 white and 3 red balls in order is same as arranging two balls at first 2 places, 4 white balls at next 4 places, (3rd to 6th place) and 3 red balls at next 3 places (7th to 9th place), i.e., $B_1B_2W_1W_2W_3W_4R_2R_2R_3$, which can be done in $2! \times 4! \times 3!$ ways. And total number of ways of arranging all $2 + 4 + 3 = 9$ balls is $9!$ Therefore the required probability is

$$\frac{2! \times 4! \times 3!}{9!} = \frac{1}{1260}$$

2. (i) The number of ways in which all the 6 girls sit together is $6! \times 7!$ (considering all 6 girls as one person). Therefore, the probability of all girls sitting together is $(6! \times 7!)/12! = 720/(12 \times 11 \times 10 \times 9 \times 8) = (1/132)$.

(ii) Starting with a boy, boys can sit in $6!$ ways leaving one place between every two boys and one at last

$B\_B\_B\_B\_B\_$

These places can be occupied by girls in $6!$ ways. Therefore, if we start with a boy, number of ways of boys and girls sitting alternately is $6! \times 6!$

$G\_G\_G\_G\_G\_G\_$

Thus total number of ways of alternative sitting arrangements is $6! \times 6! + 6! \times 6! = 2 \times 6! \times 6!$

Therefore, the probability of making alternative sitting arrangement for 6 boys and 6 girls is

$$\frac{2 \times 6! \times 6!}{12!} = \frac{2 \times 720}{12 \times 11 \times 10 \times 9 \times 8 \times 7} = \frac{1}{462}$$

3. a. Let us define the events as follows:

$E_1 \equiv$ First shot hits the target plane,

$E_2 \equiv$ Second shot hits the target plane

$E_3 \equiv$ Third shot hits the target plane

$E_4 \equiv$ Fourth shot hits the target plane

Given

$P(E_1) = 0.4, P(E_2) = 0.3, P(E_3) = 0.2, P(E_4) = 0.1$

$\Rightarrow \quad P(\bar{E}_1) = 0.6, P(\bar{E}_2) = 0.7, P(\bar{E}_3) = 0.8, P(\bar{E}_4) = 0.9$

Now the gun hits the plane if at least one of the four shots hit the plane. Therefore,

P(at least one shot hits the plane)

$= 1 - P$ (none of the shots hits the plane)

$= 1 - P(\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3 \cap \bar{E}_4)$

$= 1 - P(\bar{E}_1)\, P(\bar{E}_2)\, P(\bar{E}_3)\, P(\bar{E}_4)$

$= 1 - 0.6 \times 0.7 \times 0.8 \times 0.9$

$= 1 - 0.3024 = 0.6976$

4. $P(A) = 0.5, P(A \cap B) \le 0.3$

So,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

gives

$P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$\le 1 + 0.3 - 0.5$

$= 0.8 \quad [\because\ P(A \cup B) \le 1]$

Hence, $P(B) = 0.9$ is not possible.

5. We must have one ace in $n - 1$ attempts and one ace in the n^{th} attempt. The probability of one ace in first $n - 1$ attempts is

$$\frac{{}^4C_1 \times {}^{48}C_{n-2}}{{}^{52}C_{n-1}}$$

and one ace in the n^{th} attempt is

$$\frac{{}^3C_1}{[52-(n-1)]} = \frac{3}{53-n}$$

Hence, the required probability is

$$\frac{4 \times 48!}{(n-2)!\,(50-n)!} \times \frac{(n-1)!\,(53-n)!}{52!} \times \frac{3}{53-n}$$

$$= \frac{(n-1)\,(52-n)\,(51-n)}{50 \times 49 \times 17 \times 13}$$

6. Since $P(A \cup B \cup C) \ge 0.75$, therefore,

$0.75 \le P(A \cup B \cup C) \le 1$

$\Rightarrow \quad 0.75 \le P(A) + P(B) + P(C)$

$\qquad - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \le 1$

$\Rightarrow \quad 0.75 \le 0.3 + 0.4 + 0.8 - 0.08 - P(B \cap C) - 0.28 + 0.09 \le 1$

$\Rightarrow \quad 0.75 \le 1.23 - P(B \cap C) \le 1$

$\Rightarrow \quad -0.48 \le -P(B \cap C) \le -0.23$

$\Rightarrow \quad 0.23 \le P(B \cap C) \le 0.48$

7. Given that A and B are independent events.

$\therefore \quad P(A \cap B) = P(A)\, P(B) \quad (1)$

Also given that

$P(A \cap B) = \frac{1}{6} \quad (2)$

and

$P(\bar{A} \cap \bar{B}) = \frac{1}{3} \quad (3)$

Also,

$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

$\Rightarrow \quad P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B) + P(A \cap B)$

$\Rightarrow \quad \frac{1}{3} = 1 - P(A) - P(B) + \frac{1}{6}$

$\Rightarrow \quad P(A) + P(B) = \frac{5}{6} \quad (4)$

From Eqs. (1) and (2), we get

$P(A)\, P(B) = \frac{1}{6} \quad (5)$

Let $P(A) = x$ and $P(B) = y$. Then Eqs. (4) and (5) become

$x + y = \frac{5}{6}$

$xy = \frac{1}{6}$

Solving, we get $x = 1/2$ and $y = 1/3$ or $x = 1/3$ and $y = 1/2$. Thus, $P(A) = 1/2$ or $1/3$.

8. Let $P(A)$ and $P(B)$ be the percentage of the population in a city who read newspapers A and B, respectively. Therefore, $P(A) = 25/100 = 1/4$, $P(B) = 20/100 = 1/5$ and $P(A \cap B) = 8/100 = 2/25$.

Therefore, percentage of those who read A but not B is

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$= 1/4 - 2/25$

$= 17/100 = 17\%$

Similarly,

$P(B \cap \bar{A}) = P(B) - P(A \cap B) = 3/25 = 12\%$

Therefore, percentage of those who read advertisements is

30% of $P(A \cap \bar{B}) + 40\%$ of $P(B \cap \bar{A}) + 50\%$ of $P(A \cap B)$

$$= \frac{30}{100} \times \frac{17}{100} + \frac{40}{100} \times \frac{3}{25} + \frac{50}{100} \times \frac{2}{25} = \frac{139}{1000} = 13.9\%$$

Hence, the percentage of the population who read an advertisement is 13.9%.

9. The total number of ways of ticking one or more alternatives out of 4 is ${}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15$. Out of these 15 combinations, only one combination is correct. The probability of ticking the alternative correctly at the first trial is 1/15 that at the second trial is (14/15)(1/14) = 1/15 and that at the third trial is (14/15)(13/14)(1/13) = 1/15.

Thus, the probability that the candidate will get marks on the question if he is allowed up to three trials is 1/15 + 1/15 + 1/15 + = 1/5.

10. Let E_i denote the event that the lot contains 2 defcetive articles and E_2 the event that the lot contains 3 defective articles. Suppose that A denotes the event that all the defective articles are found by the twelfth testing. we have,

$$P(E_1) = 0.4, P(E_2) = 0.6$$

Now,

$$P(E_1) + P(E_2) = 0.4 + 0.6 = 1$$

Therefore there can be no other possibility. Now,

$$P(A/E_1) = \frac{({}^2C_1)({}^{18}C_{10})}{({}^{20}C_{11})} = \frac{1}{9} = \frac{11}{190}$$

(Here up to 11th draw, 1 defective and 10 non-defective articles are drawn and the second (i.e. last) defective article is drawn at twelfth draw.)

Also,

$$P(A/E_2) = \frac{({}^3C_2)({}^{17}C_9)}{({}^{20}C_{11})} \cdot \frac{1}{9} = \frac{11}{128}$$

[Here up to 11th draw, 2 defective and 9 non-defective articles are drawn and the third (i.e. last) defective articles is drawn at the twelfth draw.]

We have,

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$$

$$= 0.4 \times \frac{11}{190} + 0.6 \times \frac{11}{228} = \frac{99}{1990}$$

11. The man can be one step away from the starting point if (i) either he is one step ahead or (ii) one step behind the starting point. Now if at the end of the 11 steps, the man is one step ahead of the starting point, then out of 11 steps, he must have taken six forward steps and five backward steps. The probability of this event is

$${}^{11}C_6 \times (0.4)^6 \times (0.6)^5 = 462 \times (0.4)^6 \times (0.6)^5$$

Again if at the end of the 11 steps, the man is one step behind the starting point, then out of 11 steps, he must have taken six backward steps and five forward steps. The probability of this event is

$${}^{11}C_6 \times (0.6)^6 \times (0.4)^5 = 462 \times (0.6)^6 \times (0.4)^5$$

Since the events (i) and (ii) are mutually exclusive, therefore the required probability that one of these events happens is

$$[462 \times (0.4)^6 \times (0.6)^5] + [462 \times (0.6)^6 \times (0.4)^5]$$

$$= 462 \times (0.4)^5 \times (0.6)^5\,(0.4 + 0.6)$$

$$= 462 \times (0.4 \times 0.6)^5$$

$$= 462(0.24)^5$$

12. For the first two draws, following events may occur.

E_1: Both the balls are white

E_2: First is white and second is black

E_3: First is black and second is white

E_4: Both the balls are black

Let E represent the event that the third ball is black. Then,

$$P(E_1) = \frac{2}{4} \times \frac{1}{2} = \frac{1}{6}$$

$$P(E_2) = \frac{2}{4} \times \frac{2}{3} = \frac{1}{3}$$

$$P(E_3) = \frac{2}{4} \times \frac{2}{5} = \frac{1}{5}$$

$$P(E_4) = \frac{2}{4} \times \frac{3}{5} = \frac{3}{10}$$

The four events E_1, E_2, E_3 and E_4 are mutually exclusive and exhaustive. Using the theorem of total probability,

$$P(E) = P(E_1)\,P(E/E_1) + P(E_2)\,P(E/E_2) + P(E_3)\,P(E/E_3) + P(E_4)\,P(E/E_4)$$

$$= \frac{1}{6} \times \frac{3}{2} + \frac{1}{3} \times \frac{3}{4} + \frac{1}{5} \times \frac{3}{4} + \frac{3}{10} \times \frac{4}{6} = \frac{23}{30}$$

13. Here the total number of coins is $N + 7$. Therefore, the total number of ways of choosing 5 coins out of $N + 7$ is ${}^{n+7}C_5$.

Let E denote the event that the sum of the values of the coins is less than 1 rupee and 50 paise. Then, E' denotes the event that the total values of the five coins is equal to or more than 1 rupee and 50 paise. The number of cases favourable to E' is

$${}^2C_1 \times {}^5C_4 \times {}^NC_0 + {}^2C_2 \times {}^5C_3 \times {}^NC_0 + {}^2C_2 \times {}^5C_2 \times {}^NC_1$$

$$= 2 \times 5 + 10 + 10N$$

$$= 10(N + 2)$$

$$\therefore \quad P(E') = \frac{10\,(N + 2)}{{}^{N+7}C_5}$$

$$\Rightarrow \quad P(E) = 1 - P(E') = 1 - \frac{10\,(N + 2)}{{}^{N+7}C_5}$$

14. The probability p_1 of winning the best of three games is equal to the sum of the probability of winning two games + the probability of wining three games, which is given by

$${}^3C_2\,(0.6)\,(0.4)^2 + {}^3C_3\,(0.4)^3$$

$$= 0.288 + 0.064 = 0.352 \qquad \text{[Using binomial distribution]}$$

Similarly, the probability of winning the best five games is

p_2 = probability of winning three games
+ probability of winning four games
+ probability of winning 5 games

$$= {}^5C_3\,(0.6)^2\,(0.4)^3 + {}^5C_4\,(0.6)\,(0.4)^4 + {}^5C_5\,(0.4)^5$$

$$= 0.2304 + 0.0768 + 0.01024$$

$$= 0.31744$$

As $p_1 > p_2$, therefore A must choose the first offer, i.e., best of three games.

15. Let $A = \{a_1, a_2, \ldots, a_n\}$. Let S be the sample space and E be the event that $P \cap Q = \phi$. The number of subsets of A is 2^n. Each one of P and Q can be selected in 2^n ways. Hence, the total number of ways of selecting P and Q is $2^n\,2^n = 4^n$.

Now for $P \cap Q = \phi$, element of A either does not belong to any of subsets or it belongs to at most one of them. Therefore, for each element, there are 3 choices, namely

(i) $a_i \notin P, a_i \notin Q$

(ii) $a_i \notin P, a_i \notin Q$

(iii) $a_i \notin P, a_i \notin Q$

Therefore, the total number of ways of selecting P and Q such that

$P \cap Q = \phi$ are $(3)^n$ [3 for each element of A].

$\therefore \quad n(E) = 3^n$

Hence,

$$P(E) = \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$$

16. Let G denote the event the examinee gueses, C denote the event the examinee copies and K the event the examinee knows. Then given that $P(G) = 1/3$, $P(C) = 1/6$ and $P(K) = 1 - 1/3 - 1/6 = 1/2$. Here, it has been assumed that the events G, C and K are mutually exclusive and totally exhaustive. If R denotes the event that the answer is right, then

$P(R/G) = 1/4$, as out of the four choices only one is correct. $P(R/C) = 1/8$ (given). Also, $P(R/K) = 1$ since the probability of answering correctly when one knows the answer is equal to 1.

Now by Bayes's theorem, the probability that he knows the answer, given that he answered correctly is given by

$$P(K/R) = \frac{P(K)\,P(R/K)}{P(G)\,P(R/G) + P(C)\,P(R/C) + P(K)\,P(R/K)}$$

$$= \frac{(1/2)\times 1}{(1/3)\,(1/4) + (1/6)\,(1/8) + (1/2)\times 1} = \frac{24}{29}$$

17. Let X = defective and Y = non-defective. Then, all possible outcomes are $\{XX, XY, YX, YY\}$. Also,

$$P(XX) = \frac{50}{100} \times \frac{50}{100} = \frac{1}{4}$$

$$P(XY) = \frac{50}{100} \times \frac{50}{100} = \frac{1}{4}$$

$$P(YX) = \frac{50}{100} \times \frac{50}{100} = \frac{1}{4}$$

$$P(YY) = \frac{50}{100} \times \frac{50}{100} = \frac{1}{4}$$

Here,

$$A = (XX) \cup (XY);\ B = (XY) \cup (YY);\ C = (XX) \cup (YY)$$

$$\therefore \quad P(A) = P(XX) + P(XY) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\therefore \quad P(B) = P(XY) + P(YX) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Now,

$$P(A\ B) = P(XY) = \frac{1}{4} = P(A)\,P(B)$$

Therefore, A and B are independent events.

$$P(B\ C) = P(YX) = \frac{1}{4} = P(B)\,P(C)$$

Therefore, B and C are independent events.

$$P(C\ A) = P(X\ Y) = \frac{1}{4} = P(C)\,P(A)$$

Therefore, C and A are independent events.

$$P(A\ B\ C) = 0 \text{ (impossible events)}$$

$$\neq P(A)\,P(B)\,P(C)$$

Therefore, A, B, C are dependent events. Thus, we can conclude that A, B, C are pairwise independent, but A, B, C are dependent.

18. The given numbers are 00, 01, 02, …, 99. There are total 100 numbers, out of which the numbers, the product of whose digits is 18, are 29, 36, 63 and 92.

$$\therefore \quad p = P(E) = \frac{4}{100} = \frac{1}{25}$$

$$\Rightarrow \quad q = 1 - p = \frac{24}{25}$$

From binomial distribution,

$P(E$ occurring at least 3 times$) = P(E$ occurring 3 times$) + P(E$ occurring 4 times$)$

$$= {}^4C_3\, p^3 q + {}^4C_4\, p^4$$

$$= 4 \times \left(\frac{1}{25}\right)^3 \left(\frac{24}{25}\right) + \left(\frac{1}{25}\right)^4$$

$$= \frac{97}{(25)^4}$$

19. $E_1 \equiv$ number noted is 7

$E_2 \equiv$ number noted is 8

$H \equiv$ getting head on coin

$T \equiv$ getting tail on coin

Then, by total probability theorem,

$$P(E_1) = P(H)\,P(E_1/H) + P(T)\,P(E_1/T)$$

$$P(E_2) = P(H)\,P(E_2/H) + P(T)\,P(E_2/T),$$

where $P(H) = (1/2)$, $P(T) = (1/2)$ and $P(E_1/H)$ is the probability of getting a sum of 7 on two dice.

Here, favourable cases are $\{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4)\ (4, 3)\}$.

$$\therefore \quad P(E_1/H) = \frac{6}{36} = \frac{1}{6}$$

Also, $P(E_1/T)$ is the probability of getting '7' numbered card out of 11 cards

$$\therefore \quad P(E_1/T) = 1/11.$$

$P(E_2/H)$ is the probability of getting a sum of 8 on two dice.

Here, favourable cases are $\{(2, 6)\ (6, 2)\ (4, 4), (5, 3), (3, 5)\}$.

$$\therefore \quad P(E_2/H) = \frac{5}{36}$$

The probability of getting '8' numbered card out of 11 cards is 1/11.

$$\therefore \quad P(E_1) = \frac{1}{2} \times \frac{1}{6} + \frac{1}{2} \times \frac{1}{11} = \frac{1}{12} + \frac{1}{22} = \frac{11+6}{132} = \frac{17}{132}$$

$$P(E_2) = \frac{1}{2} \times \frac{5}{36} + \frac{1}{2} \times \frac{1}{11} = \frac{1}{2}\left[\frac{55+36}{396}\right] = \frac{91}{792}$$

Now, E_1 and E_2 are mutually exclusive events, therefore

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$
$$= \frac{17}{132} + \frac{91}{792}$$
$$= \frac{102 + 91}{792}$$
$$= \frac{193}{792} = 0.2436$$

20. We have 14 seats in two vans and there are 9 boys and 3 girls. The number of ways of arranging 12 people on 14 seats without restriction is

$$^{14}P_{12} = \frac{14!}{2!} = 7(13!)$$

Now, the number of ways of choosing a van is 2. The number of ways of arranging girls on three adjacent seats is 2(3!). The number of ways of arranging 9 boys on the remaining 11 seats is $^{11}P_9$. Therefore, the required number of ways is

$$2 \times (2 \times 3!) \times {}^{11}P_9 = \frac{4!\,11!}{2!}\,12!$$

Hence, the probability of the required events is

$$\frac{12!}{7 \times 13!} = \frac{1}{91}$$

21. a. The probability of S_1 to be among the eight winners is equal to the probability of S_1 winning in the group, which is given by 1/2.

b. If S_1 and S_2 are in the same pair-then exactly one wins.

If S_1 and S_2 are in two separate pairs, then for exactly one of S_1 and S_2 to be among the eight winners, S_1 wins and S_2 loses or S_1 loses and S_2 wins.

Now the probability of S_1, S_2 being in the same pair and one wins is (Probability of S_1, S_2 being in the same pair) × (Probability of any one winning in the pair). And the probability of S_1, S_2 being in the same pair is

$$\frac{n(E)}{n(S)}$$

The number of ways 16 players are divided into 8 pairs is

$$n(S) = \frac{16!}{(2!)^8 \times 8!}$$

The number of ways in which 16 persons can be divided in 8 pairs so that S_1 and S_2 are in same pair is

$$n(E) = \frac{14!}{(2!)^7 \times 7!}$$

Therefore, the probability of S_1 and S_2 being in the same pair is

$$\frac{\dfrac{14!}{(2!)^7 \times 7!}}{\dfrac{16}{(2!)^8 \times 8!}} = \frac{2! \times 8}{16 \times 15} = \frac{1}{15}$$

The probability of any one winning in the pair of S_1, S_2 is P(certain event) = 1.

Hence, the prbability tha the pair of S_1, S_2 being in two pairs separately and any one of S_1, S_2 wins is given by the probability of S_1, S_2 being in two pairs separately and S_1 wins, S_2 loses + the probability of S_1, S_2 being in two pairs separately and S_1 loses, S_2 wins. It is given by

$$\left[1 - \frac{1}{15}\right] \times \frac{1}{2} \times \frac{1}{2} + \left[1 - \frac{1}{15}\right] \times \frac{1}{2} \times \frac{1}{2}$$
$$= \frac{1}{2} \times \frac{14}{15} = \frac{7}{15}$$

Therefore, the required probability is (1/15) + (7/15) + (8/15).

22. The required probability is 1 – (probability of the event that the roots of $x^2 + px + q = 0$ are non-real). The roots of $x^2 + px + q = 0$ will be non-real if and only if

$$p^2 - 4q < 0 \text{ or } p^2 < 4q.$$

We enumerate the possible values of p and q, for which this can happen, in the following table.

q	p	Number of pairs of pq
1	1	1
2	1, 2	2
3	1, 2, 3	3
4	1, 2, 3	3
5	1, 2, 3, 4	4
6	1, 2, 3, 4	4
x7	1, 2, 3, 4, 5	5
8	1, 2, 3, 4, 5	5
9	1, 2, 3, 4, 5	5
10	1, 2, 3, 4, 5, 6	6

Thus, the number if possible pairs is 38. Also, the total number of possible pairs is 10 × 10 = 100. Therefore, the required probability is $1 - (38/100) = 1 - 0.38 = 0.62$.

23. Given that p is the probability that the coin shows a head, then $1 - p$ will be the probability that the coin shows a tail. Now,

$\alpha = P(A$ gets the 1st head in 1st try) + P (A gets the 1st head in 2nd try) + ⋯

$$\Rightarrow \alpha = P(H) + P(T)\,P(T)\,P(T)\,P(H) + P(T)P(T) \times P(T)P(T)P(T)P(T)P(H)$$

$$= p + (1-p)^3 + (1-p)^6\, p + \cdots$$

$$= p\,[1 + (1-p)^3 + (1-p)^6 + \cdots]$$

$$= \frac{p}{1-(1-p)^3} \qquad (1)$$

Similarly,

$\beta = P(B$ gets the 1st head in 1st try) + $P(B$ gets the 1st head in 2nd try) + ⋯

$$= P(T)P(H) + P(T)\,P(T)\,P(T)\,P(T)\,P(H) + \cdots$$

$$= (1-p)\,p + (1-p)^4\,p + \cdots$$

$$= \frac{(1-p)\,p}{1-(1-p)^3} \qquad (2)$$

From Eqs. (1) and (2), we get

$\beta = (1-p)\alpha$

Also, Eqs. (1) and (2) give expression for α and β in terms of p.

Also,

$\alpha + \beta + \gamma = 1$ (exhaustive events and mutually exclusive events)

$$\Rightarrow \gamma = 1 - \alpha - \beta$$
$$= 1 - \alpha - (1-p)\alpha$$
$$= 1 - (2-p)\alpha$$
$$= 1 - (2-p)\alpha$$
$$= 1 - (2-p)\frac{p}{1-(1-p)^3}$$
$$= \frac{1-(1-p)^3-(2p-p^2)}{1-(1-p)^3}$$
$$= \frac{1-1+p^3++3p(1-p)-2p+p^2}{1-(1-p)^3}$$
$$= \frac{p^3-2p+p}{1-(1-p)^3}$$
$$= \frac{p(p^2-2p+1)}{1-(1-p)^3}$$
$$= \frac{p(1-p)^2}{1-(1-p)^3}$$

24.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
0	0	0	0	0	0	0	0

Given that if P_i, P_j play with $i < j$, then P_i will win. For the first round, P_4 should be paired with any one from P_5 to P_8. It can be done in ${}^4C_1$ ways. Then P_4 to be the finalist, at least one player from P_5 to P_8 should reach in the second round. Therefore, one pair should be from remaining 3 from P_5 to P_8 in ${}^3C_2$. Then favourable pairings in first round is ${}^4C_1\,{}^3C_2\,{}^3C_2$. Then in the 2nd round, we have four players. Favourable ways is 1. Now total possible pairings is

$$\frac{{}^8C_2\times{}^6C_2\times{}^4C_2\times{}^2C_2}{4!}\times\frac{{}^4C_2\times{}^2C_2}{2!}$$

Therefore, the probability is

$$\frac{{}^4C_1\;{}^3C_2\;{}^3C_2\;4!\,2!}{{}^8C_2\;{}^6C_2\;{}^4C_2\;{}^2C_2\;{}^4C_2\;{}^2C_2} = \frac{4}{35}$$

25. Given that the probability of showing head by a coin when tossed is p. Therefore the orobability of coin that no two or more consecutive heads occur when tossed n times is given as follows.

The probability of getting one or more or no head, i.e., probability of getting H or T is $P_i = 1$.

Also, the probability of getting one H or number H is

$$p_2 = P(HT) + P(TH) + P(TT)$$
$$= p(1-p) + p(1-p)\,P + (1-p)\,(1-p)$$
$$= 1 - p^2$$

For $n \geq 3$:

The probability that (0) two or more consecutive heads occur when tossed n times is

$p_n = p$(last outcome is T) P(two or more consecutive heads in $(n-1)$ throws) + P(last outcome is H) $P((n-1)^{\text{th}}$ throw results in a T) P(number two or more consecutive heads in $(n-2)n$ throws)

$$= (1-p)\,P_{n-1} + p(1-p)\,p_{n-2}$$

Hence, proved.

26. Let W_1 (B_1) be the event that a white (a black) ball is drawn in the first draw and let W be the event that white ball is drawn in the second draw. Then,

$$P(W) = P(B_1)\,P(W/B_1) + P(W_1)\,P(W/W_1)$$
$$= \frac{n}{m+n}\,\frac{m}{m+n+k} + \frac{m}{m+n}\,\frac{m+k}{m+n+k}$$
$$= \frac{m(n+m+k)}{(m+n)\,(m+n+k)}$$
$$= \frac{m}{m+n}$$

27. The total number of outcomes is 6^n. We can choose three numbers out of 6 in ${}^6C_3$ ways. Now these three numbers must appear at least once in n throws which is equivalent to number of ways of filling 3 boxes with n different objects if no box remains empty which is done in $3^n - {}^3C_1 2^n + {}^3C_2$ ways. Hence, favourable number of cases is ${}^6C_3 \times (3^n - {}^3C_1 2^n + {}^3C_2)$. Hence, the required probability is

$$\frac{{}^6C_3\,[3^n - 3(2^n) + 3]}{6^n}$$

28. Let E_1 be the event that the coin drawn is fair and E_2 be the event that the coin drawn is biased.

$$\therefore\; P(E_1) = \frac{m}{N} \text{ and } P(E_2) = \frac{N-m}{N}$$

A is the event that on tossing the coin, the head appears first and then appears the tail.

$$\therefore\; P(A) = P(E_1 \cap A) + P(E_2 \cap A)$$
$$= P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$$
$$= \frac{m}{N}\left(\frac{1}{2}\right)^2 + \left(\frac{N-m}{N}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) \quad (1)$$

We have to find the probability that A has happened because of E_1.

$$\therefore\; P(E_1/A) = \frac{P(E_1 \cap A)}{P(A)}$$
$$= \frac{\frac{m}{N}\left(\frac{1}{2}\right)^2}{\frac{m}{N}\left(\frac{1}{2}\right)^2 + \frac{N-m}{N}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)} \quad \text{[Using Eq. (1)]}$$
$$= \frac{m/4}{m/4 + \frac{2(N-m)}{9}} = \frac{9m}{m+8N}$$

29. Let us consider the following events.

E_1: event of passing I exam

E_2: event of passing II exam

E_3: event of passing III exam

Then a student can qualify in anyone of following ways.

1. He passes first and second exam.
2. He passes first, fails in second but passes third exam.
3. He fails in first, passes second and third exam.

Therefore, the required probability is

$$P(E_1)P(E_2/E_1) + P(E_1)\,P(E_2/E_1)\,P(E_3/E_2) + P(E_1)\,P(E_2/E_1)\,P(E_3/E_2)$$

[as an event is dependent on previous one]

$$= p\,p + p\,(1-p)\frac{p}{2} + (1-p)\frac{p}{2}p$$

$$= p^2 + \frac{p^2}{2} - \frac{p^3}{2} + \frac{p^2}{2} - \frac{p^3}{2}$$

$$= 2p^2 - p^3$$

30. Let us consider the following events:

E_1: A hits B; $P(E_1) = 2/3$

E_2: B hits A; $P(E_2) = 1/2$

E_3: C hits A; $P(E_3) = 1/3$

E: A is hit

$$\therefore\quad P(E) = P(E_2 \cup E_3)$$

$$= 1 - P(\bar{E}_2 \cap \bar{E}_3)$$

$$= 1 - P(\bar{E}_2)\cdot P(\bar{E}_3)$$

$$= 1 - \frac{1}{2}\cdot\frac{2}{3}$$

$$= \frac{2}{3}$$

Now,

$$P((E_2 \cap \bar{E}_3)/E) = \frac{P(E_2 \cap \bar{E}_3)}{P(E)}$$

$[\because P(E_2 \cap \bar{E} \cap E) = P(E_2 \cap \bar{E}_3)$, i.e., B hits A and A is hit $= B$ hits $A]$

$$= \frac{P(E_2)\,P(\bar{E}_3)}{P(E)}$$

$$= \frac{1/2 \times 2/3}{2/3} = \frac{1}{2}$$

31. Given that A and B are two independent events. C is the event in which exactly one of A and B occurs. Let $P(A) = x$, $P(B) = y$. Then,

$$P(C) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= P(A)\,P(\bar{B}) + P(\bar{A})\,P(B)$$

$[\because$ if A and B are independent so are 'A and $\bar{B}$' and $\bar{A}$ and $B]$

$$\Rightarrow\quad P(C) = x(1-y) + y(1-x) \qquad (1)$$

Now consider,

$$P(A \cup B)\,P(\bar{A} \cap \bar{B}) = [P(A) + P(B) - P(A)\,P(B)]\,[P(\bar{A})\,P(\bar{B})]$$

$$= (x + y - xy)\,(1-x)\,(1-y)$$

$$= (x+y)\,(1-x)\,(1-y) - xy(1-x)\,(1-y) \cdots \times (x+y)\,(1-x)\,(1-y) \qquad [\because x, y \in (0, 1)]$$

$$\le x(1-x)\,(1-y) + y(1-x)\,(1-y)$$

$$\le x(1-y) + y(1-x) - x^2(1-y) - y^2(1-x) + \cdots + x(1-y) + y(1-x)$$

$$\le P(C) \qquad [\text{Using Eq. (1)}]$$

Thus $P(C) \ge P(A \cup B)P(\bar{A} \cap \bar{B})$ is proved.

32. Let us define the following events

A: 4 white balls are drawn in first six draws

B: 5 white balls are drawn in first six draws

C: 6 white balls are drawn in first six draws

E: exactly one white ball is drawn in next two draws (i.e. one white and one red)

Then

$$P(E) = P(E/A)\,P(A) + P(E/B)\,P(B) + P(E/C)\,P(C)$$

But

$P(E/C) = 0$ [as there are only 6 white balls in the bag]

$$\therefore\quad P(E) = P(E/A)\,P(A) + P(E/B)\,P(B)$$

$$= \frac{{}^{10}C_1 \times {}^{2}C_1}{{}^{12}C_2}\,\frac{{}^{12}C_2 \times {}^{6}C_4}{{}^{18}C_6} + \frac{{}^{11}C_1 \times {}^{1}C_1}{{}^{12}C_1}\,\frac{{}^{12}C_1 \times {}^{6}C_5}{{}^{18}C_6}$$

33. Let us define the following events

C: person goes by car

S: person goes by scooter

B: person goes by bus

T: person goes by train

L: person reaches late

Then, we are given in the question

$$P(C) = \frac{1}{7},\ P(S) = \frac{3}{7},\ P(B) = \frac{2}{7},\ P(T) = \frac{1}{7}$$

$$P(L/C) = \frac{2}{9},\ P(L/S) = \frac{1}{9},\ P(L/B) = \frac{4}{9},\ P(L/T) = \frac{1}{9}$$

We have to find $P(C/\bar{L})$ [Since reaches in time ≡ not late]. Using Bayes's theorem,

$$P(C/\bar{L}) = \frac{P(\bar{L}/C)\,P(C)}{P(\bar{L}/C)P(C) + P(\bar{L}/S)\,P(S) + P(\bar{L}/B) + A(\bar{L}/T)P(T)} \qquad (1)$$

Now,

$$P(\bar{L}/C) = 1 - \frac{2}{9} = \frac{7}{9},\ P(\bar{L}/S) = 1 - \frac{1}{9} = \frac{8}{9}$$

$$P(\bar{L}/B) = 1 - \frac{4}{9} = \frac{5}{9},\ P(\bar{L}/T) = 1 - \frac{1}{9} = \frac{8}{9}$$

Substituting these values in Eq. (1), we get

$$P(C/\bar{L}) = \frac{\frac{7}{9} \times \frac{1}{7}}{\frac{7}{9} \times \frac{1}{7} + \frac{8}{9} \times \frac{3}{7} + \frac{5}{9} \times \frac{2}{7} + \frac{8}{9} \times \frac{1}{7}}$$

$$= \frac{7}{7 + 24 + 10 + 8} = \frac{7}{49} = \frac{1}{7}$$

Objective Type

Fill in the blanks

1. Let E_1 be the event that face 1 has turned up and E_2 be the event that face 1 or 2 has turned up. By the given data,

$$P(E_2) = 0.1 + 0.32 = 0.42,\ P(E_1 \cap E_2) = 0.1$$

Given that E_2 has happened and we have to find then the probability of happening of E_1. Therefore, by conditional probability theorem, we have

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{0.1}{0.42} = \frac{10}{42} = \frac{5}{21}$$

2. Given that

$$P(A \cup B) = P(A \cap B)$$

$\Rightarrow$ $P(A) + P(B) - P(A \cap B) = P(A \cap B)$

$\Rightarrow$ $[P(A) - P(A \cap B)] + [P(B) - P(A \cap B)] = 0$

But

$P(A) - P(A \cap B) \geq 0$ and $P(B) - P(A \cap B) \geq 0$

$[\because P(A \cap B) \leq P(A), P(B)]$

$\Rightarrow$ $P(A) - P(A \cap B) = 0$ and $P(B) - P(A \cap B) = 0$

[since sum of two non-negative numbers can be zero only when these numbers are zeros.]

$\Rightarrow$ $P(A) = P(B) = P(A \cap B)$

which is the required relationship.

3. Let A be the event that maximum number on the two chosen tickets is not more than 10, and B be the event that minimum number on them is 5. We have to find $P(B/A)$. We know that

$$P(B/A) = \frac{P(B \cap A)}{P(A)}$$

The total ways of selecting two tickets out of 100 is ${}^{100}C_2$. The number of ways favourable to A is the number of ways of selecting any 2 numbers from 1 to 10, i.e., ${}^{10}C_2 = 45$. $A \cap B$ contains one number 5 and other greater than 5 and $\leq$ 10. So, number of ways favourable to $A \cap B$ is ${}^5C_1 = 5$. Therefore,

$$P(A) = \frac{45}{{}^{100}C_2}$$

$$P(B \cap A) = P(B \cap A) = \frac{5}{{}^{100}C_2}$$

Thus,

$$P(B/A) = \frac{5/{}^{100}C_2}{45/{}^{100}C_2} = \frac{5}{45} = \frac{1}{9}$$

4. Let,

$$P(A) = \frac{1+3p}{3}, P(B) = \frac{1-p}{4}, P(C) = \frac{1-2p}{2}$$

A, B and C are three mutually exclusive events.

$\therefore$ $P(A) + P(B) + P(C) \leq 1$

$\Rightarrow$ $\dfrac{1+3p}{3} + \dfrac{1-p}{4} + \dfrac{1-2p}{2} \leq 1 \leq 1$

$\Rightarrow$ $4 + 12p + 3 - 3p + 6 - 12p \leq 12$

$\Rightarrow$ $p \leq 1/3$ (1)

Also,

$$0 \leq P(A) \leq 1 \Rightarrow 0 \leq \frac{1+3p}{3} \leq 1$$

$$0 \leq 1 + 3p \leq 3$$

$\Rightarrow$ $-\dfrac{1}{3} \leq p \leq \dfrac{2}{3}$ (2)

$$0 \leq P(B) \leq 1 \Rightarrow 0 \leq \frac{1-p}{4} \leq 1$$

$\Rightarrow$ $0 \leq 1 - p \leq 4$

$\Rightarrow$ $-3 \leq p \leq 1$ (3)

$$0 \leq P(C) \leq 1 \Rightarrow 0 \leq \frac{1-2p}{2} \leq 1$$

$\Rightarrow$ $-\dfrac{1}{2} \leq p \leq \dfrac{1}{2}$ (4)

Combining Eqs. (1), (2), (3) and (4), we get

$$\frac{1}{3} \leq p \leq \frac{1}{2}$$

5. First draw is from P, second draw is from Q and third draw is from P. There may be following cases:

Case I:

$R \to R \to R$

The required probability is (6/10) × (5/11) × (6/10) = (18/110).

Case II:

$R \to B \to R$

The required probability is (6/10) × (6/11) × (6/10) = (18/110).

Case III:

$B \to R \to R$

The required probability is (4/10) × (4/11) × (7/10) = (56/550).

Case IV:

$B \to B \to R$

The required probability is (4/10) × (7/11) × (6/10) = (84/550).

Therefore, the total probability is

$$\frac{18}{110} + \frac{18}{110} + \frac{56}{550} + \frac{84}{550} = \frac{90+90+56+84}{550}$$

$$= \frac{320}{550} = \frac{32}{55}$$

6. Probability of getting a sum of 5 is 4/36 = 1/9 = $P(A)$ as favourable cases are {(1, 4), (4, 1), (2, 3) (3, 2)}. Similarly, favourable cases of getting a sum of 7 are {(1, 6). (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)}. The total number of such cases is 6. Therefore, the probability of getting a sum of 7 is (6/36) = (1/6). The probability of getting a sum of 5 or 7 is (1/6) + (1/) + (5/18) (As events are mutually exclusive).

Hence, the probability of getting neither a sum of 5 nor a sum of 7 is 1 – (1/18) = (13/18).

Now, we get a sum of 5 before a sum of 7 if either we get a sum of 5, in first chance or we get neither a sum of 5 nor a sum of 7 in first chance and a sum of 5 in second chance and so on. Therefore, the required probability is

$$\frac{1}{9} + \frac{13}{18} \times \frac{1}{9} + \frac{13}{18} \times \frac{13}{18} \times \frac{1}{9} + \cdots \infty$$

$$= \frac{1/9}{1 - 13/18}$$

$$= \frac{1}{9} \times \frac{18}{5} = \frac{2}{5}$$

7. $P(A \cup B) = 0.8$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)\,P(B)$$

[as A and B are independent events]

$\Rightarrow \ 0.8 = 0.3 + P(B) - 0.3\,P(B)$

$\Rightarrow \ 0.5 = 0.7\,P(B)$

$\Rightarrow \ P(B) = 5/7$

8. Sample space is $\{Y, Y, Y, R, R, B\}$, where Y stands for yellow colour, R for red and B for blue.

The probability that the colour yellow, red and blue appear in the first, second and third tosses, respectively, is $(3/6) \times (2/6) \times (1/6) = (1/36)$.

9. Given that $P(A^c) = 0.3$, $P(B) = 0.4$ and $P(A \cap B^C) = 0.5$.

Then,

$$P[B/(A \cup B^C)] = \frac{P[B \cap (A \cup B^C)]}{P(A \cup B^c)}$$

$$= \frac{P((B \cap A) \cup (B \cap B^c))}{P(A \cup B^c)}$$

$$= \frac{P(A \cap B)}{P(A) + (B^c) - P(A \cap B^c)}$$

$$= \frac{P(A) - P(A \cap B^c)}{1 - P(A^c) + 1 - (B) - P(A \cap B^c)}$$

$$= \frac{1 - 0.3 - 0.5}{1 - 0.3 + 1 - 0.4 - 0.5}$$

$$= \frac{0.2}{0.8} = \frac{1}{4}$$

10. Let E_1 be the event of getting minimum number 3, E_2 be the event of getting maximum number 7. Then,

$P(E_1) = P$(getting one number 3 and other 2 from numbers 4 to 10)

$$= \frac{{}^1C_1 \times {}^7C_2}{{}^{10}C_3} = \frac{7}{40}$$

Similarly,

$P(E_2) = P$(getting one number 7 and other 2 from numbers 1 to 6)

$$= \frac{{}^1C_1 \times {}^6C_2}{{}^{10}C_3} = \frac{1}{8}$$

$P(E_1 \cup E_2) = P$ (getting one number 3, second number 7 and third from numbers 4 to 6)

$$= \frac{{}^1C_1 \times {}^1C_1\ {}^3C_1}{{}^{10}C_3} = \frac{1}{40}$$

$\therefore\ P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \frac{7}{40} + \frac{1}{8} - \frac{1}{40}$$

$$= \frac{7 + 5 - 1}{40}$$

$$= \frac{11}{40}$$

True or false

1. Let E be the event 'no two S's occur together'. A, A, I, N can be arranged is $4!/2! = 12$ ways.

— A — A — I — N —

In the arrangement shown above there are 5 places for four S.

Out of 5 places, 4 can be selected in ${}^5C_4 = 5$ ways. Therefore, no two S's occur together in $12 \times 5 = 60$ ways. Hence, the total number of arranging all letters of word ASSASSIN is $8!/(4!\ 2!) = 840$. Therefore, the required probability is $(60/840) = (1/14)$. Now,

$$P(A) + P(B) - P(A)\,P(B) = 0.2 + 0.3 - 0.2 \times 0.3$$

$$= 0.5 - 0.06 = 0.44$$

$$\neq 0.5$$

Hence, the statement is false.

2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Multiple choice question with one correct answer

1. d. The two events can happen simultaneously, e.g., (2, 3). Therefore, they are not mutually exclusive. Also, the two events are not dependent on each other.

2. a. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.25 + 0.50 - 0.14$$

$$= 0.61$$

$\therefore\ P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B)$

$$= 1 - 0.61 = 0.39$$

3. b. $p = 0.4,\ q = 0.6$

$\therefore\ P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^3C_0\,(0.4)^0\,(0.6)^3$$

$$= 1 - 0.216 = 0.784$$

4. c. $P(\bar{A}/\bar{B}) = \dfrac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$

$$= \frac{P(\overline{A \cup B})}{P(\bar{B})}$$

$$= \frac{1 - (A \cup B)}{P(\bar{B})}$$

5. d. Since there are 15 possible cases for selecting a coupon and seven coupons are selected, the total number of cases of selecting seven coupons is 15^7. It is given that the largest number on the selected coupon is 9. Therefore the selection is to be made from the coupons numbered 1 to 9. This can be made in 9^7 ways. Out of these 9^7 cases, 8^7 cases do not contain the number 9. Thus, the favourable number of cases is $9^7 - 8^7$. Hence, the required probability is $(9^7 - 8^7)/(15^7)$.

6. d. Let p be the probability of one coin showing head. Then the probability of one coin showing tail is $1 - p$. According to question, the coin is tossed 100 times and probability of 50 coins showing head is equal to the probability of 51 coins showing head.

Using binomial probability distribution $P(X = r) = {}^nC_r p^r q^{n-r}$, we get

$${}^{100}C_{50}p^{50}(1-p)^{50} = {}^{100}C_{51}p^{51}(1-p)^{49}$$

$$\Rightarrow \quad \frac{1-p}{p} = \frac{{}^{100}C_{51}}{{}^{100}C_{50}} = \frac{50!\,50!}{51!\,49!} = \frac{50}{51}$$

$$\Rightarrow \quad 51 - 51p = 50p$$

$$\Rightarrow \quad 101p = 51 \Rightarrow p = \frac{51}{101}$$

7. b. P(at least 7 points) = P(7 points) + P(8 points)

[$\because$ at most 8 points can be scored]

Now, 7 points can be scored by scoring 2 points in 3 matches and 1 point in one match. Similarly, 8 points can be scored by scoring 2 points in each of 4 matches. Therefore, the required probability is

${}^4C_1 \times [P(2 \text{ points})]^3\, P(1 \text{ point}) + [P(2 \text{ points})]^4 = 4(0.5)^3 \times 0.05 + (0.50)^4$

$= 0.0250 + 0.0625 = 0.0875$

8. a. The minimum face value is not less than 2 and maximum face value is not greater than 5 if we get any of the members 2, 3, 4, 5, while total possible outcomes are 1, 2, 3, 4, 5 and 6. Therefore, in one throw of die, probability of getting any number out of 2, 3, 4 and 5 is $4/6 = 2/3$.

If the die is rolled four times, then all these events being independent, the required probability is $(2/3)^4 = 16/81$.

9. b. Given that

P(India wins) $= p = 1/2$

$\therefore$ P(India loses) $= p' = 1/2$

Out of 5 matches, India's second win occurs at third test. Hence, India wins third test and simultaneously it has won one match from first two and lost the other. Therefore, the required probability is

$$P(LWW) + P(WLW) = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3$$

$$= \frac{1}{4}$$

10. b. Out of 6 vertices, 3 can be chosen in ${}^6C_3$ ways. The triangle will be equilateral if it is ΔACE or ΔBDF (2 ways).

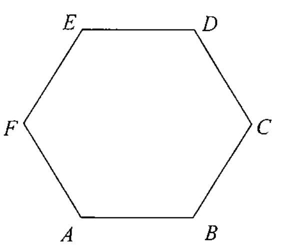


Fig. 9.19

Therefore, the required probability is

$$\frac{2}{{}^6C_3} = \frac{2}{20} = \frac{1}{10}$$

11. a. We know that

P(exactly one of A or B occurs) $= P(A) + P(B) - 2P(A \cap B)$

Therefore,

$$P(A) + P(B) - 2P(A \cap B) = p \qquad (1)$$

Similarly,

$$P(B) + P(C) - 2P(B \cap A) = p \qquad (2)$$

and

$$P(C) + P(A) - 2P(C \cap A) = p \qquad (3)$$

Adding Eqs. (1), (2) and (3) we get

$$2[P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A)] = 3p$$

$$\Rightarrow \quad P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) = 3p/2 \qquad (4)$$

It is also given that

$$P(A \cap B \cap C) = p^2 \qquad (5)$$

Now,

P(at least one of A, B and C) $= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$$= \frac{3p}{2} + p^2 \quad \text{[Using Eqs. (4) and (5)]}$$

$$= \frac{3p + 2p^2}{2}$$

12. a. We know that $7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$, $7^5 = 16807$.

Therefore, 7^k (where $k \in Z$) results in a number whose unit's digit is 7 or 9 or 3 or 1.

Now, $7^m + 7^n$ will be divisible by 5 if unit's place digit in the resulting number is 5 or 0. Clearly, it can never be 5. But it can be 0 if we consider values of m and n such that the sum of unit's place digits become 0. And this can be done by choosing

$$\left.\begin{array}{l} m = 1,\ 5,\ 9, \ldots,\ 97 \\ n = 3,\ 7,\ 11, \ldots,\ 99 \end{array}\right\} \quad \text{(25 options each) } [7 + 3 = 10]$$

or

$$\left.\begin{array}{l} m = 2,\ 6,\ 10, \ldots,\ 98 \\ n = 4,\ 8,\ 12, \ldots,\ 100 \end{array}\right\} \quad \text{(25 options each) } [9 + 3 = 10]$$

Therefore, the total number of selections of m, n such that $7^m + 7^n$ is divisible by 5 is $(25 \times 25 + 25 \times 25) \times 2$ (since we can interchange values of m and n).

Also the number of total possible selections of m and n out of 100 is 100×100. Therefore, the required probability is

$$\frac{2(25 \times 25 + 25 \times 25)}{100 \times 100} = \frac{1}{4}$$

13. d. The minimum of two numbers will be less than 4 or at least one of the numbers is less than 4.

$\therefore$ P(at least one numbers < 4) = 1 − P(both the numbers ≥ 4)

$$= 1 - \frac{3}{6} \times \frac{2}{5}$$

$$= 1 - \frac{6}{30}$$

$$= = 1 - \frac{1}{5} = \frac{4}{5}$$

14. a. Given that $P(B) = 3/4$, $P(A \cap B \cap \bar{C}) = 1/3$

$P(\bar{A} \cap B \cap \bar{C}) = 1/3$. From Venn's diagram, we have

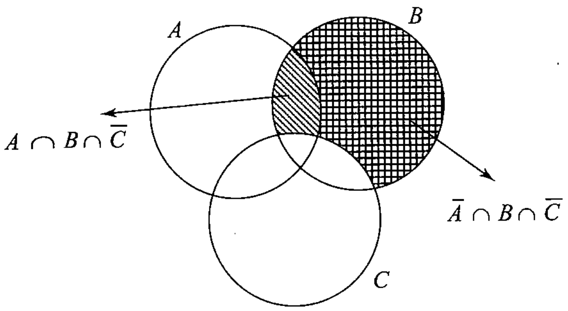


Fig. 9.20

$$P(B \cap C) = P(B) - P(A \cap B \cap \bar{C}) - P(\bar{A} \cap B \cap \bar{C})$$

$$\Rightarrow \quad P(B \cap C) = \frac{3}{4} - \frac{1}{3} - \frac{1}{3} = \frac{9-4-4}{12} = \frac{1}{12}$$

15. d. If a number is to be divisible by both 2 and 3, it should be divisible by their L.C.M. L.C.M. of 2 and 3 is 6. The numbers are 6, 12, 18, ..., 96. The total number is 16. Hence, the probability is

$$\frac{^{16}C_3}{^{100}C_3} = \frac{4}{1155}$$

16. a. In single throw of a dice, probability of getting 1 is 1/6 and probability of not getting 1 is 5/6.

Then, getting 1 in even number of chances is getting 1 in 2nd chance or in 4th chance or in 6th chance and so on. Therefore, the required probability is

$$\frac{5}{6} \times \frac{1}{6} + \left(\frac{5}{6}\right)^3 \times \frac{1}{6} + \left(\frac{5}{6}\right)^5 \times \frac{1}{6} + \cdots + \infty$$

$$= \frac{5}{36}\left[\frac{1}{1-\frac{25}{36}}\right]$$

$$= \frac{5}{36} \times \frac{36}{11}$$

$$= \frac{5}{11}$$

17. b. Favourable cases are {(1, 1, 1), (2, 2, 2), ..., (6, 6, 6)}. The number of favourable cases is 6. The total number cases is 6 × 6 × 6 = 216. Therefore, the required probability is 6/216 = 1/36.

18. c. The probability of getting a white ball in a single draw is $p = 12/24 = 1/2$. The probability of getting a white ball 4th time in the 7th draw is

P(getting 3W in 6th draws and W in 7th draw)

$$= {}^6C_3 \left(\frac{1}{2}\right)^6 \frac{1}{2} = \frac{5}{32}$$

19.(a) $P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$

$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$

$= P(A)\,P(B) + P(A)\,P(C) - P(A)\,P(B)\,P(C)$

$= P(A)\,[P(B) + P(C) - P(B \cap C)] = P(A)\,P(B \cup C)$

Therefore, S_1 is true.

$$P(A \cap (B \cap C)) = P(A)\,P(B)\,P(C) = P(A)\,P(B \cup C)$$

Therefore, S_2 is also true.

20. c. Let E_1 be the event that the Indian man is seated adjacent to his wife and E_2 be the even that each American man is seated adjacent to his wife. Then,

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

Now, $E_1 \cap E_2$ is the evetn that all men are seated adjacent to their wives.

Therefore, we can consider the 5 couples as single–single objects which can be arranged in a circle in 4! ways. But for each couple, husband and wife can interchange their places in 2! ways.

Hence, the umber of ways when all men are seated adjacent to their wives is $4! \times (2!)^5$. Also in all, 10 persons can be seated in a circle in 9! ways.

$$\therefore \quad P(E_1 \cap E_2) = \frac{4! \times (2!)^5}{9!}$$

Similarly, if each American man is seated adjacent to his wife, considering each American couple as single object and Indian woman and man as separate objects, there are 6 different objects which can be arranged in a circle in 5! ways. Also for each American couple, husband and wife can interchange their places in 2! ways.

So, the number of ways in which each American man is seated adjacent to his wife is $5! \times (2!)^4$.

$$\therefore \quad P(E_2) = \frac{5! \times (2!)^4}{9!}$$

Hence,

$$P(E_1/E_2) = \frac{(4! \times (2!)^5)/9!}{(5! \times (2!)^4)/9!} = \frac{2}{5}$$

21. c. We have,

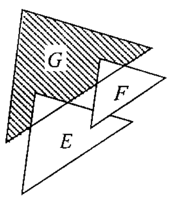


Fig. 9.21

$$E \cap F \cap G = \phi$$

$$P(E^c \cap F^c / G) = \frac{P(E^c \cap F^c \cap G)}{P(G)}$$

$$= \frac{P(G) - P(E \cap G) - P(G \cap F)}{P(G)}$$

[From Venn diagram, $E^c \cap F^c \cap G = G - E \cap G - F \cap G$]

$$= \frac{P(G) - P(E)P(G) - P(G)P(F)}{P(G)} \quad [\because E, F, G \text{ are pairwise independent}]$$

$$= 1 - P(E) - P(F) = P(E^c) - P(F)$$

22. c.

Even G = original signal is green

E_1 = A receives the signal correct

E_2 = B receives the signal correct

E = Signal received by B is green

P (Signal received by B is green)

$$= P(GE_1E_2) + P(G\bar{E}_1\bar{E}_2) + P(\bar{G}E_1\bar{E}_2) + P(\bar{G}\bar{E}_1E_2)$$

$$P(E) = \frac{46}{5\times16}$$

$$P(G/E) = \frac{40/5\times16}{46/5\times16} = \frac{20}{23}$$

23. c.

$r_1, r_2, r_3 \in \{1,2,3,4,5,6\}$

r_1, r_2, r_3 are of the form $3k$, $3k+1$, $3k+2$

$$\text{Required probability} = \frac{3!\times {}^2C_1\times{}^2C_1\times{}^2C_1}{6\times6\times6} = \frac{6\times8}{216} = \frac{2}{9}$$

Multiple choice questions with one or more than one correct answer

1. a, c, d.

a. $P(M) + P(N) - 2\,P(M\cap N)$

$= [P(M) + P(N) - P(M\cap N)] - P(M\cap N)$

$= P(M\cup N) - P(M\cap N)$

= Probability that exactly one of M and N occurs

b. $P(M) + P(N) - P(M\cap N)$

$= P(M\cup N)$

= Probability that at least of M and N occurs

c. $P(M^c) + P(N^c) - 2\,P(M^c\cap N^c)$

$= 1 - P(M) + 1 - P(N) - 2P(M\cup N)^c$

$= 2 - P(M) - P(N) - 2[1 - P(M\cup N)]$

$= P(M\cup N) + P(M\cup N) - P(M) - P(N)$

$= P(M\cup N) - P(M\cap N)$

= Probability that exactly one of M and N occurs

d. $P(M\cap N^c) + P(M^c\cap N)$

= Probability that M occurs but N does not or probability that M does not occurs but N occurs

= Probability that exactly one of M and N occurs

Thus, (a), (c) and (d) are the correct options.

2. c. Let A, B, C be the events that the student passes tests I, II, III, respectively. Then, according to question, $P(A) = p$, $P(B) = q$, $P(C) = 1/2$.

Now the student is successful if A and B happen or A and C happen or A, B and C happen.

$$\therefore \quad P(AB\bar{C}) + P(AC\bar{B}) + P(ABC) = \frac{1}{2}$$

$$\Rightarrow \quad pq\left(1-\frac{1}{2}\right) + p\,\frac{1}{2}(1-q) + p\,q\,\frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2}pq + \frac{1}{2}p - \frac{1}{2}pq + \frac{1}{2}pq = \frac{1}{2}$$

$$\Rightarrow \quad p + pq = 1$$

$$\Rightarrow \quad p(1+q) = 1$$

which holds for $p = 1$ and $q = 0$.

3. c. Given that

$P(A\cup B) = 0.6;\ P(A\cap B) = 0.2$

$\therefore \quad P(\bar{A}) + P(\bar{B}) = 1 - P(A) + 1 - P(B)$

$= 2 - (P(A) + P(B))$

$= 2 - [P(A\cup B) + P(A\cap B)]$

$= 2 - [0.6 + 0.2]$

$= 2 - 0.8$

$= 1.2$

4. a, b, c.

We know that

$P(A\cap B) = P(A) + P(B) - P(A\cup B)$ (1)

Also,

$P(A\cup B) \le 1$

$\Rightarrow \quad -P(A\cup B) \ge -1$ (2)

$\therefore \quad P(A\cap B) \ge P(A) + P(B) - 1$ [Using Eqs. (1) and (2)]

Therefore, option (a) is correct. Again,

$P(A\cup B) \ge 0$

$\Rightarrow \quad -P(A\cup B) \le 0$ (3)

$\Rightarrow \quad P(A\cap B) \le P(A) + P(B)$ [Using Eqs. (1) and (3)]

Therefore, option (b) is also correct.

From Eq. (1), option (c) is correct and (d) is not correct.

5. b, c, d.

$P(E\cap F) = P(E)\,P(F)$

Now,

$P(E\cap \bar{F}) = P(E) - P(E\cap F) = P(E)\,[1 - P(F)]$

$= P(E)P(F^c)$

and

$P(E^c\cap F^c) = 1 - P(E\cup F) = 1 - [P(E) + P(F) - P(E\cap F)$

$= [1 - P(E)]\,[1 - P(F)] = P(E^c)\,P(F^c)$

Also,

$P(E/F) = P(E)$ and $P(E^c/F^c) = P(E^c)$

$\Rightarrow \quad P(E/F) + P(E^c/F^c) = 1$

6. a, c.

a. For any two events A and B,

$$P(A/B) = \frac{P(A\cap B)}{P(B)}$$

Now we know that

$P(A\cup B) \le 1$

$\Rightarrow \quad P(A) + P(B) - P(A\cap B) \le 1$

$\Rightarrow \quad P(A\cap B) \ge P(A) + P(B) - 1$

$$\Rightarrow \quad \frac{P(A\cap B)}{P(B)} \ge \frac{P(A) + P(B) - 1}{P(B)} \quad [\text{as } P(B)\ne 0 \therefore P(B) > 0]$$

$$\Rightarrow \quad P(A/B) \ge \frac{P(A) + P(B) - 1}{P(B)}$$

Therefore, option (a) is correct.

b. From Venn's diagram, we can clearly conclude that

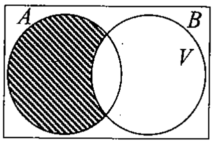


Fig. 9.22

$P(A \cup \bar{B}) = P(A) - P(A \cap B)$

Therefore, option (b) is incorrect

c. $P(A \cap B) = P(A) + P(B) - P(A \cap B)$

$= 1 - P(\bar{A}) + 1 - P(\bar{B}) - P(A)\, P(B)$

[$\because$ A and B are independent events]

$= 2 - P(\bar{A}) - P(\bar{B}) - [1 - P(\bar{A})][1 - P(\bar{B})]$

$= 2 - P(\bar{A}) - P(\bar{B}) - 1 + P(\bar{A}) + P(\bar{B}) - P(\bar{A})P(\bar{B})$

$= 1 - P(\bar{A} \cap \bar{B})$ [$\because$ if A and B are independent, $\bar{A}$ and $\bar{B}$ are also independent]

Therefore, option (c) is the correct statement.

d. For disjoint events,

$P(A \cup B) = P(A) + P(B)$

Therefore, option (d) is he incorrect.

7. a, b.

Let $P(E) = x$ and $P(F) = y$. According to the question,

$$P(E \cap F) = \frac{1}{12}$$

As E and F are independent events, we have

$$P(E \cap F) = P(E)\, P(F)$$

$$\Rightarrow \frac{1}{12} = xy \qquad (1)$$

Also,

$$P(\bar{E} \cap \bar{F}) = P(\overline{E \cup F})$$

$$= 1 - P(E \cup F)$$

$$\Rightarrow \frac{1}{2} = 1 - [P(E) + P(F) - P(E)\, P(F)]$$

$$\Rightarrow x + y = \frac{7}{12} \qquad (2)$$

Solving Eqs. (1) and (2), we get either $x = 1/3$ and $y = 1/4$ or $x = 1/4$ and $y = 1/3$.

Therefore, options (a) and (b) are correct.

8. a. $P(2 \text{ white and } 1 \text{ black}) = P(W_1\, W_2\, B_3 \text{ or } W_1\, B_2\, W_3 \text{ or } B_1\, W_2\, W_3)$

$$= P(W_1\, W_2\, B_3) + P(W_1\, B_2\, W_3) + P(B_1\, W_2\, W_3)$$

$$= P(W_1)\, P(W_2)\, P(B_3) + P(W_1)\, P(B_2)\, P(W_3) + P(B_1)\, (W_2)\, (W_3)$$

$$= \frac{3}{4}\times\frac{2}{4}\times\frac{3}{4} + \frac{3}{4}\times\frac{2}{4}\times\frac{1}{4} + \frac{1}{4}\times\frac{2}{4}\times\frac{1}{4}$$

$$= \frac{1}{32}(9 + 3 + 1)$$

$$= \frac{13}{32}$$

9. a, d.

We have,

$$P(E/F) + P(\bar{E}/F) = \frac{P(E \cap F)}{P(F)} + \frac{P(\bar{E} \cup F)}{P(F)}$$

$$= \frac{P(E \cap F) + P(\bar{E} \cap F)}{P(F)}$$

$$= \frac{P(F)}{P(F)} = 1$$

Therefore, option (a) holds. Also,

$$P(E/F) + P(\bar{E}/F) = \frac{P(E \cap F) + P(\bar{F} \cap E)}{P(F)}$$

$$= \frac{P(E)}{P(F)} \neq 1$$

Therefore option (b) does not hold.

Similarly, we can show that option (c) does not hold but option (d) holds.

10. a. The probability that only two tests are needed is (probability that the first machine tested is faulty) × (probability that the second machine tested is faulty given the first machine tested is faulty), which is given by (2/4) × (1/3) = 1/6.

11. d. Given that $P(E) \le P(F)$ and $P(E \cap F) > 0$. It does not necessarily mean that E is the subset of F. Therefore, the choices (a), (b), (c) do not hold in general. Hence, option (d) is the right choice here.

12. a. The event that the fifth toss result in a head is independent of the event that the first four tosses result in tails. Therefore, the probability of the required event is 1/2.

13. b, c.

According to the problem,

$$m + p + c - mp - mc - pc + mpc = 3/4 \qquad (1)$$

$$mp\,(1 - c) + mc(1 - p) + pc(1 - m) = 2/5 \qquad (2)$$

Also,

$$mp + pc + mc - 2mpc = 1/2 \qquad (3)$$

From Eqs. (2) and (3),

$$mpc = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$$

$$\therefore \quad mp + mc + pc = \frac{2}{5} + \frac{3}{10} = \frac{7}{10}$$

$$\therefore \quad m + p + c = \frac{3}{4} + \frac{7}{10} - \frac{1}{10} = \frac{15 + 14 - 2}{20} = \frac{27}{20}$$

14. b. The number of ways of arranging 10 balls without any restriction is 10!. As for no two black balls are placed adjacently, first arrange 7 white balls in 7! ways.

$$- W - W - W - W - W - W - W -$$

Now white balls must be placed in three of eight gaps created in ${}^8C_3 3!$ ways. Hence, number of favorable ways is ${}^8C_3\, 3!\, 7!$.

Therefore, the required probability is

$$\frac{{}^8C_3 3! 7!}{10!} = \frac{7}{15}$$

15. c., d.

$$P(A \cup B)' = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - P(A) - P(B) + P(A)\,P(B)$$
$$= [1 - P(A)]\,[1 - P(B)]$$
$$= P(A')\,P(B')$$

Also,

$$P(A \cup B) = P(A) + P(B) - P(A)\,P(B)$$
$$\Rightarrow \quad P(A \cap B) = P(A)\,P(B)$$
$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\quad P(B)}{P(B)} = P(A)$$

Also,

$$P(B/A) = P(B)$$
$$P(A' - B') = P(A') - P(A' \cap B')$$
$$= P(A') - P(A')\,P(B')$$
$$= P(A')(1 - P(B'))$$
$$= P(A')P(B)$$

16. a, d.
Let $P(E) = e$ and $P(F) = f$

$$P(E \cup F) - P(E \cap F) = \frac{11}{25}$$
$$\Rightarrow \quad e + f - 2ef = \frac{11}{25} \qquad (1)$$
$$P(\bar{E} \cap \bar{F}) = \frac{2}{25}$$
$$\Rightarrow \quad (1-e)(1-f) = \frac{2}{25}$$
$$\Rightarrow \quad 1 - e - f + ef = \frac{2}{25} \qquad (2)$$

From (1) and (2)

$$ef = \frac{12}{25} \text{ and } e + f = \frac{7}{5}$$

Solving, we get

$$e = \frac{4}{5}, f = \frac{3}{5} \text{ or } e = \frac{3}{5}, f = \frac{4}{5}$$

Comprehension

1. b. $P(u_i) \propto i \Rightarrow P(u_i) = K_i$,
But

$$\Sigma P(u_i) = 1$$
$$\Rightarrow \quad \Sigma k_i = 1 \Rightarrow k\,\Sigma i = 1 \Rightarrow K = \frac{2}{n(n+1)}$$
$$\Rightarrow \quad P(u_i) = \frac{2i}{n(n+1)}$$

By the total probability theorem,

$$P(w) = \sum_{i=1}^{n} P(u_i)\,P(w/u_i)$$
$$= \sum_{i=1}^{n} \frac{2i}{n(n+1)} \times \frac{i}{n+1}$$
$$= \frac{2}{n(n+1)^2}\,\frac{n(n+1)(2n+1)}{6}$$
$$= \frac{2n+1}{3n+3}$$
$$\therefore \quad \lim_{n\to\infty} P(w) = \lim_{n\to\infty} \frac{2n+1}{3n+3} = \lim_{n\to\infty} \frac{2+1/n}{3+3/n} = \frac{2}{3}$$

2. c. $P(u_i) = c$

Using Bayes's theorem,

$$P(u_n/w) = P(u_m/w)\,\frac{P(w/u_n)\,P(u_n)}{\sum_{i=1}^{n} P(w/u_i)\,P(u_i)}$$
$$= \frac{c \times \frac{n}{n+1}}{c\left[\frac{1}{n+1} + \frac{2}{n+1} + \cdots + \frac{n}{n+1}\right]}$$
$$= \frac{n}{n+1} \times \frac{n+1}{\frac{n(n+1)}{2}} = \frac{2}{n+1}$$

3. b. $P(w/E) = \dfrac{P(w \cap E)}{P(E)}$

$$= \frac{\frac{1}{n} \times \frac{2}{n+1} + \frac{1}{n} \times \frac{4}{n+1} + \frac{1}{n} \times \frac{6}{n+1} + \cdots + \frac{1}{n} \times \frac{n}{n+1}}{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\left(\frac{n}{2} \text{ times}\right)}$$
$$= \frac{\frac{2}{n+1}\left[1 + 2 + 3 + \cdots + \frac{n}{2}\right]}{\frac{1}{n} \times \frac{n}{2}} \quad (n \text{ being even})$$
$$= \frac{4}{n+1}\left[\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2}\right]$$

4. a. $P(X = 3) = \left(\dfrac{5}{6}\right)\left(\dfrac{5}{6}\right)\dfrac{1}{6} = \dfrac{25}{216}$

5. b. Given that

$$P(X \le 2) = \frac{1}{6} + \frac{5}{6} \times \frac{1}{6} = \frac{11}{36}$$

Hence, the required probability is $1 - (11/36) = 25/36$.

6. d. For $X \ge 6$, the probability is

$$\frac{5^5}{6^6} + \frac{5^5}{6^7} + \cdots \infty = \frac{5^5}{6^6}\left(\frac{1}{1-5/6}\right) = \left(\frac{5}{6}\right)^5$$

For $X > 3$,

$$\frac{5^3}{6^4} + \frac{5^4}{6^5} + \frac{5^5}{6^6} + \cdots \infty = \left(\frac{5}{6}\right)^3$$

Hence, the conditional probability is

$$\frac{(5/6)^6}{(5/6)^3} = \frac{25}{36}$$

7. b. $H \to 1$ ball form U_1 to U_2
$T \to 2$ ball form U_1 to U_2

E : 1 ball drawn from U_2

$P(W \text{ from } U_2)$

$$= \frac{1}{2} \times \left(\frac{3}{5} \times 1\right) + \frac{1}{2} \times \left(\frac{2}{5} \times \frac{1}{2}\right) + \frac{1}{2} \times \left(\frac{^3C_2}{^5C_2} \times \frac{1}{3}\right)$$

$$+ \frac{1}{2} \times \left(\frac{^3C_1 \cdot {^2C_1}}{^5C_2} \times \frac{2}{3}\right) = \frac{23}{30}$$

8. d. $$P\left(\frac{H}{W}\right) = \frac{P(W/H) \times P(H)}{P(W/T) \cdot P(T) + (W/H) \cdot P(H)}$$

$$= \frac{\frac{1}{2}\left(\frac{3}{2} \times 1 + \frac{2}{5} \times \frac{1}{2}\right)}{23/30} = \frac{12}{23}$$

Assertion and reason

1. d. We know that

$$P(H_i/E) = \frac{P(H_i \cap E)}{P(E)}$$

$$= \frac{P(E/H_i)\, P(H_i)}{P(E)}$$

$$\Rightarrow P(H_i/E)\, P(E) = P(E/H_i)\, P(H_i)$$

$$\Rightarrow P(E) = \frac{P(E/H_i)\, P(H_i)}{P(H_i/E)}$$

Now given that

$$0 < P(E) < 1$$

$$\Rightarrow 0 < \frac{P(E/H_i) P(H_i)}{P(H_i/E)} < 1$$

$$\Rightarrow P(E/H_i)\, P(H_i) < P(H_i/E)$$

But if $P(H_i \cap E) = 0$, then $P(H_i/E) = P(E/H_i) = 0$. Then $P(E/H_i)$ $(PH_i) < P(H_i/E)$ is not true. Hence, statement 1 is not always true.

Also as $H_1, H_2, \ldots, H_n$ are mutually exclusive and exhaustive events, therefore

$$\sum_{i=1}^{n} P(H_i) = 1$$

Hence, statement 2 is true.

2. b. For unique solution,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

where $a, b, c, d \in \{0, 1\}$.

The total number of cases is 16. Favourable number of cases is 6 (either $ad = 1$, $bc = 0$ or $ad = 0$, $bc = 1$). The probability that system of equations has unique solution is 6/16 = 3/8. Since $x = 0$ satisfies both the equations, the system has at least one solution.

Appendix

Solutions to Concept Application Exercises

Chapter 1

Exercise 1.1

1. $f(0) = 0 + 3 = 3$ (using $f(x) = x + 3$)
$f(3) = 3^2 = 9$ (using $f(x) = x^2$)
$f(4) = 2 - 3(4) = -10$ (using $f(x) = 2 - 3x$)
$f(2) = 2^2 = 4$
Hence, $f(3)$ is greatest.

2. Let $f(x) = ax^2 + bx + c$
$f(0) = -4 \Rightarrow c = -4$
$f(1) = -5$
$\Rightarrow a + b - 4 = -5$ (1)
$f(-1) = -1$
$\Rightarrow a - b - 4 = -1$ (2)
Solving (1) and (2) we get $a = 1$ and $b = -2$
$\Rightarrow f(x) = x^2 - 2x - 4$
$\Rightarrow f(3) = 9 - 6 - 4 = -1$

3. (i)
$-5 \le x \le -1$
$\Rightarrow x^2 \in [1, 5]$

(ii) $3 < x < 6$
$\Rightarrow x^2 \in (9, 36)$

(iii) $-2 < x \le 3$
$\Rightarrow -2 < x < 0$ or $0 \le x \le 3$
For $-2 < x < 0$, $x^2 \in (0, 4)$ (1)
and for $0 \le x \le 3$, $x^2 \in [0, 9]$ (2)
From (1) and (2), $x^2 \in [0, 9]$
Here $x \in (-2, 3]$, now least value of x^2 is 0 which occurs when $x = 0$
Greatest value of x^2 is 9 for $x = 3$
$\Rightarrow x^2 \in [0, 9]$

(iv) $(-3, \infty)$
Here least value of x^2 is 0 for $x = 0$ and when x goes upto infinity x^2 also goes upto infinity
$\Rightarrow x^2 \in (0, \infty)$

(v) $(-\infty, 4]$
Here least value of x^2 is 0 for $x = 0$ and $x^2 \to \infty$ when $x \to -\infty$
Hence $x^2 \in (0, \infty)$

4. (i)
$2 < x < 5$
$\Rightarrow \frac{1}{2} > \frac{1}{x} > \frac{1}{5}$
$\Rightarrow \frac{1}{x} \in \left(\frac{1}{5}, \frac{1}{2}\right)$

(ii) $-5 \le x < -1$
$\Rightarrow -\frac{1}{5} \ge \frac{1}{x} > -1$
$\Rightarrow \frac{1}{x} \in \left(-1, -\frac{1}{5}\right]$

(iii) $x > 3$ or $3 < x < \infty$
$\Rightarrow \frac{1}{3} > \frac{1}{x} > 0$
$\Rightarrow \frac{1}{x} \in \left(0, \frac{1}{3}\right)$

(iv) $x \le -2$ or $-\infty < x \le -2$
$\Rightarrow 0 > \frac{1}{x} \ge -\frac{1}{2}$
$\Rightarrow \frac{1}{x} \in \left[-\frac{1}{2}, 0\right)$

(v) $x \in [-3\ 4]$ or $-3 \le x \le 4$
Now for $\frac{1}{x}$ to get defined we must have $-3 \le x < 0$ or $0 < x \le 4$
$\Rightarrow -\frac{1}{3} \ge \frac{1}{x} > -\infty$ or $\frac{1}{4} \le \frac{1}{x} < \infty$
$\Rightarrow \frac{1}{x} \in \left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{1}{4}, \infty\right)$

5. (a) is not always true : for $a = -4$ and $b = 3$ given statement is not true
(b) is not always true : for $a = -3$ and $b = 2$ given statement is not true
(c) is not always true : for $a = -5$ and $b = 3$ given statement is not true

6. $-3 < 2x - 1 < 19$

$\Rightarrow \quad -2 < 2x < 20$

$\Rightarrow \quad -1 < x < 10 \qquad (1)$

$-1 \le \dfrac{2x+3}{5} \le 3$

$\Rightarrow \quad -5 \le 2x + 3 \le 15$

$\Rightarrow \quad -8 \le 2x \le 12$

$\Rightarrow \quad -4 \le x \le 6 \qquad (2)$

Now from (1) and (2) common values are $(-1, 6]$

7. (i)

$y = \dfrac{2-5x}{3x-4}$

$\Rightarrow \quad 3yx - 4y = 2 - 5x$

$\Rightarrow \quad x = \dfrac{4y+2}{3y+5}$

Hence $y \in R - \{-5/3\}$

(ii)

$\sqrt{x^2 - 7x + 6} = \sqrt{(x-6)(x-1)}$

Minimum value of expression 0 when $x = 1$ or $x = 6$

Hence expression takes vales $[0, \infty)$

(iii)

$\dfrac{x^2-x-6}{x-3} = \dfrac{x^2-x-6}{x-3} = \dfrac{(x-3)(x+2)}{x-3}$

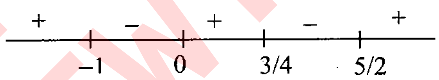
$= x + 2, x \ne 3$

Hence expression takes values $R - \{5\}$

8. $\dfrac{x(3-4x)(x+1)}{(2x-5)} < 0$

$\Rightarrow \quad -\dfrac{x(4x-3)(x+1)}{(2x-5)} < 0$

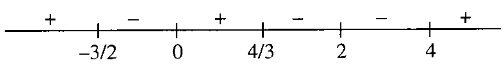
$\Rightarrow \quad \dfrac{x(4x-3)(x+1)}{(2x-5)} > 0$



$\Rightarrow \quad x \in (-\infty, -1) \cup (0, 3/4) \cup (5/2, \infty)$

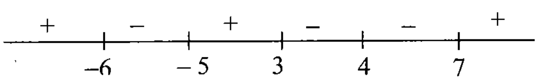
9. $\dfrac{(2x+3)(4-3x)^3(x-4)}{(x-2)^2x^5} \le 0$

$\Rightarrow \quad \dfrac{(2x+3)(3x-4)^3(x-4)}{(x-2)^2x^5} \ge 0$



From the sign scheme we have $x \in (-\infty, -3/2) \cup (0, 4/3] \cup [4, \infty)$

10. Signe of expression does not change while crossing $x = 4$ as there is $|x - 4|$.



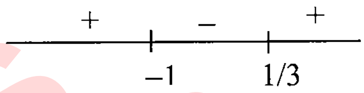
From the sign scheme we have $x \in (-6, -5] \cup [3, 4) \cup (4, 7]$

11. $f(x) = y = \dfrac{1-x^2}{x^2+3}$

$\Rightarrow \quad yx^2 + 3y = 1 - x^2$

$\Rightarrow \quad x^2 = \dfrac{1-3y}{y+1}$

Now $x^2 \ge 0 \quad \Rightarrow \quad \dfrac{1-3y}{y+1} \ge 0 \quad \Rightarrow \quad \dfrac{3y-1}{y+1} \le 0$



$\Rightarrow \quad$ Hence $y \in (1, 1/3]$

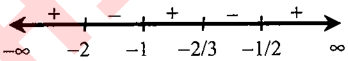
12. We are given $\dfrac{2x}{2x^2+5x+2} > \dfrac{1}{x+1}$

$\Rightarrow \quad \dfrac{2x}{2x^2+5x+2} - \dfrac{1}{x+1} > 0$

$\Rightarrow \quad \dfrac{2x^2+2x-2x^2-5x-2}{(2x^2+5x+2)(x+1)} > 0$

$\Rightarrow \quad \dfrac{-3x-2}{(2x+1)(x+1)(x+2)} > 0$

$\Rightarrow \quad \dfrac{(3x+2)}{(x+1)(x+2)(2x+1)} < 0$



From the sign scheme solution is $x \in (-2, -1) \cup (-2/3, -1/2)$

13. (i) $\dfrac{\sqrt{x-1}}{x-2} < 0$ is meaningful for $x - 1 \ge 0$ or $x \ge 1 \qquad (1)$

Also $\dfrac{\sqrt{x-1}}{x-2} < 0 \Rightarrow x - 2 < 0$ or $x < 2 \qquad (2)$

$1 \le x < 2$

(ii) $\sqrt{x-2} \le 3$ is meaningful for $x - 2 \ge 0$ or $x \ge 2 \qquad (1)$

Also $\sqrt{x-2} \le 3 \quad \Rightarrow x - 2 \le 9 \quad \Rightarrow \quad x \le 11 \qquad (2)$

From (1) and (2), we have $2 \le x \;\; 11$

14. $x^2 - |x| - 2 = 0$

$\Rightarrow \quad |x|^2 - |x| - 2 = 0$

$\Rightarrow \quad (|x| - 2)(|x| + 1) = 0$

$\Rightarrow \quad |x| = 2\ x = \pm 2.$

$x^2 - 2|x| + 3 = 0$

$\Rightarrow \quad |x|^2 - 2|x| + 3 = 0$ has no roots as discriminant is negative.

$x^2 - 3|x| + 2 = 0$

$\Rightarrow \quad |x|^2 - 3|x| + 2$

$\Rightarrow \quad (|x| - 2)(|x| - 1) = 0$

$\Rightarrow \quad |x| = 1, 2\ x = \pm 1, \pm 2.$

$x^2 + 3|x| + 2 = 0$

$\Rightarrow \quad |x|^2 + 3|x| + 2$

$\Rightarrow \quad (|x| + 2)(|x| + 1) = 0$

$\Rightarrow \quad |x| = -1, -2$ (not possible)

Thus the equation has maximum real roots.

15. Case I: Let $x \ge 3$

$\therefore \quad \left.\begin{matrix} x + 2y = 6 \\ x - 3 = y \end{matrix}\right\} \Rightarrow 3x = 12 \Rightarrow x = 4, y = 1$

Case II : Let $x < 3$

$$\left.\begin{array}{l} x + 2y = 6 \\ 3 - x = y \end{array}\right\} \Rightarrow -x = 0 \text{ or } x = 0, y = 3$$

Hence, the only solutions are (0, 3) and (4, 1)

16. $|x - 2| = \begin{cases} x-2, & x \ge 2 \\ 2-x, & x < 2 \end{cases}$

$\Rightarrow$ $|x - 2| - (x - 2) = \begin{cases} 0, & x \ge 2 \\ 4-2x, & x < 2 \end{cases}$

$\Rightarrow$ given expression is defined for $(-\infty, 2)$

17. $f(x) = x + \sqrt{x^2} = x + |x|$

$= \begin{cases} x+x, & x \ge 0 \\ x-x, & x < 0 \end{cases}$

$= \begin{cases} 2x, & x \ge 0 \\ 0, & x < 0 \end{cases}$

$\Rightarrow$ $f(x)$ takes values $[0, \infty)$

18. $\left|\dfrac{x+2}{x-1}\right| = 2$

$\Rightarrow$ $\dfrac{x+2}{x-1} = \pm 2$

$\Rightarrow$ $x + 2 = 2x - 2$ or $x + 2 = 2 - 2x$

$\Rightarrow$ $x = 4$ or $x = 0$

Here both $x = 0$ and 4 satisfies the parent equation

19. $|x^2 - 7| \le 9$

$\Rightarrow$ $-9 \le x^2 - 7 \le 9$

$\Rightarrow$ $-2 \le x^2 \le 16$

$\Rightarrow$ $x^2 \le 16$

$\Rightarrow$ $x \in [-4, 4]$

20. We must have $5 - |2x - 3| \ge 0$

$\Rightarrow$ $|2x - 3| \le 5$

$\Rightarrow$ $-5 \le 2x - 3 \le 5$

$\Rightarrow$ $x \in [-1, 4]$

21. $||x - 2| - 3| < 5$

$\Rightarrow$ $-5 < |x - 2| - 3 < 5$

$\Rightarrow$ $-2 < |x - 2| < 8$

$\Rightarrow$ $|x - 2| < 8$

$\Rightarrow$ $-8 < x - 2 < 8$

$\Rightarrow$ $-6 < x < 10$

22. $|x + y| = |x| + |y| \Rightarrow x$ and y have same sign or at least one of x and y is zero.

$\Rightarrow$ (x, y) lies in 1st or 3rd quadrant or any of the x-axis or y-axis

$|x + y| < |x| + |y| \Rightarrow x$ and y have opposite sign $\Rightarrow (x, y)$ lies in 2nd or 4th quadrant

$|x - y| = |x| + |y| \Rightarrow x$ and $-y$ have same sign $\Rightarrow x$ and y have opposite sign $\Rightarrow (x, y)$ lies in 2nd or 4th quadrant at least one of x and y is zero.

23. $|x^2 - x - 2| + |x + 6| = |x^2 - 2x - 8|$

$\Rightarrow$ $|x^2 - x - 2| + |x + 6| = |(x^2 - x - 2) - (x + 6)|$

$\Rightarrow$ $|x^2 - x - 2| + |-x - 6| = |(x^2 - x - 2) + (-x - 6)|$

$\Rightarrow$ $(x^2 - x - 2)(-x - 6) \ge 0$

$\Rightarrow$ $(x^2 - x - 2)(x + 6) \le 0$

$\Rightarrow$ $(x - 2)(x + 1)(x + 6) \le 0$

$\Rightarrow$ $x \in (-\infty, -6] \cup [-1, 2]$

24. $2x - 1 = x$ if $x \ge 0$

For which $x = 1$,

Also $2x - 1 = -x$ if $x < 0$

For which $x = 1/3$ which is not the solution as $x < 0$

Hence the only solution is $x = 1/3$

25. $2^x + 1 > 0, x \in R$

$\therefore$ given equation is $|2^x - 1| + 2^x + 1 = 2$

$\therefore$ $|2^x - 1| = 1 - 2^x$

$\therefore$ $2^x - 1 \le 0$

$\therefore$ $2^x \le 1$

$\therefore$ $x \le 0$

26. $|x^2 + 4x + 3| = x + 1$

$\Rightarrow$ $|(x + 1)(x + 3)| = x + 1$

$|(x + 1)(x + 3)| = (x + 1)(x + 3)$, when $(x + 1)(x + 3) \ge 0$

or $x \le -3$ or $x \ge -1$

Hence given equation reduces to $(x + 1)(x + 3) = x + 1$

$\Rightarrow$ $x = -1$ ($x = -2$ is rejected as $x \le -3$ or $x \ge -1$)

$|(x + 1)(x + 3)| = -(x + 1)(x + 3)$, when $(x + 1)(x + 3) < 0$

or $-3 < x < -1$

Hence given equation reduces to $-(x + 1)(x + 3) = x + 1$

$\Rightarrow$ $x = -4$ which is rejected as $-3 < x < -1$

27. We have $|x^2 - 1| + |x^2 - 4| > 3$

$\Rightarrow$ $|x^2 - 1| + |4 - x^2| > |x^2 - 1 + 3 - x^2|$

$\Rightarrow$ $(x^2 - 1)(4 - x^2) < 0$

$\Rightarrow$ $(x^2 - 1)(x^2 - 4) > 0$

$\Rightarrow$ $x^2 < 1$ or $x^2 > 4$

$\Rightarrow$ $-1 < x < 1$ or $x < -2$ or $x > 2$

28. Let $f(x) = |x - 1| - |2x - 5|$

A.	**B.** $f(x)$	**C.** $f(x) \ge 4$	**D.** $A \cap C$
$x < 1$	$1 - x - (5 - 2x)$	$x - 4 = 2x \Rightarrow x = -4$	$x = -4$
$1 \le x \le 5/2$	$x - 1 - (5 - 2x)$	$3x - 6 = 2x \Rightarrow x = 6$	No such x exists
$x > 5/2$	$x - 1 - (2x - 5)$	$4 - x = 2x \Rightarrow x = 4/3$	No such x exists

Hence solutions set is $\{-4\}$

Exercise 1.2

1. Graph of $y = x^2 + 2$ and $y = 3x + 4$ intersect when $x^2 + 2 = 3x - 4$ or $x^2 - 3x + 6 = 0$ which has no real roots, hence graph never intersect.

2. When graphs $y = -14$ and $y = -x(x^2 + x + 1) = 0$, intersect we have

$-(x^3 + x^2 + x) = -14$ or $x^3 + x^2 + x - 14 = 0$

Now $x = 2$ satisfies the equation, then one root is $x = 2$.

Dividing $x^3 + x^2 + x - 14$ by $x - 2$ we have $(x - 2)(x^2 + 3x + 7) = 0$

$\Rightarrow$ $x = 2$ or $x^2 + 3x + 7 = 0$

Now $x^2 + 3x + 7 = 0$ has non-real roots.

Hence graphs cuts in only one real point.

3.

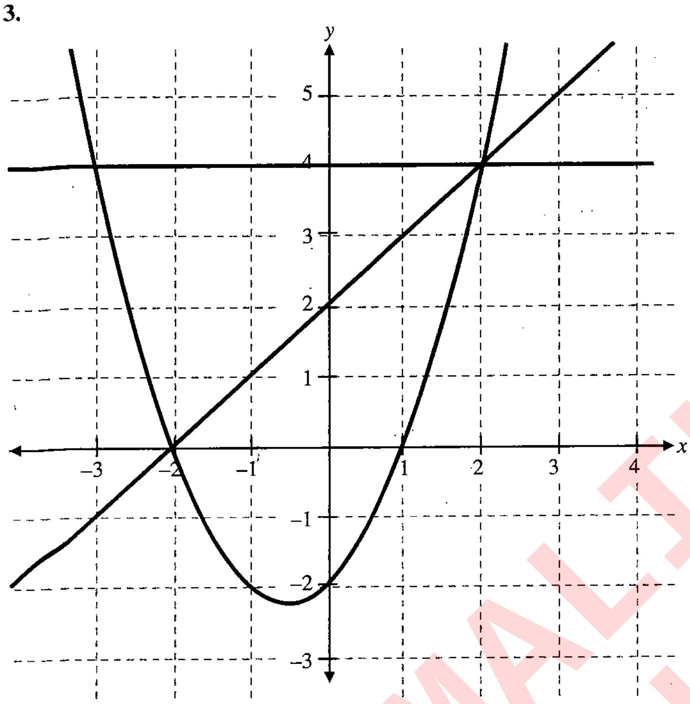


Fig. A-1.1

(i) Roots of the equation $f(x) = 0$ occurs where the graph of $y = f(x)$ and $y = 0$ intersect.

From the diagram points of intersection are $x = -2$ and $x = 1$. Hence sum or roots is -1

(ii) Roots of the equation $f(x) = 4$ occurs where the graph of $y = f(x)$ and $y = 4$ intersect

From the diagram points of intersection are $x = -3$ and $x = 2$. Hence product of roots is -6.

(iii) Roots of the equation $f(x) = x + 2$ occurs where the graph of $y = f(x)$ and $y = x + 2$ intersect

From the diagram points of intersection are $x = -2$ and $x = 2$. Hence difference of roots is 4.

4. $\dfrac{x^2+3x+2}{x^2-6x-7} = 0$ or $\dfrac{(x+1)(x+2)}{(x-7)(x+1)} = 0$ is solvable over $R - \{7, -1\}$

Hence from given equation $x = -2$, which is the only root of the equation.

5. $\sqrt{x-2} + \sqrt{4-x} = 2$

Squaring we get $x - 2 + 4 - x + 2\sqrt{(x-2)(4-x)} = 4$

$\Rightarrow \sqrt{(x-4)(2-x)} = 1$

$\Rightarrow -(x^2 - 6x + 8) = 1$

$\Rightarrow x^2 - 6x + 9 = 0$

$\Rightarrow (x - 3)^2 = 0$ or $x = 3$

6. Given equation is solvable for $x \in [2, \infty)$

Now $\sqrt{x-2}(x^2 - 4x - 5) = 0$

$\Rightarrow \sqrt{x-2}(x-5)(x+1) = 0$

$\Rightarrow x = 2$ or 5 ($x \neq -1$, as $x \in [2, \infty)$

7. We have,

$x(x + 2)(x^2 - 1) = -1$

$\Rightarrow x(x + 2)(x + 1)(x - 1) = -1$

$\Rightarrow (x(x + 1))(x + 2)(x - 1) = -1$

$\Rightarrow (x^2 + x)(x^2 + x - 2) = -1$

$\Rightarrow (x^2 + x)^2 - 2(x^2 + x) + 1 = 0$

Putting

$x^2 + x = y$, we get $y^2 - 2y + 1 = 0$

$\Rightarrow (y - 1)^2 = 0$

$\Rightarrow y = 1$

$\Rightarrow x^2 + x = 1$

$\Rightarrow x^2 + x - 1 = 0$

$\Rightarrow x = \dfrac{-1 \pm \sqrt{1+4}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$

Hence, the roots of the given equation are

$\dfrac{-1+\sqrt{5}}{2}, \dfrac{-1+\sqrt{5}}{2}, \dfrac{-1-\sqrt{5}}{2}$ and $\dfrac{-1-\sqrt{5}}{2}$

8. Let the given expression be equal to x. Then,

$x = 2 + \dfrac{1}{x}$

$\Rightarrow x^2 - 2x - 1 = 0$

$\Rightarrow x = \dfrac{2 \pm \sqrt{4+4}}{2} = \dfrac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

But, the given expression is positive. Therefore, $x = 1 + \sqrt{2}$ is the value of the given expression.

9. $4^x + 6^x = 9^x$

$\Rightarrow \left(\dfrac{4}{9}\right)^x + \left(\dfrac{2}{3}\right)^x = 1$

Putting $(2/3) = y$, we have

$y^2 + y - 1 = 0$

$\Rightarrow y = \dfrac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \left(\dfrac{2}{3}\right)^x = \dfrac{\sqrt{5}-1}{2}$

$\Rightarrow \Rightarrow x = \log_{2/3}\left(\dfrac{\sqrt{5}-1}{2}\right)$

10. $3^{2x^2-7x+7} = 9$

$\Rightarrow 3^{2x^2-7x+7} = 3^2$

$\Rightarrow 2x^2 - 7x + 7 = 2$

$\Rightarrow 2x^2 - 7x + 5 = 0 \Rightarrow x = 1, 5/2$

11. The equation has no real root, because L.H.S. is always positive while R.H.S. is zero.

12. $\sqrt{3x^2 - 7x - 30} + \sqrt{2x^2 - 7x - 5} = x + 5$

$\Rightarrow \sqrt{3x^2 - 7x - 30} = (x + 5) - \sqrt{2x^2 - 7x - 5}$

On squaring,

$3x^2 - 7x - 30 = x^2 + 10x + 25 - 2(x + 5) \times \sqrt{2x^2 - 7x - 5} + 2x^2 - 7x - 5$

$\Rightarrow 10x + 50 = 2(x + 5)\sqrt{2x^2 - 7x - 5}$

$\Rightarrow x = -5$ or $\sqrt{2x^2 - 7x - 5} = 5$

$\Rightarrow \quad x = -5$ or $2x^2 - 7x - 30 = 0$

$\Rightarrow \quad x = -5$ or $x = 6$ or $x = -5/2$

But $x = -5$ does not satisfy the original equation.

13. We have,

$$x = \sqrt{7+4\sqrt{3}}$$

$$\Rightarrow \quad \frac{1}{x} = \frac{1}{\sqrt{7+4\sqrt{3}}}$$

$$= \frac{\sqrt{7-4\sqrt{3}}}{\sqrt{7+4\sqrt{3}}\cdot\sqrt{7-4\sqrt{3}}}$$

$$= \sqrt{7-4\sqrt{3}}$$

$$\Rightarrow \quad x + \frac{1}{x} = \sqrt{7+4\sqrt{3}} + \sqrt{7-4\sqrt{3}}$$

$$= (\sqrt{3}+2) + (2-\sqrt{3})$$

$$= 4$$

14. Let $L = \sqrt{5x^2 - 6x + 8}$ and $M = \sqrt{5x^2 - 6x - 7}$. Hence,

$L - M = 1$ and $L^2 - M^2 = 15$

$\Rightarrow \quad L + M = 15$

Adding,

$2L = 16$

$\Rightarrow \quad L^2 = 64$

$\Rightarrow \quad 5x^2 - 6x + 8 = 64$

$\Rightarrow \quad 5x^2 - 6x - 56 = 0$

$\Rightarrow \quad (x-4)(5x+14) = 0$

$\Rightarrow \quad x = 4, -14/5$

15. We have,

$$\sqrt{x^2+4x-21} + \sqrt{x^2-x-6} = \sqrt{6x^2-5x-39}$$

$$\Rightarrow \quad \sqrt{(x+7)(x-3)} + \sqrt{(x-3)(x+2)} = \sqrt{(x-3)(6x+13)}$$

$$\Rightarrow \quad \sqrt{x-3}\,(\sqrt{x-7} + \sqrt{x+2} - \sqrt{6x+13}) = 0$$

$$\Rightarrow \quad \sqrt{x-3} = 0 \text{ or } \sqrt{x+7} + \sqrt{x+2} - \sqrt{6x+13} = 0$$

$$\Rightarrow \quad x = 3 \text{ or } \sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$$

Now,

$$\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$$

$$\Rightarrow \quad (\sqrt{x+7} + \sqrt{x+2})^2 = 6x+13$$

$$\Rightarrow \quad x+7+x+2+2\sqrt{(x+7)(x+2)} = 6x+13$$

$$\Rightarrow \quad 2x+9+2\sqrt{(x+7)(x+2)} = 6x+13$$

$$\Rightarrow \quad 2\sqrt{(x+7)(x+2)} = 4x+4$$

$$\Rightarrow \quad \sqrt{(x+7)(x+2)} = 2(x+1)$$

$\Rightarrow \quad (x+7)(x+2) = 4(x+1)^2$ (squaring both sides)

$\Rightarrow \quad x^2 + 9x + 14 = 4(x^2 + 2x + 1)$

$\Rightarrow \quad 3x^2 - x - 10 = 0$

$\Rightarrow \quad (x-2)(3x+5) = 0$

$$\Rightarrow \quad x = 2 \text{ or } x = \frac{-5}{3}$$

But $x = -5/3$ does not satisfy the given equation. Hence, the roots of the given equation are 2 and 3.

16. Given equation is $3x^4 + 6x^3 + x^2 + 6x + 3 = 0$

Dividing equation both sides by x^2

we have $3x^2 + 6x + 1 + \dfrac{6}{x} + \dfrac{3}{x^2} = 0$

$$\Rightarrow \quad 3\left(x^2 + \frac{1}{x^2}\right) + 6\left(x + \frac{1}{x}\right) + 1 = 0$$

$$\Rightarrow \quad 3\left(x + \frac{1}{x}\right)^2 + 6\left(x + \frac{1}{x}\right) - 5 = 0$$

$$\Rightarrow \quad x + \frac{1}{x} = \frac{-6 \pm \sqrt{96}}{6} = \frac{-3 \pm \sqrt{24}}{3}$$

$$\Rightarrow \quad x + \frac{1}{x} = \frac{-3 - \sqrt{24}}{3}\left(\because x + \frac{1}{x} \le -2 \text{ or } x + \frac{1}{x} \ge 2\right)$$

Hence equation has two real roots

17. Let $f(x) = x^3 - 3x + a$

Let $f'(x) = 0$

$\Rightarrow \quad 3x^2 - 3 = 0 \quad \Rightarrow \quad x = \pm 1$

For three distinct roots, $f(1) \cdot f(-1) < 0$

$\Rightarrow \quad (1 - 3 + a)(-1 + 3 + a) < 0$

$\Rightarrow \quad (a+2)(a-2) < 0$

$\Rightarrow \quad -2 < a < 2$

18. Let $f(x) = (x-1)^3 + (x-2)^3 + (x-3)^3 + (x-4)^3 + (x-5)^3$

$\Rightarrow \quad f'(x) = 3(x-1)^2 + 3(x-2)^2 + 3(x-3)^2 + 3(x-4)^2 + 3(x-5)^2$

Now for $f'(x) = 0$ we have $(x-1)^2 + (x-2)^2 + (x-3)^2 + (x-4)^2 + (x-5)^2 = 0$, which has no real roots.

Also coefficient of x^3 is 5, hence when $x \to \infty$, $f(x) \to \infty$

and when $x \to -\infty$, $f(x) \to -\infty$

Hence graph of $y = f(x)$ meets x – axis only once, hence equation $f(x) = 0$ has one real root.

19. $f(x) = x^3 + 2x^2 + 3x + 4$

$\Rightarrow \quad f'(x) = 3x^2 + 4x + 3$

Now $f'(x) = 3x^2 + 4x + 3 = 0$ has non – real roots.

Hence graph has no turning point.

Also when $x \to \infty$, $f(x) \to \infty$ and when $x \to -\infty$, $f(x) \to -\infty$

Hence graph of $y = f(x)$ meets x – axis only once.

Exercise 1.3

1. Let $f(x) = 2x^3 + 3px^2 - 4x + p$. Given,

$f(-2) = 2(-2)^3 + 3(-2)^2p - 4(-2) + p = 5$

$\Rightarrow \quad 13p - 8 = 5$

$\Rightarrow \quad p = 1$

2. Let $f(x) = (x+1)^7 + (2x+k)^3$. Now

$f(-2) = (-2+1)^7 + (-4+k)^3 = 0^3$

$\Rightarrow \quad (k-4)^3 - 1 = 0$

$\Rightarrow \quad k = 5$

3. Putting $x = -1$, we get

$-6p + 24 = 0$

$\therefore\ p = 4$

$x^4 + x^3 - 7x^2 - x + 6$

$= (x + 1)(x^3 - 7x + 6)$

$= (x + 1)(x - 1)(x^2 + x - 6)$

$= (x + 1)(x - 1)(x + 3)(x - 2)$

The other factors are $x - 1, x - 2, x + 3$.

4. $x^2 + ax + 1$ must divide $ax^3 + bx + c$. Now,

$$\frac{ax^3 + bx + c}{x^2 + ax + 1} = a(x - a) + \frac{(b - a + a^3)x + c + a^2}{x^2 + ax + 1}$$

The remainder must be zero. Hence,

$b - a + a^3 = 0,\ a^2 + c = 0$

5. $f(x)$ is divisible by $x - 1$

$\Rightarrow\ f(1) = 0$

$\Rightarrow\ 1 - 3 + 2 + a = 0$

$\Rightarrow\ a = 0$

$\Rightarrow\ f(x) = x^3 - 3x^2 + 2x$

Now remainder when $f(x)$ is divided by $x - 2$ is $f(2)$

$\Rightarrow\ f(2) = (2)^3 - 3(2)^2 + 2(2) = 0$

6. $f(x)$ is divisible by $x^2 - x$ or $x(x - 1)$

$\Rightarrow\ f(0) = 0$ or $b = 0$

Also $f(1) = 0$ or $1 - 1 + a + b = 0$ or $a = 0$

$\Rightarrow\ f(x) = x^3 - x^2$

$\Rightarrow\ f(2) = (2)^3 - (2)^2 = 4$

Exercise 1.4

1. Roots of the equation $x^2 - 8x + a^2 - 6a = 0$ are real. Therefore,

$D \geq 0$

$\Rightarrow\ 64 - 4(a^2 - 6a) \geq 0$

$\Rightarrow\ a^2 - 6a - 16 \leq 0$

$\Rightarrow\ a \in [-2, 8]$

2. The equations $ax^2 + 2bx + c = 0$ and $bx^2 - 2\sqrt{ac}\ x + b = 0$ have real roots. Therefore,

$b^2 \geq ac$ and $b^2 \leq ac$

$\Rightarrow\ b^2 = ac$

3. The discriminant of the given equation is

$D = 4(a + b - 2c)^2 - 4(a - b)^2$

$= 4(a - c + b - c)^2 - 4(a - c + c - b)^2$

$= 4[(a - c) + (b - c)]^2 - 4[(a - c) - (b - c)]^2$

$= 16(a - c)(b - c) < 0 \quad [\because a < c < b]$

Hence, the roots of the given equation are complex.

4. For the given equation $4ax^2 + 3bx + 2c = 0$, we have

$D = (3b)^2 - 4(4a)(2c)$

$= 9b^2 - 32ac$

$= 9(-a - c)^2 - 32ac$

$= 9a^2 - 14ac + 9c^2$

$$= 9c^2\left(\left(\frac{a}{c}\right)^2 - \frac{14}{9}\frac{a}{c} + 1\right)$$

$$= 9\left(\left(\frac{a}{c} - \frac{7}{9}\right)^2 - \frac{49}{81} + 1\right)$$

which is always positive. Hence, the roots are real and distinct.

5. For real root,

$(\lambda - 1)^2 - 64 \geq 0$ and $64 - 4(\lambda + 4) \geq 0$

$\Rightarrow\ (\lambda - 1)^2 \geq 64$ and $48 - 4\lambda \geq 0$

$\Rightarrow\ \lambda - 1 \geq 8$ or $\lambda - 1 \leq -8$ and $12 \geq \lambda$

$\Rightarrow\ \lambda \geq 9$ or $\lambda \leq -7$ and $\lambda \leq 12$

Hence, the greatest value of λ is 12.

Exercise 1.5

1. It is given that

$\alpha\beta = 2$

$\Rightarrow\ \dfrac{3a + 4}{a + 1} = 2$

$\Rightarrow\ 3a + 4 = 2a + 2$

$\Rightarrow\ a = -2$

Also,

$\alpha + \beta = -\dfrac{2a + 3}{a + 1}$

Putting the value of a, we get sum of roots,

$\alpha + \beta = -\dfrac{2a + 3}{a + 1} = -\dfrac{-4 + 3}{-2 + 1} = -1$

2. Let α, β be the roots of the given equation. Then,

$\alpha + \beta = a - 2$ and $\alpha\beta = -(a + 1)$ Now,

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (a - 2)^2 + 2(a + 1)$

$= a^2 - 2a + 6$

$= (a - 1)^2 + 5$

Clearly, $\alpha^2 + \beta^2 \geq 5$. So, the minimum value of $\alpha^2 + \beta^2$ is 5 which it attains at $a = 1$.

3. $x_1 + x_2 = (1 - \sin\theta),\ x_1x_2 = -\dfrac{1}{2}\cos^2\theta$

Now,

$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2$

$= (1 - \sin\theta)^2 + \cos^2\theta$

$= 2 - 2\sin\theta$

$\Rightarrow\ x_1^2 + x_2^2|_{max} = 2 + 2 = 4$

4. $\tan\theta + \sec\theta = -\dfrac{b}{a}$ (1)

$\tan\theta\sec\theta = \dfrac{c}{a}$ (2)

Now,

$\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow\ \sec\theta - \tan\theta = -\dfrac{a}{b}$

$\Rightarrow\ \sec\theta = -\dfrac{(a^2 + b^2)}{2ab}$ and $\tan\theta = \dfrac{(a^2 - b^2)}{2ab}$

Substituting these values in Eq. (2), we have

$\dfrac{(a^2 + b^2)(b^2 - a^2)}{4a^2b^2} = \dfrac{c}{a}$

$\Rightarrow\ b^4 - a^4 = 4acb^2$

$\Rightarrow \quad a^4 = b^2(b^2 - 4ac)$

5. Let α, $\alpha + 1$ be the roots of $x^2 - bx + c = 0$, where $\alpha \in Z$.

$\therefore \quad \alpha + (\alpha + 1) = b \qquad (1)$

$\alpha(\alpha + 1) = c \qquad (2)$

Form (1),

$$\alpha = \frac{b-1}{2}$$

Putting in Eq. (2),

$$\left(\frac{b-1}{2}\right)\left(\frac{b-1}{2}+1\right) = c$$

$\Rightarrow \quad \dfrac{b^2-1}{4} = c$

$\Rightarrow \quad b^2 - 1 = 4c$

$\Rightarrow \quad b^2 - 4c = 1$

6. Let the roots be α, β. Then,

$$\frac{\alpha}{\beta} = \frac{2}{3} \Rightarrow \alpha = \frac{2\beta}{3}$$

Now,

$$\alpha + \beta = \frac{m}{12}$$

$\Rightarrow \quad \dfrac{2\beta}{3} + \beta = \dfrac{m}{12}$

$\Rightarrow \quad \dfrac{5\beta}{3} = \dfrac{m}{12} \qquad (1)$

and $\quad \alpha\beta = \dfrac{5}{12}$

$\Rightarrow \quad \dfrac{2\beta}{3}\beta = \dfrac{5}{12}$

$\Rightarrow \quad \beta^2 = \dfrac{5}{8}$

$\Rightarrow \quad \beta = \sqrt{5/8}$

Putting the value of β in (1),

$$\frac{5}{3}\sqrt{\frac{5}{8}} = \frac{m}{12} \Rightarrow m = 5\sqrt{10}$$

7. Since γ, δ are roots of $x^2 + qx + 1 = 0$, we have

$\gamma^2 + q\gamma + 1 = 0 \Rightarrow \gamma^2 + 1 = -q\gamma \qquad (1)$

and

$\delta^2 + 1 = -q\delta \qquad (2)$

Now,

$(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta) = \{\alpha\beta - \gamma(\alpha + \beta) + \gamma^2\}\{(\alpha\beta + \delta(\alpha + \beta) + \delta^2\}$

$= (1 + p\gamma + \gamma^2)(1 - p\delta + \delta^2)$

$= (p\gamma - q\gamma)(-p\delta - q\delta) \quad$ [from (1) and (2)]

$= -\gamma\delta(p - q)(p + q) = q^2 - p^2$

8. Let α, β be the roots of the given equation $ax^2 + bx + c = 0$.

Then,

$$\alpha + \beta = -\frac{b}{a};\ \alpha\beta = \frac{c}{a}$$

Now roots of equation $2x^2 + 8x + 2 = 0$ are $\alpha - 1, \beta - 1$. Their sum is

$$\alpha + \beta - 2 = -\frac{b}{a} - 2 = -\frac{8}{2} = -4$$

$\Rightarrow \quad \dfrac{b}{a} = 2$

Their product is

$(\alpha - 1)(\beta - 1) = \alpha\beta - (\alpha + \beta) + 1$

$= \dfrac{c}{a} + \dfrac{b}{a} + 1 = 1 \quad (\because$ New equation is $2x^2 + 8x + 2 = 0)$

$\Rightarrow \quad c + b = 0$

$\Rightarrow \quad b = -c$

$\Rightarrow \quad \dfrac{a}{1} = \dfrac{b}{2} = \dfrac{c}{-1}$

9. Let α, β be the roots of the equation $x^2 - a(x - 1) + b = 0$. Then,

$\alpha^2 - a\alpha + a + b = 0$ and $\beta^2 - a\beta + a + b = 0$

$\therefore \quad \alpha^2 - a\alpha = \beta^2 - a\beta = -a - b$

$\Rightarrow \quad \dfrac{1}{\alpha^2 - a\alpha} + \dfrac{1}{\beta^2 - a\beta} + \dfrac{2}{a+b}$

$$= \frac{1}{-(a+b)} + \frac{1}{-(a+b)} + \frac{2}{a+b} = 0$$

10. Let α, β be the roots of $375x^2 - 25x - 2 = 0$. Then

$$\alpha + \beta = \frac{25}{375} = \frac{1}{15} \text{ and } \alpha\beta = -\frac{2}{375}$$

$\therefore \quad \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} S_r = \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} (\alpha^r + \beta^r)$

$= (\alpha + \alpha^2 + \alpha^3 + \cdots \infty) + (\beta + \beta^2 + \beta^3 + \cdots \infty)$

$= \dfrac{\alpha}{1-\alpha} + \dfrac{\beta}{1-\beta}$

$= \dfrac{\alpha - \alpha\beta + \beta - \alpha\beta}{(1-\alpha)(1-\beta)}$

$= \dfrac{\alpha + \beta - 2\alpha\beta}{1 - (\alpha + \beta) + \alpha\beta}$

$= \dfrac{\frac{1}{15} + \frac{4}{375}}{1 - \frac{1}{15} - \frac{2}{375}}$

$= \dfrac{25 + 4}{375 - 25 - 2} = \dfrac{29}{348} = \dfrac{1}{12}$

11. $(\alpha + \beta)^2 = (a + b)^2$

and

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$= (a + b)^2 - 4\left(\dfrac{a^2 + b^2}{2}\right)$

$= 2ab - (a^2 + b^2)$

$= -(a - b)^2$

Now, the required equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$ is

$x^2 - \{(\alpha + \beta)^2 + (\alpha - \beta)^2\}x + (\alpha + \beta)^2(\alpha - \beta)^2 = 0$

$\Rightarrow \quad x^2 - \{(a + b)^2 - (a - b)^2\}x - (a + b)^2(a - b)^2 = 0$

$\Rightarrow \quad x^2 - 4abx - (a^2 - b^2)^2 = 0$

12. Let the roots be α and β. Then,

$\alpha+\beta=2$ and $\alpha^3+\beta^3=98$

Now,

$\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

$\Rightarrow$ $98=2[(\alpha+\beta)^2-3\alpha\beta]$

$\Rightarrow$ $49=(4-3\alpha\beta)$

$\Rightarrow$ $\alpha\beta=-15$

Thus, the equation is $x^2-2x-15=0$.

13. Since α,β are roots of $x^2+bx+1=0$, therefore $\alpha+\beta=-b$, $\alpha\beta=1$. We have,

$$\left(-\alpha-\frac{1}{\beta}\right)+\left(-\beta-\frac{1}{\alpha}\right)=-(\alpha+\beta)-\left(\frac{1}{\beta}+\frac{1}{\alpha}\right)$$

$=b+b=2b$

and

$$\left(-\alpha-\frac{1}{\beta}\right)\left(-\beta-\frac{1}{\alpha}\right)=\alpha\beta+2+\frac{1}{\alpha\beta}=1+2+1=4$$

Thus, the equation whose roots are $-\alpha-1/\beta$ and $-\beta-1/\alpha$ is $x^2-x(2b)+4=0$.

Exercise 1.6

1. Subtracting the given equations, we get

$(a-b)x+c(b-a)=0 \Rightarrow x=c$ is the common root

Thus, roots of $x^2+ax+bc=0$ are b and c and that of $x^2+bx+ca=0$ are c and a. Also, $a+b=-c$. Thus, the required equation is

$x^2-(a+b)x+ab=0$

$\Rightarrow$ $x^2+cx+ab=0$

2. $y=mx$ is a factor of $ax^2+bxy+cy^2$

$\Rightarrow$ $ax^2+bxy+cy^2$ will be zero when $y-mx=0$ or $y=mx$

$\Rightarrow$ $ax^2+bx\cdot mx+cm^2x^2=0$

or $cm^2+bm+a=0$ (1)

Since $my-x$ is a factor of $a_1x^2+b_1xy+c_1y^2$, so

$a_1x^2+b_1xy+c_1y^2=0$ when $my-x=0$

$\Rightarrow$ $a_1m^2y^2+b_1.my.y+c_1y^2=0$ [putting $x=my$]

$\Rightarrow$ $a_1m^2+b_1m+c_1=0$ (2)

Eliminating m from (1) and (2), we get

$(bc_1-ab_1)(cb_1-ba_1)=(aa_1-cc_1)^2$

3. Given equations are

$x^2+2x+9=0$ (1)

and

$ax^2+bx+c=0$ (2)

Clearly, roots of Eq. (1) are imaginary. Since Eq. (1) and (2) have a common root, therefore common root must be imaginary and hence both the roots will be common. Therefore, Eqs. (1) and (2) are identical.

$$\therefore\quad \frac{a}{1}=\frac{b}{2}=\frac{c}{9}\Rightarrow a:b:c=1:2:9$$

4. Let α be a common root of the given two equations. Then,

$2a\alpha^3+b\alpha^2+c\alpha+d=0$ (1)

$2a\alpha^2+3b\alpha+4c=0$ (2)

Multiplying (2) with α and then subtracting (1) from it, we get

$2b\alpha^2+3c\alpha-d=0$ (3)

Now, Eqs. (2) and (3) are quadratic having a common root α, so

$(-2ad-8bc)^2=(-3bd-12c^2)(6ac-6b^2)$

$$\Rightarrow\quad (ad+4bc)^2=\frac{9}{2}(bd+4c^2)(b^2-ac)$$

5. Let, $f(x)=a_1(x-\alpha)(x-\beta)$

$g(x)=a_2(x-\beta)(x-\gamma)$

$h(x)=a_3(x-\gamma)(x-\alpha)$

where a_1, a_2, a_3 are positive.

Let, $f(x)+g(x)+h(x)=F(x)$

$\Rightarrow$ $F(\alpha)=a_2(\alpha-\beta)(\alpha-\gamma)$

$F(\beta)=a_3(\beta-\gamma)(\beta-\alpha)$

$F(\gamma)=a_1(\gamma-\alpha)(\gamma-\beta)$

$\Rightarrow$ $F(\alpha)F(\beta)F(\gamma)=-a_1a_2a_3(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$, which is negative

Therefore, the roots of $F(x)=0$ are real and distinct.

Exercise 1.7

1. Let α,β,γ be the roots of $ax^3+bx^2+cx+d=0$. Then,

$$\alpha+\beta+\gamma=-\frac{b}{a}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$$

$$\alpha\beta\gamma=\frac{-d}{a}$$

$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$$=\frac{b^2}{a^2}-\frac{2c}{a}$$

$$=\frac{b^2-2ac}{a^2}$$

$\Rightarrow$ $\alpha^2+\beta^2+\gamma^2<0$

which is not possible if all α,β,γ are real. So at least one root is non-real, but complex roots occur in pair. Hence, given cubic equation has two non-real roots and one real root.

2. Let the roots be $x_1, -x_1, x_2$. Then,

$x_1-x_1+x_2=a$

$\Rightarrow$ $x_2=a$

Hence, $x=a$ is a root of the given equation.

$\therefore$ $a^3-a^3+ab-c=0$

$\Rightarrow$ $ab=c$

3. Let $y=x^2$. Then $x=\sqrt{y}$

$\therefore$ $x^3+8=0\Rightarrow y^{3/2}+8=0$

$\Rightarrow$ $y^3=64\Rightarrow y^3-64=0$

Thus, the equation having roots α^2,β^2 and γ^2 is $x^3-64=0$.

4. Let,

$$y=\frac{\alpha}{1+\alpha}\Rightarrow\alpha=\frac{y}{1-y}$$

Since α is a root of the equation $x^3-px+q=0$, so

$\alpha^3-p\alpha+q=0$

$$\Rightarrow\quad \frac{y^3}{(1-y)^3}-p\frac{y}{1-y}+q=0$$

$\Rightarrow$ $y^3-py(1-y)^2+q(1-y)^3=0$

$\Rightarrow$ $(1-p-q)y^3+(2p+3q)y^2-(p+3q)y+q=0$

Therefore, the required cubic equation is

$(p+q-1)x^3-(2p+3q)x^2+(p+3q)x-q=0$

5. $\dfrac{\alpha_1\alpha_2+\alpha_2\alpha_3}{\alpha_1\alpha_2\alpha_3}=\dfrac{-\alpha_1\alpha_3}{\alpha_1\alpha_2\alpha_3}=-\dfrac{1}{\alpha_2}$

Therefore, the required equation has roots $-1/\alpha_1, -1/\alpha_2, -1/\alpha_3$.

$\therefore\quad y=-\dfrac{1}{x}$ or $x=-\dfrac{1}{y}$

Hence, the required equation is

$$\left(-\frac{1}{y}\right)^3+a\left(-\frac{1}{y}\right)^2+b=0$$

$\Rightarrow\quad by^3+ay-1=0$

$\Rightarrow\quad bx^3+ax-1=0$

Exercise 1.8

1. $f(x)=x^2-x-3=\left(x-\dfrac{1}{2}\right)^2-\dfrac{1}{4}-3$

$=\left(x-\dfrac{1}{2}\right)^2-\dfrac{13}{4}$

Now,

$\left(x-\dfrac{1}{2}\right)^2\geq 0,\ \forall\ x\in R$

$\Rightarrow\quad \left(x-\dfrac{1}{2}\right)^2-\dfrac{13}{4}\geq\dfrac{-13}{4},\ \forall\ x\in R$

Hence, the range is $[-13/4,\infty)$.

2. (i). Let

$\dfrac{x^2+34x-71}{x^2+2x-7}=y$

$\Rightarrow\quad x^2(1-y)+2(17-y)x+(7y-71)=0$

For real value of x,

$b^2-4ac\geq 0$

$\Rightarrow\quad y^2-14y+45\geq 0$

$\Rightarrow\quad y\leq 5$ or $y\geq 9$

Hence, the range is $(-\infty,5]\cup[9,\infty)$.

(ii). Let,

$y=\dfrac{x^2-x+1}{x^2+x+1}$

$\Rightarrow\quad (1-y)x^2-(1+y)x+1-y=0$

Now, if x is real, then

$D\geq 0$

$\Rightarrow\quad (1+y)^2-4(1-y)^2\geq 0$

$\Rightarrow\quad (1+y-2+2y)(1+y+2-2y)\geq 0$

$\Rightarrow\quad (3y-1)(3-y)\geq 0$

$\Rightarrow\quad 3\left(y-\dfrac{1}{3}\right)(y-3)\leq 0$

$\Rightarrow\quad \dfrac{1}{3}\leq y\leq 3$

Hence, the range is $[1/3,3]$.

3. Let

$y=\sqrt{x-1}+\sqrt{5-x}$

$\Rightarrow\quad y^2=x-1+5-x+2\sqrt{(x-1)(5-x)}$

$\Rightarrow\quad y^2=4+2\sqrt{-x^2-5+6x}$

$\Rightarrow\quad y^2=4+2\sqrt{4-(x-3)^2}$

Then, y^2 has minimum value 4 [when $4-(x-3)^2=0$] and maximum value 8 when $x=3$.

$\therefore\quad y\in[2,2\sqrt{2}]$

4. $f(x)=6^x+3^x+6^{-x}+3^{-x}+2$

$=(\sqrt{6^x}-\sqrt{6^{-x}})^2+(\sqrt{3^x}-\sqrt{3^{-x}})^2+6\geq 6$

Hence, the range is $[6,\infty)$.

5. $x^2-4x+6=(x-2)^2+2$, which is always positive.

Hence, domain is R. Now,

$f(x)=\sqrt{(x-2)^2+2}$

The least value of $f(x)$ is $\sqrt{2}$ when $x-2=0$. Hence, the range is $[\sqrt{2},\infty]$.

Exercise 1.9

1. $f(x)=\sqrt{x^2+ax+4}$ is defined for all x. Hence,

$x^2+ax+4\geq 0$ for all x

$\Rightarrow\quad D=a^2-16\leq 0$

$\Rightarrow\quad a\in[-4,4]$

2. Let $f(x)=ax^2+bx+c$. Since $f(x)$ has no real zeroes either $f(x)>0$ or $f(x)<0$ for all $x\in R$. Since $f(0)=c<0$, we get $f(x)<0$, $\forall\ x\in R$. Therefore, $a<0$ as the parabola $y=f(x)$ opens downwards. Also,

$f(i)=a+b+c<0$

3. $ax^2+bx+c=0$ has imaginary roots. Hence,

$ax^2+bx+c<0,\ \forall\ x\in R$, if $a<0$

or $ax^2+bx+c>0,\ \forall\ x\in R$, if $a>0$

But, given

$a+c<b$

or

$a-b+c<0$ or $f(-1)<0$

$\Rightarrow\quad f(x)=ax^2+bx+c<0\ \forall\ x\in R$

$\Rightarrow\quad f(-2)=4a-2b+c<0$

$\Rightarrow\quad 4a+c<2b$

4. $x,y,z\in R$

$x+y+z=6$ and $xy+yz+zx=7$

$\Rightarrow\quad y(6-y-z)+yz+z(6-y-z)=7$

$\Rightarrow\quad -y^2+(6-z+z-z)y+z(6-z)-7=0$

$\Rightarrow\quad y^2+(z-6)y+7+z(z-6)=0$

Now, y is real. Therefore,

$(z-6)^2-4[7+z(z-6)]\geq 0$

$\Rightarrow\quad 3z^2-12z-8\leq 0$

$\Rightarrow\quad \dfrac{12-\sqrt{144+96}}{6}\leq z\leq\dfrac{12+\sqrt{144+96}}{6}$

$$\Rightarrow \frac{6-2\sqrt{15}}{3} \le z \le \frac{6+2\sqrt{15}}{3}$$

From symmetry, x and y have same range.

5. Let,

$$y = \frac{x^2+2x+c}{x^2+4x+3c}$$

$\Rightarrow$ $(y-1)x^2 + (4y-2)x + 3cy - c = 0$

Now, x is real. Hence,

$D = (4y-2)^2 - 4(y-1)(3cy-c) \ge 0 \ \forall\ y \in R$

$\Rightarrow$ $(2y-1)^2 - (y-1)(3cy-c) \ge 0, \forall\ y \in R$

$\Rightarrow$ $(4-3c)y^2 + (-4+c+3c)y + 1 - c \ge 0, \forall\ y \in R$

$\Rightarrow$ $4 - 3c > 0$ and $(4c-4)^2 - 4(4-3c)(1-c) \le 0$

$\Rightarrow$ $c < \frac{4}{3}$ and $4(c-1)^2 - (4-3c)(1-c) \le 0$

$\Rightarrow c < \frac{4}{3}$ and $(c-1) \times (4c-4+4-3c) \le 0$

$\Rightarrow c < \frac{4}{3}$ and $(c-1)(c) \le 0$

$\Rightarrow$ $c < \frac{4}{3}$ and $0 \le c \le 1$

$\Rightarrow$ $0 \le c \le 1$

6. Let,

$$y = \frac{(x-a)(x-c)}{(x-b)}\ (x \ne b)$$

$\Rightarrow$ $x^2 + (-a-c-y)x + ac + by = 0$

Now, x is real. Hence,

$D = (a+c+y)^2 - 4(ac+by) \ge 0, \forall\ y \in R$

$\Rightarrow$ $y^2 + 2(a+c)y + (a+c)^2 - 4ac - 4by \ge 0, \forall\ y \in R$

$\Rightarrow$ $y^2 + 2(a+c-2b)y + (a-c)^2 \ge 0, \forall\ y \in R$

$\Rightarrow$ $D \le 0$

$\Rightarrow$ $4(a+c-2b)^2 - 4(a-c)^2 \le 0$

$\Rightarrow$ $(a+c-2b+a-c)(a+c-2b-a+c) \le 0$

$\Rightarrow$ $(a-b)(c-b) \le 0$

$\Rightarrow$ $a-b<0$ and $c-b>0$ or $a-b>0$ and $c-b<0$

$\Rightarrow$ $a<b<c$ or $c<b<a$

7. Let $y = \frac{x^2-x}{1-ax}$

$\Rightarrow$ $x^2 - x = y - axy$

$\Rightarrow$ $x^2 + x(ay-1) - y = 0$

Since x is real, so

$(ay-1)^2 + 4y \ge 0$

$\Rightarrow$ $a^2y^2 + 2y(2-a) + 1 \ge 0, \forall\ y \in R$

$\Rightarrow$ $a^2 > 0, 4(2-a)^2 - 4a^2 \le 0 \Rightarrow 4 - 4a \le 0 \Rightarrow a \in [1, \infty)$

8. For the equation $ax^2 + bx + 6 = 0$, roots are not real. Hence

$$D < 0 \Rightarrow b^2 - 24a < 0 \Rightarrow a > \frac{b^2}{24}$$

Also,

$f(-1) > 0 \Rightarrow a - b + 6 > 0$

$\Rightarrow$ $b < a + 6$

$$\Rightarrow a > \max\left\{\frac{b^2}{24}, b-6\right\}$$

9. Given, roots of $ax^2 + bx + c = 0$ are imaginary. Hence,

$b^2 - 4ac < 0$ (1)

Let us consider $f(x) = a^2x^2 + abx + ac$. Here, coefficient of $f(x)$ is a^2 which is +ve, which makes graph concave upward. Also,

$D = (ab)^2 - 4a^2(ac) = a^2(b^2 - 4ac) < 0$

Hence,

$f(x) > 0, \forall\ x \in R$

Exercise 1.10

1. Let, $f(x) = x^2 - 2(a-1)x + (2a+1)$. Then, $f(x) = 0$ will have both roots positive, if

(i) $D > 0$

$\Rightarrow$ $4(a-1)^2 - 4(2a+1) \ge 0$

$\Rightarrow$ $a^2 - 4a \ge 0$

$\Rightarrow$ $a \le 0$ or $a \ge 4$ (1)

(ii) sum of the roots > 0

$\Rightarrow$ $2(a-1) > 0 \Rightarrow a > 1$ (2)

(iii) product of roots > 0

$\Rightarrow$ $(2a+1) > 0 \Rightarrow a > -\frac{1}{2}$ (3)

From (i), (ii) and (iii), we get $a \ge 4$.

2. Roots of $(a-5)x^2 + 2(a-10)x + a + 10 = 0$ are opposite in sign. Then, product of roots is negative. That is, $\frac{a+10}{a-5} < 0$

$\Rightarrow$ $-10 < a < 5$ (using sign scheme method)

3. Both the roots of $x^2 - ax + a = 0$ are greater than 2. Compare this equation with $Ax^2 + Bx + C = 0$. Then, the required conditions are

(i) $D = a^2 - 4a \ge 0 \Rightarrow a \in (-\infty, 0) \cup [4, \infty)$

(ii) $Af(2) > 0 \Rightarrow a \in (-\infty, 4)$

(iii) $-\frac{B}{2A} > 2 \Rightarrow \frac{a}{2} > 2 \Rightarrow a > 4$

Hence, no such a can be obtained.

4. Both roots of $ax^2 + ax + 1 = 0$ are less than 1. Compare this equation with $Ax^2 + Bx + C = 0$. Then, the required conditions are

(i) $D = a^2 - 4a \ge 0 \Rightarrow a \in (-\infty, 0] \cup [4, \infty)$ (1)

(ii) $-\frac{B}{2A} > 1 \Rightarrow -\frac{a}{2a} < 1$ (which is always true as $a \ne 0$)

(iii) $Af(1) > 0$

$\Rightarrow$ $a(2a+1) > 0$

$\Rightarrow$ $a \in \left(-\infty, -\frac{1}{2}\right) \cup (0, \infty)$ (2)

From (1) and (2), $a \in (-\infty, -1/2) \cup [4, \infty)$.

5. Both roots of $x^2 + ax + 2 = 0$ lies in the interval (0, 3). Compare this equation with $Ax^2 + Bx + C = 0$. Then, the required conditions are

(i) $D = a^2 - 8 \ge 0 \Rightarrow a \in (-\infty, -2\sqrt{2}\,] \cup [\,2\sqrt{2}, \infty)$ (1)

(ii) $Af(0) > 0$ and $Af(3) > 0$

$\Rightarrow$ $2 > 0, 9 + 3a + 2 > 0$

$\Rightarrow$ $a \in \left(-\frac{11}{3}, \infty\right)$ (2)

(iii) $0 < -\frac{B}{2A} < 3$

$\Rightarrow \quad 0 < -\frac{a}{2} < 3$

$\Rightarrow \quad -6 < a < 0$ (3)

From (1), (2) and (3), $a \in (-11/3, -2\sqrt{2}$.

6. According to question, 1 lies between the roots, therefore,
$f(1) < 0$

$\Rightarrow \quad 1 - 3 + a < 0$

$\Rightarrow \quad a < 2$

7. Let $f(x) = x^2 - bx - 1$ $(b \in R^+)$

$f(-1) = b = +\text{ve}$

$f(0) = -1 = -\text{ve}$

$f(1) = -b = -\text{ve}$

Clearly, one root lies in $(-1, 0)$ and the other in $(1, \infty)$. So, α (having the least absolute value) $\in (-1, 0)$.

8. $a < b < c < d$

$\mu(x - a)(x - c) + \lambda(x - b)(x - d) = 0$

Let the corresponding expression be

$f(x) = \mu(x - a)(x - c) + \lambda(x - b)(x - d)$

$f(a) = \lambda(a - b)(a - d)$

$f(c) = \lambda(c - b)(c - d)$

$\Rightarrow \quad f(a)f(c) = \lambda^2(a - b)(a - d)(c - b)(c - d) < 0$

$\Rightarrow \quad f(a)$ and $f(c)$ are of opposite signs

Hence, root of the equation lies between a and c. Therefore, the roots are real for all real μ and λ.

Chapter 2

Exercise 2.1

1. The said computation is not correct, because -2 and -3 both are negative and is $\sqrt{ab} = \sqrt{a}\sqrt{b}$ true when at least one of a and b is positive or zero. The correct computation is

$$(\sqrt{-2})(\sqrt{-3}) = (i\sqrt{2})(i\sqrt{3}) = i^2\sqrt{6} = -\sqrt{6}$$

2. a. $\dfrac{i^{584}(i^8 + i^6 + i^4 + i^2 + 1)}{i^{574}(i^8 + i^6 + i^4 + i^2 + 1)} - 1 = \dfrac{i^{584}}{i^{574}} - 1$

$= i^{10} - 1 = -1 - 1 = -2$

b. $(1 + i)^6 + (1 - i)^6 = [(1 + i)^2]^3 + [(1 - i)^2]^3$

$= (2i)^3 + (-2i)^3$

$= (8 - 8)i^3 = 0$

3. We have,

$x = -5 + 4i$

$\Rightarrow \quad (x + 5)^2 = -16$

$\Rightarrow \quad x^2 + 10x + 41 = 0$ (i)

Now,

$x^4 + 9x^3 + 35x^2 - x + 4$

$= x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$

$= 0x^2 - 0x + 4 \times 0 - 160 = -160$ [Using (i)]

4. Let $z = i^{[1+3+5+\cdots+(2n+1)]}$. Now, $1 + 3 + 5 + \cdots + (2n + 1)$ are $n + 1$ terms of an A.P. whose sum is

$$\frac{n+1}{2}[1 + 2n + 1] = (n + 1)^2$$

Hence, $z = i^{(n+1)^2}$. Now put $n = 1, 2, 3, 4, 5, \ldots$.

$n = 1, z = i^4 = 1, n = 2, z = i^9 = i, n = 3, z = i^{16} = 1, n = 4,$
$z = i^{25} = i, n = 5, z = i^{12} = 1, \ldots$

Thus,

$$z = \begin{cases} 1, \text{ if } n \text{ is odd} \\ i, \text{ if } n \text{ is even} \end{cases}$$

5. $z^2 - az + a - 1 = 0$ (1)

Putting $z = 1 + i$ in the equation, we get

$a = 2 + i$

$\Rightarrow \quad z^2 - (2 + i)z + 1 + i = 0$ is the equation

$\Rightarrow \quad z = 1$ is the other root

6. $E = \left(\dfrac{1-i}{1+i}\right)^{n-2}(1 - i)^2 = \left(-\dfrac{2i}{2}\right)^{n-2}(-2i) = 2(-i)^{n-1}$

$= 2[(-i)^2]^{(n-1)/2} = 2(-1)^{(n-1)/2}$

Since E is to be real and positive therefore

$\dfrac{n-1}{2} = 2\lambda$

$\therefore \quad n = 4\lambda + 1$

i.e. odd of this type (but not any odd).

7. $(x + iy)(p + iq) = (x^2 + y^2)i$

$\Rightarrow \quad (xp - yq) + i(xq + yp) = (x^2 + y^2)i$

$\Rightarrow \quad xp = yp = 0$ and $xq + yp = x^2 + y^2$

$\Rightarrow \quad \dfrac{x}{q} = \dfrac{y}{p}$ and $xq + yp = x^2 + y^2$

Let $x/q = y/p = \lambda$, then $x = \lambda q$, $y = \lambda p$.

$\therefore \quad xq + yp = x^2 + y^2 \Rightarrow 1 = \lambda^2 \Rightarrow \lambda = 1$

$\therefore \quad x = q, y = p$

8. $\dfrac{(\sqrt{5+12i} + \sqrt{5-12i})(\sqrt{5+12i} + \sqrt{5-12i})}{(\sqrt{5+12i} - \sqrt{5-12i})(\sqrt{5+12i} + \sqrt{5-12i})}$

$= \dfrac{5 + 12i + 5 - 12i + 2\sqrt{5+12i}\sqrt{5-12i}}{5 + 12i - 5 + 12i}$

$= \dfrac{10 + 2 \times 13}{24i}$

$= -\dfrac{3}{2}i$

9. Let $\sqrt{9 + 40i} = x + iy$. Then

$(x + iy)^2 = 9 + 40i$

$\Rightarrow \quad x^2 - y^2 = 9$ (1)

and

$xy = 20$ (2)

Squaring (1) and adding with 4 times the square of (2), we get

$x^4 + y^4 - 2x^2y^2 + 4x^2y^2 = 81 + 1600$

$\Rightarrow \quad (x^2 + y^2)^2 = 1681$

$\Rightarrow \quad x^2 + y^2 = 41$ (3)

From (1) + (3), we get

$x^2 = 25 \Rightarrow x = \pm 5$

and

$y = \pm 4$

From Eq. (2), we can see that x and y are of same sign.

$\therefore \quad x + iy = (5 + 4i)$ or $-(5 + 4i)$

Exercise 2.2

1. a. $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \dfrac{(6+6)+i(-4+9)}{(2+2)+i(4-1)}$

$= \dfrac{12+5i}{4+3i}$

$= \dfrac{12+5i}{4+3i}\,\dfrac{4-3i}{4-3i}$

$= \dfrac{(48+15)+i(-36+20)}{16-9i^2}$

$= \dfrac{63}{25} - \dfrac{16}{25}i$

b. $\dfrac{2-\sqrt{-25}}{1-\sqrt{-16}} = \dfrac{2-5i}{1-4i}$

$= \dfrac{2-5i}{1-4i} \times \dfrac{1+4i}{1+4i}$

$= \dfrac{(2+20)+i(8-5)}{1-16i^2}$

$= \dfrac{22+3i}{17} = \dfrac{22}{17} + \dfrac{3}{17}i$

2. $z_1 = \bar{z}_2$

$\Rightarrow \quad 9y^2 - 4 - 10ix = 8y^2 + 20i$

$\Rightarrow \quad (y^2 - 4) - 10i(x+2) = 0$

Since complex number is zero, so

$y^2 - 4 = 0$ and $x + 2 = 0$

$\Rightarrow \quad x = -2$ and $y = \pm 2$

Thus,

$z = x + iy = -2 \pm 2i$

3. $\left(\dfrac{2i}{1+i}\right)^2 = \left(\dfrac{2i(1-i)}{(1+i)(1-i)}\right)^n$

$= \left(\dfrac{2(i-i^2)}{2}\right)^n$

$= (i+1)^n$

$= (2i)^{n/2}$

Hence $n = 8$ is the least positive integer for which the given complex number is a positive integer.

4. $e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta} = e^{\cos\theta}\,[e^{i\sin\theta}]$

$= e^{\cos\theta}\,[\cos(\sin\theta) + i\sin(\sin\theta)]$

Therefore, the real part is $e^{\cos\theta}\,[\cos(\sin\theta)]$.

5. $z = i^i$

$= \left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)^i$

$= (e^{i\pi/2})^i$

$= e^{-\frac{\pi}{2}}$

$\Rightarrow \quad \text{Re}(z) = e^{-\frac{\pi}{2}}$

6. $z^2 + |z| = 0$

$\Rightarrow \quad x^2 - y^2 + i(2xy) + \sqrt{x^2+y^2} = 0$

$\Rightarrow \quad x^2 - y^2 + \sqrt{x^2+y^2} = 0 \qquad (1)$

and

$2xy = 0 \qquad (2)$

From (2), let $x = 0$. From (1),

$-y^2 + \sqrt{y^2} = 0$

$\Rightarrow \quad -|y|^2 + |y| = 0$

$\Rightarrow \quad |y| = 0$ or 1

$\Rightarrow \quad y = 0$ or $y = \pm 1$

From (2), if $y = 0$, then from (1),

$x^2 + \sqrt{x^2} = 0$

$\Rightarrow \quad |x|^2 + |x| = 0$

$\Rightarrow \quad x = 0$

Hence, complex numbers are $0 + i0$, $0 + i$, $0 - i$.

Exercise 2.3

1. $|z_1| = \left|\dfrac{1-i}{1+i\sqrt{3}}\right|^{\frac{1}{6}} = \left|\dfrac{\sqrt{2}}{2}\right|^{\frac{1}{6}} = 2^{-\frac{1}{12}}$

Similarly,

$|z_2| = 2^{-\frac{1}{12}};\ |z_3| = 2^{-\frac{1}{12}}$

Hence, the result.

2. $(z^3 + 3)^2 = 16i^2$

$z^3 + 3 = 4i$ or $-4i$

$z^3 = -3 + 4i$ or $-3 - 4i$

$|z^3| = |-3 + 4i| = 5$

$|z|^3 = 5$

$\Rightarrow \quad |z| = 5^{1/3}$

3. Given,

$\dfrac{1-ix}{1+ix} = a - ib$

Taking modulus of both sides

$\left|\dfrac{1-ix}{1+ix}\right| = |a - ib|$

$\Rightarrow \quad \dfrac{|1-ix|}{|1+ix|} = |a-ib|$

$\Rightarrow \quad \dfrac{\sqrt{1+x^2}}{\sqrt{1+x^2}} = \sqrt{a^2+b^2}$

$\Rightarrow \quad a^2+b^2=1$

4. We have,

$|1-i|^x = 2^x$

$\Rightarrow \quad (\sqrt{2})^x = 2^x$

$\Rightarrow \quad 2^{x/2} = 2^x$

$\Rightarrow \quad 2^{x/2} = 1$

$\Rightarrow \quad 2^{x/2} = 2^0 \quad \Rightarrow \dfrac{x}{2} = 0 \quad \Rightarrow x = 0$

Hence, the given equation has no solution.

5. $\dfrac{1+z+z^2}{1-z+z^2} = \dfrac{1-z+z^2+2z}{1-z+z^2}$

$= 1 + \dfrac{2}{z-1(1/z)}$

Hence,

$z+\dfrac{1}{z} \in R$

$\Rightarrow \quad z+\dfrac{1}{z} = \bar{z}+\dfrac{1}{\bar{z}}$

$\Rightarrow \quad (z-\bar{z}) = \dfrac{1}{\bar{z}} - \dfrac{1}{z} = \dfrac{z-\bar{z}}{|z|^2}$

$\Rightarrow \quad (z-\bar{z})(|z|^2-1) = 0$

$\Rightarrow \quad |z| = 1$ ($z=\bar{z}$ is not possible as z is not real)

6. $|a| = |b| = |c| = r$

Again

$az^2+bz=-c$

$\Rightarrow \quad |c| = |-az^2-bz|$

$\leq |a||z^2| + |b||z|$

$\Rightarrow \quad r \leq r|z|^2 + r|z|$

$\Rightarrow \quad |z|^2+|z|-1 \geq 0$

$\Rightarrow \quad |z| \geq \dfrac{\sqrt{5}-1}{2} \qquad (1)$

Also from $az^2=-bz-c$,

$|z|^2-|z|-1 \leq 0$

$\Rightarrow \quad 0 < |z| \leq \dfrac{\sqrt{5}+1}{2} \qquad (2)$

From (1) and (2),

$\dfrac{\sqrt{5}-1}{2} \leq |z| \leq \dfrac{\sqrt{5}+1}{2}$

7. $\dfrac{a}{|z_1-z_2|} = \dfrac{b}{|z_2-z_3|} = \dfrac{c}{|z_3-z_1|} = \lambda$ (say)

$\Rightarrow \quad a^2 = \lambda^2(z_1-z_2)(\bar{z}_1-\bar{z}_2)$

$b^2 = \lambda^2(z_2-z_3)(\bar{z}_2-\bar{z}_3)$

$c^2 = \lambda^2(z_3-z_1)(\bar{z}_3-\bar{z}_1)$

$\Rightarrow \quad \dfrac{a^2}{z_1-z_2} + \dfrac{b^2}{z_2-z_3} + \dfrac{c^2}{z_3-z_1}$

$= \lambda^2(\bar{z}_1-\bar{z}_2+\bar{z}_2-\bar{z}_3+\bar{z}_3-\bar{z}_1) = 0$

8. Given

$|z_1+z_2|^2 = |z_1|^2+|z_2|^2$

$\Rightarrow \quad |z_1|^2+|z_2|^2+z_1\bar{z}_2+\bar{z}_1z_2 = |z_1|^2+|z_2|^2$

$\Rightarrow \quad z_1\bar{z}_2+\bar{z}_1z_2 = 0$

$\Rightarrow \quad \dfrac{z_1}{z_2}+\dfrac{\bar{z}_1}{\bar{z}_2} = 0$

$\Rightarrow \quad \dfrac{z_1}{z_2}+\overline{\left(\dfrac{z_1}{z_2}\right)} = 0$

$\Rightarrow \quad \dfrac{z_1}{z_2}$ is purely imaginary

9. $z_1^2+z_2^2+2z_1z_2\cos\theta = 0$

$\Rightarrow \quad \left(\dfrac{z_1}{z_2}\right)^2 + 2\left(\dfrac{z_1}{z_2}\right)\cos\theta + 1 = 0$

$\Rightarrow \quad \left(\dfrac{z_1}{z_2}+\cos\theta\right)^2 = -(1-\cos^2\theta) = -\sin^2\theta$

$\Rightarrow \quad \dfrac{z_1}{z_2} = -\cos\theta \pm i\sin\theta$

$\Rightarrow \quad \left|\dfrac{z_1}{z_2}\right| = \sqrt{(-\cos\theta)^2+\sin^2\theta} = 1$

$\Rightarrow \quad |z_1| = |z_2|$

$\Rightarrow \quad |z_1-0| = |z_2-0|$

Thus, triangle with vertices O, z_1, z_2 is iscosceles.

10. As given, let

$\dfrac{2z_1}{3z_2} = iy$

or

$\dfrac{z_1}{z_2} = \dfrac{3}{2}iy$

so that

$\left|\dfrac{z_1-z_2}{z_1+z_2}\right| = \left|\dfrac{\dfrac{z_1}{z_2}-1}{\dfrac{z_1}{z_2}+1}\right| = \left|\dfrac{\dfrac{3}{2}iy-1}{\dfrac{3}{2}iy+1}\right| = \left|\dfrac{1-\dfrac{3}{2}iy}{1+\dfrac{3}{2}iy}\right| = 1 \quad [\because |z| = |\bar{z}|]$

11. $|iz+3-4i| \leq |iz|+|3-4i|$

$= |z|+5$

$\leq 4+5 = 9$

Hence, $|z|_{\max} = 9$.

12. We have,

$\sqrt{3}+i = (a+ib)(c+id)$

$\Rightarrow \quad ac - bd = \sqrt{3}$ and $ad + bc = 1$

Now,

$$\tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{d}{c}\right) = \tan^{-1}\left(\frac{\frac{b}{a}+\frac{d}{c}}{1-\frac{b}{a}\frac{d}{c}}\right)$$

$$= \tan^{-1}\left(\frac{bc+ad}{ac-bd}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= n\pi + \frac{\pi}{6} \quad n \in Z$$

13. We have,

$$\arg(z) + \arg(\overline{z}) = \arg(z\overline{z}) = \arg(|z|^2) = 0$$

[$\because |z|^2$ is a positive real number]

14.

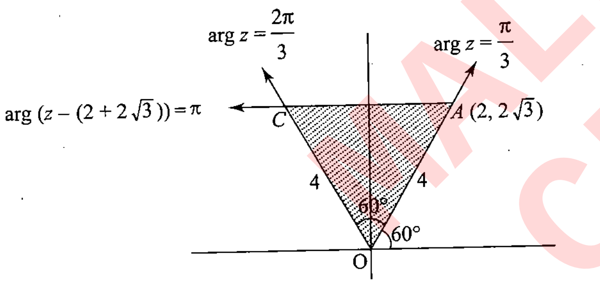


Fig. A-2.1

Note that $(2, 2\sqrt{3})$ lies on the line $y = \sqrt{3}x$ and OAC will be an equilateral triangle of side 4.

$$\text{Area} = \frac{\sqrt{3}}{4} \times 16 = 4\sqrt{3}$$

15. $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is valid only when at least one of a and b is non-negative, but given that $a < 0$, so b must be positive. Hence $z = b + ai$ lies in fourth quadrant for which

$$\arg(b + ai) = -\tan^{-1}\left|\frac{a}{b}\right|.$$

16. Here,

$$\left|\frac{z_2 - z_0}{z_0 - z_1}\right| = 1$$

and

$$\text{amp}\left(\frac{z_2 - z_0}{z_0 - z_1}\right) = \frac{\pi}{2}$$

$$\therefore \quad \frac{z_2 - z_0}{z_0 - z_1} = 1\left\{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right\} = i$$

$\Rightarrow \quad z_2 - z_0 = iz_0 - iz_1$

$\Rightarrow \quad z_2 + iz_1 = (i + 1)z_0$

$$\Rightarrow \quad z_0 = \frac{z_2 + iz_1}{1+i}$$

$$= \frac{(z_2 + iz_1)(1-i)}{1^2 + 1^2}$$

$$= \frac{1}{2}\{(i - i^2)z_1 + (1-i)z_2\}$$

17. $z\overline{z} = 1 \Rightarrow z = \frac{1}{\overline{z}}$

Now,

$$z = (z_1 + z_2 + z_3 + \cdots + z_n)\left(\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n}\right)$$

$$= (z_1 + z_2 + \cdots + z_n)(\overline{z}_1 + \overline{z}_2 + \cdots + \overline{z}_3)$$

$$= (z_1 + z_2 + \cdots + z_n)\,\overline{(z_1 + z_2 + \cdots + z_n)}$$

$= |z_1 + z_2 + \cdots + z_n|^2$ which is real (a)

$\leq |z_1| + |z_2| + \cdots + |z_n| = n^2$

$\therefore \quad 0 < z \leq n^2$

Also, z is real number.

Exercise 2.4

1. a.
$$\frac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^5} = \frac{\cos 4\alpha + i\sin 4\alpha}{i^5(\cos\beta - i\sin\beta)^5}$$

$= -i(\cos 4\alpha + i\sin 4\alpha)(\cos\beta - i\sin\beta)^{-5}$

$= -i[\cos 4\alpha + i\sin 4\alpha][\cos 5\beta + i\sin 5\beta]$

$= -i[\cos(4\alpha + 5\beta) + i\sin(4\alpha + 5\beta)]$

$= \sin(4\alpha + 5\beta) - i\cos(4\alpha + 5\beta)$

b.
$$\left(\frac{1+\cos\phi + i\sin\phi}{1+\cos\phi - i\sin\phi}\right)^n = \left[\frac{2\cos^2(\phi/2) + i\sin(\phi/2)\cos(\phi/2)}{2\cos^2(\phi/2) - 2i\sin(\phi/2)\cos(\phi/2)}\right]^n$$

$$= \left[\frac{\cos(\phi/2) + i\sin(\phi/2)}{\cos(\phi/2) - i\sin(\phi/2)}\right]^n$$

$$= \left[\frac{(\cos\phi + i\sin\phi)^{1/2}}{(\cos\phi + i\sin\phi)^{-1/2}}\right]^n$$

$$= [\cos\phi + i\sin\phi]^n$$

$$= \cos n\phi + i\sin n\phi$$

c.
$$\frac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{(\cos\gamma + i\sin\gamma)(\cos\delta + i\sin\delta)}$$

$$= \cos(\alpha + \beta - \gamma - \delta) + i\sin(\alpha + \beta - \gamma - \delta)$$

2. $x + \frac{1}{x} = 2\cos\theta$

$\Rightarrow \quad x^2 - 2x\cos\theta + 1 = 0 \Rightarrow x = \cos\theta \pm i\sin\theta$

$\Rightarrow \quad x^n = \cos n\theta \pm i\sin n\theta$

Also,

$$\frac{1}{x} = \cos\theta \mp i\sin\theta \Rightarrow \frac{1}{x^n} = \cos n\theta \mp i\sin n\theta$$

Thus,

$$x^n + \frac{1}{x^n} = 2\cos n\theta$$

3. Let,

$$\cos\frac{\pi}{10} - i\sin\frac{\pi}{10} = z$$

$$\Rightarrow \quad \cos\frac{\pi}{10} + i\sin\frac{\pi}{10} = \frac{1}{z}$$

$$\Rightarrow \left[\frac{1-\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}}{1-\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}}\right]^{10} = \left(\frac{1-z}{1-\frac{1}{z}}\right)^{10}$$

$$= \left\{\frac{-(z-1)z}{(z-1)}\right\}^{10}$$

$$= (-z)^{10}$$

$$= z^{10}$$

$$= \left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{10}$$

$$= \cos\pi - i\sin\pi$$

$$= -1$$

4. $iz^4 = -1$

$$z^4 = \frac{-1}{i} \Rightarrow z^4 = i \Rightarrow z = (i)^{1/4}$$

$$\Rightarrow z = (0+i)^{1/4}$$

$$\Rightarrow z = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/4}$$

$$\Rightarrow z = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$$

5. $(1+i)^n + (1-i)^n$

$$= (2)^{n/2}\left\{\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right\}$$

$$= 2^{\frac{n}{2}}\, 2\cos\frac{n\pi}{4} = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4} = (\sqrt{2})^{n+2}\cos\frac{n\pi}{4}$$

6. Let,

$z_1 = \cos\alpha + i\sin\alpha,\ z_2 = \cos\beta + i\sin\beta,\ z_3 = \cos\gamma + i\sin\gamma$

$\therefore\ z_1 + z_2 + z_3 = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i \times 0 = 0$ (1)

a. Now,

$$\frac{1}{z_1} = (\cos\alpha + i\sin\alpha)^{-1} = \cos\alpha - i\sin\alpha$$

$$\frac{1}{z_2} = \cos\beta - i\sin\beta$$

$$\frac{1}{z_3} = \cos\gamma - i\sin\gamma$$

$\therefore\ \dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} = (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$ (2)

$= 0 - i \times 0 = 0$

$$z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$$

$$= 0 - 2z_1z_2z_3\left(\frac{1}{z_3} + \frac{1}{z_1} + \frac{1}{z_2}\right)$$

$= 0 - 2z_1z_2z_3 \times 0 = 0$ [Using (1) and (2)]

$\Rightarrow\ (\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$

$\Rightarrow\ (\cos 2\alpha + i\sin 2\alpha) + (\cos 2\beta + i\sin 2) + (\cos 2\gamma + i\sin 2\gamma)$
$= 0 + i \times 0$

Equating real and imaginary parts on both sides,

$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$ and $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

b. $z_1^3 + z_2^3 + z_3^3 = (z_1 + z_2)^3 - 3z_1z_2(z_1 + z_2) + z_3^3$

$= (-z_3)^3 - 3z_1z_2(-z_3) + z_3^3$ [Using (1)]

$= 3z_1z_2z_3$

$\Rightarrow\ (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$
$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$\Rightarrow\ \cos 3\alpha + i\sin 3\alpha + \cos 3\beta + i\sin 3\beta + \cos 3\gamma + i\sin 3\gamma$
$= 3\{\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)\}$

Equating imaginary parts on both sides,

$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

Exercise 2.5

1. Complex cube roots of unity are 1, ω, ω^2. Let $\alpha = \omega$, $\beta = \omega^2$. Then

$\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = \omega^4 + (\omega^2)^4 + (\omega^{-1})(\omega^2)^{-1}$

$= \omega + \omega^2 + 1 = 0$

2. $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\cdots$ up to $2n$ factors

$= (-\omega^2)(-\omega)(1+\omega)(1+\omega^2)\cdots$ up to $2n$ factors

$= 1 \times 1 \times 1 \times$ L up to n factors $= 1$

3.

a. Here, $-1/2 + (1/2)\,i\sqrt{3}$ is one of the two imaginary cube roots of unity. If we denote it by ω, then

$\omega^{1000} = \omega^{999}\,\omega = (\omega^3)^{333}\,\omega$

$$= \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

b. $$z = \frac{(\sqrt{3}+i)^{17}}{(1-i)^{50}}$$

$$= \frac{1}{i^{17}}\frac{(i\sqrt{3}+i^2)^{17}}{[(1-i)^2]^{25}}$$

$$= \frac{2^{17}}{i}\frac{\left(\frac{i\sqrt{3}-1}{2}\right)^{17}}{(-2i)^{25}}$$

$$= \frac{1}{2^8}(\omega)^{17}$$

$$= \frac{1}{2^8}(\omega)^2$$

$$= \frac{1}{2^8}\frac{-1-i\sqrt{3}}{2}$$

c. $(i+\sqrt{3})^{100} + (i-\sqrt{3})^{100} + 2^{100}$

$$= \left(\frac{i^2+i\sqrt{3}}{i}\right)^{100} + \left(\frac{i^2-i\sqrt{3}}{i}\right)^{100} + 2^{100}$$

$$= \frac{2^{100}}{i^{100}}\left(\frac{-1+i\sqrt{3}}{2}\right)^{100} + \frac{2^{100}}{i^{100}}\left(\frac{-1-i\sqrt{3}}{2}\right)^{100} + 2^{100}$$

$$= 2^{100}(\omega)^{100} + 2^{100}(\omega^2)^{100} + 2^{100}$$

$$= 2^{100}(\omega^{100} + \omega^{200} + 1)$$

$$= 2^{100}(\omega + \omega^2 + 1) = 0$$

4. $z + z^{-1} = 1$

$\Rightarrow\ z^2 - z + 1 = 0$

$\Rightarrow \quad z = -\omega$ or $-\omega^2$

For $z = -\omega$,

$$z^{100} + z^{-100} = (-\omega)^{100} + (-\omega)^{-100}$$
$$= \omega + \frac{1}{\omega} = \omega + \omega^2 = -1$$

For $z = -\omega^2$,

$$z^{100} + z^{-100} = (-\omega^2)^{100} + (-\omega^2)^{-100}$$
$$= \omega^{200} + \frac{1}{\omega^{200}}$$
$$= \omega^2 + \frac{1}{\omega^2} = \omega^2 + \omega = -1$$

5. $x^{12} - 1 = (x^6 + 1)(x^6 - 1) = (x^6 + 1)(x^2 - 1)(x^4 + x^2 + 1)$

Common roots are given by $x^4 + x^2 + 1 = 0$

$$\therefore \quad x^2 = \frac{-1 \pm i\sqrt{3}}{2} = \omega, \omega^2 \text{ or } \omega^4, \omega^2 \quad (\because \omega^3 = 1)$$

or

$$x = \pm\omega^2, \pm\omega$$

6. Operating $R_1 \to R_1 + R_3$, we get

$$\Delta = \begin{vmatrix} 1-i & \omega^2+\omega & \omega^2-1 \\ 1-i & -1 & \omega^2-1 \\ -i & -1+\omega & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-i & -1 & \omega^2-1 \\ 1-i & -1 & \omega^2-1 \\ -i & -1+\omega & -1 \end{vmatrix} \qquad [\because \omega^2 + \omega = -1]$$

$$= 0 \qquad [\because R_1 \text{ and } R_2 \text{ are identical}]$$

Exercise 2.6

1.

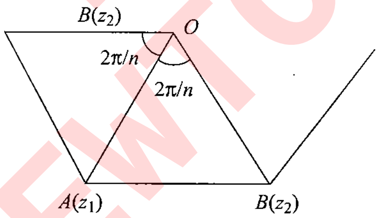


Fig. A-2.2

Let A be the vertex with affix z_1. There are two possibilities of z_2z_2 can be obtained by rotating z_1 through $2\pi/n$ either in clockwise or in anticlockwise direction. Therefore,

$$\frac{z_2}{z_1} = \left|\frac{z_2}{z_1}\right| e^{\pm\frac{i2\pi}{n}}$$

$$\Rightarrow \quad z_2 = z_1 e^{\pm\frac{i2\pi}{n}} \qquad [\because |z_2| = |z_1|]$$

2. We have,

$$\frac{z_1}{z_2} + \frac{z_2}{z_1} = 1$$

$$\Rightarrow \quad z_1^2 + z_2^2 = z_1 z_2$$

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3, \text{ where } z_3 = 0$$

Hence, z_1, z_2 and the origin form an equilateral triangle.

3. Inscribed triangle is clearly equilateral. Let the other vertex be z.

$$\therefore \quad \frac{i - z}{3 - 3i - i} = e^{\pm i\frac{2\pi}{3}}$$

$$\Rightarrow \quad i - z = (3 - 4i)e^{\pm i\frac{2\pi}{3}}$$

$$\Rightarrow \quad z = i - (3 - 4i)\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)$$

4. Let $A(z_a)$, $B(z_b)$ and $C(z_c)$ be the vertices of the triangle.

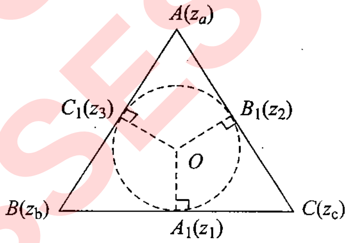


Fig. A-2.3

Considering the rotation about points A_1 and B_1, respectively, we get

$$\frac{z_c - z_1}{0 - z_1} = \frac{|z_c - z_1|}{r} e^{-i\pi/2}$$

and

$$\frac{z_c - z_2}{0 - z_2} = \frac{|z_c - z_2|}{r} e^{i\pi/2}$$

$$\therefore \quad \frac{(z_c - z_1)z_2}{z_1(z_c - z_2)} = e^{-i\pi} = -1$$

$$\Rightarrow \quad (z_c - z_1)z_2 = -z_1(z_c - z_2)$$

$$\Rightarrow \quad z_c = \frac{2z_1z_2}{(z_1 + z_2)}$$

Similarly,

$$z_a = \frac{2z_2z_3}{(z_2 + z_3)} \text{ and } z_b = \frac{2z_1z_3}{(z_1 + z_3)}$$

5.

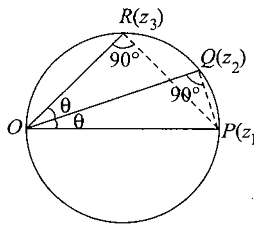


Fig. A-2.4

$$z_2 = \frac{OQ}{OP} z_1 e^{i\theta} = \cos\theta\, z_1 e^{i\theta} \qquad (1)$$

$$z_3 = \frac{OR}{OP} z_1 e^{i2\theta} = \cos 2\theta\, z_1 e^{i2\theta} \qquad (2)$$

From (1),

$$z_2^2 = \cos^2\theta z_1{}^2 e^{2i\theta} \qquad (3)$$

Dividing (iii) by (ii), we get

$$\frac{z_2^2}{z_3} = \frac{\cos^2\theta\, z_1}{\cos 2\theta}$$

Hence,

$$z_2^2 \cos^2\theta = z_1 z_3 \cos^2\theta$$

Exercise 2.7

1. We have,

$$\log_{\sqrt{3}}\left(\frac{|z|^2 - |z| + 1}{2 + |z|}\right) > 2$$

$\Rightarrow \quad \dfrac{|z|^2 - |z| + 1}{2 + |z|} > (\sqrt{3})^2$

$\Rightarrow \quad |z|^2 - 4|z| - 5 > 0$

$\Rightarrow \quad (|z| + 1)(|z| - 5) > 0$

$\Rightarrow \quad |z| - 5 > 0$

$\Rightarrow \quad |z| > 5$

$\Rightarrow \quad z$ lies outside the circle of radius 5 and centre origin

2. Let P, A, B represent complex numbers z, $1 + i0$, $-1 + i0$, respectively. Then,

$$|z - 1| + |z + 1| \le 4$$

$\Rightarrow \quad PA + PB \le 4$

Hence, P moves in such a way that the sum of its distances from two fixed points is always less than or equal to 4. So, Locus of P is the interior and boundary of the ellipse having foci at (1, 0) and (−1, 0) and eccentricity 1/2.

3. $\omega = \dfrac{z}{z - \frac{1}{3}i}$

$\Rightarrow \quad \Rightarrow |\omega| = \left|\dfrac{z}{z - \frac{1}{3}i}\right|$

$\Rightarrow \quad |z| = |\omega|\left|z - \dfrac{1}{3}i\right|$

$\Rightarrow \quad |z| = \left|z - \dfrac{1}{3}i\right| \qquad [\because |\omega| = 1]$

Hence, locus of z is perpendicular bisector of the line joining $0 + 0i$ and $0 + 1/3i$. Hence z lies on a straight line.

4. We have,

$$\log_{1/2}|z - 2| > \log_{1/2}|z|$$

$\Rightarrow \quad |z - 2| < |z|$

Hence, z lies on the right-hand side of the perpendicular bisector of the segment joining (0, 0) and (2, 0). So, Re(z) > 1.

5.

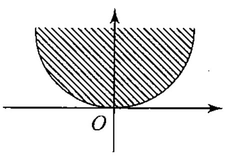


Fig. A-2.5

$$z^2 + \bar{z}^2 + 2z\bar{z} < 8i(\bar{z} - z)$$

$\Rightarrow \quad (z + \bar{z})^2 < 8i(\bar{z} - z)$

$\Rightarrow \quad 4(\text{Re } z)^2 < 16(\text{Im } z)$

$\Rightarrow \quad (\text{Re } z)^2 < 4(\text{Im } z)$

$\Rightarrow \quad x^2 < 4y$ where $z = x + iy$

Shaded part is the required region (i.e., interior of the parabola $x^2 = 4y$).

6.

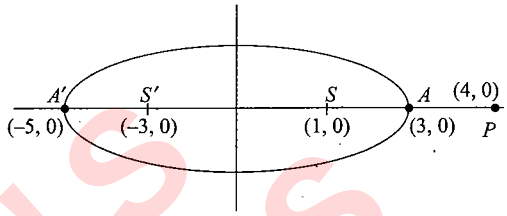


Fig. A-2.6

Given $|z - 1| + |z + 3| \le 8$. Then z lies inside or on the ellipse whose foci are (1, 0) and (−3, 0) and vertices are (−5, 0) and (3, 0). Clearly, the minimum and maximum values of $|z - 4|$ are 1 and 9, respectively, representing the distances PA and PA'. Thus, $1 \le |z - 4| \le 9$.

Exercise 2.8

1. As α is the fifth non-real root of unity, therefore

$$\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$$

β is the fourth non-real root of unity.

$\therefore \quad \beta^3 + \beta^2 + \beta + 1 = 0$

Now,

$$(1 + \alpha)(1 + \alpha^2)(1 + \alpha^4)(1 + \beta)(1 + \beta^2)(1 + \beta^3)$$
$$= (1 + \alpha + \alpha^2 + \alpha^3)(1 + \alpha^4)(1 + \beta + \beta^2 + \beta^3)(1 + \beta^3)$$
$$= 0$$

2. $x^6 = -64$

$\Rightarrow \quad \left(\dfrac{x}{2i}\right)^6 = 1$

$\Rightarrow \quad \dfrac{x}{2i} = \cos\left(\dfrac{2n\pi}{6}\right) + i\sin\left(\dfrac{2n\pi}{6}\right)$, where $n = 0, 1, 2, 3, 4, 5$

$\Rightarrow \quad x = 2i\cos\left(\dfrac{n\pi}{3}\right) - 2\sin\left(\dfrac{n\pi}{3}\right)$, where $n = 0, 1, 2, 3, 4, 5$

For $n = 4$ and 5 we have positive real part. Hence the required product is

$$\left[2i\cos\left(\frac{4\pi}{3}\right) - 2\sin\left(\frac{4\pi}{3}\right)\right]\left[2i\cos\left(\frac{5\pi}{3}\right) - 2\sin\left(\frac{5\pi}{3}\right)\right]$$

$$= 4\left[-i\frac{1}{2} + \frac{\sqrt{3}}{2}\right]\left[i\frac{1}{2} + \frac{\sqrt{3}}{2}\right]$$

$$= 4$$

3. $E = \dfrac{1}{z_1 - 1} + \dfrac{1}{z_2 - 1} + \cdots + \dfrac{1}{z_{50} - 1}$

where $z_1, z_2, \ldots, z_{50}$ are the roots of the equation $z^{51} - 1 = 0$ other than 1. Let,

$$1 + z + z^2 + \cdots + z^{50} = (z - z_1)(z - z_2)\cdots(1 - z_{50})$$

$\Rightarrow \quad \log(1 + z + z^2 + \cdots + z^{50}) = \log[(z - z_1)(z - z_2)\cdots(1 - z_{50})]$

Differentiating both sides w.r.t. z and putting $z = 1$,

$$\frac{1 + 2z + 3z^2 + \cdots + 50z^{49}}{1 + z + z^2 + \cdots + z^{50}} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_{50}}$$

$$\Rightarrow \frac{50\times 51}{2\times 51}=\left[\frac{1}{z_1-1}+\frac{1}{z_2-1}+\cdots+\frac{1}{z_{50}-1}\right]$$

$$\therefore \sum\frac{1}{z_r-1}=-25$$

4. We have already proved that

$$(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{n-1})=n$$

$$\Rightarrow |1-\alpha_1||1-\alpha_2|\cdots|1-\alpha_{n-1}|=n$$

$$\Rightarrow \prod_{r=1}^{n-1}|1-\alpha_r|=n$$

$$\Rightarrow \prod_{r=1}^{n-1}\sqrt{\left(1-\cos\frac{2r\pi}{n}\right)^2+\left(\sin\frac{2\pi r}{n}\right)^2}=n$$

$$\Rightarrow \prod_{r=1}^{n-1}\sqrt{\left(2\sin^2\frac{r\pi}{n}\right)^2+\left(2\sin\frac{\pi r}{n}\cos\frac{\pi r}{n}\right)^2}$$

$$\Rightarrow \prod_{r=1}^{n-1}2\sin\frac{r\pi}{n}\sqrt{\sin^2\frac{r\pi}{n}+\cos^2\frac{r\pi}{n}}=n$$

$$\Rightarrow \prod_{r=1}^{n-1}2\sin\frac{r\pi}{n}=n$$

$$\Rightarrow \prod_{r=1}^{n-1}\sin\left(\frac{r\pi}{n}\right)=\frac{n}{2^{n-1}}$$

5. $z^n=(z+1)^n$

$\Rightarrow |z^n|=|(z+1)^n|$

$\Rightarrow |z|^n=|z+1|^n$

$\Rightarrow |z|=|z+1|$

$\Rightarrow |z-0|=|z-(-1)|$

Hence, distance of z from 0 and -1 is equal. So, z lies on the perpendicular bisector of 0 and -1 or on the line $x=-1/2$. Hence the roots are collinear.

Chapter 3

Exercise 3.1

1. $a+(p-1)d=q$ and $a+(q-1)d=p$

$\Rightarrow (p-q)d=q-p \Rightarrow d=-1$

$T_r=a+(r-1)d=a+(p-1)d+(r-p)d$

$=q-(r-p)=p+q-r$

2. We have,

$$\frac{1}{1-x}-\frac{1}{1+\sqrt{x}}=\frac{1+(1-\sqrt{x})}{1-x}=\frac{\sqrt{x}}{(1-x)}$$

and

$$\frac{1}{1-\sqrt{x}}-\frac{1}{1-x}=\frac{1+\sqrt{x}-1}{1-x}=\frac{\sqrt{x}}{1-x}$$

Therefore the given numbers are in A.P. with common difference $\sqrt{x}/(1-x)$.

3. $5T_5=8T_8$

$\Rightarrow 5(a+4d)=8(a+7d)$

$\Rightarrow 3a+36d=0$

$\Rightarrow a+12d=0$

$\Rightarrow T_{13}=0$

4. $S_n=nP+\dfrac{n(n-1)}{2}Q$

$=\dfrac{n}{2}[2P+(n-1)Q]$

Comparing with

$S_n=\dfrac{n}{2}[2a+(n-1)d]$

$\Rightarrow d=Q$

5. The given sequence is an A.P. in which first term $a=20$ and common difference $d=-3/4$. Let the n^{th} term of the given A.P. be the first negative term. Then,

$a_n<0$

$\Rightarrow a+(n-1)d<0$

$\Rightarrow 20+(n-1)\times(-3/4)<0$

$\Rightarrow \dfrac{83}{4}-\dfrac{3n}{4}<0$

$\Rightarrow 83-3n<0$

$\Rightarrow 3n>83$

$\Rightarrow n>27\dfrac{2}{3}$

$\Rightarrow n=28$

Thus, 28^{th} term of the given sequence is the first negative term.

6. We have,

$(x+1)+(x+4)+\cdots+(x+28)=155$

L.H.S. has 10 terms in A.P. with common difference 3.

$\Rightarrow \dfrac{10}{2}[(x+1)+(x+28)]=155$

$\Rightarrow x=1$

7. Let A and D be the first term and common difference, respectively, of the given A.P. Then,

$a=p^{\text{th}}$ term $\Rightarrow a=A+(p-1)D$ (1)

$b=q^{\text{th}}$ term $\Rightarrow b=A+(q-1)D$ (2)

Subtracting (2) from (1), we get

$$D=\frac{a-b}{p-q}$$

Adding (1) and (2), we get

$a+b=2A+(p+q-2)D$

$\Rightarrow a+b=2A+(p+q-1)D-D$

$\Rightarrow (a+b)+D=2A+(p+q-1)D$

$\Rightarrow (a+b)+\dfrac{a-b}{p-q}=2A+(p+q-1)D$ (3)

Now, sum of $p+q$ terms is

$S_{p+q}=\dfrac{p+q}{2}[2A+(p+q-1)D]$ [Using (3)]

$=\dfrac{p+q}{2}\left[a+b+\dfrac{a-b}{p-q}\right]$ [Using (3)]

8. Let a be the first term and d be the common difference of the given A.P. Then, sum of n terms is

$$S_1 = \frac{n}{2}[2a+(n-1)d] \quad (1)$$

Sum of $2n$ terms,

$$S_2 = \text{Sum of } 2n \text{ terms} = \frac{2n}{2}[2a+(2n-1)d] \quad (2)$$

Sum of $3n$ terms,

$$S_3 = \frac{3n}{2}[2a+(3n-1)d] \quad (3)$$

Now,

$$S_2 - S_1 = \frac{2n}{2}[2a+(2n-1)d] - \frac{n}{2}[2a+(n-1)d]$$

$$= \frac{n}{2}[2\{2a+(2n-1)d\}-\{2a+(n-1)d\}]$$

$$= \frac{n}{2}[2a+(3n-1)d]$$

$$\therefore \quad 3(S_2 - S_1) = \frac{3n}{2}[2a+(3n-1)d] = S_3 \quad \text{[Using (3)]}$$

9. Let a be the first term and d the common difference of the given A.P. Then, the sums of m and n terms are given by

$$S_m = \frac{m}{2}[2a+(m-1)d]$$

and

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

Given,

$$\frac{S_m}{S_n} = \frac{m^2}{n^2}$$

$$\Rightarrow \frac{\frac{m}{2}[2a+(m-1)d]}{\frac{n}{2}[2a+(n-1)d]} = \frac{m^2}{n^2}$$

$$\Rightarrow \frac{2a+(m-1)d}{2a+(n-1)d} = \frac{m}{n}$$

Replacing m by $2m-1$ and n by $2n-1$, we have

$$\frac{2a+(2m-1-1)d}{2a+(2n-1-1)d} = \frac{2m-1}{2n-1}$$

$$\Rightarrow \frac{a+(m-1)d}{a+(n-1)d} = \frac{2m-1}{2n-1}$$

$$\Rightarrow \frac{T_m}{T_n} = \frac{2m-1}{2n-1}$$

10. Series 17, 21, 25, ..., 417 has common difference 4. Series 16, 21, 26, ..., 466 has common difference 5. Hence the series with common terms has common difference equal to the L.C.M. of 4 and 5 which is 20. The first common term is 21. Hence the series is 21, 41, 61, ..., 401 which has 20 terms.

11. $a + 3k = a^2 + (a+k)^2 + (a+2k)^2$ (1)
(where k is the common difference of A.P.)

$\Rightarrow \quad 5k^2 + 3(2a-1)k + 3a^2 - a = 0$ (2)

$\Rightarrow \quad 9(2a-1)^2 - 20(3a^2 - a) \geq 0 \quad (\because k \text{ is real})$

$\Rightarrow \quad 24a^2 + 16a - 9 \leq 0$

$\Rightarrow \quad -\frac{1}{3} - \frac{\sqrt{70}}{12} < a < -\frac{1}{3} + \frac{\sqrt{70}}{12}$

$\Rightarrow \quad a = -1, 0 \quad (\because a \in \text{I})$

From (1), when $a = 0$,

$5k^2 - 3k = 0$

$\Rightarrow \quad k = 0, 3/5$ [Not possible from (i)]

When $a = -1$,

$5k^2 - 9k + 4 = 0$

$\Rightarrow \quad k = 1, \frac{4}{5} \Rightarrow k = 1$ (since k is an integer)

$\therefore \quad a = -1, b = 0, c = 1, d = 2$

$\Rightarrow \quad a + b + c + d = 2$

12. Given,

$S_{2n} = 3S_n$

$$\Rightarrow \frac{2n}{2}[2a+(2n-1)d] = 3\frac{n}{2}[2a+(n-1)d]$$

$\Rightarrow \quad 4a + (4n-2)d = 6a + (3n-3)d$

$\Rightarrow \quad 2a = (n+1)d$ (1)

Now,

$$S_{3n}/S_n = \frac{\frac{3n}{2}[2a+(3n-1)d]}{\frac{n}{2}[2a+(n-1)d]}$$

$$= \frac{3[(n+1)d+(3n-1)d]}{[(n+1)d+(n-1)d]}$$

$$= \frac{3[4nd]}{[2nd]} = 6$$

13. Let the four numbers in an A.P. be $a-3d$, $a-d$, $a+d$, $a+3d$. Sum of the terms is

$4a = 20 \Rightarrow a = 5$

Sum of their squares is

$4a^2 + 20d^2 = 120$

$\Rightarrow \quad 20d^2 = 120 - 4 \times 25 = 20$

$\Rightarrow \quad d^2 = 1$ or $d = \pm 1$

Hence, the numbers are 2, 4, 6, 8 or 8, 6, 4, 2.

14. Here,

$4a = 28 \Rightarrow a = 7$

Also,

$$\frac{(a-3d)(a+d)}{(a-d)(a+3d)} = \frac{8}{15}$$

$\Rightarrow \quad 15[a^2 - 3d^2 - 2ad] = 8[a^2 - 3d^2 + 2ad]$

$\Rightarrow \quad 7(a^2 - 3d^2) = 46ad$

$\Rightarrow \quad 7(49 - 3d^2) = 46 \times 7 \times d$

$\Rightarrow \quad 49 - 3d^2 = 46d$

$\Rightarrow \quad 3d^2 + 46d - 49 = 0$

$\Rightarrow \quad (d-1)(3d+49) = 0$

$\Rightarrow \quad d = 1$

Therefore, the required numbers are 4, 6, 8, 10.

15.

$$\frac{1}{c-a}-\frac{1}{b-c}=\frac{1}{a-b}-\frac{1}{c-a}$$

$$\frac{(a+b-2c)}{b-c}=\frac{(c+b-2a)}{a-b}$$

or

$$(a-b)(a+b-2c)=(b-c)(b+c-2a) \quad (1)$$

Above is true by given condition as shown below by (2). We are given by

$$(c-a)^2-(b-c)^2=(a-b)^2-(c-a)^2$$

$\Rightarrow$ $(c-a+b-c)(c-a-b+c)$

$$=(a-b+c-a)(a-b-c+a)$$

$\Rightarrow$ $(b-a)(2c-a-b)=(c-b)(2a-b-c)$

$\Rightarrow$ $(b-c)(b+c-2a)=(a-b)(a+b-2c)$ (2)

Now according to question $2+38+20n=200$

$\Rightarrow$ $n=8$

$\Rightarrow$ a, b, c are in A.P.

16. Sum of n arithmetic means inserted is $(n/2)(2+38)=20n$.

17. Since a, b, c, d, e, f are six A.M.'s between 2 and 12, therefore,

$$a+b+c+d+e+f=\frac{6}{2}(a+f)=\frac{6}{2}(2+12)=42$$

18. $a, x_1, x_2, \ldots, x_n, b$ are in A.P. Hence,

$b=T_{n+2}=a+(n+1)d$

$\Rightarrow$ $\frac{b-a}{n+1}=d$

$\Rightarrow$ $x_r=T_{r+1}=a+rd$

$$=a+\frac{r(b-a)}{n+1}$$

$$=\frac{a(n-r+1)+rb}{n+1} \quad (1)$$

Put $a=x$ and $b=2y$ and then again put $a=2x$ and $b=y$ and equate the results as the two means are equal. Then

$$\frac{x(n-r+1)+2yr}{n+1}=\frac{2x(n-r+1)+yr}{n+1}$$

$\Rightarrow$ $x(n-r+1)=yr$

$\Rightarrow$ $\frac{x}{y}=\frac{r}{n-r+1}$

Exercise 3.2

1. Given,

$$r=\frac{T_2}{T_1}=\frac{x^n}{x^{-4}}=x^{n+4}$$

and

$$T_8=ar^7=x^{-4}\times(x^{n+4})^7=x^{52}$$

$\Rightarrow$ $7n+24=52$

$\Rightarrow$ $n=4$

2. $S_{2n}=5(t_1+t_3+t_5+\cdots+n)$

$\Rightarrow$ $\frac{a\left(1-r^{2n}\right)}{1-r}=5\times\frac{a\left\{1-\left(r^2\right)^n\right\}}{1-r^2}$

$\Rightarrow$ $1=\frac{5}{1+r}$

$\Rightarrow$ $1+r=5$

$\Rightarrow$ $r=4$

3. $S_n=3-\frac{3^{n+1}}{4^{2n}}$

Putting $n=1, 2$, we get

$$T_1=S_1=3-\frac{9}{16}=\frac{39}{16}$$

$$S_2=3-\frac{27}{256}=T_1+T_2$$

$\Rightarrow$ $T_2=S_2-T_1$

$\Rightarrow$ $T_2=\left(3-\frac{27}{256}\right)-\left(3-\frac{9}{16}\right)$

$\Rightarrow$ $T_2=\frac{117}{256}$

$\Rightarrow$ $r=\frac{T_2}{T_1}=\frac{\frac{117}{256}}{\frac{39}{16}}=\frac{117\times16}{256\times39}=\frac{3}{16}$

4. $S_1=\underbrace{6666\cdots}_{n\text{ digits}}$

$$=6+6\times10^1+6\times10^2+\cdots+6\times10^{n-1}$$

$$=6\times\frac{\left(10^n-1\right)}{10-1}$$

$$=\frac{2}{3}\left(10^n-1\right)$$

Similarly,

$$S_2=\frac{8}{9}\left(10^n-1\right)$$

$\Rightarrow$ $S_1^2+S_2=\frac{4}{9}(10^n-1)^2+\frac{8}{9}\left(10^n-1\right)$

$$=\frac{4}{9}\left(10^n-1\right)[10^n-1+2]$$

$$=\frac{4}{9}[10^{2n}-1]$$

Also,

$$\underbrace{4444\cdots4}_{2n\text{ digits}}=4+4\times10+4\times10^2+\cdots+4\times10^{2n-1}$$

$$=4\frac{\left(10^{2n}-1\right)}{10-1}$$

Hence proved.

5. $(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\cdots n$ terms

$$=\frac{1}{x-y}[(x+y)(x-y)+(x^2+xy+y^2)(x-y)+\cdots n\text{ terms}]$$

$$=\frac{1}{x-y}[(x^2-y^2)+(x^3-y^3)+\cdots n\text{ terms}]$$

$$=\frac{1}{x-y}[(x^2+x^3+\cdots n\text{ terms})-(y^2+y^3+\cdots n\text{ terms})]$$

$$=\frac{1}{x-y}\left[\frac{x^2(x^n-1)}{x-1}-\frac{y^2(y^n-1)}{y-1}\right]$$

6. $\frac{4}{3}+\frac{10}{9}+\frac{28}{27}+\cdots$ up to n terms

$$= \left(1+\frac{1}{3}\right)+\left(1+\frac{1}{9}\right)+\left(1+\frac{1}{27}\right)+\cdots$$

$$= n + \left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots \text{ up to } n \text{ terms}\right)$$

$$= n + \frac{\frac{1}{3}\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}}$$

$$= n + \frac{1}{2}\left(1-\frac{1}{3^n}\right)$$

$$= n + \frac{3^n-1}{2\times 3^n}$$

$$= \frac{2\cdot n\, 3^n + 3^n - 1}{2\times 3^n}$$

$$= \frac{(2n+1)\times 3^n - 1}{2(3^n)}$$

7. Here,

$a = ar + ar^2$

$\Rightarrow \quad r^2 + r - 1 = 0$

$$\Rightarrow \quad r = \frac{-1\pm\sqrt{1+4}}{2}$$

$$= \frac{-1+\sqrt{5}}{2} = \frac{\sqrt{5}-1}{2} \qquad [\because r > 0]$$

8. Let $b = ar$, $c = ar^2$ and $d = ar^3$. Then,

$$\frac{1}{a^3+b^3} = \frac{1}{a^3(1+r^3)}$$

$$\frac{1}{b^3+c^3} = \frac{1}{a^3r^3(1+r^3)}$$

$$\frac{1}{c^3+d^3} = \frac{1}{a^3r^6(1+r^3)}$$

Clearly, $(a^3+b^3)^{-1}$, $(b^3+c^3)^{-1}$ and $(c^3+d^3)^{-1}$ are in G.P. with common ratio $1/r^3$.

9. $$s = \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots \text{ to } \infty$$

$$= \frac{1}{1-r}$$

$$\Rightarrow \quad r = 1 - \frac{1}{s} = \frac{s-1}{s}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} r^{2n} = \frac{1}{1-r^2} = \frac{1}{1-\frac{(s-1)^2}{s^2}} = \frac{s^2}{2s-1}$$

10. Given that

$$T_m = AR^{m-1} = \frac{1}{n^2} \text{ and } T_n = AR^{n-1} = \frac{1}{m^2}$$

$$\Rightarrow \quad A^2R^{m+n-2} = \frac{1}{m^2n^2}$$

$$\Rightarrow \quad AR^{\frac{m+n}{2}-1} = \frac{1}{mn}$$

$$\Rightarrow \quad T_{\frac{m+n}{2}} = \frac{1}{mn}$$

11. $b = ar$, $c = ar^2$

$$\Rightarrow \quad \frac{1}{a^2-b^2}+\frac{1}{b^2} = \frac{1}{a^2-a^2r^2}+\frac{1}{a^2r^2}$$

$$= \frac{a^2}{a^2r^2(a^2-a^2r^2)}$$

$$= \frac{1}{(a^2r^2-a^2r^4)}$$

$$= \frac{1}{b^2-c^2}$$

12. The given product is

$$(32)^{1+\frac{1}{6}+\frac{1}{36}+\cdots\infty} = (32)^s$$

where

$$s = 1 + \frac{1}{6} + \frac{1}{36} + \cdots \infty = \frac{1}{1-\frac{1}{6}} = \frac{6}{5}$$

Hence, given product is $(2^5)^{6/5} = 2^6 = 64$.

13. Let A be the first term and R the common ratio of the given G.P. Then,

$a = p^{\text{th}}$ term

$\Rightarrow \quad a = AR^{p-1}$

$\Rightarrow \quad \log a = \log A + (p-1)\log R \qquad (1)$

Similarly,

$\log b = \log A + (q-1)\log R \qquad (2)$

and

$\log c = \log A + (r-1)\log R \qquad (3)$

Now,

$(q-r)\log a + (r-p)\log b + (p-q)\log c$

$= (q-r)\{\log A + (p-1)\log R\} + (r-p)\{\log A + (q-1)\log R\} + (p-q)\{\log A + (r-1)\log R\}$

[Using (1), (2) and (3)]

$= \log A\{(q-r)+(r-p)+(p-q)\} + \log R\{(p-1)\times(q-r)+(q-1)(r-p)+(r-1)(p-q)\}$

$= \log A\times 0 + \log R\{p(q-r)+q(r-p)+r(p-q) - (q-r)-(r-p)-(p-q)\}$

$= \log A \times 0 + \log R\times 0 = 0$

14. $(a^2+b^2+c^2)p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \le 0$

$\Rightarrow \quad (ap-b)^2 + (bp-c)^2 + (cp-d)^2 \le 0 \qquad (1)$

Since a, b, c, d and p are real, the inequality (1) is possible only when each of the factors is zero, i.e.,

$ap - b = 0,\ bp - c = 0,\ cp - d = 0$

$$\Rightarrow \quad p = \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

$\Rightarrow \quad a, b, c, d$ are in G.P.

15. Let three terms in G.P. be a/r, a, ar. Given that the product is 125.

$\therefore \quad a = 5$

Also,

$$a^2\left(r\times 1+1\times\frac{1}{r}+r\times\frac{1}{r}\right)=\frac{175}{2}$$

$$\Rightarrow \quad 25(r^2+r+1)=\frac{175}{2}r$$

$$\Rightarrow \quad 2(r^2+r+1)=7r$$

$$\Rightarrow \quad 2r^2-5r+2=0$$

$$\Rightarrow \quad (r-2)(2r-1)=0$$

$$\Rightarrow \quad r=2, \frac{1}{2}.$$

Hence, the numbers are 10, 5, 5/2 or 5/2, 5, 10.

16. $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}=\sqrt{ab}=a^{1/2}b^{1/2}$ by given condition

$$\Rightarrow \quad a^{n+1}+b^{n+1}=a^n a^{1/2}b^{1/2}+b^n b^{1/2}a^{1/2}$$

$$=a^{n+1/2}\sqrt{b}+b^{n+1/2}\sqrt{a}$$

$$\Rightarrow \quad a^{n+1/2}(\sqrt{a}-\sqrt{b})=b^{n+1/2}(\sqrt{a}-\sqrt{b})$$

$$\Rightarrow \quad a^{n+1/2}=b^{n+1/2}$$

The above is possible only when

$n+1/2=0$

$$\Rightarrow \quad n=-\frac{1}{2}$$

Exercise 3.3

1. Let the H.P. be

$$\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d},\frac{1}{a+3d},\ldots$$

Then,

$$\frac{1}{a}=\frac{2}{5} \text{ and } \frac{1}{a+d}=\frac{12}{13}$$

$$\Rightarrow \quad a=\frac{5}{2} \text{ and } d=-\frac{17}{12}$$

Now, n^{th} term of the H.P. is

$$\frac{1}{a+(n-1)d}=\frac{12}{47-17n}$$

So, the n^{th} term is largest when $47-17n$ has the positive least value. Clearly, $12/(47-17n)$ is least for $n=2$. Hence, 2^{nd} term is the largest term.

2. a, b, c in G.P.

$\Rightarrow \quad b=ar,\ c=ar^2$

$$\Rightarrow \quad \frac{1}{a-b},\frac{1}{c-a},\frac{1}{b-c} \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{2}{a(r^2-1)}=\frac{1}{a(1-r)}+\frac{1}{ar(1-r)}=\frac{1+r}{ar(1-r)}$$

$\Rightarrow \quad -2r=(r+1)^2$

$\Rightarrow \quad r^2+4r+1=0 \qquad (1)$

Also,

$c+4b+a=a(r^2+4r+1)=0$ [Using (1)]

3. By the given conditions, we have

$2y=x+z \qquad (1)$

$b^2y^2=ax\times cz \qquad (2)$

$b=2ac/(a+c) \qquad (3)$

It is clear from the question that we have to eliminate b and y.

Substituting for y and b from (1) and (3) in (2), we get

$$\frac{4a^2c^2}{(a+c)^2}\times\frac{(x+z)^2}{4}=axcz$$

$$\Rightarrow \quad \frac{(x+z)^2}{xz}=\frac{(a+c)^2}{ac}$$

$$\Rightarrow \quad \frac{x^2+z^2+2xz}{xz}=\frac{a^2+c^2+2ac}{ac}$$

$$\Rightarrow \quad \frac{x}{z}+\frac{z}{x}+2=\frac{a}{c}+\frac{c}{a}+2$$

$$\Rightarrow \quad \frac{x}{z}+\frac{z}{x}=\frac{a}{c}+\frac{c}{a}$$

4. Let $1/a=A$, $1/b=A+D$, $1/c=A+2D$, $1/d=A+3D$. Then,

$a^{-2}-d^{-2}=A^2-(A+3D)^2=-3D(2A+3D)$ and

$b^{-2}-c^{-2}=-D(2A+3D)$

$$\Rightarrow \quad \frac{a^{-2}-d^{-2}}{b^{-2}-c^{-2}}=3$$

5. We have,

$$\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}=\lambda \text{ (say)}$$

$$\Rightarrow \quad p=\frac{a-x}{\lambda x},q=\frac{a-y}{\lambda y},r=\frac{a-z}{\lambda z}$$

Now, p, q, r are in A.P. Therefore,

$$\frac{a-x}{\lambda x},\frac{a-y}{\lambda y},\frac{a-z}{\lambda z} \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{a-x}{x},\frac{a-y}{y},\frac{a-z}{z} \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{a}{x}-1,\frac{a}{y}-1,\frac{a}{z}-1 \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{a}{x},\frac{a}{y},\frac{a}{z} \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{1}{x},\frac{1}{y},\frac{1}{z} \text{ are in A.P.}$$

$\Rightarrow \quad x, y, z$ are in H.P.

6. Here,

$$x=\frac{1}{1-a},y=\frac{1}{1-b},z=\frac{1}{1-c}$$

Since a, b, c are in A.P., so

$1-a$, $1-b$, $1-c$ are in A.P.

$$\Rightarrow \quad \frac{1}{1-a},\frac{1}{1-b},\frac{1}{1-c} \text{ are in H.P.}$$

$\Rightarrow \quad x, y, z$ are in H. P.

7. $\left.\begin{array}{l} x, 1, z \text{ are in A.P.} \Rightarrow 2=x+z \\ x, 2, z \text{ are in G.P.} \Rightarrow 4=xz \end{array}\right] \qquad (1)$

$\therefore \quad 4 = \dfrac{2xz}{x+z}$

Therefore, x, 4, z are in H.P.

8. Here,

$$a + b = a_1 + a_{2n} = a_2 + a_{2n-1} = \cdots = a_n + a_{n+1}$$
$$ab = g_1 \times g_{2n} = g_2 \times g_{2n-1} = \cdots = g_n \times g_{n+1}$$

$\Rightarrow$ and $h = \dfrac{2ab}{a+b}$

$\therefore$ sum of given series

$$= \frac{n(a+b)}{ab} = \frac{2n}{h}$$

9. Let a be the first term and d be the common difference of the given A.P. Then, as given the $(m+1)^{th}$, $(n+1)^{th}$ and $(r+1)^{th}$ terms are in G.P. So,

$a + md, a + nd, a + rd$ are in G.P.

$\Rightarrow \quad (a + nd)^2 = (a + md)(a + rd)$

$\Rightarrow \quad a(2n - m - r) = d(mr - n^2)$

$$\Rightarrow \quad \frac{d}{a} = \frac{2n-(m+r)}{mr-n^2} \qquad (1)$$

Next, m, n, r are in H.P. Hence,

$$n = \frac{2mr}{m+r} \qquad (2)$$

From (1) and (2),

$$\frac{d}{a} = \frac{2n-(m+r)}{mr-n^2} = \frac{2}{n}\left(\frac{2n-(m+r)}{(m+r)-2n}\right) = -\frac{2}{n}$$

10. Let α, β be roots of $ax^2 + bx + c = 0$. Then

$$\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}$$

$$\Rightarrow \quad \alpha + \beta = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha^2\beta^2}$$

$$\Rightarrow \quad \frac{-b}{a} = \frac{(-b/a)^2 - 2c/a}{(c/a)^2}$$

$$\Rightarrow \quad \frac{-b}{a} = \frac{b^2 - 2ac}{c^2}$$

$\Rightarrow \quad -bc^2 = ab^2 - 2a^2c$

$\Rightarrow \quad 2a^2c = ab^2 + bc^2$

$$\Rightarrow \quad \frac{2a}{b} = \frac{b}{c} + \frac{c}{a}$$

$\Rightarrow \quad \dfrac{c}{a}, \dfrac{a}{b}, \dfrac{b}{c}$ are in A.P.

$\Rightarrow \quad \dfrac{a}{c}, \dfrac{b}{c}, \dfrac{c}{b}$ are in H.P.

Exercise 3.4

1. Let A be the A.M. and G be the G.M. of a and b. Then the numbers a and b are roots of the equation $x^2 - 2Ax + G^2 = 0$, i.e.,

$x^2 - 4Gx + G^2 = 0 \qquad [\because A = 2G]$

$$\Rightarrow \quad x = \frac{4G \pm \sqrt{16G^2 - 4G^2}}{2}$$

$$\Rightarrow \quad x = \frac{4G \pm 2\sqrt{3}G}{2} = (2 \pm \sqrt{3})G$$

Hence, $a:b = (2+\sqrt{3}):(2-\sqrt{3})$

2. Let a, b $(a > b)$ be the two numbers. Given,

$$\frac{a+b}{2} = 2 \text{ or } a + b = 4 \qquad (1)$$

and

$\sqrt{(a+1)b} = 2$

$\Rightarrow \quad (a+1)b = 4$

$\Rightarrow \quad b^2 - 5b + 4 = 0 \qquad$ [Using (1)]

$\Rightarrow \quad b = 1, 4$

But $b \neq 4$, therefore, $b = 1$, and hence $a = 3$.

Hence, harmonic mean,

$$H = \frac{2\times3\times1}{3+1} = \frac{3}{2}$$

3. $H = 21/5, 3A + G^2 = 36$

$\Rightarrow \quad 3A + AH = 36$

$\Rightarrow \quad A = 5$

$\Rightarrow \quad a + b = 10$

Also, from $H = 2ab/(a+b) = 21/5$, $ab = 21$, we have

$$a^2 + b^2 = (a+b)^2 - 2ab = 100 - 42 = 58$$

Exercise 3.5

1. $S = \dfrac{3}{2} - \dfrac{5}{6} + \dfrac{7}{18} - \dfrac{9}{54} + \cdots \infty$

$$\Rightarrow \quad S = 3\left(\frac{1}{2}\right) + 5\left(-\frac{1}{6}\right) + 7\left(\frac{1}{18}\right) + 9\left(-\frac{1}{54}\right) + \cdots \qquad (1)$$

$$\Rightarrow \quad -\frac{1}{3}S = 3\left(-\frac{1}{6}\right) + 5\left(\frac{1}{18}\right) + 9\left(-\frac{1}{54}\right) + \cdots \qquad (2)$$

Subtracting (2) from (1), we have

$$\frac{4}{3}S = 3\left(\frac{1}{2}\right) + \left[2\left(-\frac{1}{6}\right) + 2\left(\frac{1}{18}\right) + 2\left(-\frac{1}{54}\right) + \cdots\infty\right]$$

$$= \frac{3}{2} + \frac{2\left(-\frac{1}{6}\right)}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{3}{2} - \frac{\frac{1}{3}}{\frac{4}{3}}$$

$$= \frac{3}{2} - \frac{1}{4}$$

$$= \frac{5}{4}$$

$$\Rightarrow \quad S = \frac{15}{16}$$

2. $S = \frac{1^2}{2} + \frac{3^2}{2^2} + \frac{5^2}{2^3} + \frac{7^2}{2^4} + \cdots \infty$ (1)

$\Rightarrow \quad \frac{1}{2} S = \frac{1^2}{2^2} + \frac{3^2}{2^3} + \frac{5^2}{2^4} + \cdots \infty$ (2)

Subtracting (2) from (1),

$\frac{1}{2} S = \frac{1^2}{2} + \frac{8}{2^2} + \frac{16}{2^3} + \frac{24}{2^4} + \frac{32}{2^5} + \cdots \infty$ (3)

Let

$S_1 = \frac{8}{2^2} + \frac{16}{2^3} + \frac{24}{2^4} + \frac{32}{2^5} + \cdots \infty$ (4)

$\Rightarrow \quad \frac{1}{2} S_1 = \frac{8}{2^3} + \frac{16}{2^4} + \frac{24}{2^5} + \cdots \infty$ (5)

Subtracting (5) from (4),

$$\frac{1}{2} S_1 = \frac{8}{2^2} + \frac{8}{2^3} + \frac{8}{2^4} + \frac{8}{2^5} + \cdots \infty$$

$$= \frac{\frac{8}{2^2}}{1 - \frac{1}{2}}$$

$$= 4$$

$$\Rightarrow \quad S_1 = 8$$

Hence, from (3),

$$\frac{1}{2} S = \frac{1}{2} + 16$$

$$\Rightarrow \quad S = 17$$

3. $S = 3 + (3 + d)\frac{1}{4} + (3 + 2d)\frac{1}{4^2} + \cdots \infty$ (1)

$\Rightarrow \quad \frac{1}{4} S = (3)\frac{1}{4} + (3 + d)\frac{1}{4^2} + \cdots \infty$ (2)

Subtracting (2) from (1), we have

$$\frac{3}{4} S = 3 + (d)\frac{1}{4} + (d)\frac{1}{4^2} + \cdots \infty$$

$$= 3 + \frac{\frac{d}{4}}{1 - \frac{1}{4}}$$

$$= 3 + \frac{d}{3}$$

$$\Rightarrow \quad S = 4 + \frac{4d}{9}$$

Given,

$$4 + \frac{4d}{9} = \frac{44}{9}$$

$$\Rightarrow \quad \frac{4d}{9} = \frac{8}{9}$$

$$\Rightarrow d = 2$$

4. $1^2 + 3^2 + 5^2 + \cdots = \Sigma(2n - 1)^2$

$$= \Sigma\,(4n^2 - 4n + 1)$$

$$= 4\Sigma n^2 - 4\Sigma n + n$$

$$= 4\frac{n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n$$

$$= \frac{n}{3}(4n^2 - 1) = \frac{n(2n-1)(2n+1)}{3}$$

5. We have,

$$\frac{1}{k}(1 + 2 + 3 + \cdots + k) = \frac{1}{k}\frac{k(k+1)}{2} = \frac{k+1}{2}$$

Thus,

$$S = \frac{1}{2}[2 + 3 + 4 + \cdots + 21]$$

$$= \frac{10}{2}(2 + 21) = 115$$

6. $\Sigma n^2 = 330 + \Sigma n$

$$\Rightarrow \frac{n\,(n+1)\,(2n+1)}{6} = 330 + \frac{n\,(n+1)}{2}$$

$$\Rightarrow \frac{n(n+1)}{2}\left[\frac{2n+1}{3} - 1\right] = 330$$

$$\Rightarrow \frac{n(n+1)}{2} \times \frac{(2n-2)}{3} = 330$$

$$\Rightarrow n\,(n+1)\,(n-1) = 990$$

$$\Rightarrow n = 10.$$

7. $11^2 + 12^2 + 13^2 + \cdots + 20^2 = (1^2 + 2^2 + 3^2 + \cdots + 20^2) - (1^2 + 2^2 + 3^2 + \cdots + 10^2)$

$$= \sum_{r=1}^{20} r^2 - \sum_{r=1}^{10} r^2$$

$$= \frac{20(21)(41)}{6} - \frac{10(11)(21)}{6}$$

$$= \frac{210}{6}(82 - 11)$$

$$= 35 \times 71 = 2485$$

8. $T_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$

$$T_n = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n}{6}(2n^2 + 3n + 1)$$

$$= \frac{2n^3 + 3n^2 + n}{6}$$

$$S_n = \sum T_n$$

$$= \frac{1}{6}\left[2\sum n^3 + 3\sum n^2 + \sum n\right]$$

$$= \frac{1}{6}\left\{2\left[\frac{n\,(n+1)}{2}\right]^2 + 3 \times \frac{n\,(n+1)\,(2n+1)}{6} + \frac{n(n+1)}{2}\right\}$$

$$= \frac{1}{6}\left[\frac{n^2\,(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{2} + \frac{n\,(n+1)}{2}\right]$$

$$= \frac{1}{6}\times\frac{1}{2}\,n(n+1)\left[n(n+1)+(2n+1)+1\right]$$

$$= \frac{n(n+1)}{12}\left[n^2+3n+2\right]$$

For $n = 22$

$$S_{22} = \frac{22\times 23}{12}\left[22^2+66+2\right]$$

$$= \frac{22\times 23}{12}[552]$$

$$= 22\times 23\times 46$$

$$= 23276$$

9. $S = 2+5+10+17+26+\cdots+t_n$ (1)

$S = \quad 2+5+10+17+\cdots+t_{n-1}+t_n$ (2)

Subtracting (2) from (1),

$\Rightarrow\ 0 = (2+3+5+7+9+\cdots n \text{ terms}) - t_n$

$\Rightarrow\ t_n = 2+(3+5+7+9+\cdots (n-1) \text{ terms})$

$$= 2+\frac{n-1}{2}[6+(n-2)2]$$

$$= 2+(n-1)(3+n-2)$$

$$= 2+n^2-1$$

$$= n^2+1$$

$$\Rightarrow\ S_n = \sum_{r=1}^{n} t_r$$

$$= \sum_{r=1}^{n}(r^2+1)$$

$$= \frac{n(n+1)(2n+1)}{6}+n$$

$$= \frac{n}{6}(2n^2+3n+7)$$

10. $S = 1+4+13+40+121+\cdots+t_n$ (1)

$S = 1+4+13+40+\cdots+t_{n-1}+t_n$ (2)

Subtracting (2) from (1),

$0 = 1+3+9+27+81+\cdots n \text{ terms} - t_n$

$\Rightarrow\ t_n = 1+3+9+27+81+\cdots n \text{ terms}$

$$= \frac{3^n-1}{2}$$

$$\Rightarrow\ S_n = \sum_{r=1}^{n} t_r = \sum_{r=1}^{n}\left(\frac{3^r-1}{2}\right)$$

$$= \frac{1}{2}\left((3+3^2+\cdots+3^n)-n\right)$$

$$= \frac{1}{2}\left[\frac{3(3^n-1)}{3-1}-n\right]$$

$$= \frac{1}{4}\left[3^{n+1}-3-2n\right]$$

11. From symmetry, we observe that S_{50} has 50 terms. First terms of $S_1, S_2, S_3, S_4, \ldots$ are 1, 2, 4, 7, ... Let T_n be the first term of n^{th} set. Then,

$S = T_1+T_2+T_3+\cdots+T_n$

$\Rightarrow\ S = 1+2+4+7+11+\cdots+T_{n-1}+T_n$ (1)

or

$S = 1+2+4+7+\cdots+T_{n-1}+T_n$ (2)

Subtracting (2) from (1),

$0 = 1+[1+2+3+4+\cdots+(n-1) \text{ terms}] - T_n$

$$\Rightarrow\ 0 = 1+\frac{n(n-1)}{2}-T_n$$

$$\Rightarrow\ T_n = 1+\frac{n(n-1)}{2}$$

$\Rightarrow\ T_{50} = 1226$ (which is the first term in S_{50})

Therefore, sum of the terms is

$$S_{50} = \frac{50}{2}\{2\times 1226+(50-1)\times 1\}$$

$$= 25(2452+49)$$

$$= 25(2501)$$

$$= 62525$$

12. $T_r = r(r^2-1) = (r-1)r(r+1)$

$$\Rightarrow\ \frac{1}{T_r} = \frac{1}{(r-1)r(r+1)}$$

$$= \frac{r+1-(r-1)}{2(r-1)r(r+1)}$$

$$= \frac{1}{2}\left(\frac{1}{r(r-1)}-\frac{1}{r(r+1)}\right)$$

$$= -\frac{1}{2}\left(\frac{1}{r(r+1)}-\frac{1}{r(r-1)}\right)$$

$$= -\frac{1}{2}(V(r)-V(r-1))$$

$$\Rightarrow\ \sum_{r=2}^{n}\frac{1}{T_r} = -\frac{1}{2}(V(n)-V(1))$$

$$= -\frac{1}{2}\left(\frac{1}{n(n+1)}-\frac{1}{2}\right)$$

$$= \frac{1}{4}\left(1-\frac{2}{n(n+1)}\right)$$

$$\Rightarrow\ \sum_{r=2}^{\infty}\frac{1}{T_r} = \frac{1}{4}$$

13. $$T(r) = \frac{1}{(2r-1)(2r+1)(2r+3)}$$

$$= \frac{2r+3-(2r-1)}{4(2r-1)(2r+1)(2r+3)}$$

$$= \frac{1}{4}\left[\frac{1}{(2r-1)(2r+1)}-\frac{1}{(2r+1)(2r+3)}\right]$$

$$= -\frac{1}{4}\left[\frac{1}{(2r+1)(2r+3)}-\frac{1}{(2r-1)(2r+1)}\right]$$

$$= -\frac{1}{4}[V(r)-V(r-1)]$$

Therefore, the sum of n terms is

$$S_n = \sum_{r=1}^{n}T(r) = -\frac{1}{4}[V(n)-V(0)]$$

$$= -\frac{1}{4}\left[\frac{1}{(2n+1)(2n+3)}-\frac{1}{3}\right]$$

$$= \frac{1}{4}\left[\frac{1}{3} - \frac{1}{(2n+1)(2n+3)}\right]$$

$$\Rightarrow \quad S = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \frac{1}{4}\left[\frac{1}{3} - \frac{1}{(2n+1)(2n+3)}\right]$$

$$= \frac{1}{12}$$

$$\Rightarrow \quad 36S = 3$$

$\Rightarrow$ $[36\,S] = 2$ (as $36S$ is not exactly 3, in fact, it is slightly lesser than 3)

14. $$t_k = \sum_{r=1}^{k} t_r - \sum_{r=1}^{k-1} t_r$$

$$= \frac{k(k+1)(k+2)(k+3)}{8} - \frac{(k-1)k(k+1)(k+2)}{8}$$

$$= \frac{k(k+1)(k+2)}{2}$$

$$\frac{1}{t_k} = \frac{2}{k(k+1)(k+2)}$$

$$= \frac{k+2-k}{k(k+1)(k+2)}$$

$$= \frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} = -(v(k) - v(k-1))$$

$$\sum_{k=1}^{n} \frac{1}{t_r} = -(v(n) - v(0)) = \frac{n(n+3)}{2(n+1)(n+2)}$$

15. r^{th} term of the given series is

$$T_r = r(1-x)(1-2x)(1-3x)\cdots(1-(r-1)x)$$

$$= (1-x)(1-2x)(1-3x)\cdots(1-\overline{r-1}x)\frac{xr-1+1}{x}$$

$$= \frac{1}{x}\{(1-x)(1-2x)\cdots(1-\overline{r-1}x) - (1-x)(1-2x)\cdots(1-\overline{r-1}x)(1-rx)\}$$

$$T_r = \frac{1}{x}[(1-x)(1-2x)\cdots(1-\overline{r-1}x) - (1-x)(1-2x)\cdots(1-rx)]$$

Putting $r = 1, 2, \ldots, n$ and then adding, we get

$$T_1 + T_2 + \cdots + T_n = \frac{1}{x}[1-(1-x)(1-2x)\cdots(1-nx)]$$

Hence,

$$\sum_{r=1}^{n} T_n = \frac{1}{x}[1-(1-x)(1-2x)\cdots(1-nx)]$$

Chapter 4

Exercise 4.1

1. $\because$ A.M. > G.M.

$$\therefore \quad \frac{1}{n}\left(\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}\right)$$

$$> \left(\frac{a_1}{a_2}\frac{a_2}{a_3}\frac{a_3}{a_4}\cdots\frac{a_{n-1}}{a_n}\frac{a_n}{a_1}\right)^{1/n}$$

$$\Rightarrow \quad \left(\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}\right) > n(1)^n$$

$$\Rightarrow \quad \left(\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \cdots + \frac{a_{n-1}}{a_n}\right) > n$$

2. $$\frac{1}{b} + \frac{1}{c} \ge 2\left[\frac{1}{b}\,\frac{1}{c}\right]^{1/2}$$

$$\frac{1}{a} + \frac{1}{b} \ge 2\left[\frac{1}{a}\,\frac{1}{b}\right]^{1/2}$$

$$\frac{1}{a} + \frac{1}{c} \ge 2\left[\frac{1}{a}\,\frac{1}{c}\right]^{1/2}$$

Adding, we get

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge \frac{1}{\sqrt{(bc)}} + \frac{1}{\sqrt{(ca)}} + \frac{1}{\sqrt{(ab)}}$$

3. Since $a > b$, therefore $(a - b)$ is positive. Now,

$$\frac{a^n - b^n}{a-b} = a^{n-1} + ba^{n-2} + b^2a^{n-3} + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1}$$

Also, we have

A.M. $\ge$ G.M.

$$\Rightarrow \quad \frac{a^{n-1} + ba^{n-2} + b^2a^{n-3} + \cdots + b^{n-1}}{n}$$

$$> (a^{n-1}\,ba^{n-2}\,b^2a^{n-3}\cdots b^{n-1})$$

$$\Rightarrow \quad \frac{a^{n-1} + ba^{n-2} + b^2a^{n-3} + \cdots + b^{n-1}}{n}$$

$$> (a^{1+2+3+\cdots+(n-1)}\,b^{1+2+3+\cdots+(n-1)})^{1/n}$$

$$\Rightarrow \quad a^{n-1} + ba^{n-2} + b^2a^{n-3} + \cdots + b^{n-1} > n\left[(ab)^{\frac{n(n-1)}{2}}\right]^{1/n}$$

$$\Rightarrow \quad a^{n-1} + ba^{n-2} + b^2\,a^{n-3} + \cdots + b^{n-1} > n[(ab)]^{\frac{(n-1)}{2}}$$

4. A.M. > H.M.

$$\Rightarrow \frac{\frac{1}{a}+\frac{1}{b}}{2} > \frac{2}{a+b}, \frac{\frac{1}{b}+\frac{1}{c}}{2} > \frac{2}{b+c} \text{ and } \frac{\frac{1}{c}+\frac{1}{a}}{2} > \frac{2}{c+a}$$

Adding, we get

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} > \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{a+c}$$

5. Using A.M. ≥ G.M., we have

$$2^{\sin x}+2^{\cos x} \geq 2\sqrt{2^{\sin x}\, 2^{\cos x}}$$

$$= 2\sqrt{2^{\sin x+\cos x}}$$

Now we know that

$$\sin x+\cos x \geq -\sqrt{2}$$

$$\Rightarrow 2^{\sin x}+2^{\cos x} \geq 2\sqrt{2^{-\sqrt{2}}}$$

Hence the minimum value of $2^{\sin x}+2^{\cos x}$ is

$$2^{1-\frac{1}{\sqrt{2}}}$$

6. We have,

A.M. > G.M.

$$\Rightarrow \frac{1}{2}(a^2+x^2) > \sqrt{a^2 x^2}$$

$$\Rightarrow a^2+x^2 > 2ax$$

Similarly,

$$b^2+y^2 > 2by$$

and

$$c^2+z^2 > 2cz$$

Adding these three results, we get

$$(a^2+x^2)+(b^2+y^2)+(c^2+z^2) > 2ax+2by+2cz$$

$$\Rightarrow (a^2+b^2+c^2)+(x^2+y^2+z^2) > 2(ax+by+cz)$$

$$\Rightarrow (1+1) > 2(ax+by+cz) \quad [\because x^2+y^2+z^2 = a^2+b^2+c^2 = 1]$$

$$\Rightarrow 2 > 2(ax+by+cz)$$

$$\Rightarrow ax+by+cz < 1$$

7. Put $a = y+z$, $b = z+x$, $c = x+y$, so that $x+y+z = 1$.

As $x = 1-a$, $y = 1-b$, $z = 1-c$ and $0 < a < 1$, $0 < b < 1$, $0 < c < 1$ it follows that $x, y, z > 0$. Now,

$$\frac{a}{1-a}\,\frac{b}{1-b}\,\frac{c}{1-c} = \frac{[(y+z)(z+x)(x+y)]}{xyz} \quad (1)$$

Again,

A.M. ≥ G.M.

$$\Rightarrow \frac{y+z}{2} \geq \sqrt{yx}, \frac{z+x}{2} \geq \sqrt{zx}, \frac{x+y}{2} \geq \sqrt{xy}$$

Multiplying, we get

$$\frac{[(y+z)(z+x)(x+y)]}{(xyz)} \geq 8 \quad (2)$$

From (1) and (2), we get

$$\frac{a}{1-a}\,\frac{b}{1-b}\,\frac{c}{1-c} \geq 8$$

8. In ΔABC,

$$\tan A+\tan B+\tan C = \tan A\tan B\tan C$$

Also,

$$\frac{\tan A+\tan B+\tan C}{3} \geq \sqrt[3]{\tan A\tan B\tan C}$$

$[\because$ A.M. ≥ G.M.]

$$\Rightarrow \tan A\tan B\tan C \geq \sqrt[3]{\tan A\ \tan B\ \tan C}$$

$$\Rightarrow \tan^2 A\tan^2 B\tan^2 C \geq 27 \quad \text{[cubing both sides]}$$

$$\Rightarrow \tan A\tan B\tan C \geq 3\sqrt{3}$$

$$\Rightarrow \tan A+\tan B+\tan C \geq 3\sqrt{3}$$

9. In ΔABC, we know that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}$$

Now,

A.M. ≥ G.M.

$$\Rightarrow \frac{\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}}{3}$$

$$\geq \left(\operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}\right)^{1/3}$$

$$\Rightarrow \frac{\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}}{3} \geq \left(\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\right)^{1/3}$$

$$\Rightarrow \frac{\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}}{3} \geq (8)^{1/3}$$

$$\Rightarrow \operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2} \geq 6$$

10. $\sqrt{a_1a_2} < \dfrac{a_1+a_2}{2}$ $(\because$ G.M. < A.M.)

$$\sqrt{a_1a_3} < \frac{a_1+a_3}{2}$$

$$\vdots$$

$$\sqrt{a_{n-1}a_n} < \frac{a_{n-1}+a_n}{2}$$

Adding, we get

$$\sqrt{a_1a_2}+\sqrt{a_1a_3}+\cdots < \frac{1}{2}(n-1)(a_1+a_2+\cdots a_n)$$

since in all these inequalities in R.H.S. $a_1, a_2, \ldots, a_n$ all occur $n-1$ times each.

11. We have,

$$a^n-b^n = (a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+b^{n-1})$$

$$\therefore 3^n-2^n = 3^{n-1}+3^{n-2}\,2+3^{n-3}\,2^2+\cdots+2^{n-1}$$

Now, we have

A.M > G.M.

$$\Rightarrow \frac{3^{n-1}+3^{n-2}\,2+\cdots+2^{n-1}}{n}$$

$$> \left[(3\times3^2\times\cdots\times3^{n-1})(2\times2^2\times\cdots\times2^{n-1})\right]^{1/n}$$

$$=3^{\frac{n-1}{2}}2^{\frac{n-1}{2}}=6^{\frac{n-1}{2}}$$

$$\Longrightarrow \quad 3^n > 2^n + n6^{\frac{n-1}{2}}$$

$$\Longrightarrow \quad \frac{3^n}{2^n + n6^{(n-1)/2}} > 1$$

12. Let $z = xy$. Then,

$$z^4 = (xy)^4 = \frac{1}{(ab)^2}(a^2x^4)(b^2y^4) \qquad (1)$$

Therefore, z is maximum when z^4 is maximum. Now,

$$a^2x^4 + b^2y^4 = c^6$$

Hence, (a^2x^4) (b^2y^4) will be maximum if both the factors are equal. Therefore,

$$a^2x^4 = b^2y^4 = \frac{c^6}{2}$$

Therefore, maximum value of z^4 is

$$\frac{1}{(ab)^2}\left(\frac{c^6}{2}\right)\left(\frac{c^6}{2}\right)$$

Hence, maximum value of z is $\frac{c^3}{\sqrt{2ab}}$.

13. $$\frac{a^4+b^4+c^4}{3} > \left(\frac{a+b+c}{3}\right)^4$$

$$=\left(\frac{a+b+c}{3}\right)\left(\frac{a+b+c}{3}\right)^3$$

Now,

$$\frac{a+b+c}{3} > (abc)^{1/3} \Rightarrow \left(\frac{a+b+c}{3}\right)^3 > abc$$

$$\Rightarrow \quad \frac{a^4+b^4+c^4}{3} > \left(\frac{a+b+c}{3}\right) abc$$

$$\Rightarrow \quad a^4+b^4+c^4 > abc(a+b+c)$$

14. A.M. of $(1/2)^{\text{th}}$ powers < $(1/2)^{\text{th}}$ power of A.M.

$$\therefore \quad \frac{(C_1)^{\frac{1}{2}}+(C_2)^{\frac{1}{2}}+\cdots+(C_n)^{\frac{1}{2}}}{n} < \left(\frac{C_1+C_2+\cdots+C_n}{n}\right)^{1/2}$$

$$\Rightarrow \quad \frac{\sqrt{C_1}+\sqrt{C_2}+\cdots+\sqrt{C_n}}{n} < \left(\frac{2^n-1}{n}\right)^{1/2}$$

$$\Rightarrow \quad \sqrt{C_1}+\sqrt{C_2}+\cdots+\sqrt{C_n} < \frac{n\sqrt{(2^n-1)}}{\sqrt{n}}$$

Hence,

$$\sqrt{C_1}+\sqrt{C_2}+\cdots+\sqrt{C_n} < \sqrt{[n(2^n-1)]}$$

15. We know that A.M. of m^{th} power > m^{th} power of A.M.

$$\therefore \quad \frac{\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2}{2} > \left[\frac{\left(a+\frac{1}{a}\right)+\left(b+\frac{1}{b}\right)}{2}\right]^2, \text{ here } m=2$$

$$\Rightarrow \quad \left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 > \frac{1}{2}\left[(a+b)+\left(\frac{1}{a}+\frac{1}{b}\right)\right]^2 \qquad (1)$$

Also,

$$\frac{a^{-1}+b^{-1}}{2} > \left(\frac{a+b}{2}\right)^{-1}$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right) > \frac{2}{a+b}$$

$$\Rightarrow \quad \frac{1}{a}+\frac{1}{b} > \frac{4}{a+b}$$

$$\Rightarrow \quad \frac{1}{a}+\frac{1}{b} > 4$$

Hence, from (1),

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 > \frac{1}{2}(1+4)^2 = \frac{25}{2}$$

16. $$\frac{[a+a+\cdots a \text{ times}]+[b+b+\cdots b \text{ times}]}{a+b} \geq \left[a^a b^b\right]^{\frac{1}{a+b}}$$

$$\geq \frac{a+b}{\left(\frac{1}{a}+\frac{1}{a}+\cdots a \text{ times}\right)+\left(\frac{1}{b}+\frac{1}{b}+\cdots n \text{ times}\right)}$$

$$\Rightarrow \quad \frac{a^2+b^2}{a+b} \geq \left[a^a b^b\right]^{\frac{1}{a+b}} \geq \frac{a+b}{1+1}$$

$$\Rightarrow \quad \left(\frac{a^2+b^2}{a+b}\right)^{a+b} \geq a^a b^b \geq \left(\frac{a+b}{2}\right)^{a+b}$$

$$\Rightarrow \quad \left[\frac{a^2+b^2}{a+b}\right]^{a+b} > a^a b^b > \left\{\frac{a+b}{2}\right\}^{a+b}$$

17. Consider p quantities each equal to a and q quantities each equal to b. We know that

A.M. > G.M.

$$\therefore \quad \frac{(a+a+\cdots \text{ to } p \text{ terms})+(b+b+\cdots \text{ to } q \text{ terms})}{p+q}$$

$$> [(a \times a \times \cdots \text{ to } p \text{ factors})(b \times b \cdots \text{ to } q \text{ factors})]^{\frac{1}{p+q}}$$

$$\Rightarrow \quad \frac{ap+bp}{p+q} > (a^p b^q)^{\frac{1}{p+q}}$$

$$\Rightarrow \quad \left(\frac{ap+bq}{p+q}\right)^{p+q} > (a^p b^q)$$

18. We know that

weighted A.M. > weighed G.M.

$$\Rightarrow \frac{px^{q-r}+qx^{r-p}+rx^{p-q}}{p+q+r} > \left[(x^{q-r})^p (x^{r-p})^q (x^{p-q})^r\right]^{\frac{1}{p+q+r}}$$

$$\Rightarrow \frac{px^{q-r}+qx^{r-p}+rx^{p-q}}{p+q+r} > 1$$

$$\Rightarrow px^{q-r}+qx^{r-p}+rx^{p-q} > p+q+r$$

In case either $p = q = r$ or $x = 1$, inequality becomes equality.

19. Using weighted A.M., G.M. inequality, we have

$$\frac{ca+ab+bc}{a+b+c} \geq (a^c b^a c^b)^{\frac{1}{a+b+c}}$$

$$\Rightarrow ca+ab+bc \geq (a^c b^a c^b)$$

Similarly,

$$\frac{ba+cb+ac}{b+c+a} \geq (a^b b^c c^a)^{\frac{1}{a+b+c}} \Rightarrow ba+cb+ac \geq (a^b b^c c^a)$$

$$\frac{aa+bb+cc}{a+b+c} \geq (a^a b^b c^c)^{\frac{1}{a+b+c}} \Rightarrow aa+bb+cc \geq (a^b b^c c^a)$$

Adding together, we get

$$(a+b+c)(a+b+c) \geq a^a b^b c^c + a^b b^c c^a + a^c b^a c^b$$

$$\Rightarrow 1 \geq a^a b^b c^c + a^b b^c c^a + a^c b^a c^b$$

20. Let,

$A = x^2y^3z^4$

$$\Rightarrow A^2 = x^4y^6z^8 = 2^3 \times 3^3 \times 4^4 \left(\frac{x^2}{2}\right)^2 \left(\frac{y^2}{3}\right)^3 \left(\frac{z^2}{4}\right)^4 \quad (1)$$

Therefore, A is maximum when A^2 is maximum. Now,

$x^2 + y^2 + z^2 = 1$

$$\Rightarrow 2\left(\frac{x^2}{2}\right) + 3\left(\frac{y^2}{3}\right) + 4\left(\frac{z^2}{4}\right) = 1$$

$$\Rightarrow \left(\frac{x^2}{2}\right)^2 \left(\frac{y^2}{3}\right)^3 \left(\frac{z^2}{4}\right)^4$$ is maximum when all the factors are equal

$$\therefore \frac{x^2}{2} = \frac{y^2}{3} = \frac{z^2}{4} = \frac{x^2+y^2+z^2}{2+3+4} = \frac{1}{9} \qquad [\because x^2+y^2+z^2 = 1]$$

Hence from (1), maximum value of A^2 is

$$2^2 \times 3^3 \times 4^4 \left(\frac{1}{9}\right)^2 \left(\frac{1}{9}\right)^3 \left(\frac{1}{9}\right)^4 = \frac{2^2 \times 3^3 \times 4^4}{9^9} = \frac{2^{10}}{3^{15}}$$

Hence, maximum value of $x^2y^3z^4$ is $2^5 \times 3^{-15/2}$.

Chapter 5

Exercise 5.1

1. Since the man can go in 4 ways and come back in 3 ways, the total number of ways is $4 \times 3 = 12$.

2. Since a card can be sent by any of the three servants, so the number of ways of sending the invitation card to the first friend is 3. Similarly, invitation cards can be sent to each of the six friends in 3 ways. Hence, total number of ways is $3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6$

3. Since each question can be answered in 4 ways, the total number of ways of answering 5 questions is $4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

4. First and second prizes in mathematics can be given in (30×29) ways.

First and second prizes in physics can be given in (30×29) ways.

First prize chemistry can be given in 30 ways.

First prize english can be given in 30 ways.

Hence, the number of ways to give prizes in all the four subjects is $(30 \times 29) \times (30 \times 29) \times 30 \times 30$.

5. Let A_1, A_2, A_3, A_4, A_5 be five persons.

(i) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Similarly, each of A_2, A_3, A_4, A_5 can leave the cabin in 7 ways. Thus, the total number of ways in which each of the five persons can leave the cabin at any of the seven floors is $7 \times 7 \times 7 \times 7 \times 7 = 7^5$.

(ii) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Now, A_2 can leave the cabin at any of the remaining 6 floors. So, A_2 can leave the cabin in 6 ways. Similarly, A_3, A_4, A_5 can leave the cabin in 5, 4 and 3 ways, respectively. Thus, the total number of ways in which each of the five persons can leave the cabin at different floors is $7 \times 6 \times 5 \times 4 \times 3 = 2520$.

6. We have $q^2 - 4p \geq 0$.

If $p = 1$, then $q^2 \geq 4$ is satisfied by $q = 2, 3, 4$

If $p = 2$, then $q^2 \geq 8$ is satisfied by $q = 3, 4$

If $p = 3$, then $q^2 \geq 16$ is true for $q = 4$

Hence, there are seven equations having real roots.

7. A determinant of order 2 is of the form

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

It is equal to $ad - bc$.

Now, $\Delta \neq 0$ if $ad = 1, bc = 0, ad = 0, bc = 1$.

But $ad = 1, bc = 0$ if $a = d = 1$ and one of b, c is zero.

Therefore, $ad = 1, bc = 0$ in the following three cases:

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

Similarly, $ad = 0$, $bc = 1$ in the following three cases:

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

Therefore, total number of determinants is 6.

8. When $x - y > 5$, we have following cases:

Value of x	Value of y	Number of cases
6	0	1
7	0, 1	2
8	0, 1, 2	3
9	0, 1, 2, 3	4
10	0, 1, 2, 3, 4	5
	Total	15

Similarly, we have 15 cases for $x - y < -5$. Thus, there are 30 pairs.

9. **a.** $$\lim_{x\to 0}\left(1+\frac{a^x+b^x-2}{2}\right)^{\frac{2}{a^x+b^x-2}\lim_{x\to 0}\left(\frac{a^x-1+b^x-1}{x}\right)} = e^{\ln ab}$$

$$= ab = 6$$

The total number of possible ways in which a, b can take values is $6 \times 6 = 36$. The total possible ways are (1, 6), (6, 1), (2, 3), (3, 2). The total number of possible ways is 4.

b. $f'(x) = 3x^2 + 2ax + 9$

$y = f(x)$ is increasing.

$\Rightarrow$ $f'(x) \geq 0$, $\forall$ x and for $f'(x) = 0$ should not form an interval

$\Rightarrow$ $(2a)^2 - 4 \times 3 \times b \leq 0 \Rightarrow a^2 - 3b \leq 0$

This is true for exactly 16 ordered pairs (a, b), $1 \leq a, b \leq 6$, namely

(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 6), (4, 6)

Now, from product rule, the number of ways is number of ways b is selected $\times$ number of ways c is selected, which is given by $16 \times 6 = 96$.

10. First square can be selected in 64 ways. Second square can be selected in $65 - 15 = 49$ ways.

But when first square, say $c3$ is selected, sometimes $g5$ is selected as second square.

Similarly, when first square, say $g5$ is selected, sometimes $c3$ is selected as second square. The total number of ways is $64 \times 49/2 = 1568$

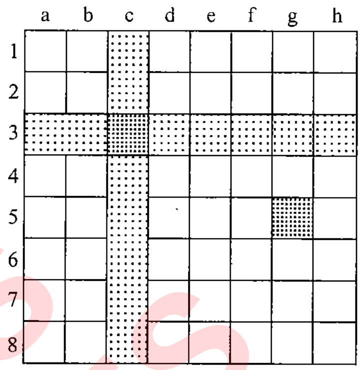


Fig. A-5.1

Exercise 5.2

1. $$\frac{(2n)!}{n!} = \frac{1\times2\times3\times4\times\cdots\times(2n-2)\times(2n-1)2n}{n!}$$

$$= \frac{\{1\times3\times\cdots\times(2n-1)\}\{2\times4\times\cdots\times(2n)\}}{n!}$$

$$= \frac{\{1\times3\times\cdots\times(2n-1)\}2^n\{1\times2\times\cdots\times(n-1)n\}}{n!}$$

$$= \frac{\{1\times3\times5\times7\times\cdots\times(2n-1)\}2^n n!}{n!}$$

$$= \{1 \times 3 \times 5 \times 7 \times \cdots \times (2n-1)\}2^n$$

2. We have,

$$1! + 2! + 3! + \cdots + n! = 1 + 2 + 6 + 24 + 5! + 6! + \cdots + n!$$
$$= 33 + 5! + 6! + \cdots + n!$$

This sum always ends with 3 or the digit in the unit's place of the sum is 3. Now, we know that square of any integer never ends with 3. Then the given sum cannot be a perfect square.

3. Let p be divisible by k and r be any natural number between 1 and k. If $p + r$ is divided by k, we obtain r as the remainder. Now,

$n! = 1 \times 2 \times 3 \times 4 \times \cdots \times (n-1)n$

Therefore, $n!$ is divisible by every natural number between 2 and n. So, $n! + 1$, when divided by any natural number between 2 and n, leaves 1 as the remainder. Hence, $n! + 1$ is not divisible by any natural number between 2 and n.

4. Let,

$N = 1! + 2! + 3! + 4! + 5! + 6! + \cdots + n!$

$$\Rightarrow \frac{N}{15} = \frac{1!+2!+3!+4!+5!+\cdots+n!}{15}$$

$$= \frac{1!+2!+3!+4!}{15} + \frac{5!+6!+\cdots+n!}{15}$$

$$= \frac{33}{15} + \text{integer (as } 5!, 6!, \ldots \text{ are divisible by 15)}$$

Hence, remainder is 3.

5. $80 = 2^4 \times 5$

To find the exponent of 80 in 200!, we find the exponent of 2 and

5. Exponent of 2 is $\left[\frac{200}{2}\right]+\left[\frac{200}{2^2}\right]+\left[\frac{200}{2^3}\right]+\left[\frac{200}{2^4}\right]+\left[\frac{200}{2^5}\right]$

$+\left[\frac{200}{2^6}\right]+\left[\frac{200}{2^7}\right]$

$= 100 + 50 + 25 + 12 + 6 + 3 + 1$

$= 197$

Exponent of 5 is

$$\left[\frac{200}{5}\right]+\left[\frac{200}{5^2}\right]+\left[\frac{200}{5^3}\right] = 40 + 8 + 1 = 49$$

Now, exponent of 16 in 200! is [197/4] = 49. Hence, coefficient of 80 is 49.

Exercise 5.3

1. $^{n-1}P_r + r\,^{n-1}P_{r-1} = \frac{(n-1)!}{(n-1-r)!} + r\frac{(n-1)!}{(n-r)!}$

$$= \frac{(n-1)!}{(n-1-r)!}\left\{1 + r\frac{1}{n-r}\right\}$$

$$= \frac{(n-1)!}{(n-1-r)!}\left(\frac{n}{n-r}\right)$$

$$= \frac{n!}{(n-r)!} = {}^nP_r$$

2. $\frac{n!}{(n-5)!} \times \frac{(n-3)!}{n!} = 20$

$\Rightarrow$ $(n-3)(n-4) = 20 \Rightarrow n = -1, 8$

But -1 is not acceptable.

3. **a.** Given,

$^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$

$\Rightarrow$ $\frac{22!}{(21-r)!} : \frac{20!}{(18-r)} = 11:52$

$\Rightarrow$ $\frac{22!}{(21-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$

$\Rightarrow$ $\frac{22\times 21\times 20}{(21-r)(20-r)(19-r)} = \frac{11}{52}$

$\Rightarrow$ $(21-r)(20-r)(19-r) = 2 \times 21 \times 52$

$\Rightarrow$ $(21-r)(20-r)(19-r) = 2 \times 3 \times 7 \times 4 \times 13$

$\Rightarrow$ $(21-r)(20-r)(19-r) = 12 \times 13 \times 14$

$\Rightarrow$ $r = 7$

b. $^{56}P_{r+6} : {}^{54}P_{r+3} = 30800:1$

$\Rightarrow$ $\frac{56!(51-r)!}{(50-r)!54!} = 30800$

$\Rightarrow$ $56 \times 55 \times (51 - r) = 30800$

$\Rightarrow$ $r = 41$

4. The number of 1-digit numbers is $^4P_1$. The number of 2-digit numbers is $^4P_2$. The number of 3-digit numbers is $^4P_3$. The number of 4-digit numbers is $^4P_4$. Hence, the required number is $^4P_1 + {}^4P_2 + {}^4P_3 + {}^4P_4$.

5. Extreme left place can be filled in 6 ways, the middle place can be filled in 6 ways and the extreme right place in only 3 ways. Since number to be formed is odd, so the required number is $6 \times 6 \times 3 = 108$.

6. Given set of numbers is {1, 2, ... 11} in which 5 are even and six are odd, which signifies that in the given product, it is not possible to arrange but only to subtract the even numbers from odd numbers. There must be at least one factor involving subtraction of an odd number form another odd number. So at least one of the factors is even. Hence, product is always even.

7. Five boys can sit in 5! ways; in this case, there are 6 vacant places where the girls can sit in $^6P_3$ ways. Therefore, the required number of ways is $^6P_3 \times 5!$

8. $G_1\ G_2\ G_3\ G_4\ G_5\ B_1, B_2, B_3, B_4, B_5$

Since five boys are always together, the group of five boys is considered as one object.

Now, the number of arrangements of five girls and the group of boys is 6!

Again, the number of arrangements of five boys within the group is 5!

Therefore, the total number of arrangements is $6! \times 5!$

9. The word 'MOBILE' has three even places and three odd places. It has 3 consonants and 3 vowels. In three odd places, we have to fix up 3 consonants, which can be done in $^3P_3$ ways.

Now, in the remaining three places, we have to fix up the remaining three, which can be done in $^3P_3$ ways.

Hence, the total number of ways is $^3P_3 \times {}^3P_3 = 36$.

10. (i) When repetition is not allowed.

There are 6 digits 0, 1, 2, 3, 4, 5 and we can use any number of digits.

Number of digits	Numbers
1	$(\neq 0)\ 5 = 5$
2	$(\neq 0)\ 5 \times 5 = 25$
3	$(\neq 0)\ 5 \times 5 \times 4 = 100$
4	$(\neq 0)\ 5 \times 5 \times 4 \times 3 = 300$
5	$(\neq 0)\ 5 \times 5 \times 4 \times 3 \times 2 = 600$
6	$(\neq 0)\ 5 \times 5 \times 4 \times 3 \times 2 \times 1 = 600$
Total	1630

For numbers greater than 3000, number of digits is greater than 4.

All the 6- and 5-digit numbers are greater than 3000.

Now for 4-digit numbers > 3000:

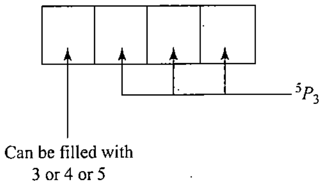


Fig. A-5.2

Hence, number greater than 3000 having 4 digits 4 is $3 \times {}^5P_3 = 180$

Total numbers >3000 is 180 + 600 + 600 = 1380.

(ii) When repetition is allowed.

It is equivalent to permutation of 6 objects (digits) when any object is repeated any number of time, which is equal to 6^6. But this includes one number 0. Hence, total numbers are $6^6 - 1 = 46655$.

Now, number of 5-digit numbers is $5 \times 6 \times 6 \times 6 \times 6 = 6480$.

And, number of 6-digit numbers is $5 \times 6 \times 6 \times 6 \times 6 \times 6 = 38880$.

Again, 4-digit numbers are to be greater than 3000.

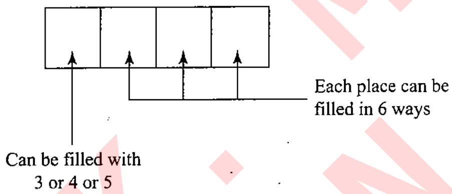


Fig. A-5.3

Hence, total number of 4-digit numbers is $3 \times 6 \times 6 \times 6 = 648$

But these 648 numbers have 3000 as a number which should be excluded. So, the number of 4-digit numbers > 3000 is 648 – 1 = 647. Therefore, total numbers >3000 are 647 + 6480 + 38880 = 46007.

Exercise 5.4

1. We have,

$^{n+2}C_8 : {}^{n-2}P_4 = 57:16$

$$\Rightarrow \frac{^{n+2}C_8}{^{n-2}P_4} = \frac{57}{16}$$

$$\Rightarrow \frac{(n+2)!}{8!(n-6)!} \cdot \frac{(n-6)!}{(n-2)!} = \frac{57}{16}$$

$$\Rightarrow (n+2)(n+1)n(n-1) = \frac{57}{16} \times 8!$$

$$= \frac{19 \times 3}{16} \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$\Rightarrow (n-1)n(n+1)(n+2) = 18 \times 19 \times 20 \times 21$$

$$\Rightarrow n - 1 = 18$$

$$\Rightarrow n = 19$$

2. Maximum value of $^{20}C_r$ is $^{20}C_{12}$ and maximum value of $^{25}C_r$ is $^{25}C_{12}$. Hence, the required ratio is $^{20}C_{10}/^{25}C_{10} = 143/4025$.

3. It is to be noted here that, when two persons shake hands, it is counted as one handshake, not two. So, this is a problem on combinations.

The total number of handshakes is same as the number of ways of selecting 2 persons among 12 persons is $^{12}C_2 = 12!/(10! \times 2!) = 66$.

4. Number of cards exchanged is 2 × (number of ways 2 students can be selected who will exchange cards), which is given by $2\ {}^{20}C_2$.

5. Let there be r white and $(15 - r)$ black balls.

Then, total number of permutations of these balls is $15!/r!(15 - r)! = {}^{15}C_r$ since r white balls are alike and $(15 - r)$ black balls are alike.

Therefore, the number of arrangement is $^{15}C_r$ which is maximum, when $r = (15 - 1)/2$ or $(15 + 1)/2$, i.e., $r = 7$ or 8.

6. We have 10 men and 7 women and a committee of 6 containing at least 3 men and 2 women is to be formed.

Men (10)	Women (7)	Number of selections
3	3	$^{10}C_3 \times {}^7C_3 = 4200$
4	2	$^{10}C_4 \times {}^7C_2 = 4410$
	Total	8610

Now, let us consider the case when 2 particular women are always there in the same committee. We have to make a selection of 4 from 10 men and 5 women. In this case, to comply with the initial condition of at least 3 men and at least 2 women, we have the following cases:

Men (10)	Women (5)	Number of selections
3	0	$^{10}C_4 \times {}^5C_0 = 210$
3	1	$^{10}C_3 \times {}^5C_1 = 600$
	Total	810

Hence, the number of committees when two particular women are never together is 8610 – 810 = 7800.

7. Now, any two months can be chosen in $^{12}C_2$ ways. The six birthdays can fall in these two months in 2^6 ways. Out of these 2^6 ways there are two ways when all the six birthdays fall in one month. So, the required number of ways is $^{12}C_2 \times (2^6 - 2)$.

8. Since $x_1 < x_2 < x_3 < x_4 < x_5$ and $x_3 = 30$, therefore, $x_1, x_2 < 30$, i.e., x_1 and x_2 should come from tickets numbered from 1 to 29 and this may happen in $^{29}C_2$ ways. Remaining ways, i.e., $x_4, x_5 > 30$, should come from 20 tickets numbered from 31 to 50 in $^{20}C_2$ ways. So, total number of ways is $^{29}C_2\ {}^{20}C_2$.

9. a. Four hotels can be chosen out of five hotels in $^5C_4$ ways and the four visitors can stay in one set of four hotels in 4! ways (one visitor in one hotel).

Therefore, the total number of ways of accommodating the visitors is $^5C_4 \times 4! = 5 \times 24 = 120$.

b. As A and B stay at the same hotel, let us consider the two as a single units (A, B). Now, we require three hotels for C, D and (A, B), which can be chosen and distributed in $^5C_3 \times 3! = 10 \times 6 = 60$ ways.

10. The total number of units to be covered is $3 + 7 + 11 = 21$. A person can choose 3 units in $^{21}C_3$ ways. A person can choose 7 units in $^{18}C_7$ ways. The rest 11 units can be chosen in 1 way.

Therefore, the total number of ways is $^{21}C_3 \times {}^{18}C_7 \times 1$.

11. Let the particular side on which 3 particular sailors can work be named A; and another side B on which 2 particular sailors can work. Thus, we are left with 3 sailors only.

The number of ways of selection of one sailor for side A is $^3C_1 = 3$ and then we are left with 2 sailors for other sides. Now, on each side 4 sailors can be arranged in 4! ways. Hence, total number of arrangements is $3 \times 4! \times 4! = 3 \times 24 \times 24 = 1728$.

12. The selection scheme is as follows:

	Category I	Category II	Category III
Group A (4)	2	2	3
Group B (5)	2	3	2
Group C (6)	3	2	2
No. of ways of selections	$^4C_2\,^5C_2\,^6C_3 = 1200$	$^4C_2\,^5C_3\,^6C_2 = 900$	$^4C_3\,^5C_2\,^6C_2 = 600$

Total number of different selections is $1200 + 900 + 600 = 2700$.

Exercise 5.5

1. The number of selection of two ladies to sit together is $^3C_2$.

Let the two seats occupied by these two (selected) ladies be numbered as 1 and 2 and remaining seats by 3, 4, 5 and 6. The 3rd lady cannot occupy seat number 3 and 6, so there are only two choices (viz., numbers 4 and 5) left with the 3rd lady who can make selection of seats in $^2C_1$ ways.

Now, the two selected ladies (in seat numbers 1 and 2) can be permuted (i.e., change their seats) in 2! ways and 3 gentlemen can be permuted on remaining 3 seats in 3! ways.

Hence, by product rule, total number of ways is $^3C_2 \times {}^2C_1 \times 2! \times 3! = 72$.

2. Since total number of members is 15, but one is to be left because of circular condition, therefore, remaining members are 14 but three special members constitute a member. Therefore, required number of arrangements are $12! \times 2$, because, chairman remains between the two specified persons and the person can sit in two ways.

3. 6 men can dine at a round table in 5! ways.

Now, there are 6 vacant spaces between any two men.

These can be occupied by 5 women in $^6P_5$ i.e., 6! ways.

Therefore, the total number of ways is $6! \times 5!$.

4. Eight different beads can be arranged in circular form in $(8 - 1)! = 7!$ ways. Since there is no distinction between the clockwise and anticlockwise arrangement, the required number of arrangements is $7!/2 = 2520$.

5. If four particular flowers are always together, flowers can be strung in $(4!/2)/4! = 288$ ways.

Exercise 5.6

1. The required number of ways is: $(10 + 1)(9 + 1)(7 + 1) - 1 = 11 \times 10 \times 8 - 1 = 879$

2. Let the number of candidates be n. Therefore, $n - 2$. are to be elected and so one can vote up to $n - 2$. Hence, the number of ways in which one can vote is

$$^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-2} = 56 \text{ (given)}$$

$$\Rightarrow \quad 2^n - ({}^nC_0 + {}^nC_{n-1} + {}^nC_n) = 56$$

$$\Rightarrow \quad 2^n - n = 58$$

$$\Rightarrow \quad 2^n = 58 + n$$

which is satisfied by $n = 6$ only.

3. For each historical monument there are two possibilities, either he visits or does not visit.

The number of ways in which he can visit 5 historical monuments is 2^5. Similarly, he can visit 6 gardens in 2^6 ways.

To visit at least one of seven shopping malls, it could be in $2^7 - 1$ ways.

Therefore, the total number of ways he can visit the city is $2^5 \times 2^6 \times (2^7 - 1)$ up and is incomplete.

4. We have

$$720 = 2^4 \times 3^2 \times 5^1$$

Now, number of divisors of 720 is equivalent to selection of zero or more integer from (2, 2, 2, 2), (3, 3), (5).

The number of ways of selection is $5 \times 3 \times 2 = 30$. This also includes '1' and '720'.

Now, in even divisor factor 2 should be there. Then from set of 2, we must have non-empty selection. Hence, the number of even divisors is $4 \times 3 \times 2 = 24$. Also, the number of divisors of 720 can be seen as number of different terms in the expansion of

$$(2^0 + 2^1 + 2^2 + 2^3 + 2^4) \times (3^0 + 3^1 + 3^2) \times (5^0 + 5^1)$$

which also gives sum of divisors.

Hence, sum of divisors is

$$(2^0 + 2^1 + 2^2 + 2^3 + 2^4) \times (3^0 + 3^1 + 3^2) \times (5^0 + 5^1)$$

$$= \frac{2^5 - 1}{2 - 1} \times \frac{3^3 - 1}{3 - 1} \times \frac{5^2 - 1}{5 - 1} = 2418$$

5. $3^p 6^m 21^n = 2^m 3^{p+m+n} 7^n$

Therefore, the required number of proper divisors is equal to the number of selections of any number of 3's and 7's [$\because$ for odd divisors 2 must not be selected] and is given by $(p + m + n + 1)(n + 1) - 1$.

6. $7056 = 2^4 . 3^2 . 7^2$

no. of divisore = $(4 + 1)(2 + 1)(2 + 1) = 5 \times 3 \times 3 =$ 45

No. of ways of resolving into 2 factors = $\dfrac{45+1}{2} = 23$

Exercise 5.7

1. First divide 22 books into 5 groups of the size 5, 5, 4, 4, 4.

The number of ways is

$$\frac{22!}{3!2!(5!)^2(4!)^3}$$

Now, number of ways of distribution of these groups among 5 students is 5! Then total number of ways of distribution is

$$\frac{22!}{3!2!(5!)^2(4!)^3} \times 5! = \frac{22!}{3!2!5!(4!)^3}$$

2. The number of ways 16 constables can be divided into 8 batches is $(16)!/8!(2!)^8$.

Now, the first batch may be assigned to patrol any of the 8 villages, the second one can then be assigned to any one of the remaining seven villages and so on.

Hence, the number of ways of assigning 8 batches to patrol 8 villages is

$$\frac{(16)!}{8!(2!)^8}(8!) = \frac{(16)!}{(2!)^8}$$

3. The number of selections of 4 prizes to the particular boy is ${}^{10}C_4$. From the remaining 6 prizes, each prize can be given to any of the four students. Therefore, the number of ways is 4^6.

Therefore, the total number of ways is ${}^{10}C_4 \times 4^6 = 210 \times 4096 =$ 860160.

4. This is equivalent to divide $3n$ different objects into 3 equal size group. Hence, number of ways is $3n!/[(n!)^3 3!]$.

5. First, give Rs. 3 to each of four persons. Now for remaining 4 rupees we have,

$$x + y + z + w = 4 \qquad (1)$$

where x, y, z, w are number of rupees gained by person 1, 2, 3, 4, respectively.

We have to find number of non-negative solutions of Eq. (1), which is ${}^{4+4-1}C_{4-1} = {}^7C_3 = 35$.

6. Let x_W, x_R, x_B be the number of white balls, red balls and blue balls being selected. We must have

$$x_W + x_R + x_B = 10$$

The required number of ways is equal to the number of non-negative integral solutions of $x_W + x_R + x_B = 10$, which is ${}^{3+10-1}C_{10} = {}^{12}C_{10} = {}^{12}C_2$.

7. Let $a = 2p + 1, b = 2q + 1, c = 2r + 1, d = 2s + 1$ where p, q, r, s are non-negative integers. Therefore,

$$2p + 1 + 2q + 1 + 2r + 1 + 2s + 1 = 20$$

$$\Rightarrow \quad p + q + r + s = 8$$

Therefore, the required number is equal to the number of non-negative integral solutions of $p + q + r + s = 8$, which is given by ${}^{8+4-1}C_{4-1} = {}^{11}C_3 = 165$.

8. We have to select 4 professors for Roorkee and 6 professors for outside. Again, 2 professors prefer Roorkee and 3 outside, so we are left with 5 professors.

The number of ways in which two more professors for Roorkee can be selected is ${}^5C_2 = 10$.

And remaining 3 professors are left for outside. Now, 6 professors outside Roorkee can be allotted to 3 centres in $6!/(2!2!2!3!) \times 3$ ways.

Now, 4 professors for 2 centres in Roorkee can be allotted in $4!/(2!2!2!) \times 2$ ways. Hence, the total number of ways is

$$10 \times \frac{6!}{2!2!2!3!} \times 3! \times \frac{4!}{2!2!2!} \times 2!$$

$$= 5400$$

9. Number of ways in which two numbers can be selected from 100 integers is ${}^{100}C_2$.

Let us find the number of ways by selecting two numbers, if their difference is more than 10.

Let the two integers selected be P and Q.

$$\begin{array}{ccccc} & P & & Q & \\ x & & y & & z \end{array}$$

Fig. A-5.3

Here, x, y, z are number of integers before P, between P and Q and after Q. Therefore, $x + y + z = 98$.

Also, we have to find solutions for $x, z \geq 0$ and $y \geq 10$.

$$\therefore \quad x + y_1 + z = 88, \text{ where } y_1 \geq 0$$

Then, number of solutions of above equations is ${}^{88+3-1}C_{3-1} = {}^{90}C_2$.

Hence, the number of ways where two different natural numbers can be selected, which differ by almost 10 is ${}^{100}C_2 - {}^{90}C_2$.

Exercise 5.8

1. Required number = Coefficient of x^{30} in $(x^2 + x^3 + \cdots + x^{16})^8$

$= \text{Coefficient of } x^{30} \text{ in } x^{16}(1 + x + \cdots + x^{14})^8$

$= \text{Coefficient of } x^{14} \text{ in } \left(\dfrac{1-x^{15}}{1-x}\right)^8$

$= \text{Coefficient of } x^{14} \text{ in } (1 - x)^{-8}$

$= {}^{21}C_{14}$

2. Let w be a non-negative integer such that $3x + y + z + w = 30$

Let $a = x - 1, b = y - 1, c = z - 1, d = w$. Then

$$3a + b + c + d = 25, \text{ where } a, b, c, d \geq 0 \qquad (1)$$

Clearly, $0 \leq a \leq 8$. If $a = k$, then

$$b + c + d = 25 - 3k \qquad (2)$$

The number of non-negative integral solutions of Eq. (2) is

$^{n+r-1}C_r = {}^{3+25-3k-1}C_{25-3k} = {}^{27-3k}C_{25-3k} = {}^{27-3k}C_2$

$$= \frac{(27-3k)(26-3k)}{2}$$

$$= \frac{3}{2}(3k^2 - 53k + 234)$$

Therefore, the required number is

$$\frac{3}{2}\sum_{k=0}^{8}(3k^2 - 53k + 234)$$

$$= \frac{3}{2}\left[3\frac{8\times9\times17}{6} - 53\frac{8\times9}{2} + 234\times9\right]$$

$$= 1215$$

3. Any number between 1 and 100000 must be less than seven digits. Therefore, it must be in the form of $a_1, a_2, a_3, a_4, a_5, a_6$, where $a_1, a_2, a_3, a_4, a_5, a_6 \in \{0, 1, 2, \ldots, 9\}$. According to question,

$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 18$ (1)

where $0 \le a_i \le 9$, $i = 1, 2, 3, \ldots, 9$

$\therefore$ Required number = Coefficient of x^{18} in $(1 + x + x^2 + \cdots + x^9)^6$

= Coefficient of x^{18} in $\left(\frac{1-x^{10}}{1-x}\right)^6$

= Coefficient of x^{18} in $[(1 - x^{10})^6 (1 - x)^{-6}]$

= Coefficient of x^{18} in $[(1 - {}^6C_1 x^{10})(1 - x)^{-6}]$

[ignoring terms having powers of x greater than 18]

$= {}^{6+18-1}C_{18} - 6 \times {}^{6+8-1}C_8$

$= {}^{23}C_5 - {}^{6\times13}C_5$

4. Number of integral solutions

= Coefficient of x^{24} in $(x + x^2 + x^3 + x^4 + x^5)(x^{12} + x^{13} + \cdots + x^{18})(x^{-1} + x^0 + x^1 + \cdots + x^{11})$

= Coefficient of x^{24} in $\frac{x(1-x^5)}{(1-x)}\frac{x^{12}(1-x^7)}{(1-x)}\frac{x^{-1}(1-x^{13})}{(1-x)}$

= Coefficient of x^{12} in $(1 - x^5)(1 - x^7)(1 - x^{13})(1 - x)^{-3}$

= Coefficient of x^{12} in $(1 - x^5 - x^7 + x^{12})(1 - x)^{-3}$

$= {}^{14}C_{12} - {}^9C_7 - {}^7C_5 + 1$

Chapter 6

Exercise 6.1

1. In the expansion of $(x - 1/x)^6$, the general term is

$$T_{r+1} = {}^6C_r\, x^{6-r}\left(-\frac{1}{x}\right)^r = {}^6C_r(-1)^r x^{6-2r}$$

For the term independent of x,

$6 - 2r = 0 \Rightarrow r = 3$

Thus, the required coefficient is $(-1)^3 \times {}^6C_3 = -20$.

2. $T_{r+1} = {}^{12}C_r\left(\frac{a}{x}\right)^{12-r}(bx)^r = {}^{12}C_r(a)^{12-r}(b)^r x^{2r-12}$

Adjust $2r - 12 = -10 \Rightarrow r = 1$. Then, the coefficient of x^{-10} is ${}^{12}C_1 (a)^{11}(b)^1 = 12a^{11}b$.

3. $T_{r+1} = {}^{15}C_r(x^4)^{15-r}\left(\frac{1}{x^3}\right)^r = {}^{15}C_r\, x^{60-7r}$

$\Rightarrow$ $60 - 7r = 4$

$\Rightarrow$ $r = 8$

Then, the required term is 9th.

4. $T_1 = {}^nC_0 = 1$

$T_2 = {}^nC_1\, ax = 6x$ (1)

$T_3 = {}^nC_2\,(ax)^2 = 16x^2$ (2)

From (2),

$na = 6$ (3)

From (3),

$\frac{n(n-1)}{2}a^2 = 16$ (4)

Eliminating a from (3) and (4),

$\frac{n-1}{2n} = \frac{4}{9} \Rightarrow n = 9$

From (3),

$a = 2/3$

5. Coefficient of x^p is ${}^{(p+q)}C_p$ and coefficient of x^q is ${}^{(p+q)}C_q$. But ${}^{(p+q)}C_p = {}^{(p+q)}C_q$ ($\because {}^nC_r = {}^nC_{n-r}$)

6. $T_4 = T_{3+1} = {}^nC_3\, a^{n-3}b^3$

$\Rightarrow$ ${}^nC_3 = 56 \Rightarrow \frac{n!}{3!(n-3)!} = 56$

$\Rightarrow$ $n(n-1)(n-2) = 56 \times 6$

$\Rightarrow$ $n(n-1)(n-2) = 8 \times 7 \times 6$

$\Rightarrow$ $n = 8$

7. $\frac{1}{6} = \frac{{}^nC_6\left(2^{1/3}\right)^{n-6}\left(3^{-1/3}\right)^6}{{}^nC_{n-6}\left(2^{1/3}\right)^6\left(3^{-1/3}\right)^{n-6}}$

$\Rightarrow$ $6^{-1} = 6^{-4} \times 6^{n/3} = 6^{n/3-4}$

$\Rightarrow$ $\frac{n}{3} - 4 = -1$

$\Rightarrow$ $n = 9$

8. Given,

${}^{15}C_{2r+2} = {}^{15}C_{r-2}$

But,

${}^{15}C_{2r+2} = {}^{15}C_{15-(2r+2)} = {}^{15}C_{13-2r}$

$\Rightarrow$ ${}^{15}C_{13-2r} = {}^{15}C_{r-2}$

$\Rightarrow$ $r = 5$

9. The $(r + 1)$th term is ${}^{2n}C_r(x^2)^{2n-r}(1/x)^r = {}^{2n}C_r\, x^{4n-3r}$.

For the coefficient of x^p,

$$4n - 3r = p \Rightarrow r = \frac{1}{3}(4n - p)$$

Therefore, the coefficient of x^p is

$$^{2n}C_{(4n-p)/3} = \frac{(2n)!}{\left[\frac{1}{3}(4n-p)\right]!\left[2n - \frac{1}{3}(4n-p)\right]!}$$

$$= \frac{(2n)!}{\left[\frac{1}{3}(4n-p)\right]!\left[\frac{1}{3}(2n+p)\right]!}$$

10. T_{r+1}, the $(r+1)^{th}$ term in the expansion of $(5^{1/6} + 2^{1/8})^{100}$ is

$$T_{r+1} = {}^{100}C_r\,(5^{1/6})^{100-r}(2^{1/8})^r$$

As 5 and 2 are relatively prime, T_{r+1} will be rational if $(100 - r)/6$ and $r/8$ are both integers, i.e., $100 - r$ is a multiple of 6 and r is a multiple of 8.

As $0 \le r \le 100$, multiples of 8 up to 100 and corresponding values of $100 - r$ are given by

$r = 0, 8, 16, 24, \ldots, 88, 96$

$100 - r = 100, 92, 84, 76, \ldots, 12, 4$

Out of $100 - r$ values, multiples of 6 are 84, 60, 36 and 12.

Hence, there are just four rational terms. Therefore, the number of irrational terms is $101 - 4 = 97$.

Exercise 6.2

1. Let,

$$(1 + x - 3x^2)^{4163} = a_0 + a_1x + a_2x^2 + \cdots$$

Putting $x = 1$ on both sides, we get

$$(1 + 1 - 3)^{4163} = a_0 + a_1 + a_2 + \cdots$$

Hence, sum of the coefficients is $(-1)^{4163} = -1$.

2. Sum of the coefficients in the expansion of $(x - 2y + 3z)^n$ is $(1 - 2 + 3)^n = 2^n$ (putting $x = y = z = 1$)

$\therefore\quad 2^n = 128 \Rightarrow n = 7$

Therefore, the greatest coefficient in the expansion of $(1 + x)^7$ is $^7C_3$ or $^7C_4$ because both are equal to 35.

3. $(1 + x - 2x^2)^6 = 1 + a_1x + a_2x^2 + \cdots + a_{12}x^{12}$

Putting $x = 1$ and $x = -1$ and adding the results

$64 = 2(1 + a_2 + a_4 + \cdots)$

$\therefore\quad a_2 + a_4 + a_6 + \cdots + a_{12} = 31$

4. $$\left(x^2 + \frac{1}{x^2} + 2\right)^n = \left(x + \frac{1}{x}\right)^{2n}$$

Hence the middle term is

$$T_{\frac{2n}{2}+1} = T_{n+1} = {}^{2n}C_n(x)^n\left(\frac{1}{x}\right)^n = {}^{2n}C_n$$

$$= \frac{(2n)!}{(n!)^2}$$

5. We have,

$$(1 + x)^{50} = \sum_{r=0}^{50} {}^{50}C_r x^r$$

Therefore, sum of the coefficients of odd powers of x is

$$^{50}C_1 + {}^{50}C_3 + \cdots + {}^{50}C_{49} = \frac{1}{2}(2^{50}) = 2^{49}$$

6. $$\frac{1}{n!} + \frac{1}{2!(n-2)!} + \frac{1}{4!(n-4)!} + \cdots$$

$$= \frac{1}{n!}\left[{}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots\right] = \frac{2^{n-1}}{n!}$$

7. There are 60 terms in the expansion of $(1 + x)^{59}$. Sum of the last 30 coefficients is

$$S = {}^{59}C_{30} + {}^{59}C_{31} + \cdots + {}^{59}C_{58} + {}^{59}C_{59}$$

$$\Rightarrow\quad S = {}^{59}C_{29} + {}^{59}C_{28} + \cdots + {}^{59}C_1 + {}^{59}C_0 \qquad [\text{Using } {}^nC_r = {}^nC_{n-r}]$$

Adding the above two expansions, we get

$$2S = {}^{59}C_0 + {}^{59}C_1 + \cdots + {}^{59}C_{59} = 2^{59}$$

$$\Rightarrow\quad S = 2^{58}$$

8. $$\sum_{j=0}^{n}\left({}^{4n+1}C_j + {}^{4n+1}C_{2n-j}\right) = ({}^{4n+1}C_0 + {}^{4n+1}C_1 + \cdots + {}^{4n+1}C_n)$$
$$+ ({}^{4n+1}C_{2n} + {}^{4n+1}C_{2n-1} + \cdots + {}^{4n+1}C_n)$$
$$= ({}^{4n+1}C_0 + {}^{4n+1}C_1 + \cdots + {}^{4n+1}C_{2n}) + {}^{4n+1}C_n$$
$$= 2^{4n} + {}^{4n+1}C_n$$

Exercise 6.3

1. Since the 7^{th} and 8^{th} terms are equal,

$$^nC_6\,x^6 = {}^nC_7\,x^7$$

$$\Rightarrow\quad x = \frac{^nC_6}{^nC_7} = \frac{7}{n-6}$$

$$\Rightarrow\quad \left(\frac{7}{x} + 6\right)^2 = n^2$$

2. $$t_r = r^2\,\frac{^nC_r}{^nC_{r-1}}$$

$$= r^2\,\frac{n-r+1}{r}$$

$$= r\,(n + 1 - r)$$

$$= (n + 1)\,r - r^2$$

$$\therefore\quad \text{Sum} = (n+1)\sum_1^n r - \sum_1^n r^2$$

$$= (n+1)\,\frac{n\,(n+1)}{2} - \frac{n\,(n+1)\,(2n+1)}{6}$$

$$= \frac{n\,(n+1)}{6}\,\{3(n+1) - (2n+1)\}$$

$$= \frac{n(n+1)\,(n+2)}{6}$$

3. Let $^nC_{r-1}$, ^nC_r, $^nC_{r+1}$ be in G.P.

$$\therefore\quad \frac{^nC_r}{^nC_{r-1}} = \frac{^nC_{r+1}}{^nC_r}$$

$$\Rightarrow\quad \frac{n-r+1}{r} = \frac{n-(r+1)+1}{r+1}$$

$\Rightarrow \quad (r+1)(n-r+1) = r(n-r)$

$\Rightarrow \quad nr - r^2 + r + n - r + 1 = nr - r^2$

$\Rightarrow \quad n + 1 = 0$

$\Rightarrow \quad n = -1$

This is not possible.

Hence not in G.P.

(ii) Let ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$ be in H.P.

$\therefore \quad \frac{1}{{}^nC_{r-1}}, \frac{1}{{}^nC_r}, \frac{1}{{}^nC_{r+1}}$ are in A.P.

$\Rightarrow \quad \frac{2}{{}^nC_r} = \frac{1}{{}^nC_{r-1}} + \frac{1}{{}^nC_{r+1}}$

$\Rightarrow \quad 2 = \frac{{}^nC_r}{{}^nC_{r-1}} + \frac{{}^nC_r}{{}^nC_{r+1}}$

$\Rightarrow \quad 2 = \frac{n-r+1}{r} + \frac{r+1}{n-r}$

$\Rightarrow \quad 2r(n-r) = (n-r)^2 + (n-r) + r^2 + r$

$\Rightarrow \quad 2rn - 2r^2 = n^2 - 2nr + r^2 + n - r + r^2 + r$

$\Rightarrow \quad n^2 + 4r^2 - 4nr + n = 0$ (1)

$\Rightarrow \quad (n-2r)^2 + n = 0$

This is not possible as both $(n-2r)^2$ and n are positive.

Hence not in H.P.

4. We know that

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}\frac{\alpha}{x}$$

$T_3 = a, T_4 = b, T_5 = c, T_6 = d$

Putting $r = 3, 4, 5$ in the above, we get

$$\frac{T_4}{T_3} = \frac{n-2}{3}\frac{\alpha}{x} = \frac{b}{a} \quad (1)$$

$$\frac{T_5}{T_4} = \frac{n-3}{4}\frac{\alpha}{x} = \frac{c}{b} \quad (2)$$

$$\frac{T_6}{T_5} = \frac{n-4}{5}\frac{\alpha}{x} = \frac{d}{c} \quad (3)$$

We have to prove

$$\frac{b^2 - ac}{c^2 - bd} = \frac{5a}{3c}$$

or

$$\frac{\frac{b}{c} - \frac{a}{b}}{\frac{c}{b} - \frac{d}{c}} = \frac{5a}{3c}$$

Now,

$$\frac{\frac{b}{c} - \frac{a}{b}}{\frac{c}{b} - \frac{d}{c}} = \frac{\frac{4x}{(n-3)\alpha} - \frac{3x}{(n-2)\alpha}}{\frac{(n-3)\alpha}{4x} - \frac{(n-4)\alpha}{5x}}$$

$$= \frac{x^2}{\alpha^2}\frac{4\times 5}{(n-3)(n-4)}\frac{4n-8-3n+9}{5n-15-4n+16}$$

$$= \frac{x^2}{\alpha^2}\frac{4\times 5}{(n-2)(n-3)}$$

Also,

$$\frac{5a}{3c} = \frac{x^2}{\alpha^2}\frac{4\times 5}{(n-2)(n-3)}$$

$\therefore \quad$ L.H.S. = R.H.S.

Exercise 6.4

1. $3^{2n+2} - 8n - 9 = 3^2(3^2)^n - 8n - 9$

$= 9(1+8)^n - 8n - 9$

$= 9[1 + {}^nC_1\,8 + {}^nC_2\,8^2 + \cdots + 8^n] - 8n - 9$

$= 64n + 9[{}^nC_2\,8^2 + {}^nC_3\,8^3 + \cdots + 8^n]$

$= 64\,[n + 9\,\{{}^nC_2 + {}^nC_3\,8 + \cdots + 8^{n-2}\}]$

$= 64\,K$ (where K is an integer)

Hence, $3^{2n+2} - 8n - 9$ is divisible by 64.

2. $49^n + 16n - 1 = (1-8)^{2n} + 16n - 1$

$= ({}^{2n}C_0 - {}^{2n}C_1 8 + {}^{2n}C_1 8^2 - \cdots + {}^{2n}C_{2n}8^{2n}) + 16n - 1$

$= (1 - 16n + {}^{2n}C_1 8^2 - \cdots + {}^{2n}C_{2n}8^{2n}) + 16n - 1$

$= ({}^{2n}C_1 8^2 - \cdots + {}^{2n}C_{2n}8^{2n})$

$= 64K$

Hence, $49^n + 16n - 1$ divisible by 64.

3. $27^{40} = 3^{120}$

$3^{119} = (4-1)^{119}$

$= {}^{119}C_0\,4^{119} + {}^{119}C_1\,4^{118} + {}^{119}C_2\,4^{117} - {}^{119}C_3\,4^{116} + \cdots + (-1)$

$\Rightarrow \quad 3^{119} = 4k - 1$

$\Rightarrow \quad 3^{120} = 12k - 3$

$= 12(k-1) + 9$

Therefore, the required remainder is 9.

4. $1 + 99^n = 1 + (100-1)^n$

$= 1 + [{}^nC_0\,100^n - {}^nC_1\,100^{n-1} + {}^nC_2\,100^{n-2} + \cdots - {}^nC_n]$ [$\because$ n is odd]

$= 100\,[{}^nC_0\,100^{n-1} - {}^nC_1\,100^{n-2} + \cdots + {}^nC_{n-1}]$

$= 100 \times$ integer, whose unit's place is different from zero ($\because$ n is odd)

Hence, the number of zeros at the end of the sum $99^n + 1$ is 2.

5. $(23)^{14} = (529)^7$

$= (530-1)^7$

$= {}^7C_0\,(530)^7 - {}^7C_1(530)^6 + \cdots - 7C_5\,(530)^2 + {}^7C_6\,530 - 1$

$= {}^7C_0\,(530)^7 - {}^7C_1\,(530)^6 + \cdots + 3710 - 1$

$= 100m + 3709$

Therefore, the last two digits are 09.

6. Given expression is

$2[1 + {}^9C_2\,(3\sqrt{2}x)^2 + {}^9C_4\,(3\sqrt{2}x)^4 + {}^9C_6\,(3\sqrt{2}x)^6 + {}^9C_8\,(3\sqrt{2}x)^8]$

Therefore, the number of non-zero terms is 5.

7. $(x+a)^n-(x-a)^n = 2[{}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + {}^nC_5x^{n-5}a^5 + \cdots]$

$\therefore \quad (\sqrt{2}+1)^6-(\sqrt{2}-1)^6$

$= 2[{}^6C_1(\sqrt{2})^5(1)^1 + {}^6C_3(\sqrt{2})^3(1)^3 + {}^6C_5(\sqrt{2})^1(1)^5]$

$\because \quad (\sqrt{2}+1)^6-(\sqrt{2}-1)^6 = 2[6\times4\sqrt{2}+20\times2\sqrt{2}+6\sqrt{2}]$

$= 2[24\sqrt{2}+40\sqrt{2}+6\sqrt{2}] = 140\sqrt{2}$

8. We have,

$$\frac{1}{\sqrt{4x+1}}\left\{\left(\frac{1+\sqrt{4x+1}}{2}\right)^n-\left(\frac{1-\sqrt{4x+1}}{2}\right)^n\right\}$$

$$= \frac{1}{2^n\sqrt{4x+1}}\{2[{}^nC_1\sqrt{4x+1}+{}^nC_3(\sqrt{4x+1})^3+\cdots]\}$$

$$= \frac{1}{2^{n-1}}[{}^nC_1+{}^nC_3(\sqrt{4x+1})^2+{}^nC_5(\sqrt{4x+1})^4+\cdots]$$

Since R.H.S. of the given expression contains x^5, so this expression should go up to $(\sqrt{4x+1})^{10}$ and hence $n = 11$ or 12.

9. Let $(\sqrt{3}+1)^{2m} = I+f$, where I is the integral and f the fractional part.

$\therefore \quad 0 < f < 1$

Also, let $\quad f' = (\sqrt{3}-1)^{2m}$. Now,

$$\begin{aligned} I+f+f' &= (\sqrt{3}+1)^{2m}+(\sqrt{3}-1)^{2m} \\ &= (4+2\sqrt{3})^m+(4-2\sqrt{3})^m \\ &= 2^m[(2+\sqrt{3})^m+(2-\sqrt{3})^m] \\ &= 2^{m+1}[2^m+{}^mC_2\,2^{m-2}(\sqrt{3})^2+\cdots] \end{aligned}$$

or

$I+f+f' =$ even integer $\quad$ (1)

$\Rightarrow \quad f+f'$ is an integer

$\because \quad 0<f<1$ and $0<f'<1$

$\therefore \quad 0<f+f'<2$

Thus, $f+f'$ is an integer between 0 and 2.

$\therefore \quad f+f' = 1$

Hence from (1), we conclude that $I+f+f'$ is an integer next above $(\sqrt{3}+1)^{2m}$ and it contains 2^{m+1} as a factor.

Exercise 6.5

1. We have,

$C_0-C_2+C_4-C_6+\cdots = 2^{n/2}\cos\dfrac{n\pi}{4} \quad$ (1)

We also know that

$C_0+C_2+C_4+C_6+\cdots = 2^{n-1} \quad$ (2)

Adding (1) and (2), we have

$$(C_0+C_4+C_8+\cdots) = \frac{1}{2}\left(2^{n/2}\cos\frac{n\pi}{4}+2^{n-1}\right)$$

2. We have,

$${}^{4n}C_0+{}^{4n}C_2x^2+{}^{4n}C_4x^4+\cdots+{}^{4n}C_{4n}x^{4n}$$
$$= \frac{1}{2}\left[(1+x)^{4n}+(1-x)^{4n}\right]$$

Putting $x=1$ and $x=i$, we get

$${}^{4n}C_0+{}^{4n}C_2+{}^{4n}C_4+\cdots+{}^{4n}C_{4n} = \frac{1}{2}[2^{4n}]$$

and ${}^{4n}C_0-{}^{4n}C_2+{}^{4n}C_4-\cdots+{}^{4n}C_{4n}$

$$= \frac{1}{2}[(1+i)^{4n}+(1-i)^{4n}]$$

Thus,

$$2[{}^{4n}C_0+{}^{4n}C_4+\cdots+{}^{4n}C_{4n}] = 2^{4n-1}+\frac{1}{2}\left[(1+i)^{4n}+(1-i)^{4n}\right]$$

Now,

$(1+i)^{4n}+(1-i)^{4n}$

$$= \left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^{4n}+\left[\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)\right]^{4n}$$

$= 2^{2n}(\cos n\pi+i\sin n\pi)+2^{2n}(\cos n\pi-i\sin n\pi)$

$= 2^{2n+1}\cos n\pi = 2^{2n+1}(-1)^n$

$\therefore \quad 2[{}^{4n}C_0+{}^{4n}C_4+\cdots+{}^{4n}C_{4n}] = 2^{4n-1}+\dfrac{1}{2}2^{2n+1}(-1)^n$

$\Rightarrow \quad {}^{4n}C_0+{}^{4n}C_4+\cdots+{}^{4n}C_{4n} = 2^{4n-2}+(-1)^n\,2^{2n-1}$

Exercise 6.6

1. $3^{50}\left(1+\dfrac{2x}{3}\right)^{50}$

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}\frac{2x}{3} = \frac{51-r}{r}\frac{2}{15}$$

As $x = 1/5$,

$\therefore \quad \dfrac{T_{r+1}}{T_r} \ge 1 \Rightarrow 102-2r \ge 15r$

$\Rightarrow \quad 102 \ge 17r$

$\Rightarrow \quad r \le 6$

Hence, the 7th and 6th terms are the greatest.

$$T_{6+1} = 3^{50}\,{}^{50}C_6\left(\frac{2}{15}\right)^6$$

and $\quad T_{5+1} = 3^{50}\,{}^{50}C_5\left(\dfrac{2}{15}\right)^5$

Also, $\quad \dfrac{T_7}{T_6} = \dfrac{{}^{50}C_6}{{}^{50}C_5}\dfrac{2}{15}$

$$= \frac{50-6+1}{6}\frac{2}{15}$$

$$= \frac{45}{6}\frac{2}{15} = 1$$

2. In the expansion of $(1+x)^n$,

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}x$$

Here, $n = 8, x = 4/3$.

$$\therefore \quad \frac{T_{r+1}}{T_r} = \frac{9-r}{r} \times \frac{4}{3}$$

$\therefore \quad T_{r+1} \geq T_r$ if

$$36 - 4r \geq 3r$$

$$\Rightarrow \quad 36 \geq 7r$$

$$\Rightarrow \quad r \leq 5\frac{1}{7}$$

Hence $r = 5$, so T_{5+1}, i.e., the 6th term is the greatest.

$$T_6 = {}^8C_5(4x)^5 = \frac{8!}{5!3!}\left(\frac{4}{3}\right)^5$$

$$= \frac{8\times7\times6}{6}\left(\frac{4}{3}\right)^5$$

$$= 56\left(\frac{4}{3}\right)^5$$

3. If n is even, the greatest coefficient is ${}^nC_{n/2}$. Therefore, the greatest term is ${}^nC_{n/2}\, x^{n/2}$.

$$\therefore \quad {}^nC_{n/2}\, x^{n/2} > {}^nC_{(n/2)-1} x^{(n-2)/2} \text{ and } {}^nC_{n/2} x^{n/2} > {}^nC_{(n/2)+1} x^{(n/2)+1}$$

$$\Rightarrow \quad \frac{n - \frac{n}{2} + 1}{\frac{n}{2}}\, x > 1 \text{ and } \frac{\frac{n}{2}}{\frac{n}{2}+1}\, x < 1$$

$$\Rightarrow \quad x > \frac{\frac{n}{2}}{\frac{n}{2}+1} \text{ and } x < \frac{\frac{n}{2}+1}{\frac{n}{2}}$$

$$\Rightarrow \quad x > \frac{n}{n+2} \text{ and } x < \frac{n+2}{n}$$

4. In the expansion of $(2x + 5)^{10}$, the middle term is T_6.

Consider the expansion of $(1 + 2x/5)^{10}$. Now,

$$\left|\frac{T_6}{T_5}\right| > 1 \text{ and } \left|\frac{T_7}{T_6}\right| < 1$$

$$\Rightarrow \quad \left|\frac{10-5+1}{5}\frac{2x}{5}\right| > 1 \text{ and } \left|\frac{10-6+1}{6}\frac{2x}{5}\right| < 1$$

$$\Rightarrow \quad \left|\frac{12}{25}x\right| > 1 \text{ and } \left|\frac{x}{3}\right| < 1$$

$$\Rightarrow \quad \frac{25}{12} < |x| < 3$$

$$\Rightarrow \quad x \in \left(-3, -\frac{25}{12}\right) \cup \left(\frac{25}{12}, 3\right)$$

Exercise 6.7

1. $$S = \sum_{r=1}^{n} \frac{1^2 + 2^2 + \cdots + r^2}{2r+1}\, {}^nC_r$$

$$= \sum_{r=1}^{n} \frac{r(r+1)(2r+1)}{6(2r+1)}\, {}^nC_r$$

$$= \frac{1}{6}\sum_{r=1}^{n} r(r+1)\, {}^nC_r$$

$$= \frac{1}{6}\sum_{r=1}^{n} (r+1)\cdot n\cdot {}^{n-1}C_{r-1}$$

$$= \frac{1}{6} n \sum_{r=1}^{n} \left((r-1)+2\right) {}^{n-1}C_{r-1}$$

$$= \frac{1}{6} n\cdot \sum_{r=1}^{n} \left((r-1)\cdot {}^{n-1}C_{r-1} + 2\cdot {}^{n-1}C_{r-1}\right)$$

$$= \frac{1}{6} n\cdot \sum_{r=1}^{n} \left((n-1)\cdot {}^{n-2}C_{r-2} + 2\cdot {}^{n-1}C_{r-1}\right)$$

$$= \frac{1}{6} n\cdot(n-1)\cdot 2^{n-2} + \frac{n}{3}\cdot 2^{n-1}$$

$$= \frac{1}{6} n(n+3)\, 2^{n-2}$$

2. $$\sum_{r=0}^{n} r^2\, {}^nC_r p^r q^{n-r}$$

$$= \sum_{r=0}^{n} nr\, {}^{n-1}C_{r-1} p^r q^{n-r}$$

$$= n\sum_{r=0}^{n} [(r-1)+1]\, {}^{n-1}C_{r-1} p^r q^{n-r}$$

$$= n\sum_{r=0}^{n} [(r-1)\, {}^{n-1}C_{r-1} + {}^{n-1}C_{r-1}]\, p^r q^{n-r}$$

$$= n\sum_{r=0}^{n} [(n-1)\, {}^{n-2}C_{r-2} + {}^{n-1}C_{r-1}]\, p^r q^{n-r}$$

$$= p^2 n(n-1)\sum_{r=0}^{n} {}^{n-2}C_{r-2} p^{r-2} q^{n-r} + np\sum_{r=0}^{n} {}^{n-1}C_{r-1} p^{r-1} q^{n-r}$$

$$= p^2 n(n-1)(p+q)^{n-2} + np(p+q)^{n-1}$$

$$= p^2 n(n-1) + np$$

$$= p^2 n^2 + np(1-p)$$

$$= p^2 n^2 + npq$$

3. $$S = 1 - {}^nC_1\left(\frac{1+x}{1+nx}\right) + {}^nC_2 \frac{1+2x}{(1+nx)^2} + \cdots$$

$$= \sum_{r=0}^{n} (-1)^r\, {}^nC_r \frac{(1+rx)}{(1+nx)^r}$$

$$= \sum_{r=0}^{n} (-1)^r \left[\frac{{}^nC_r}{(1+nx)^r} + \frac{{}^nC_r\, rx}{(1+nx)^r}\right]$$

$$= \sum_{r=0}^{n} {}^nC_r\left(-\frac{1}{1+nx}\right)^r + x\sum_{r=0}^{n} \frac{n.\, {}^{n-1}C_{r-1}}{(1+nx)^r}(-1)^r$$

$$= \left[1 - \frac{1}{1+nx}\right]^n - \left(\frac{nx}{1+nx}\right)\sum_{r=0}^{n} {}^{n-1}C_{r-1}\left(-\frac{1}{1+nx}\right)^{r-1}$$

$$= \left[1 - \frac{1}{1+nx}\right]^n - \left(\frac{nx}{1+nx}\right)\left[1 - \frac{1}{1+nx}\right]^{n-1}$$

$$= \left[1-\frac{1}{1+nx}\right]^{n-1}\left[1-\frac{1}{1+nx}-\frac{nx}{1-nx}\right]=0$$

4. $S = \dfrac{{}^nC_0}{1}+\dfrac{{}^nC_2}{3}+\dfrac{{}^nC_4}{5}+\dfrac{{}^nC_6}{7}+\cdots$

The general term of the series is

$$\frac{{}^nC_{2r}}{2r+1} = \frac{{}^{n+1}C_{2r+1}}{n+1}, \text{ where } r = 0, 1, 2, \ldots$$

$$\therefore \quad S = \frac{1}{n+1}[{}^{n+1}C_1+{}^{n+1}C_3+{}^{n+1}C_5+\cdots]$$

$$= \frac{2^n}{n+1}$$

5. $C_2+2C_3+3C_4+\cdots+14C_{15} = \sum_{r=1}^{14} r\,{}^{15}C_{r+1}$

$$= \sum_{r=1}^{14}[(r+1)-1]\,{}^{15}C_{r+1}$$

$$= \sum_{r=1}^{14}[(r+1)\,{}^{15}C_{r+1}-{}^{15}C_{r+1}]$$

$$= \sum_{r=1}^{14}(15\,{}^{14}C_r-{}^{15}C_{r+1})$$

$$= 15(2^{14}-1)-(2^{15}-{}^{15}C_0-{}^{15}C_1)$$

$$= 13\times 2^{14}+1$$

6. Given polynomial is $n + 1$ degree polynomial.

$(x+{}^nC_0)\,(x+3\,{}^nC_1)\,(x+5\,{}^nC_2)\cdots[x+(2n+1)\,{}^nC_n] = x^{n+1} + a_1x^n+\cdots+a_{n+1}$

Then,

$$-\frac{a_1}{a_0} = -{}^nC_0-3{}^nC_1-5{}^nC_2-\cdots-(2n+1){}^nC_n$$

$$\Rightarrow \quad a_1 = {}^nC_0+3{}^nC_1+5{}^nC_2+\cdots+(2n+1){}^nC_n$$

$$= \sum_{r=0}^{n}{}^nC_r(2r+1) = 2\sum_{r=0}^{n} r\,{}^nC_r+\sum_{r=0}^{n}{}^nC_r$$

$$= 2\sum_{r=0}^{n} r^{n-1}C_r+\sum_{r=0}^{n}{}^nC_r$$

$$= 2n\,2^{n-1}+2^n = (n+1)2^n$$

7. ${}^{20}C_0-\dfrac{{}^{20}C_1}{2}+\dfrac{{}^{20}C_2}{3}-\dfrac{{}^{20}C_3}{4}+\ldots$

$$= \sum_{r=0}^{20}\frac{{}^{20}C_r}{r+1}(-1)^r = \sum_{r=0}^{20}\frac{{}^{21}C_{r+1}}{20+1}(-1)^r = -\frac{1}{21}\sum_{r=0}^{20}{}^{21}C_{r+1}(-1)^{r+1}$$

$$= -\frac{1}{21}\left[-{}^{21}C_1+{}^{21}C_2-{}^{21}C_3+\ldots\right]$$

$$= -\frac{1}{21}\left[({}^{21}C_0-{}^{21}C_1+{}^{21}C_2-{}^{21}C_3+\ldots)-{}^{21}C_0\right]$$

$$= -\frac{1}{21}[(1-1)^{21}-1] = \frac{1}{21}$$

8. We have,

$$\frac{1}{81^n}-\frac{10}{81^n}{}^{2n}C_1+\frac{10^2}{81^n}{}^{2n}C_2-\frac{10^3}{81^n}{}^{2n}C_3+\cdots+\frac{10^{2n}}{81^n}$$

$$= \frac{1}{81^n}[{}^{2n}C_0-{}^{2n}C_1 10^1+{}^{2n}C_2 10^2-{}^{2n}C_3 10^3+\cdots+{}^{2n}C_{2n}10^{2n}]$$

$$= \frac{1}{81^n}[1-10]^{2n} = \frac{(-9)^{2n}}{81^n} = \frac{81^n}{81^n} = 1$$

Exercise 6.8

1. $\sum_{r=0}^{n} r(n-r)({}^nC_r)^2 = \sum_{r=0}^{n} r\,{}^nC_r(n-r)\,{}^nC_{n-r}$

$$= \sum_{r=0}^{n} n\,{}^{n-1}C_{r-1}\,n\,{}^{n-1}C_{n-r-1}$$

$$= n^2\sum_{r=0}^{n}{}^{n-1}C_{r-1}\,{}^{n-1}C_{n-r-1}$$

$= n^2\times$ coefficient of x^{n-2} in $(1+x)^{n-1}\,(1+x)^{n-1}$

$= n^2\times{}^{2n-2}C_{n-2} = n^2\,{}^{2n-2}C_n$

2. $\sum_{r=0}^{2n}(-1)^r\,({}^{2n}C_r)^2 = \sum_{r=0}^{2n}(-1)^r\,{}^{2n}C_r\,{}^{2n}C_r$

$$= \sum_{r=0}^{2n}(-1)^r\,{}^{2n}C_r\,{}^{2n}C_{2n-r}$$

= Coefficient of x^{2n} in $(1-x)^{2n}\,(1+x)^{2n}$

= Coefficient of x^{2n} is $(1-x^2)^{2n}$

$= (-1)^n\,{}^{2n}C_n$

3. ${}^nC_0\,{}^nC_0-{}^nC_1\,{}^{n+1}C_1+{}^nC_2\,{}^{n+2}C_2-\cdots$

$= {}^nC_0\,{}^nC_n-{}^nC_1\,{}^{n+1}C_n+{}^nC_2\,{}^{n+2}C_n-\cdots$

= Coefficient of x^n in $[{}^nC_0(1+x)^n-{}^nC_1(1+x)^{n+1}+{}^nC_2(1+x)^{n+2}+\cdots+(-1)^n\,{}^nC_n\,(1+x)^{2n}]$

= Coefficient of x^n in $(1+x)^n\,[{}^nC_0-{}^nC_1\,(1+x)+{}^nC_2(1+x)^2-\cdots+(-1)^n\,{}^nC_n\,(1+x)^n]$

= Coefficient of x^n in $(1+x)^n\,[1-(1+x)]^n$

= Coefficient of x^n in $(1+x)^n\,(-x)^n$

$= (-1)^n$

4. L.H.S. $= {}^nC_0\,{}^{2n}C_n-{}^nC_1\,{}^{2n-2}C_n+{}^nC_2\,{}^{2n-4}C_n-\cdots$

= Coefficient of x^n in $[{}^nC_0(1+x)^{2n}-{}^nC_1\,(1+x)^{2n-2}+{}^nC_2(1+x)^{2n-4}-\cdots]$

= Coefficient of x^n in $[(1+x)^2-1]^n$

= Coefficient of x^n in $(2x+x^2)^n$

$= \text{Coefficient of } x^n \text{ in } x^n(2+x)^n$

$= 2^n$

$=$ R.H.S.

5. $$S = \sum_{p=1}^{n}\left(\sum_{m=p}^{n} {}^nC_m\, {}^mC_p\right)$$

$$= \sum_{p=1}^{n}({}^nC_p\, {}^pC_p + {}^nC_{p+1}\, {}^{p+1}C_p + \cdots + {}^nC_n\, {}^nC_p)$$

$$= \sum_{p=1}^{n}[\text{coefficient of } x^p \text{ in } \{{}^nC_p(1+x)^p + {}^nC_{p+1}(1+x)^{p+1} + \cdots + {}^nC_n(1+x)^n\}]$$

$$= \sum_{p=1}^{n}[\text{coefficient of } x^p \text{ in } \{{}^nC_0 + {}^nC_1(1+x) + {}^nC_2(1+x)^2 + \cdots + {}^nC_{p-1}(1+x)^{p-1} + {}^nC_p(1+x)^p + \cdots + {}^nC_n(1+x)^n\}]$$

$$= \sum_{p=1}^{n}[\text{coefficient of } x^p \text{ in } \{1+(1+x)\}^n]$$

$$= \sum_{p=1}^{n}[\text{coefficient of } x^p \text{ in } (2+x)^n]$$

$$= \sum_{p=1}^{n}[{}^nC_p\, 2^{n-p}]$$

$$= {}^nC_1\, 2^{n-1} + {}^nC_2\, 2^{n-2} + \cdots + {}^nC_n$$

$$= {}^nC_0\, 2^n + {}^nC_1\, 2^{n-1} + {}^nC_2\, 2^{n-2} + \cdots + {}^nC_n - {}^nC_0\, 2^n$$

$$= (1+2)^n - 2^n$$

$$= 3^n - 2^n$$

Alternative solution:

$$\sum_{p=1}^{n}\left(\sum_{m=p}^{n} {}^nC_m\, {}^mC_p\right) = \sum_{p=1}^{n}\left(\sum_{m=p}^{n}\frac{n!}{(n-m)!\,m!}\,\frac{m!}{(m-p)!\,p!}\right)$$

$$= \sum_{p=1}^{n}\left(\sum_{m=p}^{n}\frac{n!}{(n-m)!}\,\frac{1}{(m-p)!\,p!}\right)$$

$$= \sum_{p=1}^{n}\frac{n!}{(n-p)!\,p!}\left(\sum_{m=p}^{n}\frac{(n-p)!}{(n-m)!(m-p)!}\right)$$

$$= \sum_{p=1}^{n}{}^nC_p\left(\sum_{m=p}^{n}{}^{n-p}C_{m-p}\right)$$

$$= \sum_{p=1}^{n}{}^nC_p 2^{n-p}$$

$$= {}^nC_1 2^{n-1} + {}^nC_2 2^{n-2} + \cdots + {}^nC_n 2^0$$

$$= (2+1)^n - {}^nC_0 2^n$$

$$= 3^n - 2^n$$

Also,

$$\lim_{n\to\infty}\frac{1}{3^n}\sum_{p=1}^{n}\left(\sum_{m=p}^{n}{}^nC_m\, {}^mC_p\right) = \lim_{n\to\infty}\frac{3^n-2^n}{3^n} = 1$$

Exercise 6.9

1. We have,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots$$

Given that the third term is $-1/8x^2$, hence

$$\frac{m(m-1)}{2}x^2 = -\frac{1}{8}x^2$$

$\Rightarrow\ 4m^2 - 4m = -1$

$\Rightarrow\ (2m-1)^2 = 0 \Rightarrow m = \dfrac{1}{2}$

2. $$(217)^{1/3} = (6^3+1)^{1/3} = 6\left(1+\frac{1}{6^3}\right)^{1/3}$$

$$= 6\left(1 + \frac{1}{3\times 216} - \frac{1\times 2}{3\times 3\times 2}\left(\frac{1}{216}\right)^2 + \cdots\right) = 6.01$$

3. $$\left(\frac{a+x}{a}\right)^{-1/2} + \left(\frac{a-x}{a}\right)^{-1/2} = \left(1+\frac{x}{a}\right)^{-1/2} + \left(1-\frac{x}{a}\right)^{-1/2}$$

$$= \left[1 + \left(-\frac{1}{2}\right)\left(\frac{x}{a}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2.1}\left(\frac{x}{a}\right)^2 + \cdots\right]$$

$$+ \left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{a}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2.1}\left(-\frac{x}{a}\right)^2 + \cdots\right]$$

$$= 2 + \frac{3x^2}{4a^2} + \cdots$$

Hence, the coefficient of x^2 is $3/4a^2$.

4. Given expression is

$$(1 + x + x^2 + \cdots)^2 = [(1-x)^{-1}]^2$$

$$= (1-x)^{-2}$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + (n+1)x^n + \cdots$$

Therefore, the coefficient of x^n is $n+1$.

5. Given that $|x| > 1$.

So, the given expression can be written as

$$x^{-2}\left(1+\frac{1}{x}\right)^{-2} = x^{-2}\left[1 - \frac{2}{x} + \frac{3}{x^2} - \frac{4}{x^3} + \cdots\right]$$

$$= \left[\frac{1}{x^2} - \frac{2}{x^3} + \frac{3}{x^4} - \frac{4}{x^5} + \cdots\right]$$

6. Since $1 + 2x + 3x^2 + 4x^3 + \cdots \infty = (1-x)^{-2}$, therefore, we have

$$(1 + 2x + 3x^2 + 4x^3 + \cdots \infty)^{1/2} = [(1-x)^{-2}]^{1/2}$$
$$= (1-x)^{-1}$$
$$= 1 + x + x^2 + \cdots + x^n + \cdots \infty$$

Therefore, the coefficient of x^n is 1.

7. We have,

$$T_{r+1} = \frac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)\cdots\left(\frac{7}{2}-r+1\right)x^r}{r!}$$

This will be the first negative term when

$$\frac{7}{2} - r + 1 < 0, \text{ i.e., } r > \frac{9}{2}$$

Hence, $r = 5$.

Chapter 7

Exercise 7.1

1. Since the determinant is symmetrical it is simple to expand by Sarrus rule. Hence,

$$\Rightarrow \begin{vmatrix} \tan A & 1 & 1 \\ 1 & \tan B & 1 \\ 1 & 1 & \tan C \end{vmatrix} = \tan A \tan B \tan C + 1 + 1 - \tan A$$
$$- \tan B - \tan C$$

$= 2$ [as in a triangle $\tan A + \tan B + \tan C = \tan A \tan B \tan C$)

2. Expanding by Sarrus rule,

$$\begin{vmatrix} 1 & e^{i\pi/3} & e^{i\pi/4} \\ e^{-i\pi/3} & 1 & e^{i2\pi/3} \\ e^{-i\pi/4} & e^{-i2\pi/3} & 1 \end{vmatrix} = 1 + e^{i\pi/3} \times e^{i2\pi/3} \times e^{-i\pi/4} + e^{-i\pi/3}$$
$$\times e^{-i2\pi/3} \times e^{i\pi/4} - (e^{-i\pi/4} \times e^{i\pi/4} + e^{-i\pi/3} \times e^{i\pi/3} + e^{-i2\pi/3} \times e^{i2\pi/3})$$

$= 1 + e^{i3\pi/4} + e^{-i3\pi/4} - (1 + 1 + 1)$

$= -2 + 2\cos(3\pi/4)$

$= -2 - \sqrt{2}$

3. Expanding along the first row, we get

$\Delta = (x-a)(x+b)(x-c) + (x-b)(x+a)(x+c)$

$= 2x(x^2 + ac - ab - bc)$

Now

$\Delta = 0$ gives $x = 0$

or

$x^2 = b(a+c) - ac$

If $b(a+c) > ac$, we have three roots $0, \pm\sqrt{\{b(a+c) - ac\}}$, but if $b(a+c) \le ac$, the only real root is $x = 0$.

4. $$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = (\alpha+\beta+\gamma)\frac{1}{2}[(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2] = 0$$

Since $\alpha \ne \beta \ne \gamma$

$\therefore \quad \alpha + \beta + \gamma = 0$

$\alpha + \beta - \gamma = -2\gamma$

$\beta + \gamma - \alpha = -2\alpha$

$\gamma + \alpha - \beta = -2\beta$

Let $y = -2x \Rightarrow x = \frac{-y}{2}$

Therefore, the required equation is

$$a\left(-\frac{y}{2}\right)^3 + b\left(-\frac{y}{2}\right)^2 + c\left(-\frac{y}{2}\right) + d = 0$$

$$\frac{-a}{8}y^3 + \frac{b}{4}y^2 - \frac{c}{2}y + d = 0$$

or

$$ay^3 - 2by^2 + 4cy - 8d = 0$$

or

$$ax^3 - 2bx^2 + 4cx - 8d = 0$$

Exercise 7.2

1.

a. Operating $C_2 - lC_1 - mC_3$, we get the value of the determinant is 0.

b. Operating $C_1 \to C_1 + C_2 + C_3$, we get the value of the determinant is 0.

c. $$\Delta = \begin{vmatrix} \log x & \log y & \log z \\ \log 2 + \log x & \log 2 + \log y & \log 2 + \log z \\ \log 3 + \log x & \log 3 + \log y & \log 3 + \log z \end{vmatrix}$$

Operating $R_3 \to R_3 - R_1$ and $R_2 \to R_2 - R_1$, we get

$$\Delta = \begin{vmatrix} \log x & \log y & \log z \\ \log 2 & \log 2 & \log 2 \\ \log 3 & \log 3 & \log 3 \end{vmatrix}$$

$= 0 \quad (\because R_2$ and R_3 are same)

d. Using $C_1 \to C_1 - C_2 - 4C_3$, we have

$$\begin{vmatrix} 0 & (a^x - a^{-x})^2 & 1 \\ 0 & (b^y - b^{-y})^2 & 1 \\ 0 & (c^z - c^{-z})^2 & 1 \end{vmatrix} = 0$$

e. Applying $C_3 \to C_3 - C_1$, we get

$$\begin{vmatrix} \sin^2\left(x + \frac{3\pi}{2}\right) & \sin^2\left(x + \frac{5\pi}{2}\right) & \sin(2x+5\pi)\sin(2\pi) \\ \sin\left(x + \frac{3\pi}{2}\right) & \sin\left(x + \frac{5\pi}{2}\right) & 2\cos\left(x + \frac{5\pi}{2}\right)\sin(\pi) \\ \sin\left(x - \frac{3\pi}{2}\right) & \sin\left(x - \frac{5\pi}{2}\right) & 2\cos\left(x - \frac{5\pi}{2}\right)\sin(-\pi) \end{vmatrix} = 0$$

[$\because$ all elements of C_3 are zero]

2. $\Delta = \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$

$= \begin{vmatrix} b+c & -b & a \\ a+c & -c & b \\ a+b & -a & c \end{vmatrix}$ [Applying $C_2 \to C_2 - C_3$]

$= \begin{vmatrix} c & -b & a \\ a & -c & b \\ b & -a & c \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2$]

$= \begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$ [Interchanging C_1 and C_3]

$= 3abc - a^3 - b^3 - c^3$

3. Let $\Delta = \begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$

$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$ [Applying $R_2 \to R_2 - R_3$; then take 4 common from R_2]

$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$ [$R_3 \to R_3 - R_1 + 2R_2$]

$= -4\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$= -4(a-b)(b-c)(c-a)$

Hence $k = -4$.

4. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc+a^2 & ac+b^2 & ab+c^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

Now,

$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \dfrac{1}{abc}\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ abc & abc & abc \end{vmatrix}$ [Multiplying C_1, C_2, C_3 by a, b, c, respectively]

$= \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$\Rightarrow \Delta = 2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)$

5. Let $\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$

Applying $C_2 \to C_2 - pC_1$ and $C_3 \to C_3 - qC_1$, we get

$\Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 1+3p \\ 3 & 6 & 1+6p \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 6 & 1 \end{vmatrix}$ [Applying $C_3 \to C_3 - pC_2$]

$= \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 5 & 1 \end{vmatrix}$ [Applying $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$]

$= 5 - 4 = 1$ [Expanding along R_1]

6. Operating $C_1 \to C_1 - bC_3$, $C_2 \to C_2 + aC_3$, we get

$\Delta = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$

$= (1+a^2+b^2)^2 \times \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$

$R_3 \to R_3 - bR_1 + aR_2$ gives

$\Delta = (1+a^2+b^2)^2 \times \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix}$

$= (a + a^2 + b^2)^3$ [Expanding along C_1]

7. Let $\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_2 + C_3$]

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & -b+a \\ 0 & -c+a & 2c+a \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

$$= (a+b+c)\begin{vmatrix} 2b+a & -b+a \\ -c+a & 2c+a \end{vmatrix} \qquad \text{[Expanding along } C_1]$$

$$= (a+b+c)\{(2b+a)(2c+a) - (-b+a)(-c+a)\}$$

$$= (a+b+c)\{(4bc+2ab+2ca+a^2) - (bc-ab-ac+a^2)\}$$

$$= (a+b+c)(3bc+3ab+3ca)$$

$$= 3(a+b+c)(ab+bc+ca)$$

8. Let a be the first term and d be the common difference of corresponding A.P. Then,

$$\Delta = xyz\begin{vmatrix} 1/x & 1/y & 1/z \\ p & 2q & 3r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= xyz\begin{vmatrix} a+(p-1)d & a+(2q-1)d & a+(3r-1)d \\ p & 2q & 3r \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - aR_3$, $R_2 \to R_2 - R_3$ and then taking d common from R_1, we get

$$\Delta = xyzd\begin{vmatrix} (p-1) & (2q-1) & (3r-1) \\ (p-1) & (2q-1) & (3r-1) \\ 1 & 1 & 1 \end{vmatrix} = 0$$

9. The given determinant can be written as the sum of two determinants

$$\begin{vmatrix} x_1 & a_1b_2 & a_1b_3 \\ 0 & x_2+a_2b_2 & a_2b_3 \\ 0 & a_3b_2 & x_3+a_3b_3 \end{vmatrix} + \begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & x_2+a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & x_3+a_3b_3 \end{vmatrix}$$

Expanding the first determinant along C_1, we get

$$x_1\begin{vmatrix} x_2+a_2b_2 & a_2b_3 \\ a_3b_2 & x_3+a_3b_3 \end{vmatrix} = x_1[(x_2+a_2b_2)(x_3+a_3b_3) - a_3b_2a_2b_3]$$

$$= x_1(x_2x_3 + x_3a_2b_2 + x_2a_3b_3 + a_2b_2a_3b_3 - a_3b_2a_2b_3]$$

$$= x_1x_2x_3 + x_1x_3a_2b_2 + x_1x_2a_3b_3$$

In the second determinant, taking b_1 common from C_1, and then applying $C_2 \to C_2 - b_2C_1$ and $C_3 \to C_3 - b_3C_1$, we obtain

$$b_1\begin{vmatrix} a_1 & 0 & 0 \\ a_2 & x_2 & 0 \\ a_3 & 0 & x_3 \end{vmatrix} = a_1b_1x_2x_3$$

Therefore the given determinant is

$$x_1x_2x_3 + x_1x_3a_2b_2 + x_1x_2a_3b_3 + a_1b_1x_2x_3$$

$$= x_1x_2x_3\left(1 + \frac{a_1b_1}{x_1} + \frac{a_2b_2}{x_2} + \frac{a_3b_3}{x_3}\right)$$

10. $\Sigma 2^{r-1} = 1 + 2 + 2^2 + \cdots + 2^{n-1} = 1 \times \dfrac{(2^n-1)}{2-1} = 2^n - 1$

$\Sigma 2 \times 3^{r-1} = 2(1 + 3 + 3^2 + \cdots + 3^{n-1}) = \dfrac{2(3^n-1)}{3-1} = 3^n - 1$

$\Sigma 4 \times 5^{r-1} = 4(1 + 5 + 5^2 + \cdots + 5^{n-1}) = \dfrac{4(5^n-1)}{5-1} = 5^n - 1$

$$\therefore \quad \Sigma D_r = \begin{vmatrix} 2^{n-1} & 3^{n-1} & 5^{n-1} \\ \alpha & \beta & \gamma \\ 2^{n-1} & 3^{n-1} & 5^{n-1} \end{vmatrix} = 0$$

11. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{vmatrix}$$

$$= (a+b+c-x)\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$, we get

$$\Delta = (a+b+c-x)\begin{vmatrix} 0 & c-b+x & b-a \\ 0 & b-a-x & a-c+x \\ 1 & a & c-x \end{vmatrix}$$

$$= (a+b+c-x)\{(x+c-b)(x+a-c) + (x+a-b)(b-a)\}$$

$$= (a+b+c-x)\{(x^2 + x(a-b) + (c-b)(a-c) + x(b-a) - (a-b)^2\}$$

$$= (a+b+c-x)(x^2 + ac - c^2 - ab + bc - a^2 - b^2 - 2ab)$$

$$= (a+b+c-x)(x^2 - a^2 - b^2 - c^2 + ab + bc + ca)$$

$$= (a+b+c-x)\left(x^2 - \frac{1}{2}\{(a-b)^2 + (b-c)^2 + (c-a)^2\}\right)$$

Since $\Delta = 0$,

$$\therefore \quad x = a+b+c, \pm\sqrt{\frac{1}{2}((a-b)^2 + (b-c)^2 + (c-a)^2)}$$

12. Applying $R_1 \to R_1 - R_2$, we get

$$\Delta = \begin{vmatrix} x-2 & 3(x-2) & -(x-2) \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$$

$$= (x-2)\begin{vmatrix} 1 & 3 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$$

$$= (x-2)\begin{vmatrix} 1 & 3 & -1 \\ 0 & -3(x+2) & x-1 \\ 0 & 2x+9 & x-1 \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - 2R_1, \; R_3 \to R_3 + 3R_1]$$

$$= (x-2)\{-(3x+6)(x-1) - (x-1)(2x+9)\}$$

$$= (x-2)(x-1)(5x+15)$$

$\therefore \quad \Delta = 0$ gives $x = 2, 1, -3$

13. Since $A_1B_1C_1, A_2B_2C_2, A_3B_3C_3$ are divisible by k, we have $m_1, m_2, m_3 \in \mathbb{Z}$ such that

$$100A_1 + 10B_1 + C_1 = m_1k$$

$$100A_2 + 10B_2 + C_2 = m_2k$$

$$100A_3 + 10B_3 + C_3 = m_3k$$

Now,

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} A_1 & B_1 & 100A_1+10B_1+C_1 \\ A_2 & B_2 & 100A_2+10B_2+C_2 \\ A_3 & B_3 & 100A_3+10B_3+C_3 \end{vmatrix} \quad [C_3 \to C_3 + 10C_2 + 100C_1]$$

$$= \begin{vmatrix} A_1 & B_1 & m_1 \\ A_2 & B_2 & m_2 \\ A_3 & B_3 & m_3 \end{vmatrix} = k\Delta^*$$

As elements of Δ^* are integers, Δ^* is an integer. Let $\Delta^* = p$. Then we get $\Delta = kp$, i.e. Δ is divisible by k.

Exercise 7.3

1. Consider the product

$$\begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\beta & \gamma\delta & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ \gamma+\delta & \alpha+\beta & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \gamma\delta+\alpha\beta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$$

$$= \Delta$$

Hence $\Delta = 0$ [being the product of two determinants, each equal to 0]

2. $\begin{vmatrix} a^2+b^2+c^2 & bc+ca+ab & bc+ca+ab \\ bc+ca+ab & a^2+b^2+c^2 & bc+ca+ab \\ bc+ca+ab & bc+ca+ab & a^2+b^2+c^2 \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

which is always non-negative.

3. $\begin{vmatrix} (b+x)(c+x) & (c+x)(a+x) & (a+x)(b+x) \\ (b+y)(c+y) & (c+y)(a+y) & (a+y)(b+y) \\ (b+z)(c+z) & (c+z)(a+z) & (a+z)(b+z) \end{vmatrix}$

$$= \begin{vmatrix} bc+(b+c)x+x^2 & ac+(a+c)x+x^2 & ab+(a+b)x+x^2 \\ bc+(b+c)y+y^2 & ac+(a+c)y+y^2 & ab+(a+b)y+y^2 \\ bc+(b+c)z+z^2 & ac+(a+c)z+z^2 & ab+(a+b)z+z^2 \end{vmatrix}$$

$$= \begin{vmatrix} bc & b+c & 1 \\ ac & a+c & 1 \\ ab & a+b & 1 \end{vmatrix} \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (b-c)(c-a)(a-b)(y-z)(z-x)(x-y)$$

4. $\begin{vmatrix} 3 & a+b+c & a^3+b^3+c^3 \\ a+b+c & a^2+b^2+c^2 & a^4+b^4+c^4 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^5+b^5+c^5 \end{vmatrix}$

$$= \begin{vmatrix} 1+1+1 & a+b+c & a^3+b^3+c^3 \\ a+b+c & a^2+b^2+c^2 & a^4+b^4+c^4 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^5+b^5+c^5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= [(a-b)(b-c)(c-a)][(a+b+c)(a-b)(b-c)(c-a)]$$

$$= (a-b)^2(b-c)^2(c-a)^2(a+b+c)$$

Exercise 7.4

1. $f'(x) = \Delta(R_1', R_2, R_3) + \Delta(R_1, R_2', R_3) + \Delta(R_1, R_2, R_3')$

$\Delta(R_1', R_2, R_3) = 0$ and $\Delta(R_1, R_2', R_3) = 0$, as in both all the elements in R_3 are zero. Hence,

$$f'(0) = \Delta(R_1, R_2, R_3')_{x=0}$$

$$= \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ 2\cos 2x & 0 & 4x\cos(2x^2) \end{vmatrix}_{x=0}$$

$$= \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 2$$

Hence, $f'(0) = 2$.

2. Let $a(x) = a_1x^2 + a_2x + a_3$, $g(x) = b_1x^2 + b_2x + b_3$ and $h(x) = c_1x^2 + c_2x + c_3$. Then,

$$f'(x) = 2a_1x + a_2,\ g(x) = 2b_1x + b_2 \text{ and } h'(x) = 2c_1x + c_2$$

$$f''(x) = 2a_1,\ g''(x) = 2b_1,\ h''(x) = 2c_1$$

$$f'''(x) = g'''(x) = h'''(x) = 0$$

In order to prove that $\phi(x)$ is constant polynomial, it is sufficient to show that $\phi'(x) = 0$ for all x. Now,

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

$$\Rightarrow \phi'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x)\\ f'(x) & g'(x) & h'(x)\\ f''(x) & g''(x) & h''(x)\end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x)\\ f''(x) & g''(x) & h''(x)\\ f''(x) & g''(x) & h''(x)\end{vmatrix}$$

$$+\begin{vmatrix} f(x) & g(x) & h(x)\\ f'(x) & g'(x) & h'(x)\\ f'''(x) & g'''(x) & h'''(x)\end{vmatrix}$$

$$\Rightarrow \phi'(x)=0+0+\begin{vmatrix} f(x) & g(x) & h(x)\\ f'(x) & g'(x) & h'(x)\\ 0 & 0 & 0\end{vmatrix}$$

$\Rightarrow$ $\phi'(x)=0+0+0=0$ for all x

$\Rightarrow$ $\phi(x)=$ constant for all

Hence, $\phi(x)$ is a constant polynomial.

3. By partial fractions, we have

$$g(x)=\frac{f(a)}{(x-a)(a-b)(a-c)}+\frac{f(b)}{(b-a)(x-b)(b-c)}+\frac{f(c)}{(c-a)(c-b)(x-c)}$$

$$\Rightarrow g(x)=\frac{1}{(a-b)(b-c)(c-a)}\times\left[\frac{f(a)(c-b)}{(x-a)}+\frac{f(b)(a-c)}{(x-b)}+\frac{f(c)(b-a)}{(x-c)}\right]$$

$$\Rightarrow g(x)=\begin{vmatrix} 1 & a & f(a)/(x-a)\\ 1 & b & f(b)/(x-b)\\ 1 & c & f(c)/(x-c)\end{vmatrix}\div\begin{vmatrix} 1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$

$$\Rightarrow \frac{dg(x)}{dx}=\begin{vmatrix} 1 & a & -f(a)(x-a)^{-2}\\ 1 & b & -f(b)(x-b)^{-2}\\ 1 & c & -f(c)(x-c)^{-2}\end{vmatrix}\div\begin{vmatrix} 1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$

$$=\begin{vmatrix} 1 & a & -f(a)(x-a)^{-2}\\ 1 & b & -f(b)(x-b)^{-2}\\ 1 & c & -f(c)(x-c)^{-2}\end{vmatrix}\div\begin{vmatrix} a^2 & a & 1\\ b^2 & b & 1\\ c^2 & c & 1\end{vmatrix},$$

Exercise 7.5

1. $2x+3y+1=0,\ 3x+y-2=0 \Rightarrow x=1, y=-1$

If the equations are consistent, then

$a(1)+2(-1)-b=0 \Rightarrow a-b=2$

2. Given system $x-cy-bz=0,\ cx-y+az=0,\ bx+ay-z=0.$

If the system has non-trivial solution, then

$$\begin{vmatrix} 1 & -c & -b\\ c & -1 & a\\ b & a & -1\end{vmatrix}=0$$

$\Rightarrow$ $1-abc-abc-b^2-a^2-c^2=0$

$\Rightarrow$ $a^2+b^2+c^2+2abc=1$

3. Since the equations are consistant, therefore, $D=0$

$$\Rightarrow \begin{vmatrix} (a+1)^3 & (a+2)^3 & -(a+3)^3\\ (a+1) & (a+2) & -(a+3)\\ 1 & 1 & -1\end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} (a+1)^3 & (a+2)^3 & (a+3)^3\\ (a+1) & (a+2) & (a+3)\\ 1 & 1 & 1\end{vmatrix}=0$$

$\Rightarrow$ $[(a+1)-(a+2)][(a+2)-(a+3)][(a+3)-(a+1)]$
$[a+1+a+2+a+3]=0$

$\Rightarrow$ $a=-2$

4. Here we use Cramer's rule to solve this system.

$$\Delta=\begin{vmatrix} a & b & c\\ a^2 & b^2 & c^2\\ a^3 & b^3 & c^3\end{vmatrix}$$

$$=abc\begin{vmatrix} 1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}$$

$$=abc(a-b)(b-c)(c-a)$$

Since $\Delta\neq 0 \Rightarrow$ a unique solution, therefore, if a, b, c are distinct and non-zero, then the system has a unique solution. Now,

$$\Delta_x=\begin{vmatrix} d & b & c\\ d^2 & b^2 & c^2\\ d^3 & b^3 & c^3\end{vmatrix}=dbc(d-b)(b-c)(c-d)$$

$$\therefore\ x=\frac{\Delta_x}{\Delta}=\frac{d(d-b)(c-d)}{a(a-b)(c-a)}$$

By symmetry, we have

$$y=\frac{\Delta_y}{\Delta}=\frac{d(a-d)(d-c)}{b(a-b)(b-c)}$$

$$z=\frac{\Delta_z}{\Delta}=\frac{d(b-d)(d-a)}{c(b-c)(c-a)}$$

If only two of a, b, c are equal, then $\Delta=0$ but $\Delta_x, \Delta_y, \Delta_z$ are not all zero so system has no solution, but if $a=b=c$ or any of a, b, c is zero, $\Delta=\Delta_x=\Delta_y=\Delta_z=0$ so the system has infinite solutions.

Chapter 8

Exercise 8.1

1. Given,

$$2X-Y=\begin{bmatrix} 3 & -3 & 0\\ 3 & 3 & 2\end{bmatrix}$$

Multiplying both sides by 2, we have

$$4X-2Y=2\begin{bmatrix} 3 & -3 & 0\\ 3 & 3 & 2\end{bmatrix}$$

$$4X - 2Y = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix} \quad (1)$$

Also given,

$$X + 2Y = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix} \quad (2)$$

Adding Eqs. (1) and (2), we have

$$5X = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+4 & -6+1 & 0+5 \\ 6-1 & 6+4 & 4-4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix}$$

$$\therefore \quad X = \frac{1}{5}\begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Putting the value of X in Eq. (2), we have

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} + 2Y = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

$$\Rightarrow \quad 2Y = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4-2 & 1+1 & 5-1 \\ -1-1 & 4-2 & -4-0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 4 \\ -2 & 2 & -4 \end{bmatrix}$$

$$\therefore \quad y = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

Hence,

$$X = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$

2. We have,

$$A^2 = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}\begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix} = \begin{bmatrix} w^2 & 0 \\ 0 & w^2 \end{bmatrix}$$

and

$$A^3 = A^2 A = \begin{bmatrix} w^2 & 0 \\ 0 & w^2 \end{bmatrix}\begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix} = \begin{bmatrix} w^2 & 0 \\ 0 & w^2 \end{bmatrix} = I \quad [\because w^3 = 1]$$

Now, $A^{100} = A^{99} A = (A^3)^{33} A = I^{33} A = A$

3. We have,

$$AB = I_3$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & x+y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad x + y = 0$$

4. $A^2 = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2\,(1/2) & 1 \end{bmatrix}$

$$A^3 = A^2 A = \begin{pmatrix} 1 & 0 \\ 2\,(1/2) & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3\,(1/2) & 1 \end{pmatrix}$$

Continuing in this way, we get

$$A^{100} = \begin{pmatrix} 1 & 0 \\ 100\,(1/2) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 50 & 1 \end{pmatrix}$$

5. We have,

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Now,

$$A^T + A = I_2$$

$$\Rightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\cos\theta & 0 \\ 0 & 2\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 2\cos\theta = 1$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2}$$

$$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{3}$$

$$\Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3},\ n \in Z$$

6. $A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

$(\alpha I + \beta A)^2 = \alpha^2 I^2 + \beta^2 A^2 + 2\alpha\beta A$

($\because$ αI and βA are commutative)

$$= \alpha^2 I + \beta^2(-I) + 2\alpha\beta A$$

$$= (\alpha^2 - \beta^2)\, I + 2\alpha\beta A$$

$$= (\alpha^2 - \beta^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\alpha\beta\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2 - \beta^2 & 2\alpha\beta \\ -2\alpha\beta & \alpha^2 - \beta^2 \end{bmatrix} \quad (1)$$

Given $(\alpha I + \beta A)^2 = A^2$. Hence,

$$\begin{bmatrix} \alpha^2 - \beta^2 & 2\alpha\beta \\ -2\alpha\beta & \alpha^2 - \beta^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \alpha^2 - \beta^2 = 0 \text{ and } 2\alpha\beta = 1$$

i.e., $\alpha = \pm\beta,\ \alpha\beta = \frac{1}{2}$

$\Rightarrow \quad \pm\beta^2 = \frac{1}{2}$ or $\beta^2 = \frac{1}{2}, -\frac{1}{2}$

$\Rightarrow \quad \beta = \pm\frac{1}{\sqrt{2}}, \pm\frac{i}{\sqrt{2}}$

$\therefore \quad \alpha = \pm\frac{1}{\sqrt{2}}, \pm\frac{i}{\sqrt{2}}$

7. $A \to 3\times 3,\ B \to 3\times 2,\ C \to 3\times 1$

$AB \to 3\times 2 \Rightarrow (AB)^T = 2\times 3 \Rightarrow (AB)^T C$ is defined

$C^T \to 1\times 3, \Rightarrow C^T C \to 1\times 1$

Hence $C^T C(AB)^T$ is not defined. Now, $C^T AB$ is also defined.

$A^T \to 3\times 3,\ B^T \to 2\times 3$

$A^T A \to 3\times 3$

$BB^T \to 3\times 3$

$\Rightarrow \quad A^T ABB^T \to 3\times 3$

$\Rightarrow \quad A^T ABB^T C$ is defined

8. We have $A^2 = AA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ pr & p+qr & q+r^2 \end{bmatrix}$$

$$\therefore \quad A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ pr & p+qr & q+r^2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$$

$$= \begin{bmatrix} p & q & r \\ pr & p+qr & q+r^2 \\ pq+r^2p & pr+q^2+qr^2 & p+2qr+r^3 \end{bmatrix} \quad (1)$$

and

$pI + qA + rA^2$

$$= p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + q\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} + r\begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ pr & p+qr & q+r^2 \end{bmatrix}$$

$$= \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} 0 & q & 0 \\ 0 & 0 & q \\ pq & q^2 & qr \end{bmatrix} + \begin{bmatrix} 0 & 0 & r \\ pr & qr & r^2 \\ pr^2 & pr+qr^2 & qr+r^3 \end{bmatrix}$$

$$= \begin{bmatrix} p+0+0 & 0+q+0 & 0+0+r \\ 0+0+pr & p+0+qr & 0+q+r^2 \\ 0+pq+pr^2 & q^2+pr+qr^2 & p+2qr+r^3 \end{bmatrix} \quad (2)$$

Thus, from Eqs. (1) and (2), we get

$A^3 = pI + qA + rA^2$

9. $(A+B)^2 = A^2 + B^2$

$\Rightarrow \quad AB = BA$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix} = \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix}$$

$\Rightarrow \quad a - b = a + 2$

$2 = -a - 1$

$2a - b = b - 2$

$3 = -b + 1$

$\Rightarrow \quad b = -2$ and $a = -3$

Exercise 8.2

1. We have,

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 3 & 5 \end{bmatrix}$$

$$\therefore \quad A + A^T = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 5 \\ 6 & 10 & 7 \\ 5 & 7 & 10 \end{bmatrix}$$

and

$$A - A^T = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Let,

$$P = \frac{1}{2}(A + A^T) = \begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 5 & 7/2 \\ 5/2 & 7/2 & 5 \end{bmatrix}$$

and

$$Q = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & -1 & 1/2 \\ 1 & 0 & 1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix}$$

Thus, P is symmetric and Q is skew-symmetric. Also,

$$P + Q = \begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 5 & 7/2 \\ 5/2 & 7/2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1/2 \\ 1 & 0 & 1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix} = A$$

Thus, we have expressed A as the sum of a symmetric and a skew-symmetric matrix.

2. Let A be a symmetric matrix and $n \in N$. Then,

$$A^n = AAA \ldots A \text{ up to } n \text{ times}$$

$\Rightarrow$ $(A^n)^T = (AAA\ldots A \text{ up to } n \text{ times})^T$

$\Rightarrow$ $(A^n)^T = (A^T A^T A^T \ldots A^T \text{ up to } n \text{ times})$ [By reversal law]

$\Rightarrow$ $(A^n)^T = (A^T)^n = A^n$ $[\because A^T = A]$

Hence, A^n is also a symmetric matrix.

3. A and B are two orthogonal matrices. Hence,

$$A^TA = AA^T = I \text{ and } BB^T = B^TB = I$$

AB will be orthogonal if

$$\begin{aligned}(AB)^T(AB) &= (B^TA^T)(AB)\\ &= B^T(A^TA)B\\ &= B^T IB\\ &= B^TB\\ &= I\end{aligned}$$

Similarly, we can show that $(BA)^T(BA) = I$. Therefore, BA is also orthogonal.

4. $A - \lambda I = \begin{bmatrix}1 & 3\\ 2 & 2\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} = \begin{bmatrix}1-\lambda & 3\\ 2 & 2-\lambda\end{bmatrix}$

Since $A - \lambda I$ is singular, so

$$\det(A - \lambda I) = 0$$

$\Rightarrow$ $2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$

$\Rightarrow$ $\lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow$ $\lambda = 4, -1$

Exercise 8.3

1. Since $A(\text{adj } A) = |A|I$, hence

$$\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix} = |A|\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}|A| & 0\\ 0 & |A|\end{bmatrix}$$

$\therefore$ $|A| = 10$, i.e., $\det A = 10$

2. We have,

$$|A| = \cos^2\theta + \sin^2\theta = 1$$

and

$$\text{adj } A = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$\Rightarrow$ $A^{-1} = \dfrac{\text{adj } A}{|A|} = \begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

which is multiplicative inverse of A.

3. We have,

$$AB = \begin{bmatrix}5x & 0 & 0\\ 0 & 1 & 0\\ 0 & 10x-2 & 5x\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\Rightarrow$ $x = 1/5$

4. $BAB = A^{-1}$

$\Rightarrow$ $ABAB = I$

$\Rightarrow$ $(AB)^2 = I$

Now,

$$AB = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos 2\beta & \sin 2\beta\\ \sin 2\beta & -\cos 2\beta\end{bmatrix}$$

$$= \begin{bmatrix}\cos(\alpha+2\beta) & \sin(\alpha+2\beta)\\ \sin(\alpha+2\beta) & -\cos(\alpha+2\beta)\end{bmatrix}$$

and

$$(AB)^2 = \begin{bmatrix}\cos(\alpha+2\beta) & \sin(\alpha+2\beta)\\ \sin(\alpha+2\beta) & -\cos(\alpha+2\beta)\end{bmatrix}$$

$$\times \begin{bmatrix}\cos(\alpha+2\beta) & \sin(\alpha+2\beta)\\ \sin(\alpha+2\beta) & -\cos(\alpha+2\beta)\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2(\alpha+2\beta)+\sin^2(\alpha+2\beta) & 0\\ 0 & \cos^2(\alpha+2\beta)+\sin^2(\alpha+2\beta)\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= I$$

$$BA^4B = A^{-1}$$

$\Rightarrow$ $A^4B = B^{-1}A^{-1} = (AB)^{-1} = AB$

$\Rightarrow$ $A^4 = A$ (1)

$\Rightarrow$ $A^2 = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$

$$= \begin{bmatrix}\cos 2\alpha & -\sin 2\alpha\\ \sin 2\alpha & \cos 2\alpha\end{bmatrix}$$

and

$$A^4 = \begin{bmatrix}\cos 4\alpha & -\sin 4\alpha\\ \sin 4\alpha & \cos 4\alpha\end{bmatrix}$$

Hence, from Eq. (1),

$$\begin{bmatrix}\cos 4\alpha & -\sin 4\alpha\\ \sin 4\alpha & \cos 4\alpha\end{bmatrix} = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

$\Rightarrow$ $4\alpha = 2\pi + \alpha$

$\Rightarrow$ $\alpha = \dfrac{2\pi}{3}$

5. In order to use elementary row operation we may write

$A = IA$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A \text{ (Applying } R_2 \to R_2 - 2R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix} A \text{ (Applying } R_2 \to -\frac{1}{5} R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix} A \text{ (Applying } R_1 \to R_1 - 2R_2)$$

Thus,

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$$

6. Given that $CB = D$. Now,

$AX = B$

$\Rightarrow \quad CAX = CB = D$

$\Rightarrow \quad X = (CA)^{-1}D$

Now,

$$CA = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 10 \\ 7 & 7 & 13 \\ 4 & 3 & 8 \end{bmatrix}$$

and

$$(CA)^{-1} = \begin{bmatrix} -\frac{17}{5} & 2 & 1 \\ \frac{4}{5} & 0 & -1 \\ \frac{7}{5} & -1 & 0 \end{bmatrix}$$

$$\Rightarrow (CA)^{-1}D = \begin{bmatrix} -\frac{17}{5} & 2 & 1 \\ \frac{4}{5} & 0 & -1 \\ \frac{7}{5} & -1 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Chapter 9

Exercise 9.1

1. Required probability is $1 - P$(all letters in right envelope) $= 1 - 1/n!$

(As there are total number of $n!$ ways in which letters can take envelopes and just one way in which they have corresponding envelopes.)

2. Total number of ways is 36. Favourable cases are (1, 4), (2, 3), (3, 2), (4, 1), (1, 5), (2, 4), (3, 3), (4, 2), (5, 1). Total number of favourable cases is 9.

Hence, the required probability is $9/36 = 1/4$.

3. If both integers are even, then product is even. If both integers are odd, then product is odd. If one integer is odd and other is even, then product is even. Therefore, the required probability is 2/3.

4. The total number of ways in which 2 integers can be chosen from the given 20 integers is ${}^{20}C_2$.

The sum of the selected numbers is odd if exactly one of them is given and only one is odd. Therefore, the favourable number of outcomes is ${}^{10}C_1 \times {}^{10}C_1$.

Therefore, the required probability is

$$\frac{{}^{10}C_1 \times {}^{10}C_1}{{}^{20}C_2} = \frac{10}{19}$$

5. The required probability is

$$\frac{{}^3C_3 + {}^7C_3 + {}^4C_3}{{}^{14}C_3} = \frac{1 + 35 + 4}{14 \times 13 \times 2} = \frac{40}{14 \times 26} = \frac{10}{91}$$

6. The total number of ways is $n = 6^5$. A total of 12 in 5 throw can be obtained in following two ways:

(i) one blank and four 3's can be obtained in ${}^5C_1 = 5$ ways.

(ii) three 2's and two 3's can be obtained in ${}^5C_2 = 10$ ways.

Hence, the required probability is $15/6^5 = 5/2592$.

7. Total number of ways of arranging 11 letters is (11)! The number of selection of 4 letters to be placed between R and E from remaining 9 is ${}^9C_4$. These four letters can be permuted in 4! ways. Now permutation of R and E can be interchanged in 2! ways.

$$\underbrace{(R \bullet \bullet \bullet \bullet E)}_{\text{4 letters}} \underbrace{\bullet \bullet \bullet \bullet \bullet}_{\text{5 letters}} = 6\,!$$

Fig. A-9.1

$\therefore$ Hence, the number of favourable ways is ${}^9C_4 \times 6! \times 2!$

Therefore, the required probability is

$$\frac{11!}{{}^9C_4\, 6!\, 2!}$$

8. Total number of numbers formed by the digits 1, 2, 3, 4, 5 without repetition is $5! = 120$. Now we know that a number is divisible by 4 if the number placed at its last two digits is divisible by 4. So that last two digits can be 12, 24, 32 or 52, i.e., they can be filled in 4 ways. There are $3! = 6$ ways of filling the other three places.

Therefore, the favourable number of ways is $4 \times 6 = 24$.

Hence, the required probability is

$$P = \frac{\text{Favourable no. of cases}}{\text{Exhaustive no. of cases}} = \frac{24}{120} = \frac{1}{5}$$

9. In an 8-floor house, there are 7 floors above the ground floor.

Each person can leave the cabin at any of the seven floors, i.e., each person can leave the cabin in 7 ways. Thus, total number of ways into which 5 persons can leave the cabin is 7^5. Now number of the ways of leaving the cabin by 5 person each at different floor is $^7P_5$.

Hence, the required probability is $^7P_5/7^5$.

10. Let each of the two friends have n daughters. Then probability that all the tickets go to the daughters of A is

$$\frac{^nC_3}{^{2n}C_3} = \frac{n-2}{4(2n-1)} = \frac{1}{20}$$

$\Rightarrow \quad n = 3$

11. The number of ways of selecting 4 socks from 24 socks is $n(s) = {}^{24}C_4$.

The number of ways of selecting 4 socks from different pairs is

$n(E) = {}^{12}C_4 \times 2^4$

$$\Rightarrow \quad P(E) = \frac{^{12}C_4 \times 2^4}{^{24}C_4}$$

Hence, the probability of getting at least one pair is

$$1 - \frac{^{12}C_4 \times 2^4}{^{24}C_4}$$

12. Total number of ways is 7! Favourable number of ways is $7! - 2(6!)$. Hence, the probability is

$$\frac{7! - 2(6!)}{7!} = 1 - \frac{2}{7} = \frac{5}{7}$$

13. Five tickets out of 50 can be drawn in $^{50}C_5$ ways. Since $x_1 < x_2 < x_3 < x_4 < x_5$ and $x_3 = 30$, therefore, $x_1, x_2 < 30$, i.e., x_1 and x_2 should come from tickets numbered 1 and 29 and this may happen in $^{29}C_2$ ways. Remaining ways, i.e., $x_4, x_5 > 30$, should come from 20 tickets numbered 31 to 50 in $^{20}C_2$ ways.So, favourable number of cases is $^{29}C_2\, ^{20}C_2$. Hence, required probability is

$$\frac{^{29}C_2 \times {}^{20}C_2}{^{50}C_5}$$

14. We have to find the probability that out of 26 cards drawn at random from the pack of cards, 13 will be red and 13 black. The total number of different ways to draw 26 cards from 52 is $^{52}C_{26}$. The favourable ways are those in which there will be 13 cards drawn from 26 red cards and 13 from 26 black cards. 13 red cards may be drawn in $^{26}C_{13}$ different ways and 13 black cards also in $^{26}C_{13}$ different ways. Therefore, total number of favourable ways is equal to $^{26}C_{13}\, ^{26}C_{13}$. And consequently the required probability is

$$P = \frac{^{26}C_{13}\, ^{26}C_{13}}{^{52}C_{26}}$$

$$= \frac{\left(\frac{(26)!}{(13)!(13)!}\, \frac{(26)!}{(13)!(13)!}\right)}{\left(\frac{(52)!}{(26)!(26)!}\right)}$$

$$= \frac{\{(26)!\}^4}{\{(13)!\}^4\,(52)!}$$

Exercise 9.2

1. $P(C) = p;\ P(A) = 2p;\ P(B) = 2p$

$5p = 1 \Rightarrow p = 1/5$

$$P(B \text{ or } C) = P(B) + P(C) = \frac{2}{5} + \frac{1}{5} = \frac{3}{5}$$

2. $P(A^c) = 1 - P(A) \Rightarrow P(A) = 1 - P(A^c) = 1 - 2/3 = 1/3$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad 3/4 = 1/3 + p(B) - 1/4 \Rightarrow P(B) = 2/3$

$$p(A \cap B^c) = p(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$p((A^c \cap B) = P(B) - P(A \cap B) = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

3. We have,

$$P(\bar{A} \cap \bar{B}) = \frac{1}{3}$$

$$\Rightarrow \quad P(\overline{A \cup B}) = \frac{1}{3}$$

$$\Rightarrow \quad 1 - P(A \cup B) = \frac{1}{3}$$

$$\Rightarrow \quad P(A \cup B) = \frac{2}{3}$$

$$\Rightarrow \quad P(A) + P(B) - P(A \cap B) = \frac{2}{3}$$

$$\Rightarrow \quad p + 2p = \frac{1}{2} = \frac{2}{3}$$

$$\Rightarrow \quad p = \frac{7}{18}$$

4. $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

$$= \frac{2}{3} + \frac{1}{4} + \frac{1}{6} = \frac{13}{12}$$

which is greater than 1. Hence, the statement is wrong.

5. We have,

$1 - P(A' \cap B') = 0.6,\ P(A \cap B) = 0.3,$

Hence,

$P(A' \cup B') = P(A') + P(B') - P(A' \cap B')$

$\Rightarrow \quad 1 - P(A \cap B) = P(A') + P(B') - 0.4$

$\Rightarrow \quad P(A') + P(B') = 0.7 + 0.4 = 1.1$

6. Consider the following events:

A: A student is passed in Mathematics

B: A student is passed in Statistics

Then,

$$P(A) = \frac{70}{125},\ P(B) = \frac{55}{125},\ P(A \cap B) = \frac{30}{125}$$

Required probability is

$$P(A \cap \bar{B})P(\bar{A} \cap B) = P(A) + P(B) - 2P(A \cap B)$$

$$= \frac{70}{125} + \frac{55}{125} - \frac{60}{125} = \frac{65}{125} = \frac{13}{25}$$

7. Here,

$$P(R)\ \frac{10}{100} = 0.1,\ P(F) = \frac{5}{100} = 0.05$$

$$P(F \cap R) = \frac{3}{100} = 0.03$$

Therefore, the required probability is

$$P(R) + P(F) - 2P(F \cap R) = 0.1 + 0.05 - 2(0.03)$$
$$= 0.09$$

Exercise 9.3

1. $P(A) = \frac{2}{3}$ and $P(B) = \frac{1}{2}$

$$P(A)\ P(B) = \frac{3}{8} \times \frac{1}{2} = \frac{3}{16}$$

$$P(A \cap B) = \frac{2}{8} = \frac{1}{4} \neq P(A)\ P(B)$$

Hence, A and B are dependent.

2. Probability of first card to be a king, $P(A) = 4/52$ and probability of second to be a king, $P(B) = 3/51$. Hence, required probability is

$$P(A \cap B) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

3. The probability that the coin shows the head, $P(A) = 1/2$. The probability that the dice shows 6, $P(B) = 1/6$. Hence, required probability is

$$P(A \cap B) = P(A)\ P(B)$$
$$= \left(\frac{1}{2}\right)\left(\frac{1}{6}\right) = \frac{1}{12}$$

4. Here $P(A) = 0.4$ and $P(\bar{A}) = 0.6$. Probability that A does not happen at all is $(0.6)^3$. Thus, required probability is $1 - (0.6)^3 = 0.784$.

5. According to question,

$$^nC_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^{n-6} = {}^nC_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{n-8}$$

$$\Rightarrow\quad {}^nC_6\left(\frac{1}{2}\right)^n = {}^nC_8\left(\frac{1}{2}\right)^n$$

$$\Rightarrow\quad {}^nC_6 = {}^nC_8 = {}^nC_{n-8}$$
$$\Rightarrow\quad 6 = n - 8 \text{ or } n = 14$$

6. Let p be the chance of cutting a spade and q the chance of not cutting a spade from the pack of 52 cards. Then, $p = 13/52 = 1/4$ and $q = 1 - 1/4 = 3/4$

Now A will win a prize if he cuts spade at 1st, 4th, 7th, 10th turns, etc. Note that A will get a second chance if A, B, C all fail to cut spade once and then A cuts a spade at the 4th turn.

Similarly, he will cut a spade at the 7th turn when each of A, B, C fail to cut spade twice, etc.

Hence A's chance of winning the prize is

$$p + q^3p + q^6p + q^9p + \cdots = \frac{p}{1-q^3}$$
$$= \frac{1/4}{1-(3/4)^3}$$
$$= \frac{16}{37}$$

Similarly B's chance is

$$qp + q^4p + q^7p + \cdots = q(p + q^3p + q^6p) = 12/37$$

C's chance is $1 - (16/37) - (12/37) = 9/37$.

7. (i) Without replacement

P\{six balls drawn follow the pattern $WBWBWB$\}

$$= \frac{3}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{20}$$

P\{six balls drawn follow the pattern $BWBWBW$\}

$$= \frac{3}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{20}$$

Hence, the probability that the colours of the balls are alternate is the probability that either the pattern $WBWBWB$ or $BWBWBW$ is obtained, which is given by $1/20 + 1/20 = 1/10$ (this is without replacement).

(ii) With replacement:

Here, the probability of getting the pattern $WBWBWB$ is

$$\left(\frac{3}{6}\right)^6 = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$$\therefore\quad P\{\text{colours of the balls are alternate}\} = \frac{1}{64} + \frac{1}{64} = \frac{1}{32}$$

8. Let A and B be two given events. The odds against A are 5:2, therefore $P(A) = 2/7$. The odds in favour of B are 6:5, therefore $P(B) = 6/11$. The required probability is

$$1 - P(\bar{A})\ P(\bar{B}) = 1 - \left(1 - \frac{2}{7}\right)\left(1 - \frac{6}{11}\right)$$
$$= \frac{52}{77}$$

9. Let A denote the event that a sum of 5 occurs, B the event that a sum of 7 occurs and C the event that neither a sum of 5 nor a sum of 7 occurs. We have,

$$P(A) = \frac{4}{36} = \frac{1}{9},\ P(B) = \frac{6}{36} = \frac{1}{6} \text{ and } P(C) = \frac{26}{36} = \frac{13}{18}$$

Thus,

$$P(A \text{ occurs before } B) = \frac{1}{9} + \left(\frac{13}{18}\right) \times \frac{1}{9} + \left(\frac{13}{18}\right)^2 \frac{1}{9} + \cdots$$

$$= \frac{1/9}{1-13/18} = \frac{2}{5} \text{ [sum of an infinite G.P.]}$$

10. Let E_1 be the events that man will be selected and E_2 the events that woman will be selected. Then,

$$P(E_1) = \frac{1}{4}, \text{ so } P(\bar{E})_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(E_2) = \frac{1}{3}, \text{ so } P(\bar{E})_2 = \frac{2}{3}$$

Clearly E_1 and E_2 are independent events. So,

$$P(\bar{E}_1 \cap \bar{E}_2) = P(\bar{E}_1) \times P(\bar{E}_2) = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$$

11. Required probability $= P(WBWB) + P(BWBW)$

$$= \frac{5}{8} \times \frac{3}{7} \times \frac{4}{6} \times \frac{2}{5} + \frac{3}{8} \times \frac{5}{7} \times \frac{2}{6} \times \frac{4}{5} = \frac{1}{14} + \frac{1}{14} = \frac{1}{7}$$

12. $P(K) = \frac{7}{15} \therefore P(\bar{K}) = 1 - \frac{7}{15} = \frac{8}{15}$

$$P(H) = \frac{7}{10} \therefore P(\bar{H}) = 1 - \frac{7}{10} = \frac{3}{10}$$

Hence, required probability is

$$P(\bar{K} \cap \bar{H}) = P(\bar{K})\, P(\bar{H}) = \frac{8}{15} \times \frac{3}{10} = \frac{24}{150} = \frac{4}{25}$$

13. Probability for an incorrect digit is p. Hence, probability for a correct digit is $1 - p$. Hence, probability for 8 correct digit is $(1-p)^8$. Hence required probability is $1 - (1-p)^8$.

14. Let W denote the event of drawing a white ball at any draw and B that for a black ball. Then

$$P(W) = \frac{a}{a+b} \text{ and } P(B) = \frac{b}{a+b}$$

$P(A \text{ wins the game}) = P(W \text{ or } BBW \text{ or } BBBBW \text{ or } \ldots)$

$= P(W) + P(BBW) + P(BBBBW) + \cdots$

$= P(W) + P(B)\,P(B)\,P(W) + P(B)\,P(B)\,P(B)\,P(B)\,P(W) + \cdots$

$= P(W) + P(W)\,P(B)^2 + P(W)\,P(B)^4 + \cdots$

$$= \frac{P(W)}{1-P(B)^2} = \frac{a(a+b)}{a^2+2ab} = \frac{a+b}{a+2b}$$

Also,

$$P(B \text{ wins the game}) = 1 - \frac{a+b}{a+2b} = \frac{a}{a+2b}$$

According to the given condition,

$$\frac{a+b}{a+2b} = 3\frac{b}{a+2b} \Rightarrow a = 2b \Rightarrow a:b = 2:1$$

15. Required probability is

$$P(\bar{A}_1 \cap A_2 \cap A_3) + P(A_1 \cap \bar{A}_2 \cap A_3) = P(\bar{A}_1)P(A_2)P(A_3) + P(A_1)P(\bar{A}_2)P(A_3)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$$

16. Since 4 has appeared on the first, so we are required 4 or 5 or 6 on second dice, hence, required probability is 3/6 = 1/2.

17. $P(E) = \frac{4}{8} = \frac{1}{2}$

[$\because$ favourable cases are THH, HTH, HHT, HHH]

$P(F) = \frac{4}{8} = \frac{1}{2}$

[$\because$ favourable cases are HTH, HHT, HTT, HHH]

$$P(E \cap F) = \frac{3}{8}$$

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{3/8}{1/2} = \frac{3}{4}$$

18. Given that $p = 1/6$, $q = 5/6$. Therefore, required probability is

$$P(0) + P(1) + P(2) = {}^4C_0 \left(\frac{5}{6}\right)^4 + {}^4C_1 \left(\frac{5}{6}\right)^3 \frac{1}{6} + {}^4C_2 \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)^2$$

$$= \frac{425}{432} = 0.984$$

19. Let p be the probability that a student selected at random is a swimmer, which is given by $1 - 1/5 = 4/5$. The number of students selected is 5. Probability that exactly 4 students are swimmers is

$${}^5C_4\, p^4 q = 5\left(\frac{4}{5}\right)^4 \left(\frac{1}{5}\right) = \left(\frac{4}{5}\right)^4$$

20. Given that

$P(A) = 0.5$, $P(A \cap B) \le 0.3$

So

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow$ $P(B) = P(A \cup B) - P(A) + P(A \cap B) \le 1 + 0.3 - 0.5 = 0.8$

[$\because P(A \cup B) \le 1$]

Hence, $P(B) = 0.9$ is not possible.

Exercise 9.4

1. Let A be the event to get 7 or 8 from a pair of unbiased dice. Then,

$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$ (1)

$\therefore P(A) = 11/36$

Let B be the event to get 7 or 8 from 11 cards, numbered 2, 3, ..., 12. Then,

$P(B) = 2/11$

Let H and T denote the head and tail. Then,

$P(H) = P(T) = 1/2$

Also A, H are independent and likewise B and T are also independent. Hence, required probability is

$$P(H \cap A) + P(T \cap B) = P(H)\,P(A) + P(T)\,P(B)$$

$$= \frac{1}{2}\times\frac{11}{36}+\frac{1}{2}\times\frac{2}{11}=\frac{193}{792}$$

2. Let C be the event of the selected number being composite and E be the event of there being no remainder.

$$P(C) = \frac{n(C)}{n(S)} = \frac{15}{25} = \frac{3}{5},\ P(\bar{C}) = \frac{2}{5}$$

$$P(E/C) = \frac{4}{15},\ P(E/\bar{C}) = \frac{1}{10}$$

$$P(E) = P(C)\,P(E/C) + P(\bar{C})\,P(E/\bar{C}).$$

$$= \frac{3}{5}\times\frac{4}{15}+\frac{2}{5}\times\frac{1}{10}=\frac{4}{25}+\frac{1}{25}=\frac{1}{5}=0.2$$

3. Let A_1 and A_2 be the events that the specific home is left unlocked, and is left locked, respectively. Then $P(A_1) = 0.4$, $P(A_2) = 0.6$.

Let 'A' be the event that the real estate man get into the specific home. Then

$$P(A/A_1) = 1,\ P(A/A_2) = \frac{^7C_2}{^8C_3} = \frac{3}{8}$$

$$\Rightarrow\ P(A) = P(A_1)\,P(A/A_2) + P(A_2)\,P(A/A_2)$$

$$= (0.4)\,(1) + (0.6)\,(3/8) = \frac{4}{10}+\frac{18}{80}=\frac{5}{8}$$

4. Let A_1 be the event that missing card is spade and A_2 be event that missing card is non-spade. Then,

$$P(A_1) = \frac{1}{4},\ P(A_2) = \frac{3}{4}$$

Let 'A' be the event that 2 spade cards are drawn from the remaining cards. Then,

$$P\left(\frac{A}{A_2}\right) = \frac{^{13}C_2}{^{51}C_2} \text{ and } P\left(\frac{A}{A_2}\right) = \frac{^{13}C_2}{^{51}C_2}$$

$$\Rightarrow\ P(A) = P(A_1)\,P\left(\frac{A}{A_1}\right) + P(A_2)\,P\left(\frac{A}{A_2}\right)$$

$$= \frac{1}{4}\,\frac{^{12}C_2}{^{51}C_2}+\frac{3}{4}\,\frac{^{13}C_2}{^{51}C_2}$$

$$= \frac{1}{4\times{}^{51}C_2}\,[^{12}C_2 + 3\times{}^{13}C_2]$$

Now,

$$P\left(\frac{A_1}{A}\right) = \frac{P(A_1)P\left(\frac{A}{A_1}\right)}{P(A)}$$

$$= \frac{\frac{1}{4}\,\frac{^{12}C_2}{^{51}C_2}}{\frac{1}{4\times{}^{51}C_2}[^{12}C_2 + 3\times{}^{13}C_2]}$$

$$= \frac{11}{50}$$

5. Let M be the event that man is chosen and E be the event that chosen one is employed. From the concept of reduced sample space, we immediately get

$$P(M/E) = \frac{460}{600} = \frac{23}{30}$$

Also,

$$P(E) = \frac{600}{900} = \frac{2}{3}$$

$$P(E \cap M) = \frac{460}{900} = \frac{23}{45}$$

$$\Rightarrow\ P(E/M) = \frac{23/45}{2/3} = \frac{23}{30}$$

6. Let A_1, B_1, C_1 denote the events of selecting the coins A, B and C respectively and E be the event of obtaining 2 heads and one tail in 3 tossings. We have to find $P(A_1/E)$. Given that Probability of getting head with the coins A, B, and C are 1/2, 2/3 and 1/3, respectively.

Clearly selection of any of the coins has probability 1/3. Now, the probability of obtaining 2 heads and one tail if the coin A has been selected already is

$$P(E/A_1) = {}^3C_2\,(1/2)^2\,(1/2) = 3/8$$

$$P(E/A_2) = {}^3C_2\,(2/3)^2\,(1/3) = 4/9$$

$$P(E/A_3) = {}^3C_2\,(1/3)^2\,(2/3) = 2/9$$

Using Bayes's theorem,

$$P(A_1/E) = \frac{P(A_1)\,P(E/A_1)}{[P(A_1)P(E/A_1)+P(A_2)P(E/A_2)P(A_3)P(E/A_3)]}$$

$$= \left[\frac{1}{3}\times\frac{3}{8}\right]\Big/\left[\frac{1}{3}\times\frac{3}{8}+\frac{1}{3}\times\frac{4}{9}+\frac{1}{3}\times\frac{2}{9}\right]$$

$$= 9/25$$

7. A and B will agree in a certain statement if both speak truth or both tell a lie. We define following events

$E_1 = A$ and B both speak truth $\Rightarrow P(E_1) = xy$

$E_2 = A$ and B both tell a lie $\Rightarrow P(E_2) = (1-x)(1-y)$

$E = A$ and B agree in a certain statement

Clearly, $P(E/E_1) = 1$ and $P(E/E_2) = 1$.

Using Bayes's theorem, the required probability is

$$P(E_1/E) = \frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1) + P(E_2)P(E/E_2)}$$

$$= \frac{xy\,1}{xy\,1 + (1-x)(1-y)1}$$

$$= \frac{xy}{1 - x - y + 2xy}$$

8. Let A_i $(i = 1, 2, 3, 4)$ be the event that the urn contains 2, 3, 4 or 5 white balls and E the event that two white balls are drawn. Since the four events A_1, A_2, A_3, A_4 are equally likely, therefore $P(A_i) = 1/4$, $i = 1, 2, 3, 4$. Now, the probability that the urn contains 2 white balls and both have been drawn is

$$P(E/A_1) = \frac{^2C_2}{^5C_2} = \frac{1}{10}$$

Similarly,

$$P(E/A_2) = \frac{^3C_2}{^5C_2} = \frac{3}{10},\ P(E/A_3) = \frac{^4C_2}{^5C_2} = \frac{3}{5}$$

$$P(E/A_4) = \frac{^5C_2}{^5C_2} = 1$$

Hence, the required probability is

$$P(A_4/E) = \frac{P(A_4)P(E/A_4)}{\sum_{i=1}^{4} P(A_i)P(E/A_i)}$$

$$= \frac{\frac{1}{4}\times 1}{\frac{1}{4}\left(\frac{1}{10} + \frac{3}{10} + \frac{3}{5} + 1\right)} = \frac{1}{2}$$